Noncommutative compact manifolds constructed from quivers

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Abstract

The moduli spaces of θ-semistable representations of a finite quiver can be packaged together to form a noncommutative compact manifold.

If noncommutative affine schemes are geometric objects associated to affine associative $\mathbb{C}$-algebras, affine smooth noncommutative varieties ought to correspond to *quasi-free (or formally smooth) algebras* (having the lifting property for algebra morphisms modulo nilpotent ideals). Indeed, J. Cuntz and D. Quillen have shown that for an algebra to have a rich theory of differential forms allowing natural connections it must be quasi-free [1, Prop. 8.5].

M. Kontsevich and A. Rosenberg introduced *noncommutative spaces* generalizing the notion of stacks to the noncommutative case [5, §2]. It is hard to construct noncommutative compact manifolds in this framework, due to the scarcity of faithfully flat extensions for quasi-free algebras. An alternative was outlined by M. Kontsevich in [4] and made explicit in [5, §1] (see also [7] and [6]). Here, the geometric object corresponding to the quasi-free algebra A is the collection $(\mathsf{rep}_n A)_n$ where $\mathsf{rep}_n A$ is the affine GL_n-scheme of n-dimensional representations of A. As A is quasi-free each $\mathsf{rep}_n A$ is smooth and endowed with Kapronov's formal noncommutative structure [2]. Moreover, this collection has equivariant *sum-maps* $\mathsf{rep}_n A \times \mathsf{rep}_m A \longrightarrow \mathsf{rep}_{m+n} A$.

We define a *noncommutative compact manifold* to be a collection $(Y_n)_n$ of projective varieties such that Y_n is the quotient-scheme of a smooth GL_n-scheme X_n which is locally isomorphic to $\mathsf{rep}_n A_\alpha$ for a fixed set of quasi-free algebras A_α, is endowed with a formal noncommutative structure and there are equivariant sum-maps $X_m \times X_n \longrightarrow X_{m+n}$. In this note we will construct of a large class of examples.

An illustrative example : let $M_{\mathbb{P}_2}(n;0,n)$ be the moduli space of semi-stable vectorbundles of rank n over the projective plane $\mathbb{P}_2$ with Chern-numbers $c_1 = 0$ and $c_2 = n$, then the collection $(M_{\mathbb{P}_2}(n;0,n))_n$ is a noncommutative compact manifold. In general, let Q be a quiver on k vertices *without oriented cycles* and let $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{Z}^k$. For a finite dimensional representation N of Q with dimension vector $\alpha = (a_1, \dots, a_k)$ we denote $\theta(N) = \sum_i \theta_i a_i$ and $d(\alpha) = \sum_i a_i$. A representation M of Q is called *θ-semistable* if $\theta(M) = 0$ and $\theta(N) \geq 0$ for every subrepresentation N of M. A. King studied the moduli spaces $M_Q(\alpha, \theta)$ of θ-semistable representations of Q of dimension vector α and proved that these are projective varieties [3, Prop 4.3]. We will prove the following result.

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Theorem 1 *With notations as above, the collection of projective varieties*

$$(\bigsqcup_{d(\alpha)=n} M_Q(\alpha,\theta) \ )_n$$

is a noncommutative compact manifold.

The claim about moduli spaces of vectorbundles on $\mathbb{P}_2$ follows by considering the quiver $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet$ and $\theta = (-1,1)$.
Let C be a smooth projective curve of genus g and $M_C(n,0)$ the moduli space of semi-stable vectorbundles of rank n and degree 0 over C. We expect the collection $(M_C(n,0))_n$ to be a noncommutative compact manifold.

1 The setting.

Let Q be a *quiver* on a finite set $Q_v = \{v_1,\ldots,v_k\}$ of vertices having a finite set Q_a of arrows. We assume that Q has *no oriented cycles*.
The path algebra $\mathbb{C}Q$ has as underlying $\mathbb{C}$-vectorspace basis the set of all oriented paths in Q, including those of length zero which give idempotents corresponding to the vertices v_i. Multiplication in $\mathbb{C}Q$ is induced by (left) concatenation of paths. $\mathbb{C}Q$ is a finite dimensional quasi-free algebra.
Let $\alpha = (a_1,\ldots,a_k)$ be a *dimension vector* such that $d(\alpha) = n$. Let $\mathsf{rep}_Q(\alpha)$ be the affine space of α-dimensional representations of the quiver Q. That is,

$$\mathsf{rep}_Q(\alpha) = \bigoplus_{\underset{j}{\bullet} \xleftarrow{a} \underset{i}{\bullet}} M_{a_j \times a_i}(\mathbb{C})$$

$GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$ acts on this space via basechange in the vertexspaces. For $\theta = (\theta_1,\ldots,\theta_k) \in \mathbb{Z}^k$ we denote with $\mathsf{rep}^{ss}_Q(\alpha,\theta)$ the open (possibly empty) subvariety of θ-semistable representations in $\mathsf{rep}_Q(\alpha)$. Applying results of A. Schofield [8] there is an algorithm to determine the (α,θ) such that $\mathsf{rep}^{ss}_Q(\alpha,\theta) \neq \emptyset$. Consider the diagonal embedding of $GL(\alpha)$ in GL_n and the quotient morphism

$$X_n = \bigsqcup_{d(\alpha)=n} GL_n \times^{GL(\alpha)} rep^{ss}_Q(\alpha,\theta) \xrightarrow{\pi_n} Y_n = \bigsqcup_{d(\alpha)=n} M_Q(\alpha,\theta).$$

Clearly, X_n is a smooth GL_n-scheme and the direct sum of representations induces sum-maps $X_m \times X_n \longrightarrow X_{m+n}$ which are equivariant with respect to $GL_m \times GL_n \hookrightarrow GL_{m+n}$. Y_n is a projective variety by [3, Prop. 4.3] and its points correspond to isoclasses of n-dimensional representations of $\mathbb{C}Q$ which are direct sums of θ-stable representations by [3, Prop. 3.2]. Recall that a θ-semistable representation M is called *θ-stable* provided the only subrepresentations N with $\theta(N) = 0$ are M and 0.

2 Universal localizations.

We recall the notion of *universal localization* and refer to [9, Chp. 4] for full details. Let A be a $\mathbb{C}$-algebra and $\mathsf{projmod}\ A$ the category of finitely generated projective left A-modules. Let Σ be some class of maps in this category. In [9, Chp. 4] it is shown that there exists an algebra map $A \xrightarrow{j_\Sigma} A_\Sigma$ with the universal property that the maps $A_\Sigma \otimes_A \sigma$ have an inverse for all $\sigma \in \Sigma$. A_Σ is called the universal localization of A with respect to the set of maps Σ. In the special case when A is the path algebra $\mathbb{C}Q$ of a quiver on k vertices, we can identify the isomorphism classes in $\mathsf{projmod}\ \mathbb{C}Q$ with $\mathbb{N}^k$. To each vertex v_i corresponds an *indecomposable* projective left $\mathbb{C}Q$-ideal P_i having as $\mathbb{C}$-vectorspace basis all paths in Q *starting at* v_i. We can also determine the space of homomorphisms

$$Hom_{\mathbb{C}Q}(P_i, P_j) = \bigoplus_{\substack{p \\ \underset{i}{\bullet} \longleftarrow \underset{j}{\bullet}}} \mathbb{C}p$$

where p is an oriented path in Q starting at v_j and ending at v_i. Therefore, any A-module morphism σ between two projective left modules

$$P_{i_1} \oplus \ldots \oplus P_{i_u} \xrightarrow{\sigma} P_{j_1} \oplus \ldots \oplus P_{j_v}$$

can be represented by an $u \times v$ matrix M_σ whose (p,q)-entry m_{pq} is a linear combination of oriented paths in Q starting at v_{j_q} and ending at v_{i_p}.
Now, form an $v \times u$ matrix N_σ of free variables y_{pq} and consider the algebra $\mathbb{C}Q_\sigma$ which is the quotient of the free product $\mathbb{C}Q * \mathbb{C}\langle y_{11}, \ldots, y_{uv}\rangle$ modulo the ideal of relations determined by the matrix equations

$$M_\sigma.N_\sigma = \begin{bmatrix} v_{i_1} & & 0 \\ & \ddots & \\ 0 & & v_{i_u} \end{bmatrix} \qquad N_\sigma.M_\sigma = \begin{bmatrix} v_{j_1} & & 0 \\ & \ddots & \\ 0 & & v_{j_v} \end{bmatrix}.$$

