# noncommutative geometry @$_\mathbf{n}$

Lieven Le Bruyn[*]
Universitaire Instelling Antwerpen
B-2610 Antwerp (Belgium)
lebruyn@wins.uia.ac.be

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1   smooth affine geometry @$_\mathbf{n}$.

(1.1) :   Let **cat** be a category of associative $\mathbb{C}$-algebras with unit. An algebra $A \in Ob(\textbf{cat})$ is said to be **cat**-*smooth* iff for every *test-object* (B, I) in **cat** (that is, $B \in Ob(\textbf{cat})$, $I \triangleleft B$ a nilpotent ideal such that $\frac{B}{I} \in Ob(\textbf{cat})$) and every $A \xrightarrow{\phi} \frac{B}{I} \in Mor(\textbf{cat})$, there exists a *lifting morphism* $A \xrightarrow{\tilde{\phi}} B \in Mor(\textbf{cat})$ making the diagram below commutative

$$
\begin{array}{ccc}
B & \longrightarrow & \dfrac{B}{I} \\
& \nwarrow_{\exists\tilde{\phi}} & \big\uparrow{\phi} \\
& & A
\end{array}
$$
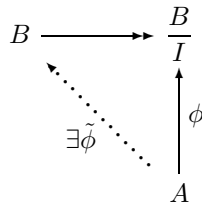
When **cat** = **commalg**, the category of all commutative $\mathbb{C}$-algebras, we recover Grothendieck's formulation of smooth (regular) commutative algebras. For this reason we call a **commalg**-smooth algebra *g-smooth*.

When **cat** = **alg**, the category of all $\mathbb{C}$-algebras, we recover Quillen's notion of *quasi-free* or *formally smooth* algebras. For this reason we call a **alg**-smooth algebra *q-smooth*.

Usually, we will assume that **cat**-smooth algebras are *affine* algebras in **cat**.

Note however, that a commutative q-smooth algebra need not be g-smooth. For example, consider the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$ and the 4-dimensional noncommutative local algebra

$$
B = \frac{\mathbb{C}\langle x, y\rangle}{(x^2, y^2, xy + yx)} = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy
$$

Consider the one-dimensional nilpotent ideal $I = \mathbb{C}(xy - yx)$ of B, then the 3-dimensional quotient $\frac{B}{I}$ is commutative and we have a morphism $\mathbb{C}[x_1, \ldots, x_d] \xrightarrow{\phi} \frac{B}{I}$ by $x_1 \mapsto x, x_2 \mapsto y$ and $x_i \mapsto 0$ for $i \geq 2$. This morphism

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[*]Research director of the FWO (Belgium)

admits no lift to B as for any potential lift $[\tilde{\phi}(x),\tilde{\phi}(y)] \neq 0$ in B. Therefore, $\mathbb{C}[x_1,\ldots,x_d]$ can only be q-smooth if $d = 1$.

(1.2) : In fact, W. Schelter proved in [15] that A is q-smooth if and only if the A-bimodule $\Omega_A^1 = Ker\ A \otimes A \xrightarrow{m_A} A$ is projective. If A is an affine commutative q-smooth algebra, A must be the coordinate ring of a finite set of points or of a smooth affine curve. Apart from semisimple algebras and some algebras which are finite modules over commutative q-smooth algebras, noncommutative q-smooth algebras are rather exotic objects such as free algebras and algebras arising from universal constructions (some of which we will encounter below). A fairly innocent class of q-smooth algebras are the *path algebras*. Let Q be a *quiver*, that is a directed graph on a finite set $Q_v = \{v_i,\ldots,v_k\}$ of vertices, having a finite set $Q_a = \{a_1,\ldots,a_l\}$ of arrows. The path algebra $\mathbb{C}\ Q$ has as $\mathbb{C}$-basis the oriented paths in Q (including those of length zero corresponding to the vertices v_i) and multiplication induced by concatenation, that is, $1 = v_1 + \ldots + v_k$ is a decomposition into orthogonal idempotents and we have

- $v_j.a$ is zero unless $\bullet \underset{v_j}{\xleftarrow{\ \ a\ \ }} \bullet,$

- $a.v_i$ is zero unless $\bullet \underset{v_i}{\xleftarrow{\ \ a\ \ }} \bullet,$

- $a_i.a_j$ is zero unless $\bullet \xleftarrow{\ a_i\ } \bullet \xleftarrow{\ a_j\ } \bullet.$

To prove that $\mathbb{C}\ Q$ is q-smooth, take a test-object (B,I) in **alg** and an algebra map $\mathbb{C}\ Q \xrightarrow{\phi} \frac{B}{I}$. The decomposition $1 = \phi(v_1) + \ldots + \phi(v_k)$ into orthogonal idempotents can be lifted modulo the nilpotent ideal to a decomposition $1 = \tilde{\phi}(v_1) + \ldots + \tilde{\phi}(v_k)$ into orthogonal idempotents. But then, taking for every arrow

$$\bullet \underset{v_j}{\xleftarrow{\ \ a\ \ }} \underset{v_i}{\bullet} \qquad \tilde{\phi}(a) \in \tilde{\phi}(v_j)(\phi(a)+I)\tilde{\phi}(v_i)$$

gives a required lift.

