

The Singularities of Quantum Groups

Lieven Le Bruyn

Research Director of the FWO (Belgium)
Dept. of Mathematics & Comp. Sci. , University of Antwerp, UIA,
Belgium

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Universiteitsplein 1, B-2610 Wilrijk-Antwerpen, Belgium

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ABSTRACT. If $\mathbb{C}[G] \hookrightarrow \mathbb{C}[H]$ is an extension of Hopf domains of degree d , then $H \twoheadrightarrow G$ is an étale map. Equivalently, the variety $X_{\mathbb{C}[H]}$ of d -dimensional $\mathbb{C}[H]$ -modules compatible with the trace map of the extension, is a smooth GL_d -variety with quotient G .

If we replace $\mathbb{C}[H]$ by a noncommutative Hopf algebra H , we construct similarly a GL_d -variety and quotient map $X_H \xrightarrow{\pi} G$. The smooth locus of H over $\mathbb{C}[G]$ is the set of points $g \in G$ such that X_H is smooth along $\pi^{-1}(g)$.

We relate this set to the separability locus of H over $\mathbb{C}[G]$ as well as to the (ordinary) smooth locus of the commutative extension $\mathbb{C}[G] \hookrightarrow Z$ where Z is the center of H .

In particular, we prove that the smooth locus coincides with the separability locus whenever H is a reflexive Azumaya algebra. This implies that the quantum function algebras $O_\epsilon(G)$ and quantised enveloping algebras $U_\epsilon(\mathfrak{g})$ are as singular as possible.

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1. INTRODUCTION

1.1. Throughout, we work over an algebraically closed field of characteristic zero and denote it with \mathbb{C} . Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} of rank r with Cartan matrix $C = (a_{ij}) \in M_r(\mathbb{Z})$ and vector $d = (d_1, \dots, d_r) \in \mathbb{N}_+^r$ of relative prime integers such that $d.C$ is symmetric.

The (simply connected form of the) *quantised enveloping algebra* $U_\epsilon(\mathfrak{g})$ is the \mathbb{C} -algebra with generators

$$\{E_i, F_i, K_i^{\pm 1} \mid 1 \leq i, j \leq r\}$$

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The author is a research director at the FWO (Belgium).

satisfying the following relations for all $1 \leq i, j \leq r$:

$$\begin{aligned}
K_i K_j &= K_j K_i & K_i K_i^{-1} &= 1 = K_i^{-1} K_i \\
K_i E_j &= \epsilon^{d_i a_{ij}} E_j K_i \\
K_i F_j &= \epsilon^{-d_i a_{ij}} F_j K_i \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{\epsilon^{d_i} - \epsilon^{-d_i}} \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s &= 0 \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s &= 0
\end{aligned}$$

where the symbols in squarebrackets denote Gaussian binomial coefficients for the parameter ϵ , see [4].

The Hopf structure on this algebra is given by defining for all $1 \leq i \leq r$

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \Delta(F) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \Delta(K_i) = K_i \otimes K_i \\
S(E_i) &= -K_i^{-1} E_i, S(F) = -F_i K_i, S(K_i) = K_i^{-1} \\
\epsilon(E_i) &= 0, \epsilon(F_i) = 0, \epsilon(K_i) = 1.
\end{aligned}$$

Now, let ϵ be a primitive l -th root of unity, where l is odd and prime to 3 if \mathfrak{g} contains components of type G_2 . We recall the following structural results on $U_\epsilon(\mathfrak{g})$ from [4].

$U_\epsilon(\mathfrak{g})$ is an order with integrally closed center Z_ϵ in a central simple algebra of dimension l^{2N} where N is the number of positive roots of \mathfrak{g} . The commutator in $U_q(\mathfrak{g})$ induces a nontrivial Poisson structure on Z_ϵ .

There exists a Poisson subalgebra Z_0 of Z_ϵ satisfying the following properties :

1. Z_0 is a sub-Hopf algebra of $U_\epsilon(\mathfrak{g})$.
2. $U_\epsilon(\mathfrak{g})$ is a free Z_0 -module of rank l^{2N+r} .
3. Z_ϵ is a free Z_0 -module of rank l^r .

That is, we have the following inclusions

$$\begin{array}{ccc}
& U_\epsilon(\mathfrak{g}) & \\
& \uparrow & \swarrow l^{2N} \\
l^{\dim \mathfrak{g}} & & Z_\epsilon \\
& \downarrow & \nwarrow l^r \\
& Z_0 &
\end{array}$$

Being an affine commutative Hopf algebra, $Z_0 \simeq \mathbb{C}[H]$ for an algebraic group H which can be described in the following way. Let T be a maximal torus of the simply connected Lie group G corresponding to \mathfrak{g} , and B_\pm the corresponding Borel subgroups. Then, we have a fiber diagram

$$\begin{array}{ccc}
H & \longrightarrow & B_+ \\
\downarrow & & \downarrow pr_+ \\
B_- & \xrightarrow{pr_-^{-1}} & T
\end{array}$$

where pr_{\pm} are the projections and $pr_{-}^{-1} = (-)^{-1} \circ pr_{-}$. The connection between H and G is given by the map

$$\sigma : H \longrightarrow G^0 \quad (h_{-}, h_{+}) \mapsto h_{-}^{-1} h_{+}$$

which is an unramified cover of the big cell G^0 of G of degree 2^r .

There is an (infinite dimensional) group \tilde{G} of analytic automorphisms of H such that its orbits in H are of the form

$$\sigma^{-1}(\mathcal{C} \cap G^0)$$

where \mathcal{C} is the conjugacy class of a non-central element in G . These orbits are also the symplectic leaves of H induced by the Poisson structure on $Z_0 = \mathbb{C}[H]$.

In this paper we investigate to what extent the inclusion of Hopf algebras $\mathbb{C}[H] \hookrightarrow U_{\epsilon}(\mathfrak{g})$ differs from a finite Hopf algebra extension between the coordinate rings of irreducible algebraic groups.

1.2. Recall the commutative situation. Let G be an irreducible algebraic group with coordinate ring $\mathbb{C}[G]$. Consider a subHopf algebra $\mathbb{C}[H]$ such that $\mathbb{C}[G]$ is a finite $\mathbb{C}[H]$ -module. Corresponding to the inclusion of Hopf algebras $\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$ is a projection of varieties (or group-schemes) $G \twoheadrightarrow H$ which is a finite étale morphism, that is, unramified and smooth.

Since there is no suitable substitute for the variety corresponding to $U_{\epsilon}(\mathfrak{g})$, we will give a different geometric description of the extension $\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$ which can be generalized to the non-commutative setting.

Consider the trace map tr on the finite Galois extension of degree d between the function fields $\mathbb{C}(H) \hookrightarrow \mathbb{C}(G)$. As $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are integrally closed, tr restricts to a linear map on $\mathbb{C}[G]$ with image $tr \mathbb{C}[G] = \mathbb{C}[H]$. Remark that tr satisfies the formal Cayley-Hamilton identity for $d \times d$ matrices, see section 2 for more details.

Consider the affine algebraic variety $X_{\mathbb{C}[G]}$ with points the trace preserving algebra maps $\mathbb{C}[G] \xrightarrow{\rho} M_d(\mathbb{C})$, that is, those having the property that the diagram

$$\begin{array}{ccc} \mathbb{C}[G] & \xrightarrow{\rho} & M_d(\mathbb{C}) \\ \downarrow tr & & \downarrow Tr \\ \mathbb{C}[G] & \xrightarrow{\rho} & M_d(\mathbb{C}) \end{array}$$

is commutative, where Tr is the ordinary trace on $M_d(\mathbb{C})$.

The variety $X_{\mathbb{C}[G]}$ has a natural GL_d -action by conjugation in $M_d(\mathbb{C})$. The orbits correspond to isomorphism classes of (trace preserving) d -dimensional $\mathbb{C}[G]$ -modules.

By a result of C. Procesi [21] (or see Theorem 2.2) we can recover $\mathbb{C}[G]$ and $\mathbb{C}[H]$ from the GL_d -variety $X_{\mathbb{C}[G]}$.

1. $\mathbb{C}[H]$ is the ring of polynomial invariants $\mathbb{C}[X_{\mathbb{C}[G]}]^{GL_d}$.
2. $\mathbb{C}[G]$ is the ring of GL_d -equivariant maps $X_{\mathbb{C}[G]} \longrightarrow M_d(\mathbb{C})$.

By the first fact, we have an algebraic quotient map

$$X_{\mathbb{C}[G]} \xrightarrow{\pi} H$$

Consider $h \in H$ with corresponding maximal ideal $m_h \triangleleft \mathbb{C}[H]$. The fiber $\pi^{-1}(h)$ contains a unique closed orbit which is, by the Artin-Voigt theorem, the orbit of the unique (trace preserving) d -dimensional semi-simple $\mathbb{C}[G]/m_h \mathbb{C}[G]$ -module. The full fiber $\pi^{-1}(h)$ consists of all d -dimensional (trace preserving) $\mathbb{C}[G]/m_h \mathbb{C}[G]$ -modules.

Because $G \longrightarrow H$ is an étale map, $\mathbb{C}[G]/m_h \mathbb{C}[G] \simeq \mathbb{C} \oplus \dots \oplus \mathbb{C}$ (d components) is a semi-simple algebra and thus $\pi^{-1}(h)$ is a single orbit isomorphic to GL_d/T_d where T_d is a maximal torus of GL_d . Hence, $X_{\mathbb{C}[G]}$ is a principal fibration over H with fibres the homogeneous space GL_d/T_d .

Proposition 1.1. *Let $\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$ be a finite Hopf algebra extension of degree d , then $X_{\mathbb{C}[G]}$ is a smooth GL_d -variety with algebraic (even geometric) quotient variety H .*

In this paper we study whether this result remains true if we replace $\mathbb{C}[G]$ by a non-commutative Hopf algebra such as $U_\epsilon(\mathfrak{g})$.

1.3. We can define a trace map on $U_\epsilon(\mathfrak{g})$ by taking the composition of the reduced trace map and the trace map of the extension $\mathbb{C}[H] \hookrightarrow Z_\epsilon$. Again, using the fact that both Z_ϵ and $Z_0 = \mathbb{C}[H]$ are integrally closed, this trace map tr is well defined with image $tr U_\epsilon(\mathfrak{g}) = \mathbb{C}[H]$. This time, tr satisfies the formal Cayley-Hamilton identity for $d \times d$ matrices where $d = l^{N+r}$.

As before, we can define the affine algebraic variety $X_{U_\epsilon(\mathfrak{g})}$ of trace preserving algebra maps $U_\epsilon(\mathfrak{g}) \xrightarrow{\rho} M_d(\mathbb{C})$. This variety has a natural GL_d -action and orbits correspond to isomorphism classes of d -dimensional trace preserving $U_\epsilon(\mathfrak{g})$ -modules. Again, we can recover $U_\epsilon(\mathfrak{g})$ and $\mathbb{C}[H]$ from the GL_d -variety $X_{U_\epsilon(\mathfrak{g})}$.

1. $\mathbb{C}[H]$ is the ring of polynomial invariants $\mathbb{C}[X_{U_\epsilon(\mathfrak{g})}]^{GL_d}$.
2. $U_\epsilon(\mathfrak{g})$ is the ring of GL_d -equivariant maps $X_{U_\epsilon(\mathfrak{g})} \longrightarrow M_d(\mathbb{C})$.

We have an algebraic quotient variety

$$X_{U_\epsilon(\mathfrak{g})} \xrightarrow{\pi} H$$

and the description of the fibers $\pi^{-1}(h)$ is as above : it consists of all d -dimensional trace preserving $\bar{U}_h = U_\epsilon(\mathfrak{g})/m_h U_\epsilon(\mathfrak{g})$ -modules and there is a unique closed orbit corresponding to a semi-simple $U_\epsilon(\mathfrak{g})$ -module which we denote by

$$M_h^{ss} = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where the S_i is a simple $U_\epsilon(\mathfrak{g})$ -module of dimension d_i occurring in M_h^{ss} with multiplicity e_i . Clearly, $d = \sum d_i e_i$.

Keeping proposition 1.1 in mind, we define.

Definition 1.2. We say that $U_\epsilon(\mathfrak{g})$ is *smooth in h* if and only if $X_{U_\epsilon(\mathfrak{g})}$ is smooth along $\pi^{-1}(h)$.

The *smooth locus* $Sm U_\epsilon(\mathfrak{g})/\mathbb{C}[H]$ of $U_\epsilon(\mathfrak{g})$ over $\mathbb{C}[H]$ is the subset of all $h \in H$ such that $U_\epsilon(\mathfrak{g})$ is smooth in h .

1.4. An alternative geometric description of $U_\epsilon(\mathfrak{g})$ is as follows. Because $U_\epsilon(\mathfrak{g})$ is a free $\mathbb{C}[H]$ -module of rank $t = l^{\dim \mathfrak{g}}$, every \bar{U}_h is a \mathbb{C} -algebra of dimension t . This determines a map

$$H \xrightarrow{\phi} Alg_t$$

where Alg_t is the variety of (structure constants of) t -dimensional associative algebras with unit, see for example [14] or [23]. One can recover $U_\epsilon(\mathfrak{g})$ from ϕ by taking ϕ^* of the generic t -dimensional algebra over Alg_t as in [23].