Repeating this procedure for every $\sigma \in \Sigma$ we obtain the universal localization $\mathbb{C}Q_\Sigma$. Observe that if Σ is a finite set of maps, then the universal localization $\mathbb{C}Q_\Sigma$ is an affine algebra.
It is easy to see that $\mathbb{C}Q_\Sigma$ is quasi-free and that the representation space $\mathsf{rep}_n\ \mathbb{C}Q_\sigma$ is an open subscheme (but possibly empty) of $\mathsf{rep}_n\ \mathbb{C}Q$. Indeed, if $m = (m_a)_a \in \mathsf{rep}_Q(\alpha)$, then m determines a point in $\mathsf{rep}_n\ \mathbb{C}Q_\Sigma$ if and only if the matrices $M_\sigma(m)$ in which the arrows are all replaced by the matrices m_a are invertible for all $\sigma \in \Sigma$. In particular, this induces numerical conditions on the dimension vectors α such that $\mathsf{rep}_n \mathbb{C}Q_\Sigma \neq \emptyset$. Let $\alpha = (a_1, \ldots, a_k)$ be a dimension vector such that $\sum a_i = n$ then every $\sigma \in \Sigma$ say with

$$P_1^{\oplus e_1} \oplus \ldots \oplus P_k^{\oplus e_k} \xrightarrow{\sigma} P_1^{\oplus f_1} \oplus \ldots \oplus P_k^{\oplus f_k}$$

gives the numerical condition $e_1 a_1 + \ldots + e_k a_k = f_1 a_1 + \ldots + f_k a_k$.

3 Local structure.

Fix $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{Z}^k$ and let $\Sigma = \cup_{z \in \mathbb{N}_+} \Sigma_z$ where Σ_z is the set of all morphisms σ

$$P_{i_1}^{\oplus z\theta_{i_1}} \oplus \ldots \oplus P_{i_u}^{\oplus z\theta_{i_u}} \xrightarrow{\sigma} P_{j_1}^{\oplus -z\theta_{j_1}} \oplus \ldots \oplus P_{j_v}^{\oplus -z\theta_{j_v}}$$

where $\{i_1, \ldots, i_u\}$ (resp. $\{j_1, \ldots, j_v\}$) is the set of indices $1 \leq i \leq k$ such that $\theta_i > 0$ (resp. $\theta_i < 0$). Fix a dimension vector α with $\langle \theta, \alpha \rangle = 0$, then θ determines a character χ_θ on $GL(\alpha)$ defined by $\chi_\theta(g_1, \ldots, g_k) = \prod det(g_i)^{\theta_i}$. With notations as before, the function $d_\sigma(m) = det(M_\sigma(m))$ for $m \in \mathsf{rep}_Q(\alpha)$ is a *semi-invariant* of weight $z\chi_\theta$ in $\mathbb{C}[\mathsf{rep}_Q(\alpha)]$ if $\sigma \in \Sigma_z$.
The open subset $X_\sigma(\alpha) = \{m \in \mathsf{rep}_Q(\alpha) \mid d_\sigma(m) \neq 0\}$ consists of θ-semistable representations which are also n-dimensional representations of the universal

localization $\mathbb{C}Q_\sigma$. Under this correspondence θ-stable representations correspond to simple representations of $\mathbb{C}Q_\sigma$. If we denote

$$X_{\sigma,n} = \bigsqcup_{d(\alpha)=n} GL_n \times^{GL(\alpha)} X_\sigma(\alpha) \hookrightarrow X_n$$

then $X_{\sigma,n} = \mathsf{rep}_n \mathbb{C}Q_\sigma$ and the restriction of π_n to $X_{\sigma,n}$ is the GL_n-quotient map $\mathsf{rep}_n \mathbb{C}Q_\sigma \twoheadrightarrow \mathsf{fac}_n \mathbb{C}Q_\sigma$ which sends an n-dimensional representation to the isomorphism class of the semi-simple n-dimensional representation of $\mathbb{C}Q_\sigma$ given by the sum of the Jordan-Hölder components, see [7, 2.3]. As the semi-invariants d_σ for $\sigma \in \Sigma$ cover the moduli spaces $M_Q(\alpha,\theta)$ this proves the local isomorphism condition for the collection $(Y_n)_n$.
A point $y \in Y_n$ determines a unique closed orbit in X_n corresponding to a representation