(1.3) : J. Cuntz and D. Quillen argue in [4, 5] that q-smooth algebras behave (for example with respect to deRham cohomology) as commutative affine smooth algebras. In [7, §9] M. Kontsevich gave a (somewhat cryptic) sketch how one might go about to develop an affine noncommutative geometry for q-smooth algebras. He suggests that one should *approximate* the noncommutative geometry of A *at level* n by the *representation space* $\mathsf{rep}_n\ A$ which is the affine scheme representing the functor

$$\mathsf{commalg} \xrightarrow{Hom_{\mathsf{alg}}(A,M_n(-))} \mathsf{sets}.$$

When A is q-smooth, it follows from the lifting property for q- and g-smooth algebras that $\mathsf{rep}_n\ A$ is a smooth affine variety. We will denote this *approximation at level* n by

$$\mathsf{space}\ A@_{\mathbf{n}} = \mathsf{rep}_n\ A.$$

Recently, Kontsevich and A. Rosenberg [8] made this proposal more explicit. They argue that in order to extend a structure struct in commutative geometry to the noncommutative affine smooth variety $\mathsf{spec}\ A$ we must be able to define it at every level

$$\mathsf{struct}(\mathsf{space}\ A) \Rightarrow \forall n\ :\ \mathsf{struct}@_{\mathbf{n}} = \mathsf{struct}(\mathsf{rep}_n\ A)$$

That is, a noncommutative structure of some kind on $\mathsf{spec}\ A$ should induce analogous commutative structures on all the representation spaces $\mathsf{rep}_n\ A$. These structures struct can either be

- *classical* , that is, ordinary commutative gadgets such as functions, vector fields and so on (see [8, §1.3.2] for more examples), or

- *non-classical*, that is, new structures on commutative schemes coming from noncommutative algebra such as the *formal structures* of Kapranov [6] to be defined below.

(1.4) : In this talk i want to indicate how one can extend this approximation proposal from q-smooth algebras to arbitrary algebras and obtain in this way a rich $\mathsf{affine\ geometry}\ \mathbf{@_n}$. Then i will briefly indicate how one can construct global objects at level n and how one might build a noncommutative geometry from these approximate objects.

2   affine geometry $\mathbf{@_n}$.

(2.1) : First we will specify $\mathbf{alg@_n}$, that is, the algebras that are level n approximations of associative algebras. A *trace map* on an associative $\mathbb{C}$-algebra A is a $\mathbb{C}$-linear map $A \xrightarrow{\ tr\ } A$ such that for all $a, b \in A$ we have $tr(ab) = tr(ba)$, $tr(a)b = btr(a)$ and $tr(tr(a)b) = tr(a)tr(b)$. Algebras with trace are the objects of a category $\mathbf{alg^{tr}}$ with morphisms the trace preserving $\mathbb{C}$-algebra maps. The forgetful functor $\mathbf{alg^{tr}} \longrightarrow \mathbf{alg}$ has a left adjoint

$$\mathbf{alg} \xrightarrow{\ \tau\ } \mathbf{alg^{tr}}$$

that is, given an algebra A we can construct an algebra A^τ with trace in a universal way by adding formally the traces.

Fix a number n and express $\prod_{i=1}^n (t - \lambda_i)$ as a polynomial in t with coefficients polynomials in the Newton functions $\nu_k = \sum_{i=1}^n \lambda_i^k$. Replacing ν_k by $tr(x^k)$ we get a formal *Cayley-Hamilton polynomial of degree* $n : \chi_x^{(n)}(t)$. Let A be an algebra with trace tr, we say that A is a *Cayley-Hamilton algebra of degree* n if $tr(1) = n$ and $\chi_a^{(n)}(a) = 0$ in A for all $a \in A$. Cayley-Hamilton algebras of degree n are the objects of a category $\mathbf{alg_n^{tr}}$ with trace preserving algebra maps as morphisms. There is a natural functor

$$\mathbf{alg^{tr}} \xrightarrow{\ n\ } \mathbf{alg_n^{tr}}$$

by sending an algebra A with trace tr to the quotient A_n by the twosided ideal generated by the elements $tr(1) - n$ and $\chi_a^{(n)}(a)$ for all $a \in A$. Starting with an arbitrary algebra A we propose to take $A\ \mathbf{@_n} = A_n^\tau$. That is,

Of course it may happen that $A\ \mathbf{@_n} = 0$ (for example if $A = A_m(\mathbb{C})$ the m-th Weyl algebra). For more details on algebras with trace and their properties (some of which we will recall below) we refer to the paper [14] of C. Procesi.

(2.2) : For A an arbitrary associative $\mathbb{C}$-algebra we define $\mathsf{space}\ A@_{\mathbf{n}}$ as before to be the representation space $\mathsf{rep}_n\ A$ which is the affine scheme representing the functor

$$\mathsf{commalg} \xrightarrow{\ Hom_{\mathsf{alg}}(A,M_n(-))\ } \mathsf{sets}$$

That is, there is a universal representation $A \xrightarrow{j_A} M_n(\mathbb{C}[\mathsf{rep}_n\ A])$ such that for every $\mathbb{C}$-algebra map $A \xrightarrow{\phi} M_n(B)$ with B a commutative algebra, we have a uniquely determined morphism $\mathbb{C}[\mathsf{rep}_n\ A] \xrightarrow{\psi} B$ such that the diagram below is commutative

$$
\begin{array}{ccc}
A & \xrightarrow{\quad\phi\quad} & M_n(B) \\
& {\scriptstyle j_A}\searrow \quad \nearrow {\scriptstyle M_n(\psi)} & \\
& M_n(\mathbb{C}[\mathsf{rep}_n\ A]) &
\end{array}
$$

