

Smoothness in Algebraic Geography

Lieven Le Bruyn

Research Director of the FWO (Belgium)
Dept. of Mathematics & Comp. Sci., University of Antwerp, UIA,
Belgium

Zinovy Reichstein

Partially supported by NSA grant MDA904-9610022
Department of Mathematics, Oregon State University
Corvallis, OR 97031

November 1997

Report no. 97-11



Division of Pure Mathematics
Department of Mathematics & Computer Science

**universitaire
instelling
antwerpen**

Universiteitsplein 1, B-2610 Wilrijk-Antwerpen , BELGIUM

Smoothness in Algebraic Geography

Lieven Le Bruyn

Research Director of the FWO (Belgium)
Dept. of Mathematics & Comp. Sci., University of Antwerp, UIA,
Belgium

Zinovy Reichstein

Partially supported by NSA grant MDA904-9610022
Department of Mathematics, Oregon State University
Corvallis, OR 97031

November 1997

Report no. 97-11

Abstract

Let V be a vector space and let $\{e_1, \dots, e_r\}$ be a basis of V . An algebra structure on V is given by r^3 structure constants c_{ij}^h , where $e_i \cdot e_j = \sum_h c_{ij}^h e_h$. We require this algebra structure to be associative with unit element e_1 . This limits the sets of structure constants (c_{ij}^h) to a subvariety of k^{r^3} , which we denote by Alg_r . Base changes in V (leaving e_1 fixed) give rise to the natural transport of structure action on Alg_r ; isomorphism classes of r -dimensional algebras are in 1-1 correspondence with the orbits under this action.

In this paper we classify the smooth closed subvarieties of Alg_r which are invariant under the transport of structure action and study the singularities which may occur. In particular, we show that if $r = n^2$ then the closure of the locus corresponding to the matrix algebra $M_n(k)$ is not smooth for $n \geq 3$. This gives a negative answer to a question of Seshadri [Se78].

Smoothness in Algebraic Geography

Lieven Le Bruyn ¹

Departement Wiskunde, Universiteit Antwerpen (UIA),
B-2610 Wilrijk (Belgium). E-mail: lebruy@hwins.uia.ac.be

Zinovy Reichstein ²

Department of Mathematics, Oregon State University,
Corvallis, OR 97031. E-mail: zinovy@math.orst.edu.

Let V be a vector space and let $\{e_1, \dots, e_r\}$ be a basis of V . An algebra structure on V is given by r^3 structure constants c_{ij}^h , where $e_i \cdot e_j = \sum_h c_{ij}^h e_h$. We require this algebra structure to be associative with unit element e_1 . This limits the sets of structure constants (c_{ij}^h) to a subvariety of k^{r^3} , which we denote by Alg_r . Base changes in V (leaving e_1 fixed) give rise to the natural transport of structure action on Alg_r ; isomorphism classes of r -dimensional algebras are in 1-1 correspondence with the orbits under this action.

In this paper we classify the smooth closed subvarieties of Alg_r which are invariant under the transport of structure action and study the singularities which may occur. In particular, we show that if $r = n^2$ then the closure of the locus corresponding to the matrix algebra $M_n(k)$ is not smooth for $n \geq 3$. This gives a negative answer to a question of Seshadri [Se78].

Contents

1	Introduction	2
2	Notation and preliminaries	5
2.1	The group G_r	6
2.2	Reduced sets of structure constants	6
2.3	Trace functions	7
2.4	Transport of structure on $W^* \otimes W^* \otimes W$	9
3	The geography of X_n	10
3.1	The generic order \mathbb{G}_n	10
3.2	Generic trace rings and universal division algebras	11
3.3	PGL_n -quotients	13
3.4	Graded algebras of GK dimension 2	14
3.5	Proof of Theorem 1.1	15

¹Research director of the FWO (Belgium)

²Partially supported by NSA grant MDA904-9610022

4	Smooth invariant subvarieties	16
4.1	The isotypical decomposition of $W^* \otimes W^* \otimes W$	16
4.2	The case of odd characteristic	17
4.3	Proof of Theorem 1.3 for $r \geq 5$	19
4.4	Smooth subvarieties of Alg_3	21
4.5	Smooth subvarieties of Alg_4	22
4.6	Lie algebra structures	23
5	Tangent spaces and singularities	25
5.1	Tangent spaces	25
5.2	Alternating algebras	28
5.3	Quasi-commutative algebras	29
6	Minimal deformations and minimal singularities	31
6.1	Proof of Theorem 1.4	31
6.2	Minimal singularities	33

1 Introduction

Throughout this paper k will be an algebraically closed field, V will be an r -dimensional vector space, and e_1, \dots, e_r will be a basis of V . We want to endow V with a structure of an associative k -algebra so that e_1 becomes the identity element. By linearity this can be done by specifying the r^3 structure constants $c_{ij}^h \in k$, where

$$e_i \cdot e_j = \sum_{h=1}^r c_{ij}^h e_h . \quad (1)$$

Equivalently, an algebra structure on V is given by a bilinear map $V \times V \rightarrow V$, i.e., an element of $V^* \otimes V^* \otimes V$. The collection of constants $(c_{ij}^h : i, j, h = 1, \dots, r)$ corresponds to the tensor

$$\sum_{i,j,h} c_{ij}^h e_h^{ij} \in V^* \otimes V^* \otimes V ,$$

where $e_h^{ij} = e_i^* \otimes e_j^* \otimes e_h$. The choice of the structure constants is not arbitrary; it must reflect our assumptions that e_1 is the identity element, i.e., $e_1 \cdot e_i = e_i = e_i \cdot e_1$ for every $i = 1, \dots, r$ and that multiplication is associative, i.e., $(e_i \cdot e_j) \cdot e_h = e_i \cdot (e_j \cdot e_h)$ for every $i, j, h = 1, \dots, r$. These conditions translate respectively, into

$$c_{1i}^j = c_{i1}^j = \delta_i^j \quad (2)$$

and

$$\sum_{l=1}^r c_{ij}^l c_{lh}^m - c_{jh}^l c_{il}^m = 0 \quad (3)$$

for all $i, j, h, m = 1, \dots, r$. Equations (2) and (3) cut out an algebraic variety in $k^{r^3} = V^* \otimes V^* \otimes V$. This variety, which we shall denote by Alg_r , is of fundamental importance in the theory of algebras and their deformations; see, e.g., [DS], [F68], [K]. Flanigan [F68] referred to the study of Alg_r as “algebraic geography”.

The basic facts of algebraic geography are as follows. A point $(c_{ij}^h) \in Alg_r$ represents an r -dimensional k -algebra A , along with a particular choice of basis (which gives the structure constants c_{ij}^h). A change of basis in A gives rise to a (possibly) different point of Alg_r . Let G_r be the group of all base changes in V , which preserve the element e_1 . This group acts on Alg_r by the so-called “transport of structure” action, which is induced by the natural action of $G_r \subset GL(V)$ on $V^* \otimes V^* \otimes V$. The G_r -orbits of this action are in 1-1 correspondence with isomorphism classes of r -dimensional associative algebras. We shall denote the orbit corresponding to an algebra A by $[A] \subset Alg_r$. If $p \in [A]$ then the stabilizer subgroup $Stab(p) \subset G_r$ is exactly the group $Aut_k(A)$ of all k -algebra automorphisms of A . If $[B]$ is contained in the Zariski closure of $[A]$ in Alg_r , we shall say that A is a deformation of B (or, alternatively, B is a degeneration of A) and write $B \leq A$. The relation \leq defined this way is a partial order on the set of r -dimensional algebras; we shall refer to it as the degeneration partial order.

We remark that our definition of Alg_r is the same as the one given in [Schaps] or [Se82]. A slightly different definition arises if one does not fix the identity element of V but simply requires that one exists; see, e.g., [Ga]. The variety obtained in this way is closely related to Alg_r . The transport of structure group is larger in this case ($GL(V)$ vs. G_r) but the degeneration picture for the orbits (or, equivalently, the degeneration partial order), remains the same.

The geometry of Alg_r is rather complicated. It is known that the number of irreducible components increases exponentially with r (see [M79] and [M82]) and that their dimensions are $\leq 4/27r^3 + O(r^{8/3})$; see [N]. However, a complete description of the orbits and the degeneration partial order is only known for $r \leq 5$; see [Ga], [M79], [Ha], and our Remarks 4.4.2 and 4.5.3. Many but not all components of Alg_r are orbit closures; see [F68], [M82]. An algebra A such that the closure of $[A]$ is a component is called generic. Every semi-simple algebra is known to be generic; see, e.g., [Ge, I.3] or [F68, Cor. 2.6].

