Optimal Filtrations on Representations of Finite Dimensional Algebras

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We present a representation theoretic description of the non-empty strata in the Hesselink stratification of the nullcone of representations of quivers. We use this stratification to define optimal filtrations on representations of finite dimensional algebras. As an application we investigate the isomorphism problem for uniserial representations.

OPTIMAL FILTRATIONS ON REPRESENTATIONS OF FINITE DIMENSIONAL ALGEBRAS

LIEVEN LE BRUYN

ABSTRACT. We present a representation theoretic description of the non-empty strata in the Hesselink stratification of the nullcone of representations of quivers. We use this stratification to define optimal filtrations on representations of finite dimensional algebras. As an application we investigate the isomorphism problem for uniserial representations.

CONTENTS

1. OPTIMAL FILTRATIONS

1.1. Let A be a finite dimensional algebra over an algebraically closed field k. In this paper we want to parameterize isomorphism classes of finite dimensional A-modules having a specific Jordan-Hölder sequence. In particular, we want to relate the recent results due to K. Bongartz and B. Huisgen-Zimmermann [3, 4, 1] on uniserial modules to the Hesselink stratification of nullcones.
By Morita theory we may reduce to the case that A is a *basic algebra*. That is, all simple A-modules are one dimensional. In this case we can write A as the quotient of the path algebra of a quiver and relate finite dimensional A-modules to representations of this quiver.

1.2. A *quiver* Q is a directed graph on a finite set of vertices $\{v_1, \ldots, v_n\}$. Let a_{ij} be the number of directed arrows from v_i to v_j (or loops if $v_i = v_j$). The *Euler-form* of Q is the bilinear form

$$\chi : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$$

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determined by the matrix $\chi = (\chi_{ij})_{i,j} \in M_n(\mathbb{Z})$ with entries

$$\chi_{ij} = \delta_{ij} - a_{ij}$$

Clearly, χ encodes the structure of the directed graph Q.
A *representation* V of a quiver Q of *dimension vector* $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$ assigns to every arrow $v_i \xrightarrow{\phi} v_j$ in Q a matrix $V(\phi) \in M_{a_j \times a_i}(k)$. The set of all α-dimensional representations form an affine space,

$$Rep(Q, \alpha) = \bigoplus_{i,j=1}^{n} M_{a_j \times a_i}(k)^{\oplus a_{ij}}$$

where a_{ij} is the number of arrows in Q from v_i to v_j.
There is a natural action of the *basechange group* $GL(\alpha) = GL_{a_1} \times \ldots \times GL_{a_n}$ on $Rep(Q, \alpha)$, the orbits correspond to isomorphism classes of representations.
The *path algebra* $k\ Q$ of the quiver Q is a vectorspace with basis the oriented paths in Q of length ≥ 0 and the multiplication is induced by concatenation of paths.If p and q are paths in Q we denote by qp the path by concatenating q after p.
Representations of Q can be viewed as finite dimensional representations of $k\ Q$. In this way, representations form an Abelian category and one defines homomorphisms, extensions etc. in the obvious way. If $V \in Rep(Q, \alpha)$ and $W \in Rep(Q, \beta)$, then

$$\chi(\alpha, \beta) = dim_k\ Hom(V, W) - dim_k\ Ext^1(V, W)$$

1.3. We fix Q to be the quiver corresponding to the basic algebra A. That is, if $\{S_1, \ldots, S_n\}$ is the set of isoclasses of simple (one-dimensional) A-modules, then Q is a quiver on n vertices $\{v_1, \ldots, v_n\}$ such that

$$a_{ij} = dim_k Ext^1_A(S_i, S_j)$$

Alternatively, let $\{e_1, \ldots, e_n\}$ be a complete set of primitive idempotents f A, then the arrows from v_i to v_j in Q form a k-basis of the vectorspace

$$e_j J e_i / e_j J^2 e_i$$

where J is the Jacobson radical of A. Given a choice of primitive idempotents and bases of these vectorspaces we can identify A with $k\ Q/I$ where I is an *admissible* ideal of $k\ Q$, that is generated by linear combinations of paths of length ≥ 2 in the quiver Q. Such a choice of idempotents and bases is called a *coordinatization* of A. We will fix a coordinatization and identify from now on A with $k\ Q/I$ and the vertex v_i of Q with the primitive idempotent e_i of A.
To a finite dimensional (left) representation M of A we associate its dimension vector $\alpha = (a_1, \ldots, a_n)$ where

$$a_i = dim_k\ e_i.M.$$

The set of all α-dimensional representations of A form an affine algebraic variety $Rep(A,\alpha)$ which is a closed subvariety of $Rep(Q,\alpha)$. Recall that a representation $V \in Rep(Q,\alpha)$ is determined by matrices $V(\phi)$ assigned to every arrow ϕ in Q. Any element of $k\ Q_+$, that is the subspace generated by paths of length ≥ 1 in Q, can be evaluated using these matrices. The closed subvariety $Rep(A,\alpha)$ is then determined as these α-dimensional representations of Q such that every $f \in I$ evaluates to the zero matrix.
Clearly, $Rep(A,\alpha)$ is a $GL(\alpha)$-subvariety of $Rep(Q,\alpha)$ and orbits correspond to isomorphism classes of A-representations.

1.4. We claim that $Rep(A,\alpha)$ is a subvariety of the *nullcone* of $Rep(Q,\alpha)$ under the action of $GL(\alpha)$. By definition, this nullcone is the subvariety of $V \in Rep(Q,\alpha)$ such that the Zariski orbit closure $\overline{GL(\alpha).V}$ contains the zero representation.
Consider the one-dimensional simple A-representation

$$S_i = e_iA/e_iJ = e_ik\ Q/e_ik\ Q_+$$

then the zero representation of $Rep(Q,\alpha)$ is contained in $Rep(A,\alpha)$ and corresponds to the α-dimensional semi-simple A-module

$$S(\alpha) = S_1^{\oplus a_1} \oplus \ldots \oplus S_n^{\oplus a_n}$$

Let $M \in Rep(A,\alpha)$ and consider a Jordan-Hölder sequence

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_a = M$$

with M_i/M_{i-1} a simple (hence one dimensional) A-module. Therefore, $a = \sum_i a_i$ and $M^{ss} = \oplus_i M_i/M_{i-1}$ is the semi-simple A-module $S(\alpha)$.
Take a k-vectorspace description $M = \oplus_{i=1}^a km_i$ where m_i is a basis vector of M_i/M_{i-1}, then there is a one-parameter subgroup λ of GL_a with respect to this basis

$$\lambda(t) = \begin{bmatrix} t^d & & & \\ & t^{d-1} & & \\ & & \ddots & \\ & & & t \end{bmatrix}$$

lies in $GL(\alpha) \hookrightarrow GL_a$ and has the property that

$$lim_{t \longrightarrow 0} \lambda(t).M = M^{ss} = S(\alpha)$$

whence M is contained in the nullcone.
For this reason we first have to study the nullcone $Null(Q,\alpha)$ of $Rep(Q,\alpha)$ under the action of the base-change group $GL(\alpha)$.