GL_n acts by conjugation on $M_n(B)$ in a functorial way making $\mathsf{rep}_n\ A$ into an affine GL_n-scheme and so there are actions of GL_n by automorphisms on $\mathbb{C}[\mathsf{rep}_n\ A]$ and also on $M_n(\mathbb{C}[\mathsf{rep}_n\ A]) = \mathbb{C}[\mathsf{rep}_n\ A] \otimes M_n(\mathbb{C})$ (tensor product action). The image of A under j_A is contained in the ring of invariants $M_n(\mathbb{C}[\mathsf{rep}_n\ A])^{GL_n}$ which is the ring of GL_n-equivariant maps $\mathsf{rep}_n\ A \longrightarrow M_n(\mathbb{C})$ with algebra structure induced by the one on the target space. The main results on this GL_n-setting are due to Procesi [14] and assert that we have a commutative functorial diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\quad j_A\quad} & M_n(\mathbb{C}[\mathsf{rep}_n\ A]) \\
{\scriptstyle @_{\mathbf{n}}}\Big\downarrow & {\scriptstyle j_A}\searrow & \Big\uparrow \\
A@_{\mathbf{n}} & \cong & M_n(\mathbb{C}[\mathsf{rep}_n\ A])^{GL_n}
\end{array}
$$

and we recover the algebra with trace $A@_{\mathbf{n}} \in Ob(\mathsf{alg}_{\mathbf{n}}^{\mathsf{tr}})$ from the GL_n-affine scheme $\mathsf{space}\ A\ @_{\mathbf{n}} = \mathsf{rep}_n\ A$. Clearly, this scheme is in general not smooth nor even reduced. Procesi studied $\mathsf{alg}_{\mathbf{n}}^{\mathsf{tr}}$-smooth algebras in [14] which we will call p-smooth from now on. In fact, he proved that $A@_{\mathbf{n}}$ is p-smooth if and only if $\mathsf{rep}_n\ A$ is a smooth GL_n-scheme. In particular, we have

$$A\ \text{q-smooth} \quad \Rightarrow \forall n : \quad A@_{\mathbf{n}}\ \text{p-smooth}$$

consistent with our approximation strategy.

(2.3) : We now come to classical structures on $\mathsf{space}\ A\ @_{\mathbf{n}} = \mathsf{rep}_n\ A$. Let us consider functions as proposed in [8, §1.3.2]. Let $a \in A$ and $A \xrightarrow{x} M_n(\mathbb{C})$ a geometric point $x \in \mathsf{rep}_n\ A$, then we can take the ordinary trace $tr(x(a))$. This gives us a linear map

$$\frac{A}{[A,A]} \longrightarrow \mathbb{C}[\mathsf{rep}_n\ A]$$

Kontsevich and Rosenberg propose to take for $\mathsf{funct}\ A\ @_{\mathbf{n}}$ the subalgebra generated by the image of this map. Observe that these functions are GL_n-invariant and recall that Procesi proved that the ring of invariant polynomial functions

$$\mathbb{C}[\mathsf{rep}_n\ A]^{GL_n} = tr\ A\ @_{\mathbf{n}}$$

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where tr is the trace map on $A\ @_{\mathbf{n}}$. That is, we have

$$\mathsf{funct}\ A\ @_{\mathbf{n}} = tr\ A\ @_{\mathbf{n}}.$$

This ring has the following representation theoretic interpretation. The GL_n-orbits in $\mathsf{rep}_n\ A$ correspond to isomorphism classes of n-dimensional representations of A and the *closed* orbits correspond to n-dimensional *semi-simple* representations. Invariant theory tells us that the closed orbits are parametrized by the maximal ideals of the ring of polynomial invariants. The inclusion $tr\ A\ @_{\mathbf{n}} \hookrightarrow \mathbb{C}[\mathsf{rep}_n\ A]$ induces a morphism of schemes

$$\mathsf{rep}_n\ A \xrightarrow{\ \pi\ } \mathsf{fac}_n\ A$$

and sends an n-dimensional representation of A to the isomorphism class of the semi-simple n-dimensional representation which is the direct sum of the Jordan-Hölder factors. Moreover, the algebra with trace $A\ @_{\mathbf{n}}$ is a finitely generated module over the subalgebra $tr\ A\ @_{\mathbf{n}} = \mathbb{C}[\mathsf{fac}_n\ A]$ and hence we can associate to $A\ @_{\mathbf{n}}$ its classical structure sheaf $\mathcal{O}_{A\ @_{\mathbf{n}}}$ which is a sheaf of non-commutative algebras over $\mathsf{fac}_n\ A$.

We propose that the classical structures on $\mathsf{space}\ A\ @_{\mathbf{n}}$ are given by GL_n-equivariant structures associated with the GL_n-invariant theoretic setting

$$\mathcal{O}_{A@_{\mathbf{n}}}$$
$$\vdots$$
$$\mathsf{rep}_n\ A \xrightarrow{\ \pi\ } \mathsf{fac}_n\ A$$

All the classical structures proposed in [8, §1.3.2] are of this general form.