The component $X_n \subset Alg_{n^2}$ given by the Zariski closure of $[M_n]$ is of particular interest. One reason is the connection between X_n and moduli spaces of vector bundles which first appeared in the work of Seshadri; see [Se78] and [Se82]. Let $U(n, d)$ be the moduli space of semi-stable vector bundles of rank n and degree d over a smooth curve. This space is singular in general. Seshadri [Se78] constructed a conjectural desingularization $N_{n,d}$ of $U(n, d)$ and proved that $N_{n,d}$ is smooth iff X_n is smooth. Moreover, X_2 (and thus $N_{2,d}$) is smooth; see [Se78, Theorem 1] or [Se82, p.112]. This construction has since been used in the study of rank 2 vector bundles. In order to extend it to vector bundles of rank n one needs to have a positive answer to the following question.

Question 1.1 (Seshadri [Se78, Introduction]) *Is X_n smooth for $n \geq 3$?*

Other consequences of the smoothness of X_n have been explored by Nori [Se78, Appendix]. Our own interest in X_n was motivated by the realization that X_n is closely related to Amitsur's universal division algebras and that the smoothness on X_n would imply a positive solution to the long-standing rationality problem for PGL_n -quotients. We will explain these connections in Section 3.

In this paper we answer Question 1.1 in the negative.

Theorem 1.2 X_n (and hence $N_{d,n}$) is singular for every $n \geq 3$.

C.S. Seshadri has recently informed us that he and A. Ramanathan have an unpublished independent proof of this theorem.

More generally, we will characterize all smooth G_r -invariant closed subvarieties of Alg_r . Let $A_0(r)$ and $A_1(r)$ be given by

$$A_0(r) = k[e_2, \dots, e_r]/(e_2, \dots, e_r)^2 \quad (4)$$

and

$$A_1(r) = k\{e_2, \dots, e_r\}/(e_2e_j - e_j, e_i e_j : i \geq 3, j \geq 2). \quad (5)$$

With these notations we have the following theorem.

Theorem 1.3 Suppose $\mathrm{char}(k) \neq 2$ and $r \geq 3$. Let Y be a smooth G_r -invariant closed subvariety of Alg_r . Then

1. $Y = [A_0] = \overline{[A_0(r)]}$ and $\dim(Y) = r - 1$, or
2. $Y = \overline{[A_1(r)]}$ and $\dim(Y) = 2(r - 1)$, or
3. $r = 3$, $Y = \overline{[k \times k \times k]}$ and $\dim(Y) = 6$, or
4. $r = 4$, $Y = \overline{[M_2(k)]}$ and $\dim(Y) = 9$.

A proof of this result is given in Section 4. We also prove an analogue of Theorem 1.3 for the variety Lie_r of all Lie algebra structures on V ; see Theorem 4.6.2.

In the last two sections we initiate a study of the singularities of orbit closures $\overline{[A]}$ at the origin. In Section 5 we classify the Zariski tangent spaces to $\overline{[A]}$ at the origin (as GL_{r-1} -modules) in terms of the algebra structure of A . In particular, we compute the dimension of the tangent space to X_n thus somewhat strengthening Theorem 1.2; see Corollary 5.1.3.

In order to describe the results of Section 6 we need the following definitions. An algebra A is called a minimal deformation of B if B immediately precedes A in the degeneration partial order. In other words, $B \leq C \leq A$ if and only if $C = B$ or $C = A$. Let $A_+(r)$ is the commutative algebra given by

$$A_+(r) = k[x_2, \dots, x_{r-1}]/(x_2^3, x_i x_j, x_j^2 : i \geq 2, j \geq 3) \quad (6)$$

and $A_-(r)$ be the algebra given by

$$A_-(r) = k\{y_2, \dots, y_{r-1}\}/(y_i^2, y_i y_j, y_j y_i, y_2 y_3 + y_3 y_2 : i \geq 2, j \geq 4). \quad (7)$$

Theorem 1.4 *Let $r \geq 4$. Then the only minimal deformations of $A_0(r)$ are $A_+(r)$, $A_-(r)$ and $A_1(r)$.*

In other words, every r -dimensional algebra $A \neq A_0(r)$ can be deformed into $A_+(r)$, $A_-(r)$, or $A_1(r)$. We also study the “minimal singularities” of Alg_r , i.e., the singularities of $\overline{[A_+(r)]}$ and $\overline{[A_-]}$ up to smooth equivalence; see Section 6.2.

2 Notation and preliminaries

The following notational conventions will be used throughout the paper; many of them were introduced in the previous section.

r	integer ≥ 3 .
k	base field.
V	k -vector space of dimension r .
e_1	multiplicative identity.
e_1, \dots, e_r	basis of V .
e_h^{ij}	$= e^i \otimes e^j \otimes e_h \in V^* \otimes V^* \otimes V$.
W	subspace of V spanned by e_2, \dots, e_r .
d	usually $= \dim(W) = r - 1$.
Alg_r	subvariety of $V^* \otimes V^* \otimes V$ given by (2) and (3).
G_r	transport of structure group; see Sect. 2.1.
π	the map $V^* \otimes V^* \otimes V \rightarrow W^* \otimes W^* \otimes W$ defined in Lemma 2.2.1.
Alg'_r	isomorphic image of Alg_r in $W^* \otimes W^* \otimes W$; see Lemma 2.2.1.
$[A]$	G_r -orbit in Alg_r corresponding to the algebra A .
X_n	Zariski closure of $[M_n]$ in Alg_{n^2} .
δ	element of $W^* \otimes W$ defined in Section 2.4.
R_+, R_-	irreducible $GL(W)$ -submodules of $W^* \otimes W^* \otimes W$; see Prop. 4.1.1.
c	contraction map $W^* \otimes W^* \otimes W \rightarrow W^*$; see (15), Sect. 2.3.
a, s, σ	endomorphisms of $W^* \otimes W^* \otimes W$ defined in (15), Sect. 2.3.
$T(A)$	tangent space to $\pi(\overline{[A]})$ at 0; see Section 5.
$A_0(r), A_1(r)$	r -dimensional algebras given by (4) and (5).
$B(s, t)$	$s + t + 2$ -dimensional algebra defined in Section 5.1.
$A_+(r), A_-(r)$	r -dimensional algebras given by (6) and (7).

2.1 The group G_r

We begin by taking a closer look at the transport of structure group G_r . In the basis e_1, \dots, e_r this group consists of elements of the form

$$g = \begin{pmatrix} 1 & a_2 & \dots & a_r \\ 0 & & & \\ \vdots & & g_0 & \\ 0 & & & \end{pmatrix}, \quad (8)$$

with $a_2, \dots, a_r \in k$ and $g_0 \in \mathrm{GL}_{r-1} = \mathrm{GL}(W)$. The transport of structure action of G_r on Alg_r is given by $g(x_{ij}^h) = (y_{ij}^h)$, where

$$g(e_i) \cdot g(e_j) = \sum_{h=1}^r y_{ij}^h g(e_h). \quad (9)$$

As we noted in the introduction, this action is induced by the natural action of $G_r \subset \mathrm{GL}(V)$ on $V^* \otimes V^* \otimes V$.

Let U be the subgroups of G_r consisting of all elements g as in (8) with $g_0 = I_{r-1}$, where I_{r-1} is the $(r-1) \times (r-1)$ -identity matrix. It is easy to see that the map $U \rightarrow W^*$ given by

$$\begin{pmatrix} 1 & a_2 & \dots & a_r \\ 0 & & & \\ \vdots & & I_{(r-1)} & \\ 0 & & & \end{pmatrix} \rightarrow \sum_{i=2}^r a_i e_i^* \quad (10)$$

is an isomorphism of algebraic groups. Identifying G_r with W^* in this way, we can write G_r as a semi-direct product

$$G_r = U \times_{\phi} \mathrm{GL}(W) \simeq W^* \times_{\phi} \mathrm{GL}(W), \quad (11)$$

where ϕ is the natural (dual) action of $\mathrm{GL}(W)$ on W^* .