1.5. By the Hilbert criterium (see for example [8, III.2.3]) we know that $V \in Rep(Q, \alpha)$ lies in the nullcone $Null(Q, \alpha)$ if and only if there is a one-parameter subgroup

$$\lambda : k^* \hookrightarrow GL(\alpha) = GL_{a_1} \times \ldots GL_{a_n} \hookrightarrow GL_a$$

such that

$$\lim_{t \longrightarrow 0} \lambda(t).V = 0 \text{ in } Rep(Q, \alpha)$$

Up to conjugation in $GL(\alpha)$, or equivalently replacing V by another point in its orbit, we may assume that λ lies in the maximal torus T_a of $GL(\alpha)$ and is of the form

$$\lambda(t) = \begin{bmatrix} t^{\lambda_1} & & \\ & \ddots & \\ & & t^{\lambda_a} \end{bmatrix} \hookrightarrow \begin{bmatrix} GL_{a_1} & & \\ & \ddots & \\ & & GL_{a_n} \end{bmatrix} \hookrightarrow GL_a$$

with the $\lambda_i \in \mathbb{Z}$.
Recall that the weights of T_a are isomorphic to $\mathbb{Z}^a$ having canonical generators π_i for $1 \leq i \leq a$. Decompose the interval

$$[1 \ldots a] = \cup_{v=1}^n I_v$$

into *vertex-intervals*

$$I_v = [\sum_{i=0}^{v-1} a_i + 1 \ldots \sum_{i=0}^{v} a_i.]$$

Then, we have the weight space decomposition

$$Rep(Q, \alpha) = \oplus_{\pi_{ij} = \pi_j - \pi_i} Rep(Q, \alpha)_{\pi_{ij}}$$

where π_{ij} occurs with a non-zero weightspace if and only if $i \in I_v$, $j \in I_{v'}$ and in the quiver Q there is an arrow $v \longrightarrow v'$.
Using this weight space decomposition we can write $V = \sum_{\pi_{ij}} V_{ij}$. The condition $\lim_{t \longrightarrow 0} \lambda(t).V = 0$ in $Rep(Q, \alpha)$ with λ determined by $(\lambda_1, \ldots, \lambda_a) \in \mathbb{Z}^a$ is then equivalent to the condition

$$\lambda_j - \lambda_i \geq 1 \text{ whenever } M_{ij} \neq 0.$$

1.6. Assume $V \in Null(Q, \alpha)$. Consider the set

$$E_V = \{(i, j) \mid V_{ij} \neq 0\}.$$

There exists a unique a-tuple $\mu_V = (\mu_1, \ldots, \mu_a) \in \mathbb{Q}^a$ such that $\mu_j - \mu_i \geq 1$ for all $(i, j) \in E_V$ and such that the *norm*

$$\mid \mu_V \mid = \mu_1^2 + \ldots \mu_a^2$$

is minimal.

There is a unique $\lambda_V = (\lambda_1, \ldots, \lambda_a) \in \mathbb{Z}^a$ satisfying

$$\lambda_V \in \mathbb{N}\mu_V \text{ and } gcd(\lambda_1, \ldots, \lambda_a) = 1.$$

The corresponding one-parameter subgroup $\lambda_V : k^* \hookrightarrow T_a$ is called the *best one-parameter subgroup for* V with respect to the maximal torus T_a.
We can repeat this procedure for any point $V' = g.V$ in the $GL(\alpha)$- orbit of V. Assume V' is such that $\mid \mu_{V'} \mid$ is minimal. We then say that

$$g^{-1}\lambda_{V'}g : k^* \hookrightarrow GL(\alpha)$$

is *an optimal one parameter subgroup for* V *in* $GL(\alpha)$. With $\Lambda(V)$ we denote the set of all optimal one-parameter subgroups for V in $GL(\alpha)$.
We recall from [9, Prop.4.3] and [6] that one-parameter subgroups of $GL(\alpha)$ correspond to filtrations. Let $\lambda : k^* \hookrightarrow GL(\alpha)$ be a one-parameter subgroup and take for any vertex v_i in Q the decomposition

$$\mathbb{C}^{a_i} = \oplus W_i^{(m)}$$

where $\lambda(t)$ acts on the weight space $W_i^{(m)}$ as multiplication by t^m. Consider the filtration

$$W_i^{(\geq m)} = \oplus_{m' \geq m} W_i^{(m')}$$

Let $W \in Rep(Q, \alpha)$, then under the action of λ the components of the maps

$$\phi(W)^{(m'm)} : W_i^{(m)} \xrightarrow{\phi(W)} W_j^{(m')}$$

are multiplied by $t^{m'-m}$. Therefore, $\underset{t \longrightarrow 0}{lim} \lambda(t).W$ exists if and only if $\phi(W)^{(m'm)} = 0$ for all $m' < m$. This in turn happens if and only if $W(\phi)$ induces a map $W_i^{(\geq m)} \longrightarrow W_j^{(\geq m)}$ for all m. That is, if and only if the subspaces $W_i^{(\geq m)}$ determine subrepresentations $W_{(m)}$ of W. Thus a one-parameter subgroup λ, for which $\underset{t \longrightarrow 0}{lim} \lambda(t).W$ exists, determines a filtration of W,

$$\ldots \supset W_{(m)} \supset W_{(m+1)} \supset \ldots$$

indexed by $\mathbb{Z}$ and such that $W_{(m)} = W$ for m small.

Definition 1.1. In particular, let λ_V be an optimal one-parameter subgroup for V in $GL(\alpha)$, then $\underset{t \longrightarrow 0}{lim} \lambda_V(t).V = 0$ and we obtain a decreasing filtration $V_{(m)}$ of V with associated graded representation the zero representation in $Rep(Q, \alpha)$. We call this an *optimal filtration* on V. If $V = M \in Rep(A, \alpha)$, then the subrepresentations $M_{(m)}$ are A-representations and we call the corresponding filtration an *optimal filtration* for the A-representation M.

1.7. Two representations $V, W \in Null(Q, \alpha)$ are said to belong to the same *blade* if and only if $\Lambda(V) = \Lambda(W)$. With $[V]$ we denote the blade determined by V.