For sufficiently general h we have that $\bar{U}_h \simeq M_N(\mathbb{C}) \oplus \dots \oplus M_N(\mathbb{C})$ (r copies) a semi-simple algebra. This entails that $\phi(H)$ is contained in the irreducible component of Alg_t which is the closure of the GL_t -orbit \mathcal{O} of $M_N(\mathbb{C}) \oplus \dots \oplus M_N(\mathbb{C})$. We define $\phi^{-1}(\mathcal{O})$ as the *separability locus* of $U_\epsilon(\mathfrak{g})$ over $\mathbb{C}[H]$ and denote it with $Sep U_\epsilon(\mathfrak{g})/\mathbb{C}[H]$. Clearly, it is the Zariski open subset of H consisting of those h such that \bar{U}_h is a semi-simple algebra (and hence isomorphic to $M_N(\mathbb{C}) \oplus \dots \oplus M_N(\mathbb{C})$).

As the terminology suggests, it is also the locus over which $U_\epsilon(\mathfrak{g})$ is a separable $\mathbb{C}[H]$ -algebra as in [12].

As in the commutative case, we see that $\pi^{-1}(h)$ consists of a single orbit when $h \in \text{Sep } U_\epsilon(\mathfrak{g})/\mathbb{C}[H]$ and that this orbit is the homogeneous space GL_d/T_r where T_r is the center of $GL_N \times \dots \times GL_N \hookrightarrow GL_d$. Hence, $X_{U_\epsilon(\mathfrak{g})}$ is a principal fibration and therefore smooth over $\text{Sep } U_\epsilon(\mathfrak{g})/\mathbb{C}[H]$. Thus,

$$\text{Sep } U_\epsilon(\mathfrak{g})/\mathbb{C}[H] \hookrightarrow \text{Sm } U_\epsilon(\mathfrak{g})/\mathbb{C}[H].$$

For general Hopf algebras, one expects the smooth locus to be larger. For example, if $U_\epsilon(\mathfrak{b})$ is the quantum Borel of \mathfrak{sl}_2 we will show in example 5.7 that its smooth locus is the whole of B . However, we will prove :

Theorem 1.3. *The quantised enveloping algebra $U_\epsilon(\mathfrak{g})$ is as singular as possible. That is,*

$$\text{Sm } U_\epsilon(\mathfrak{g})/\mathbb{C}[G] = \text{Sep } U_\epsilon(\mathfrak{g})/\mathbb{C}[H]$$

The same result also holds for $O_\epsilon(G)$, that is, the *quantum function algebra* of G . Recall from [3] that $O_\epsilon(G)$ is obtained by first constructing a suitable integral form of $U_q(\mathfrak{g})$ over $\mathbb{Q}[q, q^{-1}]$, then taking an appropriate subHopf algebra of its Hopf dual and finally specializing q to ϵ .

We will outline the strategy of proof in the case of $U_\epsilon(\mathfrak{g})$.

1.5. Assume that $U_\epsilon(\mathfrak{g})$ is smooth in h and let x_h be a point in $X_{U_\epsilon(\mathfrak{g})}$ in the orbit of the uniquely determined semi-simple d -dimensional $U_\epsilon(\mathfrak{g})$ -module M_h^{ss} . We will show that the conormal space to the orbit in x_h is the space of representations $\text{Rep}(\mathbb{B}_h, \alpha_h)$ of a certain quiver \mathbb{B}_h and dimension vector α_h .

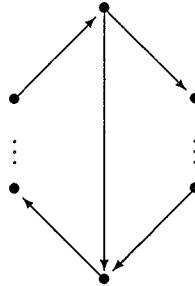
Because $X_{U_\epsilon(\mathfrak{g})}$ is smooth in x_h we can apply the Luna slice theorem (see for example [11]) to prove that the ring of polynomial invariants $\mathbb{C}[\text{Rep}(\mathbb{B}_h, \alpha_h)]^{GL(\alpha_h)}$ of this quiver setting under the action of the basechange group must be a polynomial algebra.

The problem to determine all coregular quiver situations is a hard one. However, we will prove the following crude classification.

Theorem 1.4. *Let Q be a quiver and $\alpha = (a_1, \dots, a_k)$ a dimension vector. Assume that Q is strongly connected, that is, each pair of vertices belongs to an oriented cycle.*

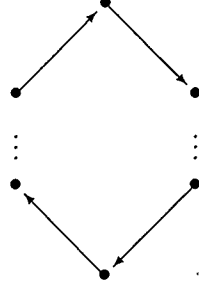
If the ring of polynomial invariants $\mathbb{C}[\text{Rep}(Q, \alpha)]^{GL(\alpha)}$ is a polynomial ring, then we are in one of the following situations :

1. type 1 : $\min_i a_i \leq 1$ or
2. type 2 : $\min_i a_i = 2$ and Q has the form $\tilde{A}_k(+1)$:



that is, the extended Dynkin diagram with cyclic orientation and one extra arrow, or

3. *type 3* : $\min_i a_i \geq 2$ and the quiver Q is the extended Dynkin diagram \tilde{A}_k with cyclic orientation



This result drastically restricts the shapes of the strongly connected components of the quiver \mathbb{B}_h and of the dimension vector α_h .

Let X_ϵ be the affine variety determined by the center Z_ϵ of $U_\epsilon(\mathfrak{g})$ and let $\{x_1, \dots, x_l\}$ be the set of points of X_ϵ lying over h . We will prove that each x_i determines a strongly connected component of \mathbb{B}_h and that the restriction of α_h to this component encodes the multiplicity of p_i over m .

We will show that only the first case can occur and only if the multiplicity of p_i is equal to one, that is, if p_i is unramified over m . Therefore, we have

Theorem 1.5. *With notations as above*

$$\text{Sep } U_\epsilon(\mathfrak{g})/\mathbb{C}[H] \hookrightarrow \text{Sm } U_\epsilon(\mathfrak{g})/\mathbb{C}[H] \hookrightarrow \text{Sm } Z_\epsilon/\mathbb{C}[H]$$

Here, $\text{Sm } Z_\epsilon/\mathbb{C}[H]$ is the usual smooth locus of the commutative extension $\mathbb{C}[H] \hookrightarrow Z_\epsilon$. Therefore, if $U_\epsilon(\mathfrak{g})$ is smooth in h , then there are precisely l' points of X_ϵ lying over h and X_ϵ is smooth in all of them.

K. Brown and K. Goodearl have proved in [1] that the smooth locus of X_ϵ coincides with the Azumaya locus of $U_\epsilon(\mathfrak{g})$ over Z_ϵ . That is, the set of points such that the corresponding maximal ideal p of Z_ϵ satisfies $U_\epsilon(\mathfrak{g})/U_\epsilon(\mathfrak{g})p \simeq M_N(\mathbb{C})$.

But then, $\overline{U}_h \simeq M_N(\mathbb{C}) \oplus \dots \oplus M_N(\mathbb{C})$ (s copies) whence $U_\epsilon(\mathfrak{g})$ is separable over $\mathbb{C}[H]$ in h .

1.6. The method of proof applies to a much larger class of Hopf algebras and algebras and will be presented in that generality. We will outline the contents briefly.

In section 2 we recall some results of Procesi on Cayley-Hamilton algebras, introduce the basic setting for our results, introduce the varieties X_A and compute the normal spaces to the orbits. Our setting will be a *triad* of algebras (A, Z, C) where A is a prime algebra which is a finite module over a subring C of the center Z and Z is a projective C -module.

In section 3 we recall some results on representations of quivers and generalize them to the setting of *marked quivers*, that is, quivers such that some of their loops get a marking. This generality is necessary as the conormal spaces are representations of marked quivers. Moreover, in this section we give the classification of coregular (marked) quiver settings mentioned before.

In section 4 we begin the structure of these marked quivers \mathbb{B}_m describing the conormal space to the orbit in X_A of the semi-simple module determined by the maximal ideal m of C . We relate their connected components and their dimension vector to the splitting behavior of the center over m .

In section 5 we will then study the extra restrictions imposed on \mathbb{B}_m and the occurring dimension vector if we assume that A is smooth in m .

Finally, in section 6 we will prove the main results of the paper. We summarize the application to Hopf algebras in the next result.

Theorem 1.6. *Let H be a prime Hopf algebra which is a finite module over a central subHopf algebra C . Let Z be the center of H .*

1. *With notations as above,*

$$\text{Sep } H/C \hookrightarrow \text{Sm } H/C \hookrightarrow \text{Sm } Z/C.$$

2. *If, in addition, H is a reflexive Azumaya algebra, then*

$$\text{Sm } H/C = \text{Sep } H/C$$

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2. CAYLEY-HAMILTON ALGEBRAS

2.1. All algebras considered will be \mathbb{C} -algebras. In this section we recall some results due to C. Procesi on Cayley-Hamilton algebras. We take [21] and [4, §2] as our primary references.

Definition 2.1. An affine algebra A is said to be a *Cayley-Hamilton algebra of degree d* if there is a linear trace map $\text{tr} : A \rightarrow \mathbb{C}$ satisfying the following properties

1. For all $a, b \in A$ we have $\text{tr}(ab) = \text{tr}(ba)$, $\text{tr}(a)b = b\text{tr}(a)$ and $\text{tr}(\text{tr}(a)b) = \text{tr}(a)\text{tr}(b)$.
2. Consider the characteristic polynomial of $d \times d$ matrices $m \in M_d(\mathbb{C})$

$$\chi_{d,m}[t] = \sum_{i=0}^d (-1)^i \sigma_i(m) t^{d-i}$$

where σ_i are polynomials with rational coefficients in $\text{Tr}(m), \dots, \text{Tr}(m^i)$. Replacing Tr by tr in the definition of the σ_i we demand that for all $a \in A$

$$\chi_{d,a}[a] = 0 \text{ and } \text{tr}(1) = d.$$

Because A is affine, it has a presentation

$$A = \mathbb{C}\langle x_1, \dots, x_m \rangle_{\text{tr}} / (r_A)$$

where $\mathbb{C}\langle x_1, \dots, x_m \rangle_{\text{tr}}$ is the *free algebra with trace* on m generators, see [4, 4.1.2] and where r_A is the ideal of relations defining A .

Recall that $\mathbb{C}\langle x_1, \dots, x_m \rangle_{\text{tr}}$ is the ordinary free algebra in the x_i over the polynomial ring in the formal variables $\text{tr}(M)$ as M runs over all non-commutative monomials in the x_i considered up to cyclic order equivalence. The trace map is then defined in the obvious way.

For every $1 \leq i \leq m$ consider the generic $d \times d$ matrix

$$y_i = (y_{jk}^{(i)})_{j,k} \in M_d(\mathbb{C}[y_{jk}^{(i)} \mid 1 \leq j, k \leq d, 1 \leq i \leq m]).$$

The subalgebra of $M_d(\mathbb{C}[y_{jk}^{(i)}])$ generated by the y_i together with all traces of monomials in the y_i is then seen to be the generic Cayley-Hamilton algebra of degree d . Often, it is denoted with $\mathbb{T}_{m,d}$ and is called the *trace ring of m generic $d \times d$ matrices*. A is then an epimorphic image of $\mathbb{T}_{m,d}$.

Let I_A be the ideal in $\mathbb{C}[y_{jk}^{(i)} \mid j, k, i]$ generated by the entries of the matrices $f(y_1, \dots, y_m)$ for all $f(x_1, \dots, x_m) \in r_A$. We define the affine commutative algebra $\mathbb{C}[X_A] = \mathbb{C}[y_{jk}^{(i)} \mid j, k, i] / I_A$ and have a canonical inclusion

$$A \xrightarrow{i_A} M_d(\mathbb{C}[X_A])$$

by sending x_i to the image of y_i under the epimorphism

$$M_d(\mathbb{C}[y_{kl}^{(m)} \mid k, l, m]) \twoheadrightarrow M_d(\mathbb{C}[X_A]).$$

2.2. The inclusion i_A is a solution to the following universal problem.

Let R be a commutative \mathbb{C} -algebra such that there is a trace preserving algebra morphism $j : A \longrightarrow M_d(R)$ (with the usual trace Tr on $M_d(R)$), then there is a unique algebra morphism $\bar{j} : \mathbb{C}[X_A] \longrightarrow R$ of commutative algebras making the diagram below commutative.

$$\begin{array}{ccc} A & \xhookrightarrow{i_A} & M_d(\mathbb{C}[X_A]) \\ & \searrow j & \downarrow M_d(\bar{j}) \\ & & M_d(R) \end{array}$$

Any $g \in GL_d$ acts on $M_d(R) = R \otimes M_d(\mathbb{C})$ by conjugation on $M_n(\mathbb{C})$. In particular, it defines a trace preserving automorphism on $M_d(\mathbb{C}[X_A])$ which by the universal property defines an automorphism $\phi(g)$ on $\mathbb{C}[X_A]$. One verifies that

$$g.c = \phi(g^{-1})(c) \quad \forall c \in \mathbb{C}[X_A]$$

defines an action of GL_d on $\mathbb{C}[X_A]$. Further, define the diagonal action of GL_d on $M_d(\mathbb{C}[X_A]) = \mathbb{C}[X_A] \otimes M_d(\mathbb{C})$ where the action on the first factor is defined above and that on the second factor is conjugation by g . By [4, §4.3] we have the following geometric description of A .