(2.4) : Let us compute all of this in the case of path algebras. Consider the semisimple subalgebra $V = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{k}$ generated by the vertex-idempotents $\{v_1, \ldots, v_k\}$. Every n-dimensional representation of V is semi-simple and determined by the multiplicities by which the factors occur. That is, we have a decomposition

$$\mathsf{rep}_n\ V = \bigsqcup_{\sum a_i = n} GL_n/(GL_{a_1} \times \ldots \times GL_{a_k}) = \bigsqcup_{\alpha} \mathsf{rep}_\alpha\ V$$

into homogeneous spaces where α runs over the *dimension vectors* $\alpha = (a_1, \ldots, a_k)$ such that $\sum_i a_i = n$. The inclusion $V \hookrightarrow \mathbb{C}\ Q$ induces a map $\mathsf{rep}_n\ \mathbb{C}\ Q \xrightarrow{\ \psi\ } \mathsf{rep}_n\ V$ and we have the decomposition of $\mathsf{rep}_n\ \mathbb{C}\ Q$ into associated fiber bundles

$$\psi^{-1}(\mathsf{rep}_\alpha\ V) = GL_n \times^{GL(\alpha)} \mathsf{rep}_\alpha\ Q.$$

Here, $GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_k}$ embedded along the diagonal in GL_n and $\mathsf{rep}_\alpha\ Q$ is the affine space of α-dimensional representations of the quiver Q. That is,

$$\mathsf{rep}_\alpha\ Q = \bigoplus_{\underset{v_j}{\bullet} \xleftarrow{\ a\ } \underset{v_i}{\bullet}} M_{a_j \times a_i}(\mathbb{C})$$

and $GL(\alpha)$ acts on this space via base-change in the vertex-spaces. That is, $\mathsf{rep}_n \, \mathbb{C} \, Q$ is the disjoint union of smooth affine components depending on the dimension vectors $\alpha = (a_1, \ldots, a_k)$ such that $\sum a_i = n$. This decomposition also translates to the algebra at level n

$$\mathbb{C} \, Q \, @_\mathbf{n} = \bigoplus_\alpha T_\alpha^Q$$

where the trace map on the component T_α^Q is determined by $tr(v_i) = a_i$. For the functions at level n we have the decomposition

$$tr \, \mathbb{C} \, Q \, @_\mathbf{n} = \bigoplus_\alpha t_\alpha^Q \quad \text{where} \quad t_\alpha^Q = tr \, T_\alpha^Q$$

and is the ring of $GL(\alpha)$-invariants $\mathbb{C}[\mathsf{rep}_\alpha \, Q]^{GL(\alpha)}$. In [9] it was proved that this ring of invariants is generated by traces along oriented cycles in the quiver Q of length $\leq n^2$. That is, for fixed α replace any arrow $\underset{v_j}{\bullet} \xleftarrow{\ a\ } \underset{v_i}{\bullet}$ by the $a_j \times a_i$ matrix of coordinate functions on $\mathsf{rep}_\alpha \, Q$ corresponding to the arrow a. Then, multiplying these matrices along a cycle produces a square matrix whose trace is a $GL(\alpha)$-invariant.

In [9] we also gave a computational description of the algebra with trace T_α^Q. It has a block decomposition

$$T_\alpha^Q = \begin{bmatrix} T_{11} & \ldots & T_{1k} \\ \vdots & & \vdots \\ T_{k1} & \ldots & T_{kk} \end{bmatrix}$$

where T_{ij} is the t_α^Q-module spanned by the paths in the quiver Q starting at vertex v_j and ending in vertex v_i (again, the length of the required paths can be bounded by n^2).

Observe that t_α^Q is positively graded and e will denote the $\mathfrak{m}$-adic completion with respect to the graded maximal ideal $\mathfrak{m} = \oplus_{i \geq 1} t_\alpha^Q(i)$ by $\hat{t}_\alpha^Q$. Analogously we denote $\hat{T}_\alpha^Q = T_\alpha^Q \otimes \hat{t}_\alpha^Q$.

(2.5) : In order for the approximation strategy for a q-smooth algebra A to succeed we need to control the local structure of $A \, @_\mathbf{n}$ and $\mathsf{fac}_n \, A$. In fact a much more general result holds. Let A be an associative algebra and x a geometric point of $\mathsf{fac}_n \, A$. We say that $A \, @_\mathbf{n}$ is *locally smooth* in x provided $\mathsf{rep}_n \, A$ is smooth along the closed orbit determined by x. Observe that if A is q-smooth (or even p-smooth), then $A \, @_\mathbf{n}$ is locally smooth in *all* x.

The point x determines an n-dimensional semi-simple representation M_x of A, say

$$M_x = S_1^{\oplus m_1} \oplus \ldots \oplus S_k^{\oplus m_k}$$

where S_i is a simple A-representation of dimension d_i occurring in M with multiplicity m_i, that is $n = \sum_i d_i m_i$.

We associate a *local quiver* $Q(x)$ to the point x. It has k vertices v_i (the number of distinct simple components S_i of M_x) and the number of arrows from v_i to v_j is given by

$$\# \ \underset{v_j}{\bullet} \longleftarrow \underset{v_i}{\bullet} = \dim_\mathbb{C} Ext_A^1(S_i, S_j).$$

Further, we define a *local dimension vector* $\alpha(x) = (m_1, \ldots, m_k)$ determined by the multiplicities of the simple components in M_x.

Denote the completion of the stalk of the structure sheaf $\mathcal{O}_{\mathsf{fac}_n\ A}$ in x by $\hat{\mathcal{O}}_x$, then with notation as before we have an isomorphism of local algebras

$$\hat{\mathcal{O}}_x \cong \hat{t}_{\alpha(x)}^{Q(x)}.$$

This is an application of the *Luna slice theorem* in invariant theory, see [11]. Further, if $\hat{\mathcal{O}}_x^{A\ @_{\mathbf{n}}}$ denotes the algebra $A\ @_{\mathbf{n}} \otimes \hat{\mathcal{O}}_x$, then it is also proved in [11] that

$$\hat{\mathcal{O}}_x^{A\ @_{\mathbf{n}}} \underset{Morita}{\sim} \hat{T}_{\alpha(x)}^{Q(x)}$$

where $\underset{Morita}{\sim}$ stands for *Morita-equivalence*, that is, equivalence of the module categories. The precise form of the Morita equivalence is determined by the embedding of $GL(\alpha(x)) = Stab_{GL_n}(M_x) \hookrightarrow GL_n$.