The representations $V, W \in Null(Q,\alpha)$ are said to belong to the same *stratum* if and only if $\Lambda(V) = \Lambda(g.W)$ for some $g \in GL(\alpha)$. Thus, $GL(\alpha).[V]$ is the stratum determined by V.
Let $V \in Null(Q,\alpha)$, take $\lambda \in \Lambda(V)$ and define

$$S(V) = \oplus_{(\pi_{ij},\lambda)\geq 1} Rep(Q,\alpha)_{\pi_{ij}}$$

where $(\pi_{ij},\lambda) = \lambda_j - \lambda_i$. Then, $S(V)$ is a linear subspace of $Null(Q,\alpha)$ and by [2, Prop. 4.2] we have that the blade $[V]$ of V is a Zariski open subset of $S(V)$ and the stratum $GL(\alpha).[V]$ is a Zariski open subset of the irreducible variety $GL(\alpha).S(V)$.
Let $P(\lambda)$ be the parabolic subgroup of $GL(\alpha)$ associated with $\lambda \in \Lambda(V)$. Recall that $P(\lambda)$ consists of those $g \in GL(\alpha)$ such that $\lim_{t\longrightarrow 0} \lambda(t).g.\lambda(t)^{-1}$ exists.
This parabolic subgroup of $GL(\alpha)$ has unipotent radical $U(\lambda)$ consisting of those elements such that the above limit is equal to the unit element and has Levi-subgroup $L(\lambda)$ which is a product of GL_j's determined by the multiplicities of the λ_i. For more details we refer to [8, III.2.5].
The parabolic subgroup $P(\lambda)$ acts on $S(V)$ and hence on $GL(\alpha) \times S(V)$ by

$$p.(g,W) = (gp^{-1}, p.W)$$

Further, there is also an action of $P(\lambda)$ on $GL(\alpha) \times Rep(Q,\alpha)$ and the natural map

$$GL(\alpha) \times Rep(Q,\alpha) \twoheadrightarrow GL(\alpha)/P(\lambda) \times Null(Q,\alpha)$$

sending (g,W) to $(gP(\lambda), g.W)$ is seen to be a geometric quotient for this action. That is, points of $GL(\alpha)/P(\lambda) \times Null(Q,\alpha)$ classify the $P(\lambda)$-orbits in $GL(\alpha) \times Rep(Q,\alpha)$.
We will denote this quotient by $GL(\alpha) \times^{P(\lambda)} Rep(Q,\alpha)$ which is a vectorbundle over the flag variety $GL(\alpha)/P(\lambda)$ with fiber $Null(Q,\alpha)$.
With $GL(\alpha) \times^{P(\lambda)} S(V)$ we denote the image in this quotient of $GL(\alpha) \times S(V)$. One verifies that it is a (not necessarily trivial) vectorbundle over the flag variety $GL(\alpha)/P(\lambda)$ with typical fiber $S(V)$.
In particular, it is a smooth variety of dimension $dim\ GL(\alpha) - dim\ P(\lambda) + dim\ S(V)$ and we have natural morphisms

$$\begin{array}{ccc} GL(\alpha) \times^{P(\lambda)} [V] & \xrightarrow{\phi'} & GL(\alpha).[V] \\ \cap\downarrow & & \cap\downarrow \\ GL(\alpha) \times^{P(\lambda)} S(V) & \xrightarrow{\phi} & GL(\alpha).S(V). \end{array}$$

From [2, Th. 4.7] we recall that ϕ is birational and a resolution of singularities. Concluding, one obtains the following

Theorem 1.2 (Hesselink). *With notations as before we have*

1. *The stratum $GL(\alpha).[V]$ is a smooth irreducible subvariety of $Null(Q,\alpha)$*
2. *The Zariski closure of this stratum is equal to $GL(\alpha).S(V)$*
3. *The desingularization of this closure is a vectorbundle over the flag variety $GL(\alpha)/P(\lambda)$ of rank the dimension of $S(V)$.*
4. *There is a natural one-to-one correspondence between $GL(\alpha)$-orbits in the stratum $GL(\alpha).[V]$ and $P(\lambda)$-orbits in the blade $[V] \hookrightarrow S(V)$*

2. Combinatorics of strata

2.1. In order to apply the Hesselink stratification of the nullcone $Null(Q,\alpha)$ we need to describe the non-empty strata explicitly. In this section we give a representation theoretic solution of this problem.
First, we will compile a finite list $List(Q,\alpha)$ of a-tuples $\mu \in \mathbb{Q}^a$ which can determine a stratum St_μ. To each $\mu \in List(Q,\alpha)$ we will associate a directed quiver Q_μ, a dimension vector α_μ and a character θ_μ and prove that $St_\mu \neq \emptyset$ if and only if there are θ_μ-semistable representations in $Rep(Q_\mu,\alpha_\mu)$ as defined in [6]. Finally, we give a combinatorial solution to this existence problem using results of A. Schofield [11].
In order to follow the construction, it may be helpful to consider the special case of the m-loop quiver discussed in [10].

2.2. We will describe the list $List(Q,\alpha)$. Fix the maximal torus T_a of $GL(\alpha)$ and let Π be the set of weights of T_a having a nontrivial weight space in $Rep(Q,\alpha)$. Recall that

$$\Pi = \{\pi_{ij} = \pi_j - \pi_i \mid i \in I_v, j \in I_{v'} \text{ and } \exists v \xrightarrow{\phi} v' \in Q\}$$

A subset $R \subset \Pi$ is said to be *unstable* if there exists a *coweight* $\mu = (\mu_1,\ldots,\mu_a) \in \mathbb{Q}^a$ such that $\mu_j - \mu_i \geq 1$ for all $\pi_{ij} \in R$.
If R is unstable, there is a unique coweight $\mu(R) \in \mathbb{Q}^a$ with this property and such that the norm $\mid \mu(R) \mid$ is minimal.
We define the *saturation* R^{sat} of R to be the subset

$$R^{sat} = \{\pi_{ij} \in \Pi \mid (\pi_{ij},\mu(R)) \geq 1\}$$

and we call R a *saturated subset* of Π whenever $R = R^{sat}$.
If R is a saturated subset of Π we have a corresponding *saturated subspace* X_R of $Null(Q,\alpha)$ by taking

$$X_R = \oplus_{\pi_{ij}\in R} Rep(Q,\alpha)_{\pi_{ij}}$$

By [2, Prop. 5.5] one has a bijective correspondence between the $GL(\alpha)$-conjugacy classes of saturated subspaces of $Null(Q,\alpha)$ and the conjugacy classes of saturated subsets $R \hookrightarrow \Pi$ under the action of the Weyl-group

$$S_\alpha = S_{a_1} \times \ldots S_{a_n}$$

of $GL(\alpha)$.
This correspondence assigns to R the subspace X_R and to a saturated subspace the set of non-zero weights of its elements.
Hence, the number of Hesselink strata of $Null(Q,\alpha)$ is smaller than the number of conjugacy classes of saturated subsets $R \hookrightarrow \Pi$ under the Weyl group.
Clearly, a saturated subset $R \hookrightarrow \Pi$ is determined by its associated coweight $\mu(R)$ and we will now describe the possible occurring coweights following [2, 6.8].
Consider a coweight $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Q}^a$. Then, we can partition $[1 \ldots a]$ into a disjoint union of *segments* I determined by the properties that there exist rational numbers $p \leq q$ such that