Theorem 2.2 (Procesi). *Let C be the central subalgebra $tr(A)$, then :*

1. *The ring of invariants $\mathbb{C}[X_A]^{GL_d} = C$.*
2. *The ring of concomitants $M_d(\mathbb{C}[X_A])^{GL_d} = A$.*

The affine scheme $X_A = Spec \mathbb{C}[X_A]$ represents the functor which assigns to a commutative algebra R the set $X_A(R)$ of algebra morphisms

$$\rho : R \otimes_{\mathbb{C}} A \longrightarrow M_d(R)$$

which are trace preserving, that is, such that the diagram below is commutative

$$\begin{array}{ccc} R \otimes_{\mathbb{C}} A & \xrightarrow{\rho} & M_d(R) \\ \downarrow id \otimes tr & & \downarrow Tr \\ R \otimes_{\mathbb{C}} A & \xrightarrow{\rho} & M_d(R) \end{array}$$

The action of GL_d on $\mathbb{C}[X_A]$ extends to an action of the groupscheme GL_d on X_A . A \mathbb{C} -valued point x of X_A is a trace preserving algebra map $\rho_x : A \longrightarrow M_d(\mathbb{C})$ and hence defines a d -dimensional left A -module M_x . The GL_d -orbits in X_A correspond to A -module isomorphism classes.

2.3. The inclusion $C = \mathbb{C}[X_A]^{GL_d} \hookrightarrow \mathbb{C}[X_A]$ determines a projection

$$X_A \xrightarrow{\pi_A} Spec C$$

which is an algebraic quotient. The \mathbb{C} -points of $Spec C$ (the maximal ideals of C) parameterize closed orbits of d -dimensional trace preserving A -modules. We have the following result, see for example [7].

Theorem 2.3 (Artin, Voigt). *The points of $Spec C$ parameterize isomorphism classes of d -dimensional trace preserving semi-simple A -modules.*

That is, for every maximal ideal m of $C = \text{tr}(A)$ there is a unique semi-simple A -module

$$M_m^{ss} = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where S_i is a simple A -module of dimension d_i occurring with multiplicity e_i . Clearly, $d = \sum d_i e_i$.

2.4. Throughout this paper, we restrict attention to the following class of Cayley-Hamilton algebras.

Definition 2.4. By a *triad of algebras* (A, Z, C) we mean a commutative diagram of algebras

$$\begin{array}{ccccc}
 A & \hookrightarrow & Q & \hookrightarrow & M_n(\overline{K}) \\
 & \searrow & \downarrow n^2 & \searrow & \downarrow \\
 & & Z & \hookrightarrow & K & \hookrightarrow & \overline{K} \\
 & \nearrow s & \nearrow & \nearrow & \nearrow \\
 C & \hookrightarrow & L & & & &
 \end{array}$$

Here, A is a prime affine algebra with integrally closed center Z . If we localize A at all its nonzero central elements we obtain $Q = A.K$ a central simple K -algebra of dimension n^2 where K is the field of fractions of Z . If \overline{K} is the algebraic closure of K , this condition on Q means that $Q \otimes_K \overline{K} \simeq M_n(\overline{K})$.

In the center we consider an integrally closed domain C such that A is a finitely generated module over C and its center Z is a projective C -module of rank s . The field of fractions of C we denote by L .

Much of this triad can be encoded in a linear trace map on A . Define

$$\text{tr} : A \longrightarrow A \quad a \mapsto \text{tr}_{L/K}(\text{Tr}(a \otimes 1))$$

where Tr is the trace on $M_n(\overline{K})$ and $\text{tr}_{L/K}$ is the trace map of the finite field extension L/K .

From the action of the Galois group $\text{Gal } \overline{K}/K$ on $M_n(\overline{K})$ and \overline{K} with invariants respectively Q and K we see that $\text{Tr}(q \otimes 1) \in K$ for all $q \in Q$. Because A is a finite module over Z , Z is integrally closed in K and Tr splits (as we are in characteristic zero), we deduce that $\text{Tr}(A \otimes 1) = Z$.

Moreover, because Z is the integral closure of C in K , we have that $\text{tr}_{L/K}(Z) = C$. Concluding, the trace map tr on A is well defined and has as its image $\text{tr}(A) = C$. We recall the following result from [4, p.46 and p.49].

Lemma 2.5. 1. A with the reduced Tr is a Cayley-Hamilton algebra of degree n with $\text{Tr}(A) = Z$.
 2. A with the trace map tr is a Cayley-Hamilton algebra of degree $d = n.s$ with $\text{tr}(A) = C$.

2.5. The next two classes of triads are of current interest :

Example 2.6. In the introduction we recalled the definition and main structural properties of the quantised enveloping algebra $U_\epsilon(\mathfrak{g})$. In particular, we have seen that the triple

$$(U_\epsilon(\mathfrak{g}), Z_\epsilon, Z_0 = \mathbb{C}[H])$$

is a triad with corresponding numbers

$$d = l^{N+r} \quad n = l^N \quad s = l^r$$

where r is the rank of \mathfrak{g} and N the number of positive roots.

Example 2.7. The quantum function algebra $O_\epsilon(G)$ at a root of unity. For definition, proofs and more details we refer the reader to [3], [6] and [10] or [1, 4.4]. We assume that ϵ is a primitive l -th root of unity, l is odd, prime to 3 if \mathfrak{g} contains factors of type G_2 and prime to each a_i where $\sum_{i=1}^r a_i \alpha_i$ is the expression of the highest root of \mathfrak{g} as a linear combination of the simple roots α_i , r is the rank of the Lie algebra \mathfrak{g} .

The Hopf algebra $A = O_\epsilon(G)$ has a central sub-Hopf algebra $C \simeq \mathbb{C}[G]$ and is a projective module over it of rank $l^{\dim G}$. Moreover, $O_\epsilon(G)$ is a Cayley-Hamilton algebra of degree l^{N+r} where N is the number of positive roots of \mathfrak{g} . That is, $(O_\epsilon(G), Z, \mathbb{C}[G])$ is a triad

$$\begin{array}{ccc} & O_\epsilon(G) & \\ & \uparrow & \swarrow l^{2N} \\ l^{\dim G} & & Z \\ & \downarrow & \searrow l^r \\ & \mathbb{C}[G] & \end{array}$$

with corresponding numbers $d = l^{N+r}$, $n = l^N$ and $s = l^r$.

2.6. Let x be a \mathbb{C} -point of X_A with corresponding d -dimensional trace preserving A -module M .

Recall the computation of self-extensions $\text{Ext}_A^1(M, M)$. Consider the vectorspace of linear maps $\lambda : A \longrightarrow \text{End}_{\mathbb{C}}(M)$ satisfying $\lambda(aa') = \rho(a)\lambda(a') + \lambda(a)\rho(a')$ where ρ is the action of A on M . Hence, λ determines an algebra map

$$\phi_\lambda : A \longrightarrow \text{End}_{\mathbb{C}}(M + M\epsilon) = M_d(\mathbb{C} + \mathbb{C}\epsilon)$$

where $\epsilon^2 = 0$ and $\phi_\lambda(a) = \rho(a) + \lambda(a)\epsilon$.

In $Z(M, M)$ we consider the subspace $B(M, M)$ of linear maps $\delta : A \longrightarrow \text{End}_{\mathbb{C}}(M)$ which are of the form

$$\delta(a) = \rho(a).m - m.\rho(a)$$

for some $m \in \text{End}_{\mathbb{C}}(M) = M_d(\mathbb{C})$. Then, $\text{Ext}_A^1(M, M) = Z(M, M)/B(M, M)$.

Let a selfextension e be determined by a linear map $\lambda \in Z(M, M)$. As the trace on all $\delta(a)$ with $\delta \in B(M, M)$ is zero, the property that $\phi_e \stackrel{\text{def}}{=} \phi_\lambda$ is trace preserving is independent of the choice of λ .

This allows us to define the space of *trace preserving extensions* $\text{Ext}_A^1(M, M)_{tr}$ as the subspace of extensions e such that ϕ_e is trace preserving.

Proposition 2.8. *Let M be a \mathbb{C} -point of X_A corresponding to the d -dimensional A -module M . There is a canonical isomorphism*

$$\text{Ext}_A^1(M, M)_{tr} \simeq T_x/T_x^0$$

where T_x is the Zariski tangentspace to the scheme X_A in x and T_x^0 is the Zariski-tangent space in x to the orbit through x .

Proof. Similar to the proof of P. Gabriel in [7, Prop. 1.1] of the result attributed to Voigt. \square

We remark that it is important to consider the scheme structure of X_A . For example, if $A = \mathbb{C}[x]/(x^2)$ and $d = 1$, then the reduced variety of X_A is one point. However, the corresponding one-dimensional representation of A has non-trivial selfextensions.

3. MARKED QUIVERS

3.1. In this section we will recall some results on representations of quivers and extend them to the slightly more general setting of *marked quivers* which are quivers of which some loops are marked. Representations of marked quivers are used to study the normal space to the orbit in X_A of a semi-simple module. Further, we give a rather crude classification for coregular quiver settings, that is, dimension vectors of quivers such that the corresponding ring of polynomial invariants is a polynomial ring.

A *quiver* Q is a directed graph on a finite set of vertices $\{v_1, \dots, v_k\}$. Let a_{ij} be the number of directed arrows from v_i to v_j (or loops if $v_i = v_j$). The *Euler-form* of Q is the bilinear form

$$\chi : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z}$$

determined by the matrix $\chi = (\chi_{ij})_{i,j} \in M_k(\mathbb{Z})$ with entries

$$\chi_{ij} = \delta_{ij} - a_{ij}$$

Clearly, χ encodes the structure of the directed graph Q .

A *representation* V of a quiver Q of *dimension vector* $\alpha = (a_1, \dots, a_k) \in \mathbb{N}^k$ assigns to every arrow $v_i \xrightarrow{\phi} v_j$ in Q a matrix $V(\phi) \in M_{a_i \times a_j}(\mathbb{C})$. The set of all α -dimensional representations form an affine space,

$$\text{Rep}(Q, \alpha) = \bigoplus_{i,j=1}^k M_{a_i \times a_j}(\mathbb{C})^{\oplus a_{ij}}$$

where a_{ij} is the number of arrows in Q from v_i to v_j .

There is a natural action of the *basechange group* $GL(\alpha) = GL_{a_1} \times \dots \times GL_{a_k}$ on $\text{Rep}(Q, \alpha)$, the orbits correspond to isomorphism classes of representations.

The *path algebra* $\mathbb{C} Q$ of the quiver Q is a vectorspace with basis the oriented paths in Q of length ≥ 0 and the multiplication is induced by concatenation of paths.

Representations of Q can be viewed as finite dimensional representations of $\mathbb{C} Q$. In this way, representations form an Abelian category and one defines homomorphisms, extensions etc. in the obvious way. If $V \in \text{Rep}(Q, \alpha)$ and $W \in \text{Rep}(Q, \beta)$, then

$$\chi(\alpha, \beta) = \dim_{\mathbb{C}} \text{Hom}(V, W) - \dim_{\mathbb{C}} \text{Ext}^1(V, W)$$

3.2. Let (A, Z, C) be a triad of algebras and m a maximal ideal of C . There is a unique semi-simple d -dimensional A -module

$$M_m^{ss} = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where S_i is a simple A -module of dimension d_i occurring with multiplicity e_i . Therefore, $d = \sum d_i e_i$.

The *ext-quiver* \mathbb{E}_m of m is the quiver on k vertices, where v_i corresponds to the simple module S_i and where

$$a_{ij} = \dim_{\mathbb{C}} \text{Ext}_A^1(S_i, S_j)$$

Consider the dimension vector $\alpha_m = (e_1, \dots, e_k)$ where e_i is the multiplicity of S_i in M_m^{ss} and the quiver Q_m , then there is a natural isomorphism

$$\text{Rep}(\mathbb{E}_m, \alpha_m) = \text{Ext}_A^1(M_m^{ss}, M_m^{ss})$$

Moreover, $\text{Aut}_A(M_m^{ss}) = GL(\alpha_m)$ and its action on $\text{Ext}_A^1(M_m^{ss}, M_m^{ss})$ coincides with the basechange action in $\text{Rep}(\mathbb{E}_m, \alpha_m)$.

After a suitable choice of basis in (which amounts to fixing a point in the orbit of M_m^{ss} in X_A) we can embed $GL(\alpha_m)$ in GL_d via

$$\begin{bmatrix} GL_{e_1} \otimes 1_{d_1} & & \\ & \ddots & \\ & & GL_{e_k} \otimes 1_{d_k} \end{bmatrix} \hookrightarrow GL_d.$$

It is clear from the definition that $\text{Ext}_A^1(M_h^{ss}, M_h^{ss})_{tr}$ is a $GL(\alpha_m)$ -submodule of $\text{Ext}_A^1(M_h^{ss}, M_h^{ss}) = \text{Rep}(\mathbb{E}_m, \alpha_m)$.