2.2 Reduced sets of structure constants

Lemma 2.2.1 (cf. [Schaps, Lemma 4]) *Let $r \geq 3$. The map $\pi : V^* \otimes V^* \otimes V \rightarrow W^* \otimes W^* \otimes W$ given by*

$$\pi\left(\sum_{i,j,h=1}^r x_{ij}^h e_h^{ij}\right) = \sum_{i,j,h=2}^r x_{ij}^h e_{ij}^h$$

restricts to an isomorphism between Alg_r and its image $\mathrm{Alg}'_r \in W^ \otimes W^* \otimes W$.*

Proof. We only need to show that the coordinate ring $k[Alg_r]$ is generated by the elements x_{ij}^h , where $i, j, h = 2, \dots, r$. In other words, we want to show that elements (i) x_{1i}^h , (ii) x_{i1}^h , and (iii) x_{ij}^1 can be expressed as polynomials in x_{ij}^h with $i, j, h \geq 2$ by means of (2) and (3). (i) and (ii) are obvious by (2). To prove (iii) set $2 \leq m = h \neq i$; then (3) yields

$$x_{ij}^1 = \sum_{l=2}^r (x_{jh}^l x_{il}^h - x_{ij}^l x_{lh}^h). \quad (12)$$

□

Lemma 2.2.2 *Let X be a G_r -invariant closed subvariety of Alg_r . Then $\pi(X)$ is a cone (and, in particular passes through $0 \in W^* \otimes W^* \otimes W$). Moreover, the following are equivalent*

1. X is smooth.
2. X is smooth at $0 \in V^* \otimes V^* \otimes V$.
3. $\pi(X)$ is a $GL(W)$ -invariant linear subspace of $W^* \otimes W^* \otimes W$.

In particular, if X is smooth, then X is rational.

Proof. Let $\pi : V^* \otimes V^* \otimes V \rightarrow W^* \otimes W^* \otimes W$ be as in Lemma 2.2.1. By our construction π is $GL(W)$ -equivariant. Thus $\pi(X)$ is invariant under the action of $GL(W)$ and, in particular, under the action of the central subtorus of $GL(W)$. This implies that $\pi(X)$ is a cone. By Lemma 2.2.1 X is isomorphic to $\pi(X)$. In particular, X is smooth at $0 \in V^* \otimes V^* \otimes V$ iff $\pi(X)$ is smooth at $0 \in W^* \otimes W^* \otimes W$. The equivalence of (1), (2) and (3) now follows from the fact that $\pi(X)$ is a cone. The last assertion is a consequence of (3). □

2.3 Trace functions

In this section we record some elementary facts regarding right and left trace functions.

Let $p \in V^* \otimes V^* \otimes V$ be a (not necessarily associative) bilinear form on V . For $v \in V$ let $R_p(v)$ and $L_p(v) : V \rightarrow V$ denote the right and the left multiplication by v . We shall denote the traces of these maps by LTr_p and RTr_p respectively. We shall also write LTr_A in place of LTr_p if A is an algebra whose tensor of structure constants equals p . Note that LTr_p and $RTr_p : V \rightarrow k$ are elements of V^* .

Lemma 2.3.1 *Let A be a finite-dimensional associative algebra. Then*

1. $LTr_A(x) = RTr_A(x) = 0$ for every $x \in \text{Rad}(A)$.
2. The bilinear form of $(x, y) \rightarrow LTr_A(x \cdot y)$ is non-singular if and only if A is semi-simple. The same is true for the form $(x, y) \rightarrow RTr(x \cdot y)$.

Proof. (1): Both L_x and R_x are nilpotent; hence their traces are 0.

(2): Part (1) says that $\text{Rad}(A)$ is contained in the kernel of both forms. \square

Next recall the definition of $\pi : V^* \otimes V^* \otimes V \longrightarrow W^* \otimes W^* \otimes W$ from Lemma 2.2.1.

Lemma 2.3.2 *If $\pi(p_1) = \pi(p_2)$ for some $p_1, p_2 \in \text{Alg}_r$ then $\text{LTr}_{p_1} = \text{LTr}_{p_2}$ and $\text{RTr}_{p_1} = \text{RTr}_{p_2}$ in V^* .*

Proof. Let $p_1 = \sum_{i,j,h=1}^r c_{ij}^h e_h^{ij}$ and $p_2 = \sum_{i,j,h=1}^r d_{ij}^h e_h^{ij}$. Our assumption that $\pi(p_1) = \pi(p_2)$ simply means

$$c_{ij}^h = d_{ij}^h \quad (13)$$

for every $i, j, h \geq 2$.

For every $p \in \text{Alg}_r$, $\text{LTr}_p(e_1) = \text{RTr}_p(e_1) = r$. Thus by linearity we only need to check $\text{LTr}_{p_1}(e_i) = \text{LTr}_{p_2}(e_i)$ and $\text{RTr}_{p_1}(e_i) = \text{RTr}_{p_2}(e_i)$ for every $i = 2, \dots, r$. An explicit calculation shows that $\text{LTr}_{p_1}(e_i) = \sum_{j=1}^r c_{ij}^j$ and $\text{RTr}_{p_1}(e_i) = \sum_{j=1}^r c_{ji}^j$. Similarly, $\text{LTr}_{p_2}(e_i) = \sum_{j=1}^r d_{ij}^j$ and $\text{RTr}_{p_2}(e_i) = \sum_{j=1}^r d_{ji}^j$. The desired equalities now follow from (13) and the fact that $c_{1j}^j = c_{j1}^j = d_{1j}^j = d_{j1}^j = 0$ for every $j \geq 2$; see (2). \square

In view of Lemma 2.3.2 we can define $\text{LTr}_q \in V^*$ when $q \in \text{Alg}_r'$ by $\text{LTr}_q \stackrel{\text{def}}{=} \text{LTr}_p$ for any $p \in \text{Alg}_r$ with $\pi(p) = q$. Similarly, we define RTr_q as RTr_p .

In the sequel we shall make use of the following $\text{GL}(W)$ -equivariant linear maps.

$$\begin{aligned} c &: W^* \otimes W^* \otimes W \longrightarrow W^* \\ \sigma &: W^* \otimes W^* \otimes W \longrightarrow W^* \otimes W^* \otimes W \\ s &: W^* \otimes W^* \otimes W \longrightarrow W^* \otimes W^* \otimes W \\ a &: W^* \otimes W^* \otimes W \longrightarrow W^* \otimes W^* \otimes W \end{aligned} \quad (14)$$

given by

$$\begin{aligned} c(w) &= w_1^*(w_3)w_2^* \\ \sigma(w) &= w_2^* \otimes w_1^* \otimes w_3 \\ s(w) &= w + \sigma(w) \\ a(w) &= w - \sigma(w) \end{aligned} \quad (15)$$

for any $w = w_1^* \otimes w_2^* \otimes w_3 \in W^* \otimes W^* \otimes W$. Here c is a contraction, s and a are, respectively, symmetrization and anti-symmetrization in the first two components; this explains our choice of notation.

Lemma 2.3.3 *Let $\sigma : W^* \otimes W^* \otimes W \longrightarrow W^* \otimes W^* \otimes W$ and $c : W^* \otimes W^* \otimes W \longrightarrow W^*$ be as in (15). Then*

1. $c(q)(w) = \text{RTr}_q(w)$ and
2. $c\sigma(q)(w) = \text{LTr}_q(w)$

for every $q \in W^* \otimes W^* \otimes W$ and every $w \in W$.

Proof. By linearity it is enough to verify (1) and (2) for $q = e_h^{ij}$ and $w = e_s$, where $i, j, h, s \geq 2$. In this case $c(q)(w) = \delta_i^h \delta_j^s = \text{RTr}_q(w)$. Similarly, $c\sigma(q)(w) = \delta_j^h \delta_i^s = \text{LTr}_q(w)$, and the lemma follows. \square

2.4 Transport of structure on $W^* \otimes W^* \otimes W$

Lemma 2.2.1 says that we lose no information about an algebra structure on V by only keeping track of the structure constants c_{ij}^h with $i, j, h \geq 2$. In this section we will show that the transport of structure action of G_r on Alg_r can also be recovered from this reduced data.

Observe that the natural isomorphism between $W^* \otimes W$ and $\text{End}(W)$ is $\text{GL}(W)$ -equivariant. Denote the element of $W^* \otimes W$ which corresponds to the identity element of $\text{End}(W)$ by δ . In other words, $\delta = \sum_{b \in B} b^* \otimes b$ for any basis B of W . In particular,

$$\delta = \sum_{i=2}^r e_i^* \otimes e_i. \quad (16)$$

We can now define an action of G_r on $W^* \otimes W^* \otimes W$ by letting $\text{GL}(W)$ act naturally and $U \simeq W^*$ act via

$$w^*(t) = t + s(w^* \otimes \delta). \quad (17)$$

It is easy to see that these two actions extend to an action of $G_r = U \times_{\phi} \text{GL}(W)$ on $W^* \otimes W^* \otimes W$.