- $\{\mu_i \mid i \in I\} = \{x \in p + \mathbb{Z} \mid p \leq x \leq q\}$
- $I = \{1 \leq i \leq a \mid \mu_i \in p + \mathbb{Z}, p - 1 \leq \mu_i \leq q + 1\}$

We define a coweight $\mu \in \mathbb{Q}^a$ to be *balanced* if and only if for every segment I of μ we have

$$\sum_{i \in I} \mu_i = 0$$

We call a coweight $\mu \in \mathbb{Q}^d$ *dominant* if and only if for every vertex v we have for all $i, j \in I_v$

$$i \leq j \Rightarrow \mu_i \leq \mu_j$$

Finally, for $\mu \in \mathbb{Q}^a$ we denote $R(\mu) = \{\pi_{ij} \in \Pi \mid \mu_j - \mu_i \geq 1\}$.

Proposition 2.1. *With notations as before.*

1. *Let R be a saturated subset of Π, then $\mu(R)$ is balanced.*
2. *If μ is balanced and for every balanced coweight μ' such that $R(\mu) \overset{\neq}{\hookrightarrow} R(\mu')$ we have $\mid \mu \mid < \mid \mu' \mid$, then $R(\mu)$ is a saturated subset of Π.*

Proof. (Compare with [2, 6.8]) (1) : Let I be a segment for μ and consider for $\epsilon \in \mathbb{Q}$,

$$\mu_\epsilon = \mu + (\delta_{1I}, \ldots, \delta_{aI})\epsilon$$

where $\delta_{iI} = 1$ if $i \in I$ and zero otherwise. By definition of a segment, there exists $\epsilon_0 > 0$ such that for all ϵ with absolute value $< \epsilon_0$ we have that

$$R(\mu) = R(\mu_\epsilon)$$

By minimality of $\mid \mu \mid$, it follows that $\mid \mu_\epsilon \mid \geq \mid \mu \mid$ whenever $\mid \epsilon \mid < \epsilon_0$. But,

$$\mid \mu_\epsilon \mid = \mid \mu \mid + 2\epsilon \sum_{i \in I} \mu_i + \epsilon^2 \# I$$

whence μ must be balanced. If not, we can take $\epsilon < 0$ and contradict minimality of μ.
Statement (2) follows from (1) and the definitions. □

Since we are interested in conjugacy classes under the Weyl group of saturated subsets of Π, we can restrict attention to dominant balanced coweights.
Denote with $List(Q,\alpha)$ the finite list of dominant balanced coweights satisfying the condition of the second part of the proposition.
Any $\mu \in List(Q,\alpha)$ determines a (conjugacy class) of a saturated subspace

$$S_\mu = \oplus_{\pi_{ij} \in R(\mu)} Rep(Q,\alpha)_{\pi_{ij}}$$

of $Null(Q,\alpha)$.
Remains the problem to determine which of these S_μ is the closure of a stratum St_μ in the Hesselink stratification of $Null(Q,\alpha)$.

2.3. Given $\mu \in List(Q,\alpha)$ we want to determine whether S_μ is of the form $S(V)$ for some representation $V \in Null(Q,\alpha)$ and if so we want to determine the Zariski-open blade $[V] \hookrightarrow S(V)$.
In order to achieve this we use some results of F. Kirwan [7, 12.18-12.26]. Let us fix $\mu \in List(Q,\alpha)$ and define

$$\mu_1(R) = \{\pi_{ij} \in \Pi \mid \mu_j - \mu_i = 1\}$$

and

$$T_\mu = \oplus_{\pi_{ij} \in \mu_1(R)} Rep(Q,\alpha)_{\pi_{ij}}$$

Then, there is a natural projection map with vectorspaces as fibers

$$S_\mu \xrightarrow{pr} T_\mu$$

Let λ be the uniquely determined one-parameter subgroup of T_a determined by μ, that is, $\lambda \in \mathbb{N}\mu \cap \mathbb{Z}^a$ with $gcd(\lambda_1,\ldots,\lambda_a) = 1$. The action of $GL(\alpha)$ on $Rep(Q,\alpha)$ induces actions of

- $P(\lambda)$ on S_μ
- $L(\lambda)$ on T_μ

There is a Zariski open (*but possibly empty*) subset T_μ^{ss} of representations $W \in T_\mu$ such that $\lambda \in \Lambda(W)$. Specializing [7, 12.24 & 12.26] to our setting we obtain

Proposition 2.2. *Let $\mu \in List(Q,\alpha)$.*
Then, $S_\mu = S(V)$ for some $V \in Null(Q,\alpha)$ if and only if $T_\mu^{ss} \neq \emptyset$.
Moreover, in that case we have

1. $[V] = \{W \in S_\mu \mid pr(N) \in T_\mu^{ss}\}$
2. *T_μ^{ss} is an $L(\lambda)$-stable subset of T_μ*
3. *The fibers of $[V] \xrightarrow{pr} T_\mu^{ss}$ are vectorspaces*
4. *$[V]$ is a $P(\lambda)$-stable subset of $S_\mu = S(V)$*

2.4. In order to verify the condition $T_\mu^{ss} \neq \emptyset$ we give an interpretation of the $L(\lambda)$-action on T_μ as a quiver situation.
Let $\mu \in List(Q,\alpha)$. Let $J_1,\ldots,J_u$ be the distinct segments of μ where

$$J_i = \{\underbrace{p_i,\ldots,p_i}_{\sum_{v=1}^n b_{tj0}^{(i)}},\underbrace{p_i+1,\ldots,p_i+1}_{\sum_{v=1}^n b_{tj1}^{(i)}},\ldots,\underbrace{p_i+k_i,\ldots,p_i+k_i}_{\sum_{v=1}^n b_{tjk_i}^{(i)}}\}$$

where $b_{tjk}^{(i)}$ is the number of entries $a \in I_v$ such that $\mu_a = p_i + k$.