If $v_i \neq v_j$, the submodule $M_{e_i \times e_j}(\mathbb{C})$ corresponding to a directed arrow from v_i to v_j is a simple $GL(\alpha_m)$ -module and is contained in $\text{Ext}_A^1(M_m^{ss}, M_m^{ss})_{tr}$.

On the other hand, the subspace $M_{e_i}(\mathbb{C})$ corresponding to a loop in v_i decomposes as a $GL(\alpha_m)$ -module into $\mathbb{C}_{triv} \oplus M_{e_i}^0(\mathbb{C})$ where \mathbb{C}_{triv} is the trivial one-dimensional representation and $M_{e_i}^0(\mathbb{C})$ is the space of trace zero $e_i \times e_i$ -matrices. As the trace preserving condition imposes linear conditions on $\text{Ext}_A^1(M_m^{ss}, M_m^{ss})$ some of the components \mathbb{C}_{triv} (but only those) can disappear in $\text{Ext}_A^1(M_m^{ss}, M_m^{ss})_{tr}$.

Therefore, we have a decomposition

$$\text{Ext}_A^1(M_m^{ss}, M_m^{ss})_{tr} = \bigoplus_{i \neq j} M_{e_i \times e_j}(\mathbb{C})^{\oplus a_{ij}} \oplus \bigoplus_i (M_{e_i}(\mathbb{C})^{\oplus u_i} \oplus M_{e_i}^0(\mathbb{C})^{\oplus m_i})$$

where $u_i + m_i = a_{ii}$.

This description motivates the introduction of marked quivers.

3.3. A marked quiver Q^* is a quiver Q such that some of the loops in Q are marked. A *representation* of the marked quiver Q^* is a representation of the underlying quiver Q such that the matrices corresponding to marked loops have trace equal to zero. Let Q^* be a marked quiver on k -vertices $\{v_1, \dots, v_k\}$ with a_{ij} directed arrows from v_i to v_j , u_i unmarked loops in v_i and m_i marked loops in v_i . The *Euler-form* of Q^* is the bilinear form

$$\chi : \mathbb{Z}^k \times \mathbb{Z}^k \longrightarrow \mathbb{Z}$$

where $\chi = \chi_1 + \chi_2$ with defining matrices

$$\chi_1 = (\delta_{ij} u_i)_{i,j} - (a_{ij})_{i,j} \text{ and } \chi_2 = (-\delta_{ij} m_i)_{i,j}$$

Again, the Euler-form contains all graph-information of the marked quiver.

If $\alpha = (a_1, \dots, a_k) \in \mathbb{N}^k$, then the space of all α -dimensional representations of Q^* is equal to

$$\text{Rep}(Q^*, \alpha) = \bigoplus_{i \neq j} M_{a_i \times a_j}(\mathbb{C})^{\oplus a_{ij}} \oplus \bigoplus_i M_{a_i}(\mathbb{C})^{\oplus u_i} \oplus \bigoplus_i M_{a_i}(\mathbb{C})^{\oplus m_i}$$

Again, the basechange group $GL(\alpha)$ acts on this space and its orbits are precisely the isomorphism classes of representations.

3.4. Returning to the semi-simple module M_m^{ss} we define the *m-box* \mathbb{B}_m to be the marked quiver on k vertices having

- $a_{ij} = \dim_{\mathbb{C}} \text{Ext}_A^1(S_i, S_j)$ arrows from v_i to v_j for $i \neq j$.
- u_i unmarked loops in v_i if $M_{e_i}(\mathbb{C})$ occurs with multiplicity u_i in $\text{Ext}_A^1(S_i, S_i)_{tr}$.
- $m_i = a_{ii} - u_i$ marked loops in v_i where $a_{ii} = \dim_{\mathbb{C}} \text{Ext}_A^1(S_i, S_i)$.

If we take the dimension vector $\alpha_m = (e_1, \dots, e_k)$, then there is a natural identification

$$\text{Ext}_A^1(M_m^{ss}, M_m^{ss})_{tr} = \text{Rep}(\mathbb{B}_m, \alpha_m)$$

and the action of $\text{Aut}_A(M_m^{ss}) = GL(\alpha_m)$ on both sides is the same.

In the next section we will investigate the structure of the m -box \mathbb{B}_m further. First, we need to recall and generalize some results on representations of quivers.

3.5. If Q^* is a marked quiver and $V \in \text{Rep}(Q^*, \alpha)$, we say that V is a *simple representation* if and only if V has no proper subrepresentations.

In [17, §5] a combinatorial description is given of the dimension vectors α such that $\text{Rep}(Q, \alpha)$ contains simple representations. Observe that in the slightly more general case of marked quivers the proof can be repeated verbatim as we can separate traces. That is, if $V = (V(\phi))_\phi$ is a simple representation of Q and if ψ is a loop in Q in the vertex v_i with $a_i \geq 2$, then V' with $V'(\phi) = V(\phi)$ if $\phi \neq \psi$ and $V'(\psi) = V(\psi) - \frac{1}{a_i} \text{Tr}(V(\phi)) I_{a_i}$ is also a simple representation.

A full marked subquiver Q' of Q^* is said to be *strongly connected* if and only if each couple from its set of vertices belongs to an oriented cycle in Q' .

If $\beta \in \mathbb{N}^k$ is a dimension vector we denote with $\text{supp}(\beta)$ the full marked subquiver of Q^* on the set of vertices v_i such that $\beta(i) \neq 0$. Finally, with δ_i we denote the dimension vector $(\delta_{ij})_j$.

Proposition 3.1. *The vector $\beta \in \mathbb{N}^k$ is the dimension vector of a simple representation of Q^* if and only if either of the following two situations occur :*

1. *$\text{supp}(\beta)$ is the extended Dynkin diagram \tilde{A}_z for some $z \geq 1$ with cyclic orientation and $\beta \mid \text{supp}(\beta) = (1, \dots, 1)$, or*
2. *$\text{supp}(\beta)$ is a noncyclic strongly connected subquiver of Q^* and for all v_i in $\text{supp}(\beta)$ we have*

$$\chi(\beta, \delta_i) \leq 0 \text{ and } \chi(\delta_i, \beta) \leq 0$$

Proof. We will only recall the proof of the necessity of $\text{supp}(\beta)$ being strongly connected. Assume otherwise, then we can divide $\text{supp}(\beta)$ into maximal strongly connected marked subquivers Q_1, \dots, Q_v . The direction of all arrows between two such components must be the same by maximality. Hence, there is a component Q_i having no arrows to other components.

Let $M = \oplus M(\phi)$ be a representation of Q^* with dimension vector β . Consider the subrepresentation N with dimension vector $\delta_{Q_i} \cdot \beta$ and components $N(\phi) = M(\phi)$ if ϕ is an arrow or (marked) loop in Q_i and $M(\phi) = 0$ otherwise. If M is simple, N has to be equal to M , whence $\text{supp}(\beta) = Q_i$ is strongly connected.

The other statements are proved as in [17, Thm. 4]. \square

We take this opportunity to correct an error from [17, §7]. The problem is to determine the Jordan-Hölder decomposition of a representation of Q^* of dimension vector α in general position.

We will use induction on the dimension vector α . Let Q' be the full marked subquiver of Q^* on $\text{supp}(\alpha)$. Consider the *strongly connected component quiver* $SC(Q')$ of Q' .

That is, its vertices are the maximal strongly connected components G_i of Q' and there is an arrow from G_i to G_j if and only if there is an arrow in the quiver Q' from a vertex in G_i to a vertex in G_j . Remark that there are no oriented cycles in $SC(Q')$.

Let M be a representation in $\text{Rep}(Q', \alpha)$ in general position. By the foregoing proposition, a simple subrepresentation S of M must have its support contained in

a strongly connected component G of Q which is a sink in $SC(G)$. Recall that a *sink* in a directed graph is a vertex which is not the starting point of an arrow.

Restrict attention to this strongly connected component G . As $(1, \dots, 1) \mid G$ is the dimension vector of a simple representation of G , there exists a dimension vector β with $\text{supp}(\beta) = G$ satisfying the following properties

1. β is the dimension vector of a simple representation of G .
2. $\beta \hookrightarrow \alpha' = \alpha \mid G$.
3. β is minimal among the dimensionvectors $(1, \dots, 1) \mid G \leq \beta \leq \alpha'$ satisfying these two conditions.

Here, we denote by $\beta \hookrightarrow \alpha$ the condition that a representation of a quiver Q of dimension vector α in general position has a subrepresentation of dimension vector β .

A. Schofield [24, p. 61] has given an inductive procedure to verify the condition $\beta \hookrightarrow \alpha$ in terms of the Euler form χ of the quiver Q .

Theorem 3.2 (Schofield).

$$\beta \hookrightarrow \alpha \text{ if and only if } \max_{\beta' \hookrightarrow \beta} -\chi(\beta', \alpha - \beta) = 0$$

A representation in general position of Q^* of dimension vector α will then contain a simple subrepresentation of dimension vector β .

Continuing by induction on the dimension vector $\alpha - \beta$ we will eventually obtain the *generic semi-simple representation type* $\alpha = \beta_1 + \dots + \beta_z$.

That is, the decomposition of α as a sum of dimension vectors β_i of simple representations such that a representation of Q^* of dimension vector α in general position has Jordan-Hölder components of dimension vectors β_i .

Schofield gave a procedure to determine the *canonical decomposition* in [24]. That is, let α be a dimension vector for a quiver Q and consider a representation $M \in \text{Rep}(Q, \alpha)$ in general position. Then, M decomposes into indecomposable representations

$$M = N_1 \oplus \dots \oplus N_y$$

where N_i is an indecomposable representation of dimension vector γ_i . We then say that the canonical decomposition of α is

$$\alpha = \gamma_1 + \dots + \gamma_y$$

There are strong restrictions on the dimension vectors γ_i .

Recall that if χ denotes the Euler-form for Q , we say that a dimension vector γ is *real* if and only if $\chi(\gamma, \gamma) = 1$.

From [24, Th. 2.4] we recall that among the components γ_i in the canonical decomposition of α we have : either $\gamma = \gamma_j$ is a real root or

$$\chi(\beta_i, \beta_j)\chi(\beta_j, \beta_i) = 0$$

We can compute the canonical decomposition of α in the more general situation of a marked quiver Q^* by working out the canonical decomposition of α for the underlying quiver Q (that is, forgetting the marks).

In the (marked) quiver cases that will occur in our investigation, it will turn out that the canonical decomposition coincides with the generic semi-simple representation type. It is clear from the above that this imposes stringent conditions on the dimension vector.

3.6. We now turn to the ring of polynomial invariants

$$R = \mathbb{C}[\text{Rep}(Q^*, \alpha)]^{GL(\alpha)}$$

It is a very interesting (but hard) problem to determine all *coregular* marked quivers Q^* and dimension vectors α . That is, all pairs (Q^*, α) such that the ring of polynomial invariants is a polynomial algebra.

Recall that the ring of polynomial invariants R is by [17, Thm. 1] generated by traces of oriented cycles in the quiver Q^* of length at most N^2 where $N = \sum_i a_i$ (observe that the proof generalizes to the setting of marked quivers as we can separate traces). That is, for every arrow ϕ (resp. loop or marked loop) in Q^* from v_i to v_j we take a generic $a_i \times a_j$ matrix

$$M_\phi = \begin{bmatrix} x_{11}(\phi) & \dots & x_{1a_j}(\phi) \\ \vdots & & \vdots \\ x_{a_i 1}(\phi) & \dots & x_{a_i a_j}(\phi) \end{bmatrix}$$

(resp. a generic square matrix or a generic trace zero matrix).

If $cyc = \phi_1 \phi_2 \dots \phi_k$ is an oriented cycle in Q^* , we compute the matrix

$$M_{cyc} = M_{\phi_1} M_{\phi_2} \dots M_{\phi_k}$$

over the ring $\mathbb{C}[x_{ij}(\phi)] = \mathbb{C}[Rep(Q^*, \alpha_m)]$. If the starting vertex of ϕ_1 is v_i , this is an $a_i \times a_i$ matrix and we can compute its trace

$$Tr(M_{cyc}) \in \mathbb{C}[Rep(Q^*, \alpha_m)]$$

which is clearly a polynomial invariant under the action of $GL(\alpha_m)$.

The assertion of [17, Thm. 1] is that these functions actually generate the ring of invariants R .

Similar to the case of module varieties recalled before, the inclusion $R = \mathbb{C}[Rep(Q^*, \alpha)]^{GL(\alpha)} \hookrightarrow \mathbb{C}[Rep(Q^*, \alpha)]$ induces a quotient morphism

$$Rep(Q^*, \alpha) \xrightarrow{\pi_Q} Rep(Q^*, \alpha)/GL(\alpha)$$

A point in the quotient variety determines a maximal ideal p of R . Further, the fiber $\pi_Q^{-1}(p)$ contains a unique closed orbit, which again is the orbit of a semi-simple representation of Q^* of dimension vector α .

We have given a combinatorial method to describe all dimension vectors of simple representations of Q^* hence we can determine all *representation types* $\tau = (m_1, \beta_1; \dots; m_z, \beta_z)$ with

$$\alpha = m_1 \cdot \beta_1 + \dots + m_z \cdot \beta_z$$

where β_i is the dimension vector of a simple representation occurring with multiplicity m_i .