Hence, if $A\ @_{\mathbf{n}}$ is locally smooth in x, the étale local structure of $\mathcal{O}_{A\ @_{\mathbf{n}}}$ and $\mathsf{fac}_n\ A$ near x is fully determined by the combinatorial data $Q(x)$ and $\alpha(x)$. A lot more can be said about these combinatorics. For example, $(Q(x), \alpha(x))$ determines the local quiver-data in neighbouring points, one can compute the dimensions of the strata in $\mathsf{fac}_n\ A$ consisting of points with the same local quiver-data and given the local dimension of $\mathsf{fac}_n\ A$ in x one can even classify all the possible quiver-data. For some of these we refer to [10] and [11].

In particular, this applies to a q-smooth algebra A in any point x of $\mathsf{fac}_n\ A$. That is, the the approximation $A\ @_{\mathbf{n}}$ of *any* q-smooth algebra A is locally (in the étale topology) isomorphic to the approximation of the subclass of path algebras providing a handle on the exotic class of q-smooth affine algebras[1]

(2.6) : Now, we turn to non-classical structures on $\mathsf{space}\ A\ @_{\mathbf{n}}$ such as Kapranov's *formal noncommutative structure*, see [6]. Let R be an associative $\mathbb{C}$-algebra, R^{Lie} its Lie structure and R_m^{Lie} the subspace spanned by the expressions $[r_1, [r_2, \ldots, [r_{m-1}, r_m]\ldots]$ containing $m-1$ instances of Lie brackets. The *commutator filtration* of R is the (increasing) filtration by ideals $(F^d\ R)_{d\in\mathbb{Z}}$ with $F^d\ R = R$ for $d \in \mathbb{N}$ and

$$F^{-d}\ R = \sum_m \sum_{i_1+\ldots+i_m=d} RR_{i_1}^{Lie}R\ldots RR_{i_m}^{Lie}R$$

The associated graded $gr_F\ R$ is a (negatively) graded commutative Poisson algebra with part of degree zero $R_{ab} = \frac{R}{[R,R]}$.

Denote with nil_d the category of associative $\mathbb{C}$-algebras R such that $F^{-d-1}R = 0$ (note that an algebra map is filtration preserving). Kapranov studied in [6] nil_d-smooth algebras which we will call k_d-smooth algebras from now on.

Kapranov proves [6, Thm 1.6.1] that any affine commutative smooth algebra C has a unique k_d-smooth *thickening* R with $R_{ab} \simeq C$ (up to isomorphisms identical on C). The approach of [6, §1] may be compared to that of M. Artin in [2].

For $R \in Ob(\mathsf{nil}_d)$, Kapranov introduces a sheaf $\mathcal{O}_R$ of noncommutative rings on the commutative scheme $X_{ab} = \mathsf{spec}\ R_{ab}$. Observing that the commutator filtration for such R is *Zariskian* as in [13] the approach of [6, §2] may be compared to that of F. Van Oystaeyen in [18, Chpt II].

If X is an affine smooth commutative variety and R_d the canonical k_d-smooth thickening of $\mathbb{C}[X]$, then the sheaf of noncommutative algebras on X

$$\mathcal{O}^f = \varprojlim \mathcal{O}_{R_d}$$

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[1]A. Rosenberg suggests it would be nice to obtain the quiver-data from the non-commutative space as defined in [8, §2]. I will look into this.

is Kapranov's noncommutative formal structure on X.

If A is q-smooth and affine, the representation space $\mathsf{rep}_n\, A$ is a smooth affine variety and hence is equipped with such a formal noncommutative structure sheaf $\mathcal{O}^f$. Part of the Kontsevich-Rosenberg proposal is that the level n approximation $\mathsf{spec}\, A\, @_\mathbf{n}$ should carry such a formal structure.

(2.7) : At first sight we face a serious problem as Kapranov's inductive construction of k_d-smooth thickenings of a commutative C only works when C is g-smooth (in that case one can define at each stage a *universal* central extension). Moreover, the construction of such a formal structure must be sufficiently functorial to suit our purposes. I do not know of such a structure for all commutative affine $\mathbb{C}$-algebras (or even only those having a GL_n-action).

Fortunately, we only need it for coordinate rings of representation spaces $\mathbb{C}[\mathsf{rep}_n\, A]$ and there we can apply some ringtheory of the early 70ties, in particular G. Bergman's coproduct theorems [3] (see also the lecture notes of A. Schofield [16, Chp. 2] for more details). The starting point is that for every associative algebra A the functor

$$\mathsf{alg} \xrightarrow{\;Hom_\mathsf{alg}(A,M_n(-))\;} \mathsf{sets}$$

is *representable* in alg. That is, there exists an associative $\mathbb{C}$-algebra $\sqrt[n]{A}$ such that there is a natural equivalence between the functors

$$Hom_\mathsf{alg}(A, M_n(-)) \underset{n.e.}{\sim} Hom_\mathsf{alg}(\sqrt[n]{A}, -).$$

In other words, for every associative $\mathbb{C}$-algebra B, there is a functorial one-to-one correspondence between the sets

$$\begin{cases} \text{algebra maps} & A \longrightarrow M_n(B) \\ \text{algebra maps} & \sqrt[n]{A} \longrightarrow B \end{cases}$$