Definition 2.3. A quiver Γ is said to be a *level quiver* if we can partition the set of vertices of Γ into disjoint subsets $S_1, S_2, \ldots, S_l$ such that the only arrows in Γ are from a vertex from S_i to one in S_{i+1} for all $1 \leq i \leq l-1$.

Consider for each segment J_i with $1 \leq i \leq u$ the level quiver Q_i on $n \times (k_i + 1)$ vertices $\{(v,j) \mid v \text{ a vertex in } Q \text{ and } 1 \leq j \leq k_i + 1\}$.
In Q_i there are as many arrows from (v,k) to $(v', k+1)$ as there are arrows from v to v' in Q.
For the level quiver Q_i we take the dimension vector $\alpha_i = (b_{tjk}^{(i)})_{tk}$.
The quiver Q_μ will be the disjoint union of the level quivers Q_i associated to the different segments J_i of μ where $1 \leq i \leq u$.
The dimension vector α_μ for Q_μ will be the vector obtained from the dimension vectors α_i for Q_i.

Theorem 2.4. *With notations as above we have identifications*

$$T_\mu = Rep(Q_\mu, \alpha_\mu)$$

$$L(\lambda) = GL(\alpha_\mu)$$

Moreover, the base-change action of $GL(\alpha_\mu)$ on $Rep(Q_\mu,\alpha_\mu)$ coincides under the identifications with the action of $L(\lambda)$ on T_μ.

Proof. This is a straightforward but rather tedious verification. Perhaps it is helpful to consider the special case of the m-loop quiver treated in [10]. □

2.5. Using this identification we will now give a representation theoretic interpretation of the condition $T_\mu^{ss} \neq \emptyset$.
We define the character

$$\chi_\mu : L(\lambda) = GL(\alpha_\mu) = \times_{i=1}^u \times_{v=1}^n \times_{j=0}^{k_i} GL_{b_{vj}^{(i)}} \longrightarrow k^*$$

determined by sending a tuple (with obvious notation)

$$(g_{vj}^{(i)})_{ivj} \longrightarrow \prod_{ivj} det(g_{vj}^{(i)})^{m_{vj}^{(i)}}$$

where the exponents are determined by

$$m_{vj}^{(i)} = d.(p_i + j)$$

where d is the least common multiple of all the numerators of the rational numbers p_i where i runs over all the segments of μ, that is, $1 \leq i \leq u$.
Let $G(\mu)$ be the kernel of this character and observe that the group of characters of $G(\mu) \cap T_a$ correspond to those $\chi \in \mathbb{Z}^a$ such that $(\chi, \mu) = \sum_z \chi_z \mu_z = 0$.
But then, [12, Prop 1] applied to our situation implies

Proposition 2.5. *T_μ^{ss} is the set of semi-stable points of T_μ with respect to the action of the group $G(\mu)$.*
That is, T_μ^{ss} is the open subset of T_μ consisting of points V such that there exists a χ_μ semi-invariant function $f : T_\mu \longrightarrow \mathbb{C}$ with $f(V) \neq 0$. This in turn means that for all $g \in L(\lambda)$ we have $g.f = \chi_\mu(g)^k f$ for some $k \in \mathbb{N}$.

Using the identifications of $T_\mu = Rep(Q_\mu, \alpha_\mu)$ and of $L(\lambda) = GL(\alpha_\mu)$ we will give a representation theoretic description of the set T_μ^{ss}.
Let Γ be a quiver on s vertices, then the Grothendieck group $K_0\, k\Gamma$ of the path algebra is $\mathbb{Z}^s$ and the isomorphism assigns to the class of a representation V of Γ its dimension vector.
If we have an additive function on the Grothendieck group

$$\theta : K_0\, k\Gamma \longrightarrow \mathbb{Z}$$

we can define, following A. King in [6], in analogy with the terminology of vectorbundles on projective varieties a representation $V \in Rep(\Gamma, \beta)$ to be

- *θ-semistable* if $\theta(V) = 0$ and for every subrepresentation $W \hookrightarrow V$ we have $\theta(W) \geq 0$.
- *θ-stable* if it is θ-semistable and the only subrepresentations $W \hookrightarrow V$ satisfying $\theta(W) = 0$ are 0 or N.

Returning to the case of interest, the character χ_μ describes the additive function on the Grothendieck group

$$\theta_\mu : K_0\, kQ_\mu \longrightarrow \mathbb{Z}$$

by sending a class of a representation of Q_μ with dimension vector $(a_{vj}^{(i)})$ to $\sum_{ivj} m_{vj}^{(i)} a_{vj}^{(i)}$.
Applying [6, Prop. 3.1] to our situation we obtain

Proposition 2.6. *T_μ^{ss} is the set of θ_μ-semistable representations in $Rep(Q_\mu, \alpha_\mu) = T_\mu$.*

Therefore, we need to find a method to determine when θ-semistable representations exist.

2.6. Let Γ be a quiver on s vertices and θ an additive function of the Grothendieck group. The subset of θ-semistable of $Rep(\Gamma, \beta)$ is Zariski open (*but possibly empty*). Hence it suffices to verify whether representations in general position of $Rep(\Gamma, \beta)$ are θ-semistable.

If $\beta, \gamma \in \mathbb{N}^s$ we denote, following A. Schofield in [11], $\gamma \hookrightarrow \beta$ if and only if representations in general position of $Rep(\Gamma, \beta)$ contain a subrepresentation of dimension vector γ.
A. Schofield proved in [11] an inductive procedure to verify this condition in terms of the Euler form χ of Γ.

Theorem 2.7 (Schofield).

$$\gamma \hookrightarrow \beta \text{ iff } \underset{\delta \hookrightarrow \gamma}{Max} - \chi(\delta, \beta - \gamma) = 0$$

This result is the final ingredient in our representation theoretic description of the non-empty Hesselink strata of $Null(Q, \alpha)$.

Theorem 2.8. *Let $\mu \in List(Q, \alpha)$. The saturated subspace of $Null(Q, \alpha)$*

$$S_\mu = \oplus_{\pi_{ij} \in R(\mu)} Rep(Q, \alpha)_{\pi_{ij}}$$

is the closure of a stratum in the Hesselink stratification of $Null(Q, \alpha)$ if and only if for the corresponding

- *level quiver Q_μ,*
- *dimension vector α_μ and*
- *additive function θ_μ*

the following condition is satisfied

$$\beta \hookrightarrow \alpha_\mu \Rightarrow \theta_\mu(\beta) \geq 0$$

Moreover, the condition $\beta \hookrightarrow \alpha_\mu$ can be verified in terms of the Euler form of Q_μ.
If this condition is satisfied, the stratum St_μ consists of all $W \in S_\mu$ such that under the canonical projection

$$S_\mu \xrightarrow{pr} T_\mu = Rep(Q_\mu, \alpha_\mu)$$

$pr(W)$ is a θ_μ-semistable representation.