Lemma 2.4.1 *The map $\pi : \text{Alg}_r \rightarrow \text{Alg}'_r$ of Lemma 2.2.1 is G_r -equivariant with respect to the transport of structure action of G_r on $V^* \otimes V^* \otimes V$ and the G_r -action on $W^* \otimes W^* \otimes W$ defined above.*

Proof. It is clear that π is $\text{GL}(W)$ -equivariant; thus we only need to show that it is equivariant with respect to the action of U . Let $g \in U$ be as in (10), i.e., $g(e_i) = e_i + a_i e_1$ for all $i \geq 2$. Let $p = \sum_{i,j,h=1}^r c_{ij}^h e_h^{ij} \in \text{Alg}_r$. and let $q = g(p) = \sum_{i,j,h=1}^r d_{ij}^h e_h^{ij} \in \text{Alg}_r$. Then

$$\begin{aligned} \sum_{h=1}^r d_{ij}^h g(e_h) &= g(e_i) \cdot g(e_j) = a_i a_j e_1 + a_i e_j + a_j e_i + e_i \cdot e_j = \\ &= \alpha e_1 + a_i g(e_j) + a_j g(e_i) + \sum_{h=2}^r c_{ij}^h g(e_h) \end{aligned}$$

for some $\alpha \in k$. Equating the coefficients of $g(e_h)$ for all $h \geq 2$, we see that $d_{ij}^h = c_{ij}^h + a_i \delta_j^h + a_j \delta_i^h$. Thus

$$\pi(q) = \pi(p) + \sum_{i,j=2}^r a_j e_i^{ij} + a_i e_j^{ij} = \pi(p) + s\left(\left(\sum_{i=2}^r a_i e_i\right) \otimes \delta\right),$$

as claimed. \square

We remark that the Lemma 2.4.1 does not assert that the map

$$\pi : V^* \otimes V^* \otimes V \longrightarrow W^* \otimes W^* \otimes W$$

is G_r -equivariant. The equivariance assertion only applies to the restriction of π to Alg_r .

Corollary 2.4.2 *Let X be a G_r -invariant subvariety of $W^* \otimes W^* \otimes W$. Then X can be written as $X_0 \oplus s(W^* \otimes \delta)$, where X_0 is a $GL(W)$ -invariant subvariety of $Ker(c + c\sigma)$. Moreover, $GL(W)$ -orbits in X_0 are in 1-1 correspondence with the G_r -orbits in X .*

Proof. Set X_0 be the intersection of X with $U = Ker(c + c\sigma)$. Since U is transversal to $s(W^* \otimes \delta)$, the corollary follows from Lemma 2.4.1. \square

We can now prove the following extension of Lemma 2.2.2.

Lemma 2.4.3 *Let X be a smooth G_r -invariant closed subvariety of Alg_r . Then $\pi(X)$ is a $GL(W)$ -invariant linear subspace containing $s(W^* \otimes \delta)$.*

Proof. Lemma 2.2.2 says that $\pi(X)$ is $GL(W)$ -invariant linear subspace. It contains $s(W^* \otimes \delta)$ by Corollary 2.4.2. \square

3 The geography of X_n

Recall that since $M_n(k)$ is a simple algebra, its orbit is open in Alg_{n^2} . In other words, the Zariski closure of $[M_n]$ is an irreducible component of Alg_{n^2} ; we shall continue to denote it by X_n . In the introduction we mentioned the significance of X_n in the study of Seshadri's partial desingularizations of moduli spaces of vector bundles on smooth projective curves. We shall now further motivate Theorem 1.2 by drawing connections between X_n and universal division algebras, the rationality problem for PGL_n -quotients, and graded prime algebras of Gelfand-Kirillov dimension 2. These results will not be used in the sequel. A reader who is primarily interested in the structure of Alg_r may wish to proceed directly to Section 3.5, where we give a short self-contained proof of Theorem 1.2.

In Sections 3.1 - 3.4 we will assume $\text{char}(k) = 0$. The proof of Theorem 1.2 in Section 3.5 goes through in arbitrary characteristic.

3.1 The generic order G_n

For $p \in V^* \otimes V^* \otimes V$ let

$$\Delta(p) = \det(\text{LTr}_p(e_i \cdot e_j)) \tag{18}$$

be the determinant of the Gram matrix of the (left) trace form given in Lemma 2.3.1. Note this definition makes sense for every $p \in V^* \otimes V^* \otimes V$, not just for $p \in Alg_r$. Thus $\Delta \in k[V^* \otimes V^* \otimes V]$.

Let $k[X_n] = k[x_{ij}^h]$. Then we define the generic order \mathbb{G}_n to be the $k[X_n]$ -algebra

$$\mathbb{G}_n = k[X_n]e_1 \oplus \dots \oplus k[X_n]e_{n^2}$$

with multiplication defined by $e_i \cdot e_j = \sum_{h=1}^{n^2} x_{ij}^h e_h$. By the relations which have to hold among the coordinate functions x_{ij}^h , it follows that \mathbb{G}_n is an associative algebra with unit element e_1 . Moreover, \mathbb{G}_n is a free $k[X_n]$ -module of rank n^2 .

Note that the G_r -action on $k[X_n]$ extends to \mathbb{G}_n as follows: for $g = (g_{ij})$ we have $g(e_i) = \sum_j g_{ij} e_j$; see (9). Here $g_{i1} = \delta_i^1$, and $g_{1j} = a_j$, as in (8). Restricting this action to the central torus of $\mathrm{GL}(W) \subset G_r$, we obtain a grading of \mathbb{G}_n with $\deg(e_h) = \deg(x_{ij}^h) = 1$, $\deg(e_1) = 0$, and $\deg(x_{ij}^1) = 2$; see (12).

Lemma 3.1.1 1. \mathbb{G}_n is a positively graded algebra generated in degree one.

2. \mathbb{G}_n is an order in a central simple algebra D_n of dimension n^2 .

3. The center of D_n is $k(X_n)$.

Proof. As we saw in the proof of Lemma 2.2.1, $k[X_n] = k[x_{ij}^h; i, j, h \geq 2]$. Hence, $k[X_n]$ is generated by elements of degree 1. Since $\mathbb{G}_n = k[X_n]\{e_2, \dots, e_{n^2}\}$, this implies (1).

Now let $\Delta_n \in k[X_n]$ be the determinant of the Gram matrix of the trace form on \mathbb{G}_n . In other words, if Δ is as in (18) then Δ_n is the restriction of Δ from $V^* \otimes V^* \otimes V$ to X_n . By Lemma 2.3.1(b) $\Delta_n(p) \neq 0$ for any $p \in [M_n]$. In particular, $\Delta_n \neq 0$ in $k[X_n]$. By the Artin-Procesi theorem $\mathbb{G}_n[\Delta_n^{-1}]$ is an Azumaya algebra of rank n^2 over the domain $k[X_n][\Delta_n^{-1}]$. Parts (2) and (3) follow from this fact. \square

3.2 Generic trace rings and universal division algebras

Let Z_1, \dots, Z_m be m generic $n \times n$ -matrices. In other words, the mn^2 entries of these matrices are commuting independent variables over k . Recall that the ring $k\{Z_1, \dots, Z_m\}$ is called the algebra of generic matrices. The trace ring $T_{m,n}$ is the k -algebra generated by elements of $k\{Z_1, \dots, Z_m\}$ and the coefficients of their characteristic polynomials, which are viewed as scalar $n \times n$ -matrices. We shall denote the center of $T_{m,n}$ by $C_{m,n}$. For a more detailed description of the trace ring see, e.g., [Re, Sect. 2]. Note, in particular, that $C_{m,n}$ is naturally isomorphic to the ring of (regular) invariants of the simultaneous conjugation action of PGL_n on $(M_n)^{\oplus m}$ and thus

$$\mathrm{trdeg}_k(C_{m,n}) = mn^2 - \dim(\mathrm{PGL}_n) = (m-1)n^2 + 1. \quad (19)$$

The universal division algebra $\mathrm{UD}_{m,n}$ is obtained from $T_{m,n}$ (or, alternatively, from $k\{Z_1, \dots, Z_m\}$) by inverting all non-zero central elements. For a more detailed description of these algebras see, e.g., [Ro, Ch. 3].

We shall now see that \mathbb{G}_n is closely related to $T_{m,n}$, where $m = n^2 - 1$. The first indication of a possible relationship is the following observation.

Lemma 3.2.1 1. $\text{trdeg}(C_{n^2-1,n}) = (n^2 - 1)^2$.

2. $\text{trdeg}_k k(X_n) = \dim(X_n) = (n^2 - 1)^2$.

Proof. (1): follows from (19).