To define $\sqrt[n]{A}$ consider the free algebra product $A * M_n(\mathbb{C})$ and consider the subalgebra

$$\sqrt[n]{A} = A * M_n(\mathbb{C})^{M_n(\mathbb{C})} = \{p \in A * M_n(\mathbb{C}) \mid p.(1 * m) = (1 * m).p \,\forall m \in M_n(\mathbb{C})\}$$

Before we can prove the universal property of $\sqrt[n]{A}$ we need to recall a property that $M_n(\mathbb{C})$ shares with any Azumaya algebra : if $M_n(\mathbb{C}) \xrightarrow{\phi} R$ is an algebra morphism and if $R^{M_n(\mathbb{C})} = \{r \in R \mid r.\phi(m) = \phi(m).r \,\forall m \in M_n(\mathbb{C})\}$, then we have $R \simeq M_n(\mathbb{C}) \otimes_\mathbb{C} R^{M_n(\mathbb{C})}$. In particular, if we apply this to $R = A * M_n(\mathbb{C})$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{\phi} A * M_n(\mathbb{C})$ where $\phi(m) = 1 * m$ we obtain that $M_n(\sqrt[n]{A}) = M_n(\mathbb{C}) \otimes_\mathbb{C} \sqrt[n]{A} = A * M_n(\mathbb{C})$.

Hence, if $\sqrt[n]{A} \xrightarrow{f} B$ is an algebra map we can consider the composition

$$A \xrightarrow{id_A * 1} A * M_n(\mathbb{C}) \simeq M_n(\sqrt[n]{A}) \xrightarrow{M_n(f)} M_n(B)$$

to obtain an algebra map $A \longrightarrow M_n(B)$. Conversely, consider an algebra map $A \xrightarrow{g} M_n(B)$ and the canonical map $M_n(\mathbb{C}) \xrightarrow{i} M_n(B)$ which centralizes B in $M_n(B)$. Then, by the universal property of free algebra products we have an algebra map $A * M_n(\mathbb{C}) \xrightarrow{g*i} M_n(B)$ and restricting to $\sqrt[n]{A}$ we see that this

maps factors

$$A * M_n(\mathbb{C}) \xrightarrow{g * i} M_n(B)$$

$$\sqrt[n]{A} \cdots\cdots\cdots\cdots\cdots\blacktriangleright B$$

and one verifies that these two operations are each others inverses. The algebra $\sqrt[n]{A}$ has other surprising properties. For example, no matter how bad A is, $\sqrt[n]{A}$ is a *domain* having $\mathbb{C}^*$ as its group of invertible elements by [16, Thm. 2.19]. Now, equip $\sqrt[n]{A}$ with the commutator filtration

$$\ldots \qquad\qquad -2 \qquad\qquad -1 \qquad\qquad 0 \qquad\qquad 1 \qquad\qquad \ldots$$

$$\ldots \quad \subset \quad F_{-2} \quad \subset \quad F_{-1} \quad \subset \quad \sqrt[n]{A} \quad = \quad \sqrt[n]{A} \quad = \quad \ldots$$

By the universal property of $\sqrt[n]{A}$ we see that $\frac{A}{F_{-d-1}} \in Ob(\mathsf{nil}_d)$ is the object representing the functor

$$\mathsf{nil}_d \xrightarrow{Hom_{\mathsf{alg}}(A, M_n(-))} \mathsf{sets}.$$

In particular, as $\mathsf{nil}_0 = \mathsf{commalg}$ we deduce that

$$gr_0 \ \sqrt[n]{A} = \frac{\sqrt[n]{A}}{F_{-1}} = \frac{\sqrt[n]{A}}{[\sqrt[n]{A}, \sqrt[n]{A}]} \simeq \mathbb{C}[\mathsf{rep}_n \ A]$$

as both algebras represent the same functor. The construction of the formal sheaf of noncommutative algebras is similar to that of [6]. For fixed d, the induced filtration on the quotient $\sqrt[n]{A}_d = \frac{\sqrt[n]{A}}{F_{-d-1}}$ is Zariskian (even discrete). Hence, taking the *saturation* of a multiplicatively closed subset of the associated graded $gr \ \sqrt[n]{A}$ is an Ore-set in $\sqrt[n]{A}_d$. Hence, we can construct a sheaf of algebras $\mathcal{O}_{\sqrt[n]{A}_d}$ on $\mathsf{rep}_n \ A$ as in [18]. The formal noncommutative structure sheaf on $\mathsf{rep}_n \ A$ is then the inverse limit

$$\mathcal{O}^f_{\sqrt[n]{A}} = \varprojlim \mathcal{O}_{\sqrt[n]{A}_d}$$

We now claim that the approximation at level n of $\mathsf{spec} \ A$ is given by the *vierspan*

$$\mathsf{spec} \ A \ \mathbf{@_n} \quad = \qquad\qquad
\begin{array}{ccc}
\mathcal{O}^f_{\sqrt[n]{A}} & & \mathcal{O}_{A\mathbf{@_n}} \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\mathsf{rep}_n \ A & \longrightarrow & \mathsf{fac}_n \ A
\end{array}$$

There remains to prove that the formal structure $\mathcal{O}^f_{\sqrt[n]{A}}$ defined above coincides with Kapranov's formal structure in case A is q-smooth. We have seen that

$$M_n(\sqrt[n]{A}) \simeq A * M_n(\mathbb{C})$$

hence if A is q-smooth, so is $\sqrt[n]{A}$ by the coproduct theorems, see for example [16, Thm. 2.20] for a strong version. But then clearly $\sqrt[n]{A}_d$ is k_d-smooth for all

d (or use [6, Prop. 1.4.6]). By the unicity of k_d-smooth thickenings [6, §1.6] it is then immediate that Kapranov's formal structure $\mathcal{O}^f$ on $\mathsf{rep}_n\ A$ coincides with our $\mathcal{O}^f_{\sqrt[n]{A}}$.