2.7. We can also give a representation theoretic description of the action of the parabolic subgroup $P(\lambda) \hookrightarrow GL(\alpha)$ on the saturated subspace S_μ.
Consider the level quiver Q_μ and denote its vertices by $(v, j)^{(i)}$. Recall that v runs over the vertices of Q, i over the different segments of μ and j over the length of the i-th segment J_i. Further, J_i determines a rational number p_i.
Define the *extended quiver* $\tilde{Q}_\mu$ as follows. Whenever $(v_1, j_1)^{(i_1)}$ and $(v_2, j_2)^{(i_2)}$ are two vertices of Q_μ satisfying the condition

$$p_{i_2} + j_2 - p_{i_1} - j_1 \geq 1$$

then there are as many arrows in $\tilde{Q}_\mu$ from $(v_1, j_1)^{(i_1)}$ to $(v_2, j_2)^{(i_2)}$ as there are arrows in Q from v_1 to v_2.

Observe that Q_μ is a subquiver of $\tilde{Q}_\mu$. We have a natural inclusions

$$Rep(Q_\mu, \alpha_\mu) \hookrightarrow Rep(Q_\mu, \alpha_\mu) \hookrightarrow Rep(Q, \alpha)$$

where the last inclusion is obtained by adding the vertex spaces of Q_μ (or $\tilde{Q}_\mu$) to obtain the vertex spaces for Q. A similar procedure applies to the matrices corresponding to arrows. Under this inclusion we have

Proposition 2.9. *We can identify S_μ with $Rep(\tilde{Q}_\mu, \alpha_\mu)$. Moreover, the action of $P(\lambda) \hookrightarrow GL(\alpha)$ on S_μ defines an action of $P(\lambda)$ on $Rep(\tilde{Q}_\mu, \alpha_\mu)$ such that $L(\lambda)$ acts on the subspace $Rep(Q_\mu, \alpha_\mu)$ by base change.*

3. Uniserial representations

3.1. In this section we will apply the foregoing general results to the construction of moduli spaces for uniserial representations of the finite dimensional algebra A. First, we will show that uniserial representations in $Rep(Q, \alpha)$ can only belong to very special strata St_μ. We determine the structure of the quivers Q_μ and $\tilde{Q}_\mu$ and dimension vector α_μ. In this case, the description of the θ_μ-semistable representations presents no problem.
In fact, we show that $T_\mu^{ss}/L(\lambda)$ is in this case a product of projective spaces. Over the standard open affine sets we can reduce the action of $P(\lambda)$ to that of the unipotent group $G = k_+ \times \ldots \times k_+$ (where the number of terms depends on the multiplicity with which the top-component occurs) on a slice which is $Rep(\tilde{Q}_\mu^{sl}, \alpha_\mu)$ for a specific subquiver $\tilde{Q}_\mu^{sl}$ of $\tilde{Q}_\mu$. The orbits of this reduced action can be easily parameterized.
The results of Bongartz and Huisgen-Zimmermann in [1] can then be recovered using the equivariant embedding $Rep(A, \alpha) \hookrightarrow Rep(Q, \alpha)$.
The strategy of this classification extends to more general classes of representations. Let $\mu \in List(Q, \alpha)$ be such that none of its segments contains numbers appearing with multiplicity ≥ 2. Then, one can repeat almost verbatim the method below to classify A-representations having an optimal filtration corresponding to λ.

3.2. Fix a sequence $\mathbb{S} = (S(0), \ldots, S(l))$ of simple A-modules. Let α be the dimension vector $(a_1, \ldots, a_n)$ where a_i is the number of times the simple A-module S_i occurs among the $S(j)$. Observe that $l = a - 1$ where as before $a = \sum_i a_i$.
We say that an A-module $M \in Rep(A, \alpha)$ is *filtered of type* $\mathbb{S}$ if and only if M has a decreasing Jordan-Hölder filtration

$$M = M_0 \supset M_1 \supset \ldots \supset M_l \supset M_{l+1} = 0$$

with simple factors

$$M_i/M_{i+1} \simeq S(i) \quad \forall i$$

An A-module $M \in Rep(A,\alpha)$ is said to be *uniserial* if and only if there is a unique (up to isomorphism) Jordan-Hölder filtration on M with simple factors.

3.3. We have a $GL(\alpha)$-equivariant embedding

$$Rep(A,\alpha) \hookrightarrow Null(Q,\alpha) = \sqcup_\mu GL(\alpha).St_\mu$$

and we have determined which $\mu \in List(Q,\alpha)$ actually occur.
By construction, $\tilde{Q}_\mu$ is a directed quiver. It is easy to determine all possible Jordan-Hölder filtrations on a representation of directed quivers.
If W is a representation of dimension vector β of the directed quiver Γ, then its socle $soc(W)$ is the direct sum of the spaces at the sink-vertices of the subquiver of Γ on the support of β after removing the arrows for which the corresponding matrices of W are zero.
Any (decreasing) Jordan-Hölder sequence on W ends with one of the possible Jordan-Hölder filtrations on $soc(W)$. Then, we repeat this procedure on the representation $\overline{W} = W/soc(W)$.

Proposition 3.1. *If $M \in Rep(A,\alpha)$ is uniserial of type $\mathbb{S} = (S(0),\ldots,S(l))$, then up to A-module isomorphism M belongs to S_μ where*

$$\mu = \sigma.(q, q+1, q+2, \ldots, q+l).$$

Here, $q = -\frac{l}{2}$ and σ is the permutation of S_n such that $\sigma(k) \in I_v$ if $S(k) = S_v$ and making μ dominant for the action of S_α.