$$M_y = M_1^{\oplus e_1} \oplus \ldots \oplus M_l^{\oplus e_l}$$

with the M_i θ-stable representations occurring in M_y with multiplicity e_i. The local structure of Y_n near y is completely determined by a *local quiver* Γ_y on l vertices which usually has loops and oriented cycles and a dimension vector $\beta_y = (e_1, \ldots, e_l)$. The quiver-data (Γ_y, β_y) is determined by the canonical A_∞-structure on $Ext^*_{\mathbb{C}Q}(M_y, M_y)$. As $\mathbb{C}Q$ is quasi-free, this ext-algebra has only components in degree zero (determining the vertices and the dimension vector β_y) and degree one (giving the arrows in Γ_y).
Using [9, Thm 4.7] and the correspondence between θ-stable representations and simples of universal localizations, the local structure is the one outlined in [7, 2.5]. In particular, it can be used to locate the singularities of the projective varieties Y_n.

4 Formal structure.

In [2] M. Kapranov computes the formal neighborhood of commutative manifolds embedded in noncommutative manifolds. Equip a $\mathbb{C}$-algebra R with the *commutator filtration* having as part of degree $-d$

$$F_{-d} = \sum_m \sum_{i_1+\ldots i_m=d} RR^{Lie}_{i_1} R \ldots RR^{Lie}_{i_m} R$$

where R^{Lie}_i is the subspace spanned by all expressions $[r_1,[r_2,[\ldots,[r_{i-1},r_i]\ldots]$ containing $i-1$ instances of Lie brackets. We require that for $R_{ab} = \frac{R}{F_{-1}}$ affine smooth, the algebras $\frac{R}{F_{-d}}$ have the lifting property modulo nilpotent algebras in the category of d-nilpotent algebras (that is, those such that $F_{-d}=0$). The micro-local structuresheaf with respect to the commutator filtration then defines a sheaf of noncommutative algebras on $\mathsf{spec} R_{ab}$, the *formal structure*. Kapranov shows that in the affine case there exists an essentially unique such structure. For arbitrary manifolds there is an obstruction to the existence of a formal structure and when it exists it is no longer unique. We refer to [2, 4.6] for an operadic interpretation of these obstructions.
We will write down the formal structure on the affine open subscheme $\mathsf{rep}_n \mathbb{C}Q_\Gamma$ of X_n where Γ is a finite subset of Σ. Functoriality of this construction then implies that one can glue these structures together to define a formal structure on X_n finishing the proof of theorem 1.
If A is an affine quasi-free algebra, the formal structure on $\mathsf{rep}_n A$ is given by the micro-structuresheaf for the commutator filtration on the affine algebra

$$\sqrt[n]{A} = A * M_n(\mathbb{C})^{M_n(\mathbb{C})} = \{p \in A * M_n(\mathbb{C}) \mid p.(1*m) = (1*m).p \ \forall m \in M_n(\mathbb{C})\}$$

This follows from the fact that $\sqrt[n]{A}$ is again quasi-free by the coproduct theorems, [9, §2]. Specialize to the case when $A = \sqrt[n]{\mathbb{C}Q_\Gamma}$. Consider the extended quiver $\hat{Q}(n)$ by adding one vertex v_0 and for every vertex v_i in Q we add n arrows from v_0 to v_i denoted $\{x_{i1}, \ldots, x_{in}\}$. Consider the morphism between projective left $\mathbb{C}\hat{Q}(n)$-modules

$$P_1 \oplus P_2 \oplus \ldots \oplus P_k \xrightarrow{\tau} \underbrace{P_0 \oplus \ldots \oplus P_0}_{n}$$

determined by the matrix

$$M_\tau = \begin{bmatrix} x_{11} & \ldots & \ldots & x_{1n} \\ \vdots & & & \vdots \\ x_{k1} & \ldots & \ldots & x_{kn} \end{bmatrix}.$$

Consider the universal localization $B = \mathbb{C}\hat{Q}(n)_{\Gamma \cup \{\tau\}}$. Then, $\sqrt[n]{\mathbb{C}Q_\Gamma} = v_0 B v_0$ the algebra of oriented loops based at v_0.

5 Odds and ends.

One can build a global combinatorial object from the universal localizations $\mathbb{C}Q_\Gamma$ with Γ a finite subset of Σ and gluings coming from unions of these sets. This example may be useful to modify the Kontsevich-Rosenberg proposal of noncommutative spaces to the quasi-free world.
Finally, allowing oriented cycles in the quiver Q one can repeat the foregoing verbatim and obtain a projective space bundle over the collection $(\mathsf{fac}_n \mathbb{C}Q)_n$.

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