Let p be a maximal ideal of R corresponding to a semi-simple representation of Q^* of representation type τ .

In order to describe the local structure of R in p , we will introduce a new marked quiver Q_τ^* .

The *local marked quiver* Q_τ^* is a marked quiver on z vertices $\{w_1, \dots, w_z\}$. It has

- $-\chi(\beta_i, \beta_j)$ directed arrows from w_i to w_j for $i \neq j$,
- $1 - \chi_1(\beta_i, \beta_i)$ loops in w_i and
- $-\chi_2(\beta_i, \beta_i)$ marked loops in w_i

where χ is the Euler form of Q^* that is,

$$\chi = \chi_1 + \chi_2 \text{ where } \chi_1 = (\delta_{ij} u_i)_{i,j} - (a_{ij})_{i,j} \text{ and } \chi_2 = (-\delta_{ij} m_i)_{i,j}$$

with a_{ij} the number of arrows from v_i to v_j and u_i resp. m_i the number of (resp. marked) loops in v_i in Q^* .

Finally, we define a new dimension vector

$$\alpha_\tau = (m_1, \dots, m_z) \in \mathbb{N}^z$$

Similar to the proof given in [17, Thm. 5], but taking the marked loops into consideration, one proves the following result.

Proposition 3.3. *Let p be a maximal ideal of $R = \mathbb{C}[\text{Rep}(Q^*, \alpha)]^{GL(\alpha)}$ corresponding to a semi-simple representation of type τ . Let S be the ring of invariants $\mathbb{C}[\text{Rep}(Q_\tau^*, \alpha_\tau)]^{GL(\alpha_\tau)}$ of the local marked quiver and let q be its maximal graded ideal, then there is an isomorphism between the completions*

$$\hat{R}_p \simeq \hat{S}_q$$

Therefore, if R is a polynomial ring, then S is also a polynomial ring. This often allows us to reduce to a simpler (marked) quiver situation.

3.7. It will be crucial for our purposes to have some control on the coregular marked quiver situations (Q^*, α) . We can reduce to the case of a strongly connected marked quiver.

Lemma 3.4. *The polynomial invariants of $\text{Rep}(Q^*, \alpha)$ form a polynomial algebra if and only if for every maximal strongly connected component Q' of Q^* the polynomial invariants of $\text{Rep}(Q', \alpha | Q')$ form a polynomial algebra.*

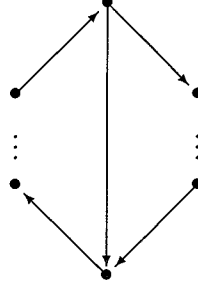
Proof. The polynomial invariants are generated by taking traces along oriented cycles in Q^* by [17, Thm. 1]. \square

We can now state and prove a crude characterization of marked quiver situations which are coregular, that is, having a polynomial ring of invariants.

Theorem 3.5. *Let Q^* be a strongly connected marked quiver and $\alpha = (a_1, \dots, a_k)$ a dimension vector.*

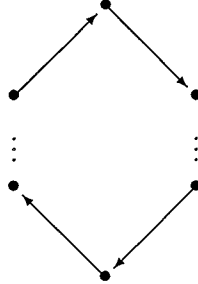
If the ring of polynomial invariants $\mathbb{C}[\text{Rep}(Q^, \alpha)]^{GL(\alpha)}$ is a polynomial ring, then we are in one of the following situations :*

1. type 1 : $\min_i a_i \leq 1$ or
2. type 2 : $\min_i a_i = 2$ and Q^* has the form $\tilde{A}_k(+1)$:



that is, the extended Dynkin diagram with cyclic orientation and one extra arrow (which may degenerate to a cycle and loop or to two loops, possibly marked), or

3. type 3 : $\min_i a_i \geq 2$ and the marked quiver Q^* is the extended Dynkin diagram \tilde{A}_k with cyclic orientation



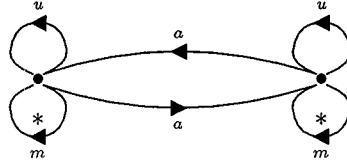
(which may degenerate to one loop possibly marked).

Proof. Let us assume that $\min_i a_i \geq 2$. Because Q^* is strongly connected $(1, \dots, 1)$ is the dimension vector of a simple representation and there are infinitely many isoclasses of such representations.

Hence, in $\text{Rep}(Q^*, \alpha)$ there are semi-simple representations of type

$$\tau = (1, (1, \dots, 1); 1, (1, \dots, 1); a_1 - 2, \delta_1; \dots; a_k - 2, \delta_k)$$

The local marked quiver Q_τ^* has at most $k + 2$ vertices and the full subquiver on the first two vertices is of the following form



where there are a directed arrows between the vertices with

$$a = -\chi((1, \dots, 1), (1, \dots, 1)) = \sum_i \left(\sum_j a_{ij} + u_i + m_i - 1 \right)$$

and u (resp. m) loops (resp. marked loops) in the vertices where

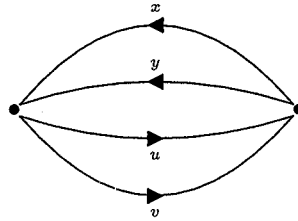
$$u = 1 - \chi_1((1, \dots, 1), (1, \dots, 1)) = \sum_{i,j} a_{ij} + \sum_i u_i - k + 1$$

$$m = -\chi_2((1, \dots, 1), (1, \dots, 1)) = \sum_i m_i$$

and the dimension vector $\alpha_\tau \mid \{w_1, w_2\} = (1, 1)$. We claim that

$$\sum_i \left(\sum_j a_{ij} + u_i + m_i - 1 \right) \leq 1$$

If not, we have the following subquiver \mathbb{B} in Q_τ^* :



$$\mathbb{B} =$$

Then, we would have in $\mathbb{C}[\text{Rep}(Q_\tau^*, \alpha_\tau)]^{GL(\alpha_\tau)}$ the invariants ux, uy, vx and vy which satisfy the relation $(ux)(vy) = (uy)(vx)$. Hence, the ring of invariants of (Q_τ^*, α_τ) cannot be a polynomial ring. But then, the same would hold for (Q^*, α) .

Therefore, at most one of the $\sum_j a_{ij} + u_i + m_i - 1$ can be equal to one. Let us consider the different possibilities.

There is just one vertex, in which case $u_1 + m_1 - 1 \leq 1$ and we have the one or two loop quiver (one or both of the loops may be marked). If there is one (marked) loop there is no restriction on the dimension vector $\alpha = (a_1)$. If there are two (marked) loops, it is well known that the ring of polynomial invariants is a polynomial ring if and only if $a_1 \leq 2$, see for example [16] or an adaptation of the argument given below.

There are at least two vertices and all terms $\sum_j a_{ij} + u_i + m_i - 1 = 0$. As Q^* is strongly connected this implies that all $u_i = m_i = 0$ and for each i there is a unique arrow to another vertex.

Hence, we are in the case $Q^* = \tilde{A}_k$ ($k \geq 2$) and there are no restrictions on the dimension vector for the invariants to be a polynomial ring.

There are at least two vertices and for a unique vertex v_i the sum $\sum_j a_{ij} + u_i + m_i - 1 = 1$, then we are in the case that $Q^* = \tilde{A}_k(+1)$ with the extra arrow possibly a (marked) loop starting in v_i .

In this case we still have to prove that $\min_j a_j \leq 2$. The case of an extra (marked) loop in v_i reduces easily to the two loop case treated before. So we may assume that there is a unique $i' \neq i$ with $v_{i'}$ the end point of the extra arrow.

First we reduce to the case of $\tilde{A}_4(+1)$. Fix the following four vertices : $z_1 = v_i$, $z_3 = v_{i'}$, z_2 the vertex v_u on the oriented path from v_i to $v_{i'}$ where a_u is minimal and likewise z_4 is the vertex $v_{u'}$ on the oriented path from $v_{i'}$ to v_i where $a_{u'}$ is minimal.

Observe that degenerate cases are possible if either a_i or $a_{i'}$ is minimal but this only simplifies the argument given below.

Let $\beta = (a_i, a_u, a_{i'}, a_{u'})$ then we claim that

$$\mathbb{C}[\text{Rep}(\tilde{A}_k(+1), \alpha)]^{GL(\alpha)} \simeq \mathbb{C}[\text{Rep}(\tilde{A}_4(+1), \beta)]^{GL(\beta)}$$

Indeed, classical invariant theory (see for example [13, Thm. II.4.1]) tells us that

$$(M_{a \times b}(\mathbb{C}) \oplus M_{b \times c}(\mathbb{C})) / GL_b \simeq M_{a \times c}(\mathbb{C})$$

if $b \geq \min(a, c)$. Iterating this reduction we obtain the claim.

Therefore, we only have to exclude the special case when Q^* is of the following form

$$Q^* = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \end{array} \quad \text{with} \quad \chi = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

and the dimension vector $\beta = (b_1, b_2, b_3, b_4)$ satisfies $\min b_i \geq 3$.

Then, one verifies that $(2, 1, 2, 2)$ and $(1, 1, 1, 1)$ are dimension vectors of simple representations and by assumption there are semi-simple representations in $\text{Rep}(Q^*, \beta)$ of type

$$\tau = (1, (2, 1, 2, 2); 1, (1, 1, 1, 1); b_1 - 3, \delta_1; b_2 - 2, \delta_2; b_3 - 3, \delta_3; b_4 - 3, \delta_4)$$

Calculating the local quiver Q_τ^* we observe that it again contains a subquiver of the form \mathbb{B} and we can repeat the argument given above to exclude this case. \square

Clearly, the first case of the foregoing theorem is the hardest to classify. Even when all the dimension components are equal to one, a full classification of the settings where the ring of invariants is a polynomial ring is unknown at the moment. We hope to return to this problem in another paper.

4. BLOCK STRUCTURE

4.1. In this section we will describe the structure of the m -box \mathbb{B}_m . We will relate its connected components to the splitting behavior of Z over C .

Throughout, we consider a triad (A, Z, C) and a maximal ideal m of C with corresponding semi-simple d -dimensional module

$$M_m^{ss} = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

where S_i is a simple A -module of dimension d_i occurring with multiplicity e_i . Clearly, $d = \sum d_i e_i$.

We want to relate these multiplicities to the structure of the finite dimensional algebra

$$\bar{A}_m = A/mA$$

With the induced trace map \bar{tr} , \bar{A}_m is a Cayley-Hamilton algebra of degree d having k distinct simple modules S_i of dimension d_i . As a trace map vanishes on the radical, the associated semi-simple algebra

$$\bar{A}_m / \text{rad } \bar{A}_m \simeq M_{d_1}(\mathbb{C}) \oplus \dots \oplus M_{d_k}(\mathbb{C})$$

is a Cayley-Hamilton algebra of degree d for the induced trace map \bar{tr} . If Tr_i denotes the usual trace on $M_{d_i}(\mathbb{C})$ then there exist by [4, Prop. 4.3] unique $e_i \geq 1$ such that

$$\bar{tr} = \sum_{i=1}^k e_i Tr_i$$

These numbers determine the multiplicities with which S_i occurs in the semi-simple A -module M_m^{ss} .

From the theory of finite dimensional algebras we recall that \bar{A}_m is Morita equivalent to a finite dimensional *basic* algebra \bar{B}_m , that is, such that all simple \bar{B}_m -modules are one dimensional.

The algebra \bar{B}_m is a quotient of the path algebra of a quiver \mathbb{S}_m by an admissible ideal of relations. We call \mathbb{S}_m the *skeleton quiver* for m .

We construct this quiver \mathbb{S}_m on k vertices $\{v_1, \dots, v_k\}$ such that the number \bar{a}_{ij} of directed arrows (possibly including loops) from v_i to v_j is given by the formula

$$\bar{a}_{ij} = \dim_{\mathbb{C}} \text{Ext}_{\bar{A}_m}^1(S_i, S_j).$$

An element $r = \sum c_i p_i \in \mathbb{C} \mathbb{S}_m$ is said to be *admissible* if all the paths p_i occurring with nonzero coefficient c_i have length ≥ 2 .

There is a twosided ideal I_B of $\mathbb{C} \mathbb{S}_m$ of admissible elements such that

$$\bar{B}_m \simeq \mathbb{C} \mathbb{S}_m / I_B$$

Clearly, we have $\bar{a}_{ij} \leq a_{ij}$, where we recall that the $a_{ij} = \dim_{\mathbb{C}} \text{Ext}_A^1(S_i, S_j)$ are the data determining the ext-quiver \mathbb{E}_m .

4.2. We will now relate the structure of \mathbb{B}_m and the dimension vector α_m to the properties of the extension of commutative algebras $C \hookrightarrow Z$.

Because Z is a projective C -module of rank s , $\bar{Z}_m = Z/mZ$ is a (commutative) s -dimensional algebra whence decomposes into local algebras :

$$\bar{Z}_m \simeq L_1 \oplus \dots \oplus L_l.$$

This means that there are l maximal ideals of Z , say $\{p_1, \dots, p_l\}$, lying over m . If $\dim_{\mathbb{C}} L_a = n_a$ we say that p_a has *multiplicity* n_a .