Proof. Up to A-isomorphism we may assume that $M \in St_\mu \hookrightarrow S_\mu = Rep(\tilde{Q}_\mu, \alpha_\mu)$. Clearly, if M is uniserial,so is the corresponding representation V if the directed quiver $\tilde{Q}_\mu$.
This means that at each step in the procedure to construct the Jordan-Hölder sequence on V the socle must be one-dimensional.
In particular, all components of the dimension vector α_μ must be equal to one or zero and for each segment J_i of μ and all $0 \leq j \leq k_i$ precisely one of the components of the vertices $(v,j)^{(i)}$ is equal to one, the others zero.
We claim that μ has only one segment. If not, there are vertices $v = (v_1,j_1)^{(i_1)}$ and $v' = (v_2,j_2)^{(i_2)}$ with

$$0 <\mid p_{i_2} + j_2 - p_{i_1} - p_1 \mid< 1$$

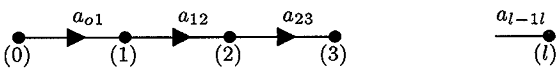
and so there are no arrows in $\tilde{Q}_\mu$ between v and v'. But then, at some stage in the construction of the Jordan-Hölder sequence the dimension of the relevant socle is ≥ 2, contradicting uniseriality.
Hence, writing the components of μ in increasing order we get

$$(q, q+1, \ldots, q+l).$$

As μ is balanced, its sum $(l+1)q + \frac{1}{2}l(l+1) = 0$ finishing the proof. □

3.4. The full subquiver of Q_μ on the support of α_μ is of the following form

$Q_\mu \mid supp(\alpha_\mu)$:



where the number a_{ii+1} of arrows from (i) to $(i+1)$ is determined as follows. In the foregoing proof we have seen that there is a unique vertex (i) among the vertices $(v,i)^{(1)}$ with dimension component equal to one. Let V_v be the set of all vertices (i) in $Q_\mu \mid supp(\alpha_\mu$ corresponding to the vertex v in Q. Observe that if $v = v_j$ then the number of elements of V_v is equal to a_j. If $(i) \in V_v$ and $(i+1) \in V_{v'}$, then a_{ii+1} is the number of arrows in Q from v to v'.
From now on we will write Q_μ for $Q_\mu \mid supp(\alpha_\mu)$ and $\alpha_\mu = (1,\ldots,1)$ for $\alpha_\mu \mid supp(\alpha_\mu)$.
If $l+1 = a$ is odd we have $\lambda = \mu = \theta_\mu = (-k,-k+1,\ldots,k)$ with $l = 2k$. If $l+1 = a$ is even, then $\lambda = \theta_\mu = (-l,-l+2,\ldots,l)$.
The Levi and parabolic subgroups of $GL(\alpha)$ are

- $L(\lambda) = T_a \hookrightarrow GL(\alpha)$,
- $P(\lambda) = B_{a_1} \times \ldots \times B_{a_n} \hookrightarrow GL(\alpha)$

where B_{a_i} is the Borel subgroup of lower triangular matrices in GL_{a_i}.
As $T_\mu = Rep(Q_\mu,\alpha_\mu)$ and $\alpha_\mu = (1,\ldots,1)$ it is easy to verify that the subvariety T_μ^{ss} of θ_μ-semistable representations consists of those representations of $Rep(Q_\mu,\alpha_\mu)$ having no subrepresentation of dimension vector

$$(1,\ldots,1,0,\ldots,0).$$

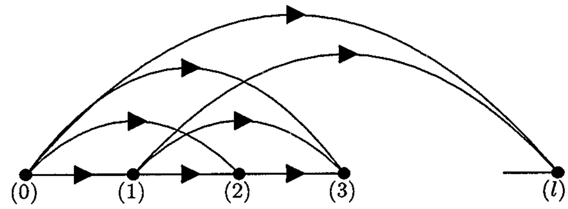
These representations are precisely those for which at least one of the a_{ii+1} maps from (i) to $(i+1)$ is non-zero for every $0 \le i < l$. Observe that these representations are then uniserial and under the canonical embedding

$$Rep(Q_\mu,\alpha_\mu) \hookrightarrow Rep(Q,\alpha)$$

correspond to uniserial Q-representations of type $\mathbb{S}$.
In particular, $St_\mu \neq \emptyset$ if and only if there is a path of length l in the quiver Q along the vertices $w_0, w_1, \ldots, w_l$ where $(i) \in V_{w_i}$.
Consequently, A can only have uniserial modules in $Rep(A,\alpha)$ of type $\mathbb{S}$ if at least one of these paths of length l does not belong to the defining ideal of A. Compare this non-vanishing path with the notion of a 'mast' in [3, 4, 1].

3.5. With conventions as for Q_μ, the extended quiver $\tilde{Q}_\mu$ is of the following form

$\tilde{Q}_\mu \mid supp(\alpha_\mu) \quad :$



where the number of arrows a_{ij} from (i) to (j) with $j > i$ is equal to the number of arrows in Q from v to v' if $(i) \in V_v$ and $(j) \in V_{v'}$.
As we have an explicit description of the open subset $T_\mu^{ss} \hookrightarrow T_\mu$, we can also describe the Hesselink stratum St_μ explicitly as $pr^{-1}(T_\mu^{ss})$ where

$$\begin{array}{ccc} St_\mu & \hookrightarrow & S_\mu = Rep(\tilde{Q}_\mu, \alpha_\mu) \\ \downarrow & & \downarrow pr \\ T_\mu^{ss} & \hookrightarrow & T_\mu = Rep(Q_\mu, \alpha_\mu) \end{array}$$

The fibers of pr over vectorspaces of dimension $\sum_{j>i+1} a_{ij}$. Clearly, all representations of St_μ are uniserial and correspond under the canonical inclusion

$$Rep(\tilde{Q}_\mu, \alpha_\mu) \hookrightarrow Rep(Q, \alpha)$$

to uniserial Q-representations of type $\mathbb{S}$.

3.6. We will now parameterize the isomorphism classes of uniserial representations of $Rep(Q, \alpha)$ of type $\mathbb{S}$. Combining the foregoing remarks with the general results on Hesselink strata, this problem amounts to parameterizing the $P(\lambda) = B(\alpha) = B_{a_1} \times \ldots \times B_{a_n}$-orbits in St_μ.
Recall that T_μ^{ss} is the set of representations in $Rep(Q_\mu, (1, \ldots, 1))$ such that for every $0 \leq i < l$ at least one of the a_{ii+1} numbers determined by the arrows from (i) to $(i+1)$ is non-zero. As $L(\lambda) = T_a$ acting via

$$(t_0, \ldots, t_l).((m_{01}, \ldots, m_{0a_{01}}), \ldots, (m_{l-11}, \ldots, m_{l-1a_{l-1l}})) =$$

$$(t_1^{-1}t_0(m_{01}, \ldots, m_{0a_{01}}), \ldots, t_l^{-1}t_{l-1}(m_{l-11}, \ldots, m_{l-1a_{l-1l}}))$$

we observe that there is a geometric quotient

$$T_\mu^{ss} \xrightarrow{b} T_\mu^{ss}/L(\lambda) \simeq \mathbb{P}^{a_{01}-1} \times \ldots \times \mathbb{P}^{a_{l-1l}-1}$$