Note also the perhaps surprising fact that for $\mathsf{rep}_n\ A$ for A a q-smooth algebra one does not need the perturbative approach of Kapranov to describe a formal neighborhood of $\mathsf{rep}_n\ A$ into a noncommutative smooth space. By the above, $\mathsf{rep}_n\ A$ is a closed subvariety of the noncommutative smooth space $\mathsf{spec}\ \sqrt[n]{A}$. Our description also clarifies the importance of this formal noncommutative structure. In order to understand the (commutative) scheme structure of $\mathsf{rep}_n\ A$ one needs to consider also representation $A \longrightarrow M_n(F)$ where F is a finite dimensional commutative algebra (rather than restrict to $\mathbb{C}$). Similarly, if one wants to understand the noncommutative scheme structure of $\mathsf{rep}_n\ A$ one ought to consider representations $A \longrightarrow M_n(F)$ where F is a finite dimensional non-commutative algebra. If F is *basic* (that is, all simple F-representations are one-dimensional over $\mathbb{C}$), then $F \in Ob(\mathbf{nil}_d)$ for some d, and then this representation is controlled by the formal structure. In general, any finite dimensional noncommutative algebra F is Morita equivalent to a basic algebra, so certainly if we vary n all the formal structures $\mathcal{O}_{\sqrt[n]{A}}$ control representation $A \longrightarrow M_m(F)$. Expressed differently, one can view the formal structures $\mathcal{O}_{\sqrt[n]{A}}$ as sewing-machines to stitch the different $\mathsf{rep}_n\ A$ together.

(2.8) : I cannot resist the temptation to add an infinite family of formal structures on $\mathsf{space}\ A\ \mathbf{@_n}$. If at layer 1 the commutator filtration is defined using the Lie bracket $[r_1, r_2]$, one can similarly define at layer m the m-commutator filtration based on the expressions

$$S_{2m}(r_1, r_2, \ldots, r_{2m}) = \sum_{\sigma \in S_{2m}} (-1)^{sgn(\sigma)} r_{\sigma(1)} r_{\sigma(2)} \ldots r_{\sigma(2m)}.$$

The associated graded algebra with respect to the m-commutator filtration on $\sqrt[n]{A}$ is no longer commutative but is a *polynomial identity algebra* of degree m (that is, it basically lives at level $\mathbf{@_m}$). As these algebras are close to commutative algebras, they have plenty of Ore sets and again the Zariskian argument provides us with new formal structure sheaves.

3 geometry $\mathbf{@_n}$.

(3.1) : In this section we will briefly indicate how one can define global objects in the approximate geometry $\mathbf{@_n}$ and afterwards how one can put approximate objects together to define a noncommutative geometry. More details will have to await another occasion.

The strategy to define a **geometry $\mathbf{@_n}$** is simple : use GL_n-equivariant (commutative) geometry as inspiration and try to define all concepts in terms of **alg $\mathbf{@_n}$**. The latter is essential in order to define formal structures as we have seen above.

In view of our local combinatorial description of p-smooth algebras, it is clear that the natural topology of **affine geometry $\mathbf{@_n}$** is the étale topology. Therefore, in order to define global objects we choose for the approach via *algebraic spaces*[2] as developed by M. Artin [1]. Algebraic spaces are defined by étale equivalence relations, so we need

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[2]V. Hinich suggests it might be more natural to extend the notion of algebraic stack in order to maintain compatibility with the Kontsevich-Rosenberg proposal of noncommutative spaces in [8, §2]. I agree and will try to work this out.

- a product in **alg @ₙ**, and

- étale morphisms in **alg @ₙ**.

As a product, the tensor-product $\otimes_{\mathbb{C}}$ is not suitable as it takes us out of **alg @ₙ** (think of tensorproducts of matrixrings). However, free algebra products provide us with a suitable definition

$$A \underset{n}{\boxtimes} B \overset{def}{=} A * B \ \textbf{@ₙ}$$

In order to define étale morphisms in **alg @ₙ** we extend the notions of formally étale (resp. formally unramified, formally smooth) from **commalg** verbatim. That is, let A be an associative algebra, then a morphism $A \ \textbf{@ₙ} \overset{\phi}{\longrightarrow} B$ in **alg @ₙ** is said to be *formally étale* iff for every test-object (T, I) in **alg @ₙ**, we have a unique lift, where all the morphisms below are A **@ₙ**-algebra maps