The central character $\chi_i = \text{Ann}(S_i) \cap Z$ of the simple A -module S_j is a maximal ideal among the $\{p_1, \dots, p_l\}$. Hence, we can reorder the S_j such that there exist numbers $0 = k_0 < k_1 < \dots < k_l = k$ such that the interval

$$[1 \dots k] = I_1 \sqcup \dots \sqcup I_l \text{ with } I_a = [k_{a-1} + 1 \dots k_a]$$

decomposes into subintervals I_a with $\chi_j = p_a$ if and only if $j \in I_a$.

Because A equipped with the reduced trace map Tr is a Cayley-Hamilton algebra of degree n , each p_a determines a semi-simple n -dimensional A -module

$$M_a = \bigoplus_{j \in I_a} S_j^{\oplus m_j} \text{ where } \sum_{j \in I_a} m_j d_j = n.$$

If we denote with μ_a the dimension vector in \mathbb{Z}^k having components m_j if $j \in I_a$ and zeroes otherwise, we obtain the relation

$$\alpha_m = \sum_{a=1}^l n_a \cdot \mu_a.$$

This determines the dimension vector α_m in terms of the ramification of Z over C .

4.3. Consider p_a , then the simple A -modules with central character p_a are exactly the S_i with $i \in I_a$. Consider A as a Cayley-Hamilton algebra of degree n under the reduced trace map and let \mathbb{B}_{p_a} be the marked quiver corresponding to the maximal ideal p_a of $Z = \text{Tr}(A)$. That is, \mathbb{B}_{p_a} is the p_a -box for the triple (A, Z, Z) . We recall the following result, see for example [8, Thm. 11.20].

Proposition 4.1 (Müller). *With notations as before, \mathbb{B}_{p_a} is a connected quiver. That is, if S_i and S_j are simple A -modules with central character p_a , then there exist simple modules T_1, \dots, T_t with central character p_a such that $T_1 = S_i$, $T_t = S_j$ and $\text{Ext}_A^1(T_i, T_{i+1}) \neq 0$ or $\text{Ext}_A^1(T_{i+1}, T_i) \neq 0$.*

The only possible difference between $\mathbf{b}_a = \mathbb{B}_m \upharpoonright I_a$ and \mathbb{B}_{p_a} is that some of the unmarked loops in \mathbb{B}_{p_a} may become marked in \mathbf{b}_a . In particular, it follows from the foregoing proposition that the \mathbf{b}_a are the connected components of \mathbb{B}_m .

Concluding, we have the following general structural results for the m -box \mathbb{B}_m and the corresponding dimensionvector α_m .

Theorem 4.2. *Consider a triad (A, Z, C) . Let m be a maximal ideal of C and $\{p_1, \dots, p_l\}$ the maximal ideals of Z lying over m . Let n_a be the multiplicity of p_a .*

1. \mathbb{B}_m is the disjoint union of $\mathbf{b}_1 \sqcup \dots \sqcup \mathbf{b}_l$ with \mathbf{b}_a a connected component.
2. $\alpha_m = \sum_{a=1}^l n_a \mu_a$ where $d_a \cdot \mu_a = n$.

Definition 4.3. For every $1 \leq a \leq l$ we call the marked quiver \mathbf{b}_a the p_a -block and combined with the dimension vector $n_a \mu_a$ the block data associated to the maximal ideal p_a lying over m .

Recall that the algebra $\overline{A}_m = A/mA$ has associated skeleton quiver \mathbb{S}_m . The connected components of \mathbb{B}_m and \mathbb{S}_m are the same (though there may be more arrows in the former). This is again evident from the result of Müller. Therefore, the above definition coincides with the usual notion of *block* of a finite dimensional algebra.

5. THE SMOOTH LOCUS

5.1. Throughout, (A, Z, C) will be a triad, m a maximal ideal of C and $\{p_1, \dots, p_l\}$ the maximal ideals of Z lying over m . In this section we will study the extra restrictions on the m -box \mathbb{B}_m and on the block data $(\mathbf{b}_a, n_a \mu_a)$ imposed by the condition that A is smooth in m .

Recall that $\pi_A : X_A \dashrightarrow \text{Spec } C$ is the algebraic quotient map corresponding to the inclusion $C = \mathbb{C}[X_A]^{GL_d} \hookrightarrow \mathbb{C}[X_A]$.

Definition 5.1. We say that A is *smooth in m* if and only if X_A is smooth along $\pi_A^{-1}(m)$. The set of all maximal ideals m such that A is smooth in m is called the *smooth locus of A over C* and is denoted by $\text{Sm } A/C$.

As the singular locus of X_A is a closed GL_d -stable subscheme of X_A , A is smooth in m if and only if X_A is smooth in a point of the unique closed orbit in $\pi_A^{-1}(m)$ determined by the semi-simple d -dimensional A -module M_h^{ss} .

For any triad (A, Z, C) the smooth locus $\text{Sm } A/C$ contains a Zariski open subset of $\text{Spec } C$.

Definition 5.2. Given a triad (A, Z, C) and a maximal ideal m of C . We say that A is *separable* in m if and only if

$$\overline{A}_m \simeq \bigoplus_{i=1}^s M_n(\mathbb{C}).$$

That is, there are s distinct maximal ideals of Z lying over m each corresponding to a single simple A -module of dimension n .

The *separability locus* of A over C is the set of all maximal ideals m where A is separable. It will be denoted by $\text{Sep } A/C$.

This notion corresponds to the usual notion of *separable extensions*, see for example [12, Chp. III]. Recall that an R -algebra Λ is said to be *separable* if and only one of the following equivalent conditions is satisfied

1. Λ is a projective $\Lambda^e = \Lambda \otimes \Lambda^{opp}$ -module.
2. Λ^e contains a *separability idempotent*, that is, an element e such that

$$(\lambda \otimes 1)e = e(1 \otimes \lambda) \quad \forall \lambda \in \Lambda$$

and $p(e) = 1$ where $p : \Lambda \otimes_R \Lambda^{opp} \longrightarrow \Lambda$ is the multiplication.

We apply this result in the following way. Assume that A_m is a separable C_m -algebra, then A_m^e contains a separability idempotent e . But then, there is an $f \in C - m$ such that $e \in A_f^e$ and hence also A_f is a separable C_f -algebra. The Zariski open subset of maximal ideals m such that A_m is a separable C_m -algebra coincides with our definition of $\text{Sep } A/C$.

To see this, observe that an algebra Λ over a local algebra R (with maximal ideal m) is separable if and only if $\Lambda/\Lambda m$ is a separable R/Rm -algebra. In our case, $C_m/C_m m \simeq \mathbb{C}$ and the only separable \mathbb{C} -algebras are products of matrices. Hence $A/\Lambda m$ is a semi-simple algebra. However, since A is a projective C -module of rank sn^2 , we have seen that the image $\text{Max } C \longrightarrow \text{Alg}_{sn^2}$ is contained in the orbit closure of $M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})$ (s copies). As any semi-simple sn^2 -dimensional algebra determines an open orbit in Alg_{sn^2} it follows that

$$\overline{A}_m = A/\Lambda m \simeq M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})$$

Lemma 5.3. Consider a triad (A, Z, C) with C a regular domain. Then, the separability locus of A over C is contained in the smooth locus of A over C .

Proof. Let U be the Zariski open subset of $\text{Spec } C$ determined by the separability locus. From the definition it is clear that X_A is a principal fibration over U with fibers isomorphic to GL_d/T^s . As both the base and the fibers are smooth, so is $\pi_A^{-1}(U)$. \square

Because any affine domain C has a Zariski open subset of smooth points, the next result follows immediately.

Proposition 5.4. For a triad (A, Z, C) we have

$$\dim X_A = \dim C + d^2 - s$$

Proof. We have seen that X_A is a principal fibration over the intersection of the smooth locus of $\text{Spec } C$ with the separability locus of A over C , with fiber isomorphic to GL_d/T_s . The formula now follows. \square

5.2. We will use the m -box \mathbb{B}_m and dimension vector α_m to give a numerical criterium to verify smoothness.

With χ_m we will denote the *Euler-form* of \mathbb{B}_m .

Theorem 5.5. A is smooth in m if and only if

$$\dim C = s - \chi_m(\alpha_m, \alpha_m) - \sum_i m_i$$

Proof. Let x_m be a point in the orbit in X_A corresponding to M_m^{ss} . Then, $GL(\alpha_m)$ is the stabilizer group in x_h . We have

$$\begin{aligned} \dim T_{x_m} &= \dim T_{x_m}/T_{x_m}^0 + \dim T_{x_m}^0 \\ &= \dim \operatorname{Ext}_A^1(M_m^{ss}, M_m^{ss})_{tr} + d^2 - \sum_{i=1}^k e_i^2 \\ &= \dim \operatorname{Rep}(\mathbb{B}_m, \alpha_m) + d^2 - \sum_{i=1}^k e_i^2 \end{aligned}$$

Knowing the dimension of X_A , the condition $\dim X_A = \dim T_{x_m}$ gives the equations

$$\begin{aligned} \dim C - s &= \dim \operatorname{Rep}(\mathbb{B}_m, \alpha_m) - \sum_{i=1}^k e_i^2 \\ &= -\chi_m(\alpha_m, \alpha_m) - \sum_{i=1}^k m_i \end{aligned}$$

proving the theorem. \square

Definition 5.6. In general, we have that $s - \chi_m(\alpha_m, \alpha_m) - \sum_i m_i \geq \dim C$. Therefore, the number

$$s_m = s - \chi_m(\alpha_m, \alpha_m) - \sum_i m_i - \dim C$$

is a measure for the singularity of A in m .

5.3. To illustrate the concepts introduced and the above theorem, we give an example of a noncommutative Hopf algebra which is smooth in all maximal ideals of a central sub-Hopf algebra.

Example 5.7. Consider the quantum Borel $A = U_q(\mathfrak{b})$ of $U_q(\mathfrak{sl}_2)$ where q is a primitive l -th root of unity, l odd. A is generated by E and K, K^{-1} satisfying the defining relation $KE = q^2 EK$.

The Hopf-structure is defined by taking K to be grouplike and E a skew-primitive with

$$\Delta(E) = E \otimes 1 + K \otimes E$$

A is a free module of rank l^2 over its center $C = \mathbb{C}[K^l, K^{-l}, E^l]$ which is a sub-Hopf algebra isomorphic to the coordinatizing of $\mathbb{C}^* \times \mathbb{C}_+$.

The reduced trace map on A is given by $\operatorname{tr}(E^a K^b) = 0$ unless both a and b are multiples of l in which case $\operatorname{tr}(E^a K^b) = lE^a K^b$. It makes A into a Cayley-Hamilton algebra of degree l . We consider the triad $(U_q(\mathfrak{b}), \mathbb{C}[B], \mathbb{C}[B])$ where $\mathbb{C}[B] = \mathbb{C}[E^l, K^l, K^{-l}]$.

Let m be the maximal ideal $(K^l - a^l, E^l - b)$ with $ab \neq 0$, then $\bar{A}_m \simeq M_l(\mathbb{C})$ as it is generated by

$$\rho(K) = \begin{bmatrix} a & & & \\ & q^2 a & & \\ & & \ddots & \\ & & & q^{2(l-1)} a \end{bmatrix} \text{ and } \rho(E) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \end{bmatrix}$$

One verifies that $\operatorname{Ext}_A^1(M_m^{ss}, M_m^{ss}) = \mathbb{C}\alpha + \mathbb{C}\beta$ where the algebra map $A \longrightarrow M_l(\mathbb{C}[\epsilon])$ is given by

$$K \mapsto \rho(K) + \alpha\epsilon I_l \text{ and } E \mapsto \rho(E) + \beta\epsilon I_l$$

and these algebra maps are trace preserving. Therefore, $\mathbb{E}_m = \mathbb{B}_m$ is the 2-loop quiver



and the dimension-vector $\alpha_m = (1)$. In this case, $s = 1$, $\chi_m = (-1)$ and one checks that

$$s - \chi_m(\alpha_m, \alpha_m) - \sum m_i = 1 - (-1) - 0 = 2 = \dim C$$

whence A is smooth in m . This corresponds to the fact that locally around m , X_A is a principal PGL_l -fibration whence smooth.