(2): Since $[M_n]$ is open and dense in X_n , we have $\dim(X_n) = \dim([M_n])$. Recall that $[M_n]$ is an orbit of the base-change group G_{n^2} and the stabilizer subgroup is $\text{Aut}(M_n) \simeq \text{PGL}_n$. Thus

$$\dim(X_n) = \dim(G_{n^2}) - \dim(\text{PGL}_n) = n^2(n^2 - 1) - (n^2 - 1) = (n^2 - 1)^2,$$

as claimed. □

Now let D_n be as in Lemma 3.1.1. Since D_n is a central simple algebra of degree n , the universal property of the trace ring (see, e.g., [Re, Prop. 2.1]) says that there exists a trace-preserving k -algebra morphism

$$\phi : T_{n^2-1,n} \longrightarrow D_n$$

such that $\phi(Z_i) = e_{i+1}$ for all $0 \leq i \leq n^2 - 1$. Here Z_0 denotes the $n \times n$ -identity matrix.

Proposition 3.2.2 1. ϕ is injective.

2. $D_n \simeq \text{UD}_{n^2-1,n}$.

3. $k(X_n)$ is isomorphic to the center of $\text{UD}_{n^2-1,n}$.

4. $\phi(T_{n^2-1,n}) \hookrightarrow \mathbb{G}_n$.

Proof. Let F be the field of fractions of $\phi(X_n)$. By Lemma 3.1.1 the center of D_n equals $k(X_n)$, whence $F \subset k(X_n)$. In order to prove equality we only need to show that every $x_{ij}^h \in F$. Since ϕ is trace preserving, the multiplication rules in \mathbb{G}_n tell us that

$$\phi(\text{tr}(Z_{i-1}Z_{j-1}Z_{t-1})) = \sum_{h=1}^{n^2} x_{ij}^h \phi(\text{tr}(Z_{h-1}Z_{t-1}))$$

for every $1 \leq i, j, t \leq n^2$; here Z_0 is the $n \times n$ -identity matrix. Now fix i and j . Then the above formula can be viewed as a system of n^2 linear equations in the variables $x_{ij}^1, \dots, x_{ij}^{n^2}$. Observe that every coefficient of this system is an element of F and that the matrix of this system is non-singular; see Lemma 2.3.1. Thus by Cramer's rule we have $x_{ij}^h \in F$ and, hence, $F = k(X_n)$.

(1): By Lemma 3.2.1, $\text{trdeg } X_n = \text{trdeg } \phi(X_n) = \text{trdeg } k(X_n)$. This implies that $\phi|_{X_n}$ is injective and thus so is ϕ ; see, e.g., [Ro, 1.6.27].

(2): As $F = k(X_n)$, we have $D_n = \phi(T_{n^2-1,n})F$. Since ϕ is injective, this means that $D_n \simeq \text{UD}(n^2 - 1, n)$.

(3): follows from (2), since $k(X_n)$ is the center of D_n .

(4): Since $\text{char}(k) = 0$, Newton's identities show that $T_{m,n}$ is generated (as a k -algebra) by elements of the form M and $\text{tr}(M)$, where M is a monomial in Z_1, \dots, Z_m . By the definition of ϕ , we have $\phi(Z_i) \in \mathbb{G}_n$ for every $i = 1, \dots, m$. Thus it is sufficient to show that \mathbb{G}_n is closed under the reduced trace tr in D_n . Since D_n is a finite-dimensional division algebra, we have $\text{LTr} = \text{RTr} = n \cdot \text{tr}$ in D_n (and, hence, in \mathbb{G}_n). Consequently, we only need to show that \mathbb{G}_n is closed under LTr or, equivalently, $\text{LTr}(e_i) \in k[X_n]$. This follows from $\text{LTr}(e_i) = \sum_{j=1}^{n^2} x_{ij}^j \in k[X_n]$. \square

Remark 3.2.3 Note that the above argument also proves the following assertion:

The center of the universal division algebra $\text{UD}_{n^2-1,n}$ is generated (as a field extension of k) by elements of the form $\text{tr}(M)$ where M is a monomial of degree ≤ 3 in the generic matrices Z_1, \dots, Z_{n^2-1} .

This result is a special case of [FGG, Thm 3.2].

3.3 PGL_n -quotients

We are now ready to relate X_n to the rationality problem for PGL_n -quotients. Let H be a linear reductive group acting almost freely on a finite dimensional k -vector space U (this means that the stabilizer subgroup of a point in general position is trivial). One of the main open problems in invariant theory is to determine for which groups H the field of rational invariants $k(U)^H$ is rational or stably rational. Stable rationality is, in fact, easier to work with, since it is independent of the choice of the (almost free) representation U . This important result (known as the ‘‘Bogomolov transfer theorem’’ and sometimes referred to as the ‘‘no-name lemma’’, due to the fact that variants and special cases of it were independently discovered by several mathematicians), allows one to prove stable rationality for a large class of groups which includes GL_n , SL_n and O_n ; see, e.g., [D]. For other groups H , most notably $H = \text{PGL}_n$, the question of stable rationality remains wide open. At the moment there are no examples where $k(U)^{\text{PGL}_n}$ is known not to be rational. On the other hand, stable rationality has only been proved if n divides 420; see [IB], [BIB].

Corollary 3.3.1 *Let U be an almost free finite-dimensional linear representation of PGL_n . Then $k(X_n)$ is stably birational to $k(U)^{\text{PGL}_n}$.*

Proof. By Proposition 3.2.2(3) $k(X_n)$ is isomorphic to the center of $\text{UD}_{m,n}$ with $m = n^2 - 1$. Recall that the center of $\text{UD}_{m,n}$ is the fraction field of $C_{m,n}$. In particular, it is isomorphic to the field $k(U)^{\text{PGL}_n}$, where $U = (M_n)^{\oplus m}$ and PGL_n acts on U by simultaneous conjugation. Note that this action is almost free for any $m \geq 2$. By the ‘‘no-name lemma’’ any other almost free finite-dimensional PGL_n -representation U will give rise to a stably isomorphic field of invariants. \square

Now suppose that X_n could be shown to be smooth for some $n \geq 2$. Then by Lemma 2.2.2 $\pi(X_n)$ would be a vector subspace of $W^* \otimes W^* \otimes W$ and thus $X_n \simeq$

$\pi(X_n) \simeq A^{(n^2-1)^2}$ would, in particular, be a rational variety. Combining this result with Corollary 3.3.1 we would then conclude that $k(U)^{\mathrm{PGL}_n}$ is stably rational, thus giving a positive solution to the rationality problem for PGL_n . It was this possibility that originally got us interested in trying to determine whether or not X_n is smooth. Unfortunately, Theorem 1.2 says that X_n is singular for all $n \geq 3$. Thus the above reasoning only applies in the case $n = 2$ (see Corollary 4.5.2). We remark that the rationality problem for PGL_2 -quotients can be easily solved by other means; see, e.g., [P, Thm. II.2.2].

The significance of Proposition 3.2.2 is that it suggests an approach to the study of the quotient space $(M_n)^{\oplus m}/\mathrm{PGL}_n$ via X_n . One indication that the geometry of X_n may be easier to understand than that of $(M_n)^{\oplus m}/\mathrm{PGL}_n$ is given by the following proposition.

Proposition 3.3.2 *Let A be a semi-simple k -algebra of dimension r , let X be the closure of $[A]$ in Alg_r and let $Y = X \setminus [A]$. Then every component of Y has codimension 1 in X . In particular, for $n \geq 2$, every component of $X_n - [M_n]$ is of codimension 1 in X_n .*

Proof. Let $\Delta \in k[V^* \otimes V^* \otimes V]$ be determinant of the Gram matrix of the (left) trace form, as in (18). Since semi-simple algebras are generic, no element of Y can define a semi-simple algebra structure on V . Thus by Lemma 2.3.1(2), Y is the intersection of X with the hypersurface $\{\Delta = 0\}$ in V . This implies that either (i) $Y = \emptyset$ or (ii) every component of Y has codimension 1 in X ; see, e.g. [Shaf, I.6.2, Cor. 1]. On the other hand, (i) is clearly absurd, since there are non-trivial degenerations of M_n . For example the orbit of the trivial algebra $A_0(n^2)$ given by (4) always lies in Y . Thus (ii) holds, as claimed. \square

We remark the description of the ‘‘boundary’’ of X_n given by Proposition 3.3.2 is in sharp contrast to the ‘‘boundary’’ of $(M_n)^{\oplus m}/\mathrm{PGL}_n$ which is of high codimension and otherwise highly irregular; see, e.g., [IBP] and [IBT].