If one replaces unicity by existence ϕ is said to be *formally smooth* and if unicity is replaced by the existence of at most one lift, ϕ is said to be *formally unramified*. These notions also have a geometric interpretation : consider an algebra map $A \overset{\phi}{\longrightarrow} B$, then $A \ \textbf{@ₙ} \overset{\phi \ \textbf{@ₙ}}{\longrightarrow} B \ \textbf{@ₙ}$ is étale if and only if the induced morphism $\mathsf{rep}_n \ B \longrightarrow \mathsf{rep}_n \ A$ is an étale morphism of commutative schemes. In order to show that this *étale topology* on $\mathsf{spec} \ A \ \textbf{@ₙ}$ is rich enough, let us give tow classes of étale maps. The first is classical, coming from commutative étale maps. Consider the diagram

where f is an étale morphism in **commalg**, then $id \otimes f$ is étale in **alg @ₙ**. The second class is more exotic, though it is the natural substitute for localizations, *universal localizations* in **alg @ₙ**. Let **projmod** A denote the category of finitely generated (left) modules over an associative algebra A and let Σ be some class of maps in this category (that is some A-module morphisms between certain projective modules). In [16, Chp. 4] it is shown that there exists an algebra map $A \overset{j_\Sigma}{\longrightarrow} A_\Sigma$ with the universal property that the maps $A_\Sigma \otimes_A \sigma$ have an inverse for all $\sigma \in \Sigma$. Using the above geometric characterization of étale maps it follows that the induced maps

$$A \ \textbf{@ₙ} \overset{j_\Sigma \ \textbf{@ₙ}}{\longrightarrow} A_\Sigma \ \textbf{@ₙ}$$

are étale in **alg @ₙ**. In particular, it may happen that a finite dimensional algebra has hugely infinite dimensional étale extensions in **alg @ₙ**. For example consider the $m + 3$-dimensional algebra

$$A = \begin{bmatrix} \mathbb{C} & V \\ 0 & \mathbb{C} \end{bmatrix}$$

where V is a $\mathbb{C}$-vectorspace of dimension $m + 1$. Then, there is a universal localization of A isomorphic to $M_2(\mathbb{C}\langle x_1, \ldots, x_m\rangle)$. We refer to the book [16] of A. Schofield for more details.

Using these étale maps and the product $\underset{n}{\boxtimes}$ one can define **algebraic spaces @n** as in [1].

(3.2) : It may be interesting to reconsider the notion of *orbit topos* introduced by R.W. Thomason in [17] in this setting. For example, one wonders how much of the orbit topos of $\mathsf{rep}_n A$ can be described using algebraic spaces in **alg @n**. In general it is not true that any GL_n-equivariant technique can be performed in **algebraic spaces @n**. A noteworthy exception is GL_n-equivariant desingularization. In [10] the following problem was handled. Let X be a smooth (commutative) surface and Δ a central simple algebra in **alg @n** over the function field $\mathbb{C}(X)$. Recall that Δ is determined by

- a divisor $D \hookrightarrow X$ and a list of its irreducible components C,

- the list of singular points p_j of D, and

- for each branch B_k of D at p_i a number $n_{i,k} \in \mathbb{Z}/n\mathbb{Z}$ such that $\sum_k n_{i,k} = 0$.

One might ask whether for each Δ there exists a smooth object in **algebraic spaces @n** having Δ as its noncommutative function algebra. An idea might be to start with a *maximal order* $\mathcal{A}$ in D over X, consider $\mathsf{rep}_n \mathcal{A}$ and construct its GL_n-equivariant desingularization. In [10, Chp. 6] it was shown that the above problem has a positive solution if and only if

$$\sum_{p_i \in C} \sum_{B_k \in C} n_{i,k} = 0 \quad \text{for all irreducible components } C \text{ of } D$$

In general, one can construct an object in **algebraic spaces @n** having D as function algebra and having only a finite number of points where it is not locally smooth. All of these singularities are locally (in the étale topology) Morita equivalent to that of the quantum plane $\mathbb{C}_q[x, y]$ with $xy = qyx$ for q an m-th root of unity where $m \mid n$.

Based on this example, our strategy to develop **geometry @n** is to use GL_n-equivariant theory as far as it can serve us, but then study the remaining cases (which will lead to interesting algebras) rather than solving the remaining problems by extending our algebraic framework.

(3.3) : Finally, how can one use **geometry @n** to develop a noncommutative geometry. Again the strategy is simple : formalize the setting $A \Rightarrow \forall n\, A\, \textbf{@n}$ and make a list of relations holding naturally among the $\mathsf{spec}\, A\, \textbf{@n}$. Then, define an object in **geometry** by a list of objects $X\, \textbf{@n}$ in **geometry @n** satisfying this list of relations. At this moment i do not have an elegant set of axioms. Let me conclude by proposing one axiom which should illustrate the principle.

If $n = \sum_i m_i$ and let V_i be an m_i-dimensional representation of A, then the direct sum $\oplus V_i$ is an n-dimensional representation of A. Hence there are morphisms

$$\underset{i}{\times}\, \mathsf{spec}\, A\, \textbf{@}_{\mathbf{m}_i} \longrightarrow \mathsf{spec}\, A\, \textbf{@n}$$

satisfying obvious compatibility relations.

To formalize this condition for objects $X = (X\, \textbf{@n})_n$ in **geometry**, let us stratify the geometric points of the underlying GL_n-algebraic space $X\, \textbf{@n}$ by

$$X\, \textbf{@n}(r) = \{x \in X\, \textbf{@n} \mid Stab_{GL_n}(x) \text{ has a maximal torus of dimension } r\}$$

Then, for each integral solution $n = \sum_i m_i$ one must have connecting morphisms

$$\underset{i}{\times} X\, @_{\mathbf{m}_i} \xrightarrow{c_{(m_i)}} X\, @_{\mathbf{n}}$$

with the additional condition that

$$X\, @_{\mathbf{n}}(r) \subset \underset{m_1+\ldots+m_r=n}{\cup} GL_n.Im(c_{(m_1,\ldots,m_r)}).$$

It is clear that a lot of additional work needs to be done.

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