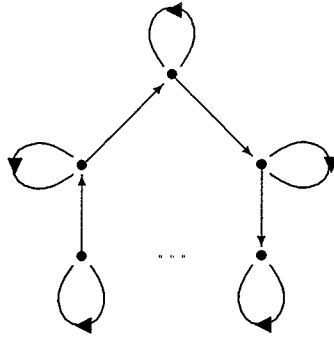
Next, let m be the maximal ideal $(K^l - a, E^l)$ with $a \neq 0$, then

$$M_m^{ss} \simeq S_1 \oplus \dots \oplus S_l$$

where S_i is the simple one-dimensional A -module determined by $\rho_i(K) = q^{2i}a$ and $\rho_i(E) = 0$. One verifies that

$$\text{Ext}_A^1(S_i, S_i) = \mathbb{C}\alpha_i \text{ and } \text{Ext}_A^1(S_i, S_j) = \delta_{i+1,j} \mathbb{C}\beta_i$$

whence the quiver \mathbb{E}_m has the following shape

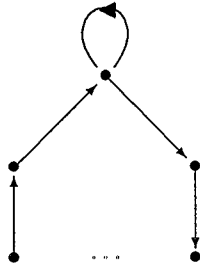


and the dimension-vector is $\alpha_m = (1, \dots, 1)$. The algebra map $A \longrightarrow M_l(\mathbb{C}[\epsilon])$ corresponding to the representation $(\alpha_i, \beta_i \mid 1 \leq i \leq l)$ is given by

$$K \mapsto \begin{bmatrix} a + \alpha_1 \epsilon & & & \\ & q^2 a + \alpha_2 \epsilon & & \\ & & \ddots & \\ & & & q^{2(l-1)} a + \alpha_l \epsilon \end{bmatrix}$$

$$E \mapsto \begin{bmatrix} 0 & \beta_1 \epsilon & 0 & 0 \\ 0 & 0 & \beta_2 \epsilon & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \beta_{l-1} \epsilon \\ \beta_l \epsilon & 0 & 0 & 0 \end{bmatrix}$$

Setting $\text{tr}(K^i) = 0$ for $1 \leq i < l$ gives $l - 1$ linear relations among the α_i leaving just a one-dimensional solution space. Therefore, \mathbb{E}_m is the quiver



In this case, the Euler-form χ_m is determined by the $l \times l$ -matrix

$$\chi_m = \begin{bmatrix} 0 & -1 & 0 & & 0 \\ & 1 & -1 & & 0 \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ -1 & & & & 1 \end{bmatrix}$$

whence

$$s - \chi_m(\alpha_m, \alpha_m) - \sum m_i = 1 - (-1) - 0 = 2 = \dim C$$

and therefore A is also smooth in these points.

Remark that it would be rather difficult to prove smoothness of $X_{U_q(b)}$ directly.

5.4. Before we can proceed we have to recall the Luna slice theorem in invariant theory. We have already given an application in proposition 3.3. For future reference we give the result in a more general setting than we need.

Recall the presentation $A = \mathbb{C}\langle x_1, \dots, x_m \rangle_{tr} / (r_A)$ which gave us the epimorphism $\mathbb{C}[y_{jk}^{(i)} \mid i, j, k] \longrightarrow \mathbb{C}[X_A]$. Therefore, X_A is a closed GL_d -stable subscheme of $M_d(\mathbb{C})^{\oplus m}$ on which the action is given by simultaneous conjugation.

Let x be a point in $M_d(\mathbb{C})^{\oplus m}$ in the orbit of M_m^{ss} . The normal space in x to the GL_d -orbit has been computed in [16, §III.1]. We recall that it can be identified with

$$\text{Rep}(Q_m^{ext}, \alpha_m)$$

where Q_m^{ext} is the quiver on $\{v_1, \dots, v_k\}$ such that the number of directed arrows from v_i to v_j is given by the formula $A_{ij} = (m-1)d_i d_j$ and the number of loops in v_i is equal to $U_i = (m-1)d_i^2 + 1$.

An element e of this space can be identified with an m -tuple of $d \times d$ matrices (E_1, \dots, E_m) in block form, the first of which $E_1 \in \text{Lie}(GL(\alpha_m)) \hookrightarrow \text{Lie}(GL_d) = M_d(\mathbb{C})$.

We can define an algebra map

$$\varepsilon : \mathbb{C}\langle x_1, \dots, x_m \rangle_{tr} \longrightarrow M_d(\mathbb{C}[\text{Rep}(Q_m^{ext}, \alpha_m)])$$

such that for all $e \in \text{Rep}(Q_m^{ext}, \alpha_m)$ we have

$$\varepsilon(x_i)(e) = \rho(x_i) + E_i$$

Definition 5.8. The *slice algebra* of A in m is the commutative affine algebra

$$S_m = \mathbb{C}[\text{Rep}(Q_m^{ext}, \alpha_m)] / (J_A)$$

where J_A is the ideal generated by the entries of $\varepsilon(r_A)$ where r_A is the ideal of relations of A as a Cayley-Hamilton algebra.

From the universal property of i_A we obtain a uniquely determined (surjective) algebra map $\bar{\varepsilon} : \mathbb{C}[X_A] \longrightarrow S_m$ such that the diagram below is commutative

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & M_d(S_m) \\ i_A \downarrow & \nearrow M_d(\bar{\varepsilon}) & \\ M_d(\mathbb{C}[X_A]) & & \end{array}$$

The slice algebra S_m determines a closed subscheme of $\text{Rep}(Q_m^{ext}, \alpha_m)$ containing the zero representation. Let z' denote the maximal ideal of S_m corresponding to the zero representation. If x_m is the maximal ideal of $\mathbb{C}[X_A]$ corresponding to the chosen point in the orbit of M_m^{ss} , then $\bar{\varepsilon}^{-1}(z') = x_m$.

The action by automorphisms of GL_d on X_A induces an action of $GL(\alpha_m) \hookrightarrow GL_d$ on the slice algebra S_m . Moreover, $GL(\alpha_m)$ acts on $\mathbb{C}[GL_d]$ by automorphisms via the action

$$GL(\alpha_m) \times GL_d \longrightarrow GL_d \text{ where } (h, g) \mapsto gh^{-1}.$$

Hence, we have a $GL(\alpha_m)$ action on $\mathbb{C}[X_A] \otimes \mathbb{C}[GL_d]$ and on $S_m \otimes \mathbb{C}[GL_d]$ and the map $\bar{\varepsilon} \otimes id$ is $GL(\alpha_m)$ -equivariant and therefore induces an algebra map between the ring of $GL(\alpha_m)$ -invariants.

If $\mathbb{C}[X_A] \xrightarrow{ac} \mathbb{C}[X_A] \otimes \mathbb{C}[GL_d]$ denotes the comodule algebra map encoding the action of GL_d on X_A , one verifies that $ac(\mathbb{C}[X_A])$ is contained in the $GL(\alpha_m)$ -invariants. Therefore we have an algebra morphism

$$\mathbb{C}[X_A] \xrightarrow{\psi} (S_m \otimes \mathbb{C}[GL_d])^{GL(\alpha_m)}.$$

On the other hand, multiplication on the left in GL_d induces an action of GL_d on $S_m \otimes \mathbb{C}[GL_d]$ by working on the second factor. As this action commutes with the $GL(\alpha_m)$ -action, it defines an action of GL_d on the invariants $(S_m \otimes \mathbb{C}[GL_d])^{GL(\alpha_m)}$ and the corresponding ring of invariants is equal to

$$((S_m \otimes \mathbb{C}[GL_d])^{GL(\alpha_m)})^{GL_d} \simeq S_m^{GL(\alpha_m)}$$

Concluding, we have the following commutative diagram of algebras

$$\begin{array}{ccc} \mathbb{C}[X_A] & \xrightarrow{\psi} & (S_m \otimes \mathbb{C}[GL_d])^{GL(\alpha_m)} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\bar{\psi}} & S_m^{GL(\alpha_m)} \end{array}$$

For more details on *étale morphisms* and their properties we refer to [22]. The next result is an application of the Luna slice theorem [19] as extended in the proof of F. Knop [11].

Theorem 5.9. *Let z be the maximal ideal of $S_m^{GL(\alpha_m)}$ corresponding to the zero representation in $\text{Rep}(Q_m^{\text{ext}}, \alpha_m)$. Then, there exists an $f \in S_m^{GL(\alpha_m)} - m$ such that in the localized diagram*

$$\begin{array}{ccc} \mathbb{C}[X_A] & \xrightarrow{\psi_f} & (S_m \otimes \mathbb{C}[GL_d])^{GL(\alpha_m)} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\bar{\psi}_f} & (S_m^{GL(\alpha)})_f \end{array}$$

the horizontal morphisms are étale and ψ_f is GL_d -equivariant. In particular, if $T = S_m^{GL(\alpha)}$ then

$$\hat{C}_m \simeq \hat{T}_z$$

Proof. Observe that by definition S_m is the scheme theoretic intersection of X_A with the normal space in $M_d(\mathbb{C})^{\oplus m}$ in the point corresponding to M_m^{ss} to the GL_d -orbit. By [11, p.112] this is the scheme on which to apply Luna's fundamental lemma. \square

We will only need a special case of this result. When A is smooth in m , we can apply [11, p. 113] and see that the slice algebra S_m can be replaced by $\mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)]$. That is, we have the following result.

Theorem 5.10. *Let A be smooth in m . The ring of polynomial invariants $R = \mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)]^{GL(\alpha_m)}$ is a positively graded algebra with maximal ideal $\mathfrak{n} = R_+$. Then, there exists an $f \in R - \mathfrak{n}$ such that in the localized diagram*

$$\begin{array}{ccc} \mathbb{C}[X_A] & \xrightarrow{\psi_f} & (\mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)] \otimes \mathbb{C}[GL_d])_f^{GL(\alpha_m)} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\bar{\psi}_f} & R_f \end{array}$$

the horizontal morphisms are étale and ψ_f is GL_d -equivariant. In particular,

$$\hat{C}_m \simeq \hat{R}_n$$

5.5. We are now in a position to state our main theorem concerning the blocks \mathfrak{b}_a of the m -box \mathbb{B}_m .

Theorem 5.11. *Let A be smooth in m and let $\{p_1, \dots, p_l\}$ be the maximal ideals of Z lying over m such that p_a occurs with multiplicity n_a .*

Let $(\mathfrak{b}_a, n_a \mu_a)$ be the block data for \mathbb{B}_a where $1 \leq a \leq l$.

1. \mathfrak{b}_a are the strongly connected components of \mathbb{B}_a .
2. The generic semi-simple representation type of $n_a \mu_a$ coincides with the canonical decomposition of $n_a \mu_a$ and is of the form :

$$n_a \mu_a = \beta_1^{(a)} + \dots + \beta_{n_a}^{(a)}$$

with $d \cdot \beta_i^{(a)} = n$ for all $1 \leq i \leq n_a$.

3. If m is the total number of marked loops in \mathbb{B}_m , then

$$\dim C = s - \sum_{a=1}^l \sum_{i=1}^{n_a} \chi_m(\beta_i^{(a)}, \beta_i^{(a)}) - m$$

Proof. Because A is smooth in m the slice algebra S_m can be taken to be $\mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)]$. Let $R = \mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)]^{GL(\alpha_m)}$, then R is a positively graded algebra with maximal graded ideal $\mathfrak{n} = R_+$. We know that there is an étale local isomorphism between an open subset of $\text{Spec } R$ containing \mathfrak{n} and one of $\text{Spec } C$ containing m .

Any open neighborhood of m contains maximal ideals u such that $\bar{A}_u \simeq M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C})$ (s copies). The maximal ideal v of R corresponding to u determines a semi-simple representation of Q_m^* of dimension vector α_m .

By GL_d -quariance of ψ_f we know that the automorphism group of this semi-simple representation is $\mathbb{C}^* \times \dots \times \mathbb{C}^*$ (s copies). Therefore, it is the direct sum of s simple representations of \mathbb{B}_m say of dimension vectors γ_j where $d \cdot \gamma_j = n$.

Again, by GL_d -equivariance of ψ_f we know that there is a unique orbit lying over v . Hence, a representation of \mathbb{B}_m of dimension vector α_m in general decomposition is semi-simple. Therefore, the generic semi-simple representation type of α_m is the same as the canonical decomposition of α .

From this it follows immediately that the blocks \mathfrak{b}_a (which we know already to be the connected components of \mathbb{B}_m) are strongly connected. If one \mathfrak{b}_a were not strongly connected, then by the construction of the generic representation type it would follow that general representations are not semi-simple.

The generic semi-simple representation type of α_m -dimensional representations of \mathbb{B}_m is

$$\alpha_m = \gamma_1 + \dots + \gamma_s$$

The support of any of the γ_j is strongly connected, hence is contained in a single \mathbf{b}_a .

Let $\beta_1^{(a)}, \dots, \beta_z^{(a)}$ be the set of γ_j belonging to \mathbf{b}_a . Clearly, the generic semi-simple representation type of $n_a \mu_a$ -dimensional representations of \mathbf{b}_a is equal to

$$n_a \mu_a = \beta_1^{(a)} + \dots + \beta_z^{(a)}.$$

Taking the scalar product with d on both sides gives the equality $n_a = z$ proving the second part of the theorem.

Modifying the formula for the dimension of the quotient variety $\text{Rep}(\mathbb{B}_m, \alpha_m)/GL(\alpha_m)$ (or equivalently, the dimension of R) proved in [17, Thm. 6] to the slightly more general setting of marked quivers we obtain

$$\dim R = \sum_{j=1}^s (1 - \chi_m(\gamma_k, \gamma_j)) - m$$

and the local isomorphism $\hat{C}_m \simeq \hat{R}_n$ finishes the proof. \square

Another consequence of the Luna slice machinery which can be proved along similar lines is that $\text{Sm } A/C$ is Zariski open in $\text{Spec } C$.

6. THE SINGULARITIES

6.1. If A is smooth in m we have seen that $\hat{C}_m \simeq \hat{R}_n$ where R is the ring of polynomial invariants $\mathbb{C}[\text{Rep}(\mathbb{B}_m, \alpha_m)]^{GL(\alpha_m)}$ and n is the maximal ideal of this graded algebra.