3.4 Graded algebras of GK dimension 2

In this section we indicate a relationship between X_n and the recent attempts to classify graded algebras of Gelfand-Kirillov dimension two. By a theorem of Artin and Stafford every graded connected domain of Gelfand-Kirillov dimension two which is generated in degree one, is a twisted homogeneous coordinate ring; see [AS]. One would now like to know if this result can be generalized to prime algebras. In this section we show that classifying degenerations of matrices (i.e., algebras A such that $[A] \subset X_n$) is a subproblem of this project.

Let X be an algebraic variety. We shall say that X is path-connected if for every $p, q \in X$ there exists a regular map $\phi : A^1 \rightarrow X$ such that $\phi(0) = p$ and $\phi(1) = q$.

Lemma 3.4.1 *Let $X \subset P^s$ be a unirational projective variety. Then*

1. (Kraft) X is path-connected.

2. The affine cone $Y \subset A^{s+1}$ over X is path-connected.

Proof. (1) is proved in [K, II.3.9]. To prove (2) choose $p, q \in Y$. If $p = 0 \in A^{s+1}$ then the assertion is obvious, since the line $\overline{p}q$ is then contained in Y . Thus we may assume without loss of generality that $p, q \neq 0$. Denote the associated projective points by \overline{p} and $\overline{q} \in X$. By part (1) there exists a regular map $\alpha : A^1 \rightarrow X$ such that $\alpha(0) = \overline{p}$ and $\alpha(1) = \overline{q}$. It is easy to see that α can be lifted to a regular map $\beta : A^1 \rightarrow Y$. Then $p = \lambda\beta(0)$ and $q = \mu\beta(1)$ for some $\lambda, \mu \in k^*$, and the map $\phi : A^1 \rightarrow Y$ given by $\phi(t) = (\mu t + \lambda(1-t))\beta(t)$ has the desired property. \square

Note that, Lemma 3.4.1 applies to $Y = \pi(X_n)$, which is a cone in $W^* \otimes W^* \otimes W$ by Lemma 2.2.2.

Proposition 3.4.2 *To every morphism $\phi : A^1 \rightarrow X_n$ one can associate a graded connected prime algebra G generated in degree one such that*

1. the Gelfand-Kirillov dimension of G equals 2
2. the central projective curve of G is P^1 .

Proof. The morphism ϕ determines a prime order $\phi^*(\mathbb{G}_n)$, which is projective of rank n^2 over $k[A^1]$. Equip $\phi^*(\mathbb{G}_n)$ with the generator filtration (as a k -algebra). Then the Rees algebra with respect to this filtration has the claimed properties. \square

3.5 Proof of Theorem 1.1

In this section we prove Theorem 1.2 by proving the following (more general) result.

Proposition 3.5.1 *Let A be an associative k -algebra of dimension $r \geq 7$. Assume that there exist elements $a, b \in A$ such that $1, a, b$, and ab are k -linearly independent. Then the orbit closure $\overline{[A]}$ is not smooth.*

Note that the conditions of this proposition are, indeed, satisfied by $A = M_n(k)$ for every $n \geq 3$.

Proof. Let T be the maximal diagonal torus in $GL(W) \subset G_r$ and let $t = (t_2, \dots, t_r) \in T$ such that $t(a_i) = t_i a_i$. Then $t.(c_{ij}^h) = (t_i t_j t_h^{-1} c_{ij}^h)$. Every element $w \in W^* \otimes W^* \otimes W$ can thus be written as $w = \sum_{\chi} w_{\chi}$, where χ ranges over the characters of T and $t.w_{\chi} = \chi(t)w_{\chi}$ for every $t \in T$.

For any triple $2 \leq \alpha, \beta, \gamma \leq r$ choose a basis $\{a_1, \dots, a_r\}$ with $a_1 = 1, a_{\alpha} = a, a_{\beta} = b$ and $a_{\gamma} = ab$. Let $w = w(\alpha, \beta, \gamma) \in W^* \otimes W^* \otimes W$ be the image under π of the structure constants of A with respect to the basis $\{a_i\}$.

Assume $\overline{[A]}$ is smooth. Then $R = \pi(\overline{[A]})$ is a $GL(W)$ -invariant subspace of $W^* \otimes W^* \otimes W$; see Lemma 2.2.2. By linear independence of characters, if $w \in R$ then every component w_{χ} also lies in R . In other words, for every $\chi(t) = t_i t_j t_h^{-1} : T \rightarrow k^*$ we have

$$w_{\chi} = c_{ij}^h e_h^{ij} + c_{ji}^h e_h^{ji} \in R$$

In particular, taking $(i, j, h) = (\alpha, \beta, \gamma)$ we see that

$$e_\gamma^{\alpha\beta} + c_{\beta\alpha}^\gamma e_\gamma^{\beta\alpha} \in R.$$

Choosing $\alpha < \beta$ and $\gamma \neq \alpha, \beta$, we obtain $\frac{1}{2}(r-1)(r-2)(r-3)$ linearly independent tensors of this form. Consequently, $\dim R \geq \frac{1}{2}(r-1)(r-2)(r-3)$. On the other hand, the orbit $[A]$ has dimension $\leq \dim G_r = r(r-1)$; see (8). Since

$$\frac{1}{2}(r-1)(r-2)(r-3) > r(r-1)$$

for $r \geq 7$, this leads to a contradiction, thus proving that $\overline{[A]}$ is singular. \square

Remark 3.5.2 One can show that the only algebras A of dimension ≥ 4 with the property that $1, a, b$, and ab are linearly dependent for every choice of $a, b \in A$, are the algebras $A = A_0(r)$ and $A = A_1(r)$; see Lemma 6.1.1. Using this fact Proposition 3.5.1 can be restated as follows:

For $r \geq 7$ the orbit closure $\overline{[A]}$ is smooth if and only if $A = A_0(r)$ or $A = A_1(r)$.

Theorem 1.3 is stronger than this assertion because it covers every $r \geq 3$ and, more significantly, because it does not a priori assume that Y is an orbit closure. The proof of Theorem 1.3 presented in the next section uses a more sophisticated version of the above argument, which is based on the representation theory of $\mathrm{GL}(W)$ rather than that of the maximal torus T .

4 Smooth invariant subvarieties

4.1 The isotypical decomposition of $W^* \otimes W^* \otimes W$

Lemma 2.4.3 reduces the study of smooth G_r -invariant subvarieties of Alg_r to the study of $\mathrm{GL}(W)$ -invariant subspaces of $W^* \otimes W^* \otimes W$ which contain $s(W^* \otimes \delta)$ and are contained in Alg'_r .

In order to describe these subspaces, we compute the isotypical decomposition of $W^* \otimes W^* \otimes W$ as $\mathrm{GL}(W)$ -module. Recall the definitions of the $\mathrm{GL}(W)$ -equivariant maps s, a , and c from (14) and (15).

Proposition 4.1.1 *Assume $W = k^d$, where $d \geq 2$. Let $R_+ = \mathrm{Ker}(a) \cap \mathrm{Ker}(c)$ and $R_- = \mathrm{Ker}(s) \cap \mathrm{Ker}(c)$. Then*

1. $\dim(R_+) = d(d-1)(d+2)/2$ and $\dim(R_-) = d(d-2)(d+1)/2$.
2. If $\mathrm{char}(k) = 0$ then $W^* \otimes W^* \otimes W$ can be written as a direct sum of irreducible $\mathrm{GL}(W)$ -representations as follows:

$$W^* \otimes W^* \otimes W = R_+ \oplus R_- \oplus (W^* \otimes \delta) \oplus \sigma(W^* \otimes \delta).$$

Proof. Write $W^* \otimes W^* \otimes W = S \oplus A$, where $S = S^2 W^* \otimes W$ is the kernel of a and $A = \wedge^2 W^* \otimes W$ is the kernel of s . Then $\dim(S) = d^2(d-1)/2$ and $\dim(A) = d^2(d+1)/2$. It is easy to see that the maps $c_{|S} : S \rightarrow W^*$ and $c_{|A} : A \rightarrow W^*$ are surjective. Hence, their kernels R_+ and R_- have dimensions $d^2(d+1)/2 - d = d(d-1)(d+2)$ and $d^2(d-1)/2 - d = d(d-2)(d+1)/2$ respectively. This proves part (1). Since $\mathrm{GL}(W)$ -representations are completely reducible, $c_{|S}$ and $c_{|A}$ can be split $\mathrm{GL}(W)$ -equivariantly. This proves the direct sum decomposition of part (2). Note that $R_- = (0)$ if $d = 2$.