We want to study the $P(\lambda) = B(\alpha)$-orbits in the open subvariety $U(i_1, \ldots, i_l)$ of $S_\mu = Rep(\tilde{Q}_\mu, (1, \ldots, 1))$ which is the inverse image under the quotient map b of the standard affine open subvariety

$$\mathbb{A}^{a_{01}-1}(i_1) \times \ldots \mathbb{A}^{a_{l-1l}-1}(i_l) \hookrightarrow \mathbb{P}^{a_{01}-1} \times \ldots \times \mathbb{P}^{a_{l-1l}-1}$$

that is, of points such that the i_j-th coordinate component of $\mathbb{P}^{a_j - 1_j - 1}$ is set equal to one.
Equivalently, $U(i_1, \ldots, i_l)$ can be identified with $Rep(\tilde{Q}^{\mathbf{i}}_{mu}, (1, \ldots, 1))$ where the quiver $\tilde{Q}^{\mathbf{i}}_{\mu}$ is the subquiver of $\tilde{Q}_\mu$ obtained by deleting these fixed arrows.
With this choice, the induced action of T_a on $U(i_1, \ldots, i_l)$ is trivial and the action of $B(\alpha)$ reduces to an action of the nilpotent group

$$U(\alpha) = U_{a_1} \times \ldots \times U_{a_n} \hookrightarrow GL(\alpha)$$

where U_z is the nilpotent radical of the Borel subgroup B_z of GL_z.

3.7. In order to simplify the action of $U(\alpha)$ on $U(i_1, \ldots, i_l)$ even further we have to describe the embedding

$$Rep(\tilde{Q}_\mu, (1, \ldots, 1)) \hookrightarrow Rep(Q, \alpha).$$

For every $0 \leq i \leq l$ there exists a unique vertex $v(i)$ of Q such that $(i) \in V_{v(i)}$. Let $j(i) = \#\{(k) \in V_{v(i)} \mid k \leq i\}$, then we will fix a basis of the vertex space of Q in $v(i)$ by taking as its $j(i)$-th basevector a fixed vector spanning the one-dimensional vertex space of $\tilde{Q}_\mu$ in (i).
Take a representation $W \in Rep(_\mu, (1, \ldots, 1))$ and consider an arrow $\gamma : (i) \longrightarrow (k)$ in $\tilde{Q}_\mu$. By definition γ determines a unique arrow $\phi : v \longrightarrow v'$ in Q where $(i) \in V_v$ and $(k) \in V_{v'}$. By our assumption on the bases of the vertex spaces of Q and $\tilde{Q}_\mu$ we have that

$$\gamma = \phi_{j(k),j(i)} \text{ where } \phi \in M_{\alpha_k \times \alpha_i}(k)$$

We will construct a slice of the form $Rep(\tilde{Q}^{sl}_\mu, (1, \ldots, 1))$ where $\tilde{Q}^{sl}_\mu$ is a subquiver of $\tilde{Q}^{\mathbf{i}}_\mu$ (and $\tilde{Q}_\mu$).
For every (i) with $i \geq 1$, let ϕ_i be the arrow in Q determined by the fixed arrow from $(i-1)$ to (i) (the value of which we gave set equal to 1). By the induced basechange action in $Rep(Q, \alpha)$ by the subgroup $U(\alpha)$ we can ensure that the $j(i)$-th column of the matrix corresponding to ϕ_i contains this 1 as its only non-zero entry.
Performing the necessary computations we see that this choice determines all components of the basechange matrix $(u_1, \ldots, u_n) \in U(\alpha)$ apart from the first column of $u_{v(0)}$. For, inductively we can determine the $j(1)$-th column of $u_{v(1)}$, then the $j(2)$-column of $u_{v(2)}$ and so on.
Consider the subquiver $\tilde{Q}^{sl}_\mu$ of $\tilde{Q}_\mu$ where we removed

- the fixed arrows from (k) to $(k+1)$ determined by i_k for every $0 \leq k < l$ and
- all arrows from (k) to (l) for $l > k$ corresponding to ϕ_k.

By the argument given above we see that every $U(\alpha)$-orbit in $U(i_1, \ldots, i_l)$ contains a representation from $Rep(\tilde{Q}^{sl}_\mu, (1, \ldots, 1))$.

Hence, we reduced the study of $U(\alpha)$-orbits in $U(i_1,\ldots,i_l)$ to that of orbits in $Rep(\tilde{Q}^{sl}_\mu,(1,\ldots,1))$ under the action of the nilpotent group

$$G = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ x_2 & 1 & 0 & \ldots & 0 \\ x_3 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ x_{a_{v(0)}} & 0 & 0 & & 1 \end{bmatrix} \hookrightarrow U(\alpha) \hookrightarrow GL(\alpha)$$

That is, $G \simeq k_+ \times \ldots \times k_+$ where the number of components is equal to $a_{v(0)} - 1$ where $a_{v(0)}$ is also the multiplicity with which the top component occurs in the Jordan-Hölder sequence.

We recall from [8, III.1.1,Satz 4] that all G-orbits in $Rep(\tilde{Q}^{sl}_\mu,(1,\ldots,1))$ are closed. Therefore, we can parameterize the orbits by stratifying according to the dimension of the stabilizer subgroup.

Each of the representing spaces can be identified to some $Rep(Q',(1,\ldots,1))$ where Q' is a subquiver of $\tilde{Q}^{sl}_\mu$ and for the corresponding stratum $U_{Q'} \hookrightarrow Rep(\tilde{Q}^{sl}_\mu,(1,\ldots,1))$ the quotient map

$$U_{Q'} \twoheadrightarrow Rep(Q',(1,\ldots,1))$$

is a vectorbundle because all fibers are isomorphic to $G/(k_+ \times \ldots \times k_+) \simeq \mathbb{A}^s$ for some $s \leq a_{v(0)} - 1$.

3.8. Finally, using the equivariant embedding $Rep(A,\alpha) \hookrightarrow Rep(Q,\alpha)$ and our solution of the classification problem for the uniserial representations of Q of type $\mathbb{S}$ we recover [1, Thm. A].

We mentioned already that the same construction can be repeated almost verbatim for representations having optimal filtrations corresponding to μ where none of the segments of μ contains numbers with multiplicity ≥ 2.

To begin the general classification problem of A-representations according to optimal filtration series, a natural idea would be to study the moduli spaces $T^{ss}_\mu/L(\lambda)$ studied by A. King in [6]. Over a suitable affine open cover of these projective varieties one might then try to construct slices reducing the acting parabolic group $P(\lambda)$ to a more manageable group.

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