In this section we will investigate what extra restrictions are imposed on the m -box (\mathbb{B}_m, α_m) and the block data $(\mathbf{b}_a, n_a \mu_a)$ when we assume in addition that C is smooth in m , that is, C_m is a regular local ring.

Then, $\hat{C}_m \simeq \mathbb{C}[[x_1, \dots, x_c]] \simeq \hat{R}_n$ where $c = \dim C$. Consequently, R must be a polynomial ring in c variables as R is a positively graded algebra of finite global dimension.

Therefore, each of the block data $(\mathbf{b}_a, n_a \mu_a)$ must be one of the three possible types appearing in theorem 3.5. A direct consequence of this classification is the next result.

Theorem 6.1. *Let A and C be smooth in m . If $\dim C > 1$, then the multiplicity of every maximal ideal p_a lying over m is $n_a = 1$. In particular, Z is smooth in every prime p_a lying over m .*

Proof. If the multiplicity n_a of p_a is equal to one, then p_a is unramified over m and is thus a smooth point of $\text{Spec } Z$.

Assume $n_a > 1$, then clearly \mathbf{b}_a cannot be of type 1 of theorem 3.5. We will also exclude the two other types.

Suppose that $\mathbf{b}_a = \tilde{A}_k(+1)$. Then $n_a = 2$ hence all dimension components are even and at least one is equal to 2. Hence, the two simple components must have one of their dimension components equal to one. Calculating the necessary and sufficient conditions for a dimension vector of $\tilde{A}_k(+1)$ to be simple we see that the only such case is $(1, \dots, 1)$. That is,

$$\mu_a = (1, \dots, 1)$$

However, if χ denotes the Euler-form of $\tilde{A}_k(+1)$ we calculate that

$$-1 = \chi((1, \dots, 1), (1, \dots, 1))$$

But then, $(1, \dots, 1)$ is not a real root so it cannot occur with higher multiplicity in a canonical decomposition by a result of Schofield [24].

Suppose that $\mathbf{b}_a = \tilde{A}_k$. Then, $n_a \mu_a = (n_a, \dots, n_a)$. As the dimension vector of a simple representation of \tilde{A}_k is either $(1, \dots, 1)$ or δ_i , and as the general representation of dimension vector $n_a \mu_a$ must split as a direct sum of n_a distinct simples, the only possibility is that $\mu_a = (1, \dots, 1)$.

Consider the p_a -box \mathbb{B} of the triad (A, Z, Z) . As all entries of $n_a \mu_a$ are > 1 it follows that $\mathbb{B} = \tilde{A}_k$ (for in this case one cannot get rid of any loops in \mathbb{B}). Let Y_A be the variety corresponding to this triple, then

$$\dim Y_A = \dim Z + n^2 - 1$$

and if y is a point of Y_A in the orbit of the semi-simple n -dimensional A -module corresponding to p_a , then

$$\dim T_y = \dim \operatorname{Rep}(\tilde{A}_k, (1, \dots, 1)) + n^2 - k = n^2$$

As always, $\dim T_y \geq \dim Y_A$ this implies that $\dim C = \dim Z \leq 1$ contradicting the assumption.

The case of one marked loop cannot occur if $n_a > 1$ as the canonical decomposition of the underlying quiver is $(n_a) = (1) + \dots + (1)$, but there is just one (1) -dimensional representation of the marked loop, the trivial representation. \square

Up till now, we have not used the assumption that A is a Hopf algebra nor that C is a commutative sub-Hopf algebra. If we do impose these conditions, then we have

1. A is a projective C -module. In fact, by [15, Thm. 1.7] we even know that A is a Frobenius extension of C . That is, A is a finitely generated projective C -module and there is an isomorphism $A \longrightarrow \operatorname{Hom}_C(A, C)$ of (C, A) -bimodules. See [9] and [2] for more details.
2. C , being the coordinate ring of an irreducible algebraic group, is smooth in all maximal ideals m .

That is, all requirements of the theorem above are satisfied and we obtain.

Theorem 6.2. *Let H be a prime Hopf algebra which is a finite module over a central subHopf algebra C . If Z is the center of H , then*

$$\operatorname{Sep} H/C \hookrightarrow \operatorname{Sm} H/C \hookrightarrow \operatorname{Sm} Z/C$$

Proof. By the theorem above, Z is an unramified cover of degree s over $\operatorname{Sm} A/C$. \square

6.2. If A and C are smooth in m , then we have seen that the m -box \mathbb{B}_m consists of exactly s blocks \mathbf{b}_a all of type 1. That is, \mathbf{b}_a is a strongly connected marked quiver, μ_a is the dimension vector of a simple representation of \mathbf{b}_a and $\min_i a_i = 1$.

In order to further restrain the structure of \mathbf{b}_a we will recall the definition and some results on *reflexive Azumaya algebras*. We refer to [18] or [20] for more details.

Let R be an integrally closed affine domain with field of fractions K and let Σ be a central simple K -algebra. A subring Λ of Σ is said to be a *reflexive Azumaya algebra over R* if and only if

1. The center of Λ is equal to R .
2. Λ is a finitely generated reflexive R -module, that is, $\Lambda^{**} \simeq \Lambda$ where $(-)^* = \operatorname{Hom}_R(-, R)$.
3. Λ_P is an Azumaya algebra over the discrete valuation ring R_P for every height one prime ideal P of R .

Equivalently, the natural map

$$\Lambda \otimes_R' \Lambda^{\circ p} \longrightarrow \operatorname{End}_R(\Lambda)$$

is an isomorphism of R -algebras, where the modified tensor product $-\otimes_R'$ is defined to be $(-\otimes_R -)^{**}$.

Two reflexive Azumaya algebras over R , Λ and Γ (possibly in different central simple algebras) are said to be equivalent if there exist finitely generated reflexive R -modules M and N such that

$$\Lambda \otimes_R' \text{End}_R(M) \simeq \Gamma \otimes_R' \text{End}_R(N)$$

as R -algebras. The set of equivalence classes of reflexive Azumaya algebras equipped with the modified tensor product as a multiplication rule is a group. This is the *reflexive Brauer group* $\beta(R)$ of R . Contrary to the ordinary Brauer group $Br(R)$, it is always a subgroup of $Br(K)$.

In [18] a cohomological interpretation was given of the reflexive Brauer group. If X_{sm} is the Zariski open set of smooth points of R , then

$$\beta(R) \simeq H_{et}^2(X_{sm}, \mathbb{G}_m)$$

Hence, at least in the étale cohomology, the reflexive Brauer group is the Brauer group of the smooth locus of R .

Clearly, one can also define reflexive Azumaya algebras locally in a maximal ideal p of R . The following result is implicit in [20] or see [1] for more details.

Lemma 6.3. *Let A be an order in a central simple algebra of dimension n^2 . Assume that A is reflexive Azumaya in a maximal ideal p of R .*

1. *The set $\text{Ram}(A)$ of maximal ideals q of R such that $A/qA \not\simeq M_n(\mathbb{C})$ is of codimension ≥ 2 in p .*
2. *If A_p is a projective R_p -module, then $A/pA \simeq M_n(\mathbb{C})$.*

Proof. As for (2) : if A_p is a projective Z_p -module, the above natural map is an isomorphism. Hence, A_p is an Azumaya algebra over Z_p meaning that A_p/pA_p is a central simple algebra over the residue field of Z_p which is \mathbb{C} , done. \square

6.3. Returning to our triad (A, Z, C) , the reflexive Azumaya condition (imposed locally) has drastic consequences on the shape of the block b_a .

Proposition 6.4. *Let A and C be smooth in m and let p_a be a maximal ideal of Z lying over m such that A is reflexive Azumaya in p_a . Assume moreover that A is a projective C -module.*

Then, the block b_a is the a one vertex quiver and $\mu_a = (1)$. In particular, $A/Ap_a \simeq M_n(\mathbb{C})$ whence p_a belongs to the Azumaya locus of A .

Proof. We know already that Z is smooth in p_a . Further, because A is a projective C -module, A_m is free over the local regular domain C_m . Hence, A_m is a Cohen-Macaulay module.

Because Z_{p_a} is regular local, it follows that A_{p_a} is a free Z_{p_a} -module. Moreover, by assumption A_{p_a} is a reflexive Azumaya algebra over Z_{p_a} . By the above lemma this entails that $A/Ap_a \simeq M_n(\mathbb{C})$ from which all the claims follow. \square

In particular we have the following application to Hopf algebras.

Theorem 6.5. *Let H be a prime Hopf algebra which is a finite module over a central subHopf algebra C . If H is a reflexive Azumaya algebra over its center Z , then*

$$\text{Sm } H/C = \text{Sep } H/C$$

Proof. If H is smooth in m , then by the foregoing proposition $A/Ap_a \simeq M_n(\mathbb{C})$ for all maximal ideals p_a of Z lying over m . But then.

$$A/A_m \simeq M_n(\mathbb{C}) \oplus \dots \oplus M_n(\mathbb{C}) \text{ (s copies)}$$

which entails that A is separable over C in m . \square

Finally, in order to apply this result to the quantised enveloping algebra $U_\epsilon(\mathfrak{g})$ or the quantum function algebra $O_\epsilon(G)$ we recall the following result proved in [1, pf. of Thm. 4.3 and Thm. 4.5].

Theorem 6.6 (Brown-Goodearl). *Let \mathfrak{g} be a finite dimensional semisimple Lie algebra with corresponding simply connected Lie group G . Let ϵ be a primitive l -th root of unity where l is odd and prime to 3 if \mathfrak{g} involves a factor of type G_2 .*

1. *The quantum function algebra $O_\epsilon(G)$ is a reflexive Azumaya algebra.*
2. *The quantised enveloping algebra $U_\epsilon(\mathfrak{g})$ is a reflexive Azumaya algebra.*

REFERENCES

- [1] K.A. Brown and K.R. Goodearl, *Homological aspects of Noetherian PI Hopf algebras, and irreducible modules of maximal dimension* Journal of Alg. (1998) to appear.
- [2] K.A. Brown, *Representation theory of Noetherian Hopf algebras satisfying a polynomial identity*, to appear
- [3] C. De Concini and V. Lyubashenko, *Quantum function algebras at roots of 1*, Adv. Math. 108 (1994) 205-262
- [4] C. De Concini and C. Procesi, *Quantum groups*, in "D-modules, representation theory and quantum groups" Lect. Notes Math. 1565 (1993) Springer-Verlag, 31-140
- [5] C. De Concini, V. Kac and C. Procesi, *Quantum coadjoint action*, Journ. AMS 5 (1992) 151-189
- [6] C. De Concini and C. Procesi, *Quantum Schubert cells and representations at roots of 1*, in "Algebraic groups and Lie groups" G.I. Lehrer editor, Australian Math. Soc. Lecture Series 9 (1997) Cambridge University Press
- [7] P. Gabriel, *Finite representation type is open*, in "Representations of algebras" Lect. Notes Math. 488 (1975) 132-155
- [8] K.R. Goodearl and R.B. Warfield, *An introduction to noncommutative Noetherian rings* London Math. Soc. Student Texts vol 16 (1989) Cambridge University Press
- [9] I. Gordon, *The representation theory of quantized function algebras*, to appear
- [10] T.J. Hodges and T. Levasseur, *Primitive ideals of $\mathbb{C}_q[SL(3)]$* , Comm. Math. Phys. 156 (1993) 581-605
- [11] F. Knop, *Beweis des Fundamentallemmas und des Scheibensatzes*, in "Algebraische Transformationsgruppen und Invariantentheorie" DMV Seminar 13 Birkhäuser (1989) 110-113
- [12] M. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lect. Notes Math. 389, Springer (1974)
- [13] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspects of mathematics D1 (1984) Vieweg
- [14] H. Kraft, *Geometric methods in representation theory* in "Representations of algebras", Lect. Notes Math. 944 (1982)
- [15] H.F. Kreimer and M. Takeuchi, *Hopf algebras and Galois extensions of an algebra* Indiana U. Math. J. 30 (1981) 675-692
- [16] L. Le Bruyn and C. Procesi, *Etale local structure of matrixinvariants and concomitants* in "Algebraic groups, Utrecht 1986" Lect. Notes in Math. 1271 (1987) 143-175
- [17] L. Le Bruyn and C. Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. 317 (1990) 585-598
- [18] L. Le Bruyn, *A cohomological description of the reflexive Brauer group* J. Alg. 105 (1987) 250-254
- [19] D. Luna, *Slices étales*, Bull. Soc. Math. France, Mémoire 33 (1973) 81-105
- [20] M. Orzech, *Brauer groups and classgroups for a Krull domain*, in "Brauer groups in ring theory and algebraic geometry" Lect. Notes Math. 917, 66-90, Springer (1982)
- [21] C. Procesi, *A formal inverse to the Cayley-Hamilton theorem*, J. Algebra 107 (1987) 63-74
- [22] M. Raynaud, *Anneaux Henséliens* Lect. Notes Math. 169 (1970)
- [23] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque 92 (1982) Société Math. de France
- [24] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. 65 (1992) 46-64