In order to complete the proof, it is enough to show that $W^* \otimes W^* \otimes W$ is a direct sum of four irreducible components if $d \geq 3$ and three irreducible components if $d = 2$. To do this we invoke standard facts about $SL(W)$ -representations, see e.g. [OV, Appendix, Table 6] or [FH, Lect. 15]. We shall use the notation of [FH]. Let S_λ be the Weyl module corresponding to the ordered partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 0$, that is, the irreducible SL_d -module with highest weight $\lambda_1 L_1 + \dots + \lambda_d L_d$. Then $W = S_{(1,0^{d-1})}$, $W^* = S_{(1^{d-1},0)}$, and we can use the Littlewood-Richardson rule (or, in this case, Pieri's formula [FH, p.225]) to obtain the following decomposition

$$W^* \otimes W^* \otimes W = S_{(2,1^{d-3},0^2)} \oplus S_{(3,2^{d-2},0)} \oplus S_{(1^{d-1},0)}^{\oplus 2}.$$

This completes the proof of part (2). □

Remark 4.1.2 Note that the last two terms in the above direct sum decomposition are isomorphic to W^* . Thus their choice is not unique. In particular, we can replace the last term by $s(W^* \otimes \delta)$. Lemma 2.4.3 shows that this is, indeed, a more convenient choice in our setting.

Theorem 4.1.3 *Assume $\mathrm{char}(k) \nmid 2(d-1)(d+1)$. Let X be a G_r -invariant closed subvariety of Alg_r , where $r \geq 3$. Then X is smooth if and only if $\pi(X) = U_{i_1, i_2, i_3}$, where*

$$U_{i_1, i_2, i_3} = R_+^{i_1} \oplus R_-^{i_2} \oplus (W^* \otimes \delta)^{i_3} \oplus s(W^* \otimes \delta), \quad (20)$$

and each of the subscripts i_1 , i_2 , and i_3 is either 0 or 1.

Proof. By Lemma 2.4.3 X is smooth if and only if it is a $\mathrm{GL}(W)$ -equivariant subspace of $W^* \otimes W^* \otimes W$ containing $s(W^* \otimes \delta)$. It remains to show that every such subspace is of the form (20). If $\mathrm{char}(k) = 0$, this follows from Proposition 4.1.1; see Remark 4.1.2. If k is a field of odd characteristic not dividing $d^2 - 1$, the desired conclusion follows from Theorem 4.2.1, which is proved in the next section. □

4.2 The case of odd characteristic

In this section we generalize the isotypical decomposition of Section 4.1 to the case where the base field k has characteristic $p \neq 2$. We show that Proposition 4.1.1 remains valid if $p \nmid 2(d-1)(d+1)$. If p is an odd prime dividing $(d-1)(d+1)$ then only two of the summands in Proposition 4.1.1 remain irreducible.

Since $\text{char}(k) \neq 2$, we have a decomposition $W^* \otimes W^* = S^2 W^* \oplus \wedge^2 W^*$ and exact sequences

$$0 \longrightarrow R_+ \longrightarrow S^2 W^* \otimes W \xrightarrow{c} W^* \longrightarrow 0 \quad (21)$$

and

$$0 \longrightarrow R_- \longrightarrow \wedge^2 W^* \otimes W \xrightarrow{c} W^* \longrightarrow 0, \quad (22)$$

where (21) splits iff $p \nmid d+1$ and (22) splits iff $p \nmid d-1$. Therefore,

$$W^* \otimes W^* \otimes W = R_+ \oplus R_- \oplus W^* \oplus W^* \text{ if } p \nmid (d-1)(d+1) \quad (23)$$

On the other hand,

$$W^* \otimes W^* \otimes W = (S^2 W^* \otimes W) \oplus R_- \oplus W^* \text{ if } p \mid d+1 \quad (24)$$

and

$$W^* \otimes W^* \otimes W = (\wedge^2 W^* \otimes W) \oplus R_+ \oplus W^* \text{ if } p \mid d-1. \quad (25)$$

We will now show that these expressions give the decomposition into indecomposables. Our proof of this theorem uses tilting modules. We thank K. Erdmann for introducing us to this theory. For details we refer the reader to [Do] and [Er].

Theorem 4.2.1 *Let k be an algebraically closed field of odd characteristic and $W = k^d$.*

(a) *If $p \nmid (d-1)(d+1)$ then (23) is a direct sum of four irreducible $\text{GL}(W)$ -modules. (Note that $R_- = (0)$ if $d = 2$.)*

(b) *If $p \mid d+1$ (resp. $p \mid d-1$) then (24) (resp. (25)) is a direct sum of three indecomposable $\text{GL}(W)$ -modules, where the last two are irreducible and $S^2 W^* \otimes W^*$ (resp. $\wedge^2 W^* \otimes W$) is a uniserial indecomposable module whose composition series is of length three with both top and socle isomorphic to W^* .*

Proof. In order to apply the theory of tilting modules we have to restrict to polynomial $\text{GL}(W)$ -representations. Therefore, we consider the dual problem and replace W^* by $W^* \otimes \det = \wedge^{d-1} W$. In other words, it is sufficient to prove that

(a') If $p \nmid d+1, d-1$ then $\wedge^{d-1} W \otimes W \otimes W$ is the sum of four irreducible modules, one occurring with multiplicity two.

(b') If $p \mid d+1$ or $p \mid d-1$ then $\wedge^{d-1} W \otimes W \otimes W$ is the sum of three distinct indecomposable modules. Two of them are irreducible. The third is uniserial; its composition series is of length three with top and socle isomorphic to $\det \otimes W$.

By [Do, Lemma 3.4(ii)] we have the decomposition into tilting modules

$$\wedge^{d-1} W \otimes W \otimes W \simeq T(3, 1^{d-2}) \oplus T(2^2, 1^{d-3})^{\oplus a} \oplus T(2, 1^{d-1})^{\oplus b_i},$$

where $a, b \geq 0$. Recall that tilting modules are always indecomposable. For every partition μ (with less than d parts) we have exact sequences

$$0 \longrightarrow Z(\mu) \longrightarrow T(\mu) \longrightarrow \nabla(\mu) \longrightarrow 0$$

where $Z(\mu)$ is filtered by $\nabla(\lambda)$'s with $\lambda < \mu$. In the case of interest to us we have $\det \otimes W = \nabla(2, 1^{d-1}) = T(2, 1^{d-1})$ an irreducible module and exact sequences

$$0 \longrightarrow \det \otimes W \longrightarrow T(3, 1^{d-2}) \longrightarrow \nabla(3, 1^{d-2}) \longrightarrow 0$$

when $p \mid d+1$. In all other cases $T(3, 1^{d-2}) = \nabla(3, 1^{d-2})$ is irreducible. Similarly, there is an exact sequence

$$0 \longrightarrow \det \otimes W \longrightarrow T(2^2, 1^{d-3}) \longrightarrow \nabla(2^2, 1^{d-3}) \longrightarrow 0$$

if $p \mid d-1$. In all other cases $T(2^2, 1^{d-3}) = \nabla(2^2, 1^{d-3})$ is irreducible. Moreover, the simple composition factors of $\nabla(\mu)$ are all of the form $L(\lambda)$ for $\lambda \leq \mu$. Therefore, if $p \mid d+1$ then $T(3, 1^{d-2})$ is uniserial with simple factor sequence

$$(L(2, 1^{d-1}), L(3, 1^{d-2}), L(2, 1^{d-1}))$$

and if $p \mid d-1$ then $T(2^2, 1^{d-3})$ is uniserial with simple factor sequence

$$(L(2, 1^{d-1}), L(2^2, 1^{d-3}), L(2, 1^{d-1})).$$

This completes the proof of (a') and (b'). \square

4.3 Proof of Theorem 1.3 for $r \geq 5$

Suppose $\text{char}(k) \nmid 2(d^2 - 1)$. Then in order to prove Theorem 1.3, we only need to determine which of the eight vector subspaces U_{i_1, i_2, i_3} given by (20) are, contained in Alg'_r . If $\text{char}(k)$ is an odd prime dividing $d^2 - 1$, then we shall use similar reasoning, with Theorem 4.1.3 taking the place of Theorem 4.2.1. We shall assume throughout that $\text{char}(k) \neq 2$.

Lemma 4.3.1 1. If $r \geq 5$ then $R_- \not\subset \text{Alg}'_r$.

2. If $r \geq 4$ then $R_+ \not\subset \text{Alg}'_r$.

Proof. (1): Let $p = e_5^{23} - e_{32}^5 + e_{45}^3 - e_{54}^3$. We claim that $p \in R_+$ but $p \notin \text{Alg}'_r$. Recall that R_+ is the intersection of the kernels of the maps a and c defined in (14) and (15). Since $a(p) = 0$ and $c(p) = 0$, we conclude that $p \in R_+$. It remains to show that p defines a non-associative algebra structure on V . Indeed, this structure is given by

$$\begin{aligned} e_2 \cdot e_3 &= c_{23}^1 e_1 + e_5 \\ e_3 \cdot e_2 &= c_{32}^1 e_1 - e_5 \\ e_4 \cdot e_5 &= c_{45}^1 e_1 + e_3 \\ e_5 \cdot e_4 &= c_{45}^1 e_1 - e_3 \\ e_i \cdot e_j &= c_{ij}^1 e_1, \end{aligned}$$

where $(i, j) \neq (2, 3), (3, 2), (4, 5), (5, 4)$ for some $c_{uv}^1 \in k$. Using these multiplication rules, we find that

$$(e_2 \cdot e_3) \cdot e_4 = c_{23}^1 e_4 + c_{54}^1 e_1 - e_3 \neq c_{34}^1 e_2 = e_2 \cdot (e_3 \cdot e_4).$$

Thus the algebra structure given by p is non-associative, i.e., $p \notin \text{Alg}'_r$.

(2): Let $p = e_2^{34} + e_2^{43} + e_3^{23} + e_3^{32} - e_4^{24} - e_4^{42}$. It is easy to see that $a(p) = 0$ and $c(p) = 0$ and, hence, $p \in R_+$. It therefore remains to show that the algebra structure on V defined by p is not associative. Assume the contrary. Then

$$0 = e_4 \cdot (e_3 \cdot e_3) - (e_4 \cdot e_3) \cdot e_3 = c_{33}^1 e_4 - c_{43}^1 e_3 - c_{23}^1 e_1 - e_3$$

and, hence, $c_{43}^1 = -1$. On the other hand,

$$0 = e_4 \cdot (e_3 \cdot e_2) - (e_4 \cdot e_3) \cdot e_2 = c_{32}^1 e_4 + e_4 \cdot e_3 - c_{43}^1 e_2 - e_2 \cdot e_2 = c_{32}^1 e_4 + c_{43}^1 e_1 + e_2 - c_{43}^1 e_2 - c_{22}^1 e_1$$

which implies $c_{43}^1 = 1$. This contradiction finishes the proof. \square

Next we will show that $U_{0,0,0}$ and $U_{0,0,1}$ are contained in Alg'_r . Recall the definitions (4) and (5) of the algebras $A_0(r)$ and $A_1(r)$ from Section 1. Note that every r -dimensional algebra degenerates to $A_0(r)$. Consequently, $[A_0(r)]$ is the unique closed orbit in Alg_r and hence is smooth. On the other hand, $A_1(r)$ is the path algebra of the quiver with two vertices (determined by the idempotents $1 - e_2$ and e_2) and $r - 2$ parallel edges; each edge joins the first vertex to the second. Thus $A_1(r)$ is an hereditary algebra; consequently, its orbit closure $\overline{[A_1(r)]}$ is an irreducible component of Alg_r .

Lemma 4.3.2 *Let $A_0(r)$ and $A_1(r)$ be as above and let U_{i_1, i_2, i_3} be as in (20). Then*

1. $U_{0,0,0} = \pi([A_0(r)]) = \pi(\overline{[A_0(r)]}) = \subset \text{Alg}'_r$.
2. $U_{0,0,1} = \pi(\overline{[A_1(r)]}) = \pi([A_0(r)]) \cup \pi([A_1(r)]) \subset \text{Alg}'_r$.

Proof. (1): The structure constants of $A_0(r)$ given by the basis e_1, \dots, e_r correspond to the origin of $W^* \otimes W^* \otimes W$. The desired conclusion now follows from the fact that $U_{0,0,0} = s(W^* \otimes \delta)$ is the G_r -orbit of 0; see Section 2.4.

(2): $U_{0,1,1}$ is G_r -equivariantly isomorphic to $W^* \oplus W^*$. Thus it is the union of two orbits: the diagonal $X_1 = U_{0,0,0}$, which is closed, and $X_2 = U_{0,0,1} \setminus U_{0,0,0}$, which is open and dense. $A_1(r)$ (with our chosen basis) corresponds to the point

$$p = e_2^* \otimes \delta \in X_2,$$

and the Lemma follows. \square

We are now ready to prove Theorem 1.3 in the case $r \geq 5$.

Proposition 4.3.3 *Suppose $\text{char}(k) \neq 2$ and $r \geq 5$. Let X be a G_r -invariant closed subvariety of Alg_r . Then X is smooth if and only if $X = [A_0(r)] = \overline{[A_0(r)]}$ or $X = \overline{[A_1(r)]}$.*

Proof. First assume $\text{char}(k) \nmid 2(d-1)(d+1)$. By Theorem 4.1.3 $\pi(X) = U_{i_1, i_2, i_3}$ for some $i_1, i_2, i_3 \in \{0, 1\}$. By Lemma 4.3.1, R_+ and R_- are not contained in Alg'_r . Hence, $i_1 = i_2 = 0$. The remaining two cases, i.e., $\pi(X) = U_{0,0,0}$ and $\pi(X) = U_{0,0,1}$ can, indeed, occur; they are covered by Lemma 4.3.2.

Next, assume $\text{char}(k) \mid d+1$. Then R_- cannot be contained in $\pi(X)$, and the cases $U_{0,0,0}$ and $U_{0,0,1}$ are covered by Lemma 4.3.2. Hence, by uniseriality of $S^2 W^* \otimes W$ any other $\pi(X)$ must contain the $\text{GL}(W)$ -submodule R_+ , which is impossible by Lemma 4.3.1. A similar argument settles the case $\text{char}(k) \mid d-1$. \square

4.4 Smooth subvarieties of Alg_3

Having proved Theorem 1.3 for all $r \geq 5$ (see Proposition 4.3.3), we now turn to the low-dimensional cases $r = 3$ and $r = 4$, which, somewhat surprisingly, require additional care. In this section we describe smooth G_3 -invariant subvarieties of Alg_3 .

Proposition 4.4.1 *Assume $\text{char}(k) \neq 2$. Let X be a smooth G_3 -invariant subvariety of Alg_3 . Then $X = \overline{[A]}$, where $A = A_0(3), A_1(3)$ or $E_3 = k \times k \times k$.*

Proof. First we show that $X = \pi(\overline{[A]})$ is a linear subspace of $W^* \otimes W^* \otimes W$ contained in Alg_3 for the three algebras A listed in the statement of the proposition. For $A = A_0(3)$ and $A_1(3)$, this follows from Lemma 4.3.2.

We claim that $\pi(\overline{[E_3]}) = U_{1,0,0}$. Indeed, let $p \in [E_3] \subset \text{Alg}'_3$. Since E_3 is commutative, $\pi(p) \in U_{1,0,0}$. On the other hand, since, the automorphism group of E_3 is finite,

$$\dim([E_3]) = \dim(G_3 p) = \dim(G_3) = 6.$$

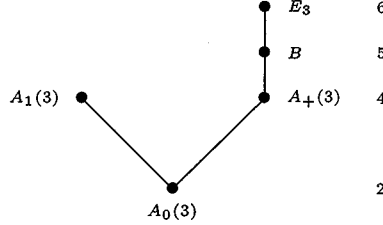
Since $U_{1,0,0}$ also has dimension 6 (see Proposition 4.1.1), we conclude that $U_{1,0,0}$ is the Zariski closure of $[E_3]$, as claimed.

Finally assume that $X \subset \text{Alg}_3$ is a smooth G_3 -invariant subvariety of Alg_3 which is not of the form described in the proposition. Note that $R_- = (0)$ for $r = 3$. Thus $\pi(X)$ is a linear subspace of $W^* \otimes W^* \otimes W$ which properly contains $U_{1,0,0}$; see Theorem 4.1.3 (if $\text{char}(k) \neq 3$) and Theorem 4.2.1 (if $\text{char}(k) = 3$). On the other hand, since E_3 is semi-simple, $U_{1,0,0} = \overline{[E_3]}$ is a component of Alg_3 . Therefore, it cannot be properly contained in $\pi(X)$. This contradiction completes the proof of Proposition 4.4.1. \square

Remark 4.4.2 An alternative method to prove Proposition 4.4.1 is to study the G_3 -orbit structure in $X = X_0 \oplus s(W^* \otimes \delta)$ (which coincides with the $\text{GL}(W)$ -orbit structure in X_0 by Corollary 2.4.2) and make use of the classification of algebras of dimension three due to Gabriel [Ga]. In this case $R_+ \simeq \text{Sym}^3 W$ as a $\text{SL}(W)$ -module. It is well known that the PGL_2 -action on the space of cubic binary forms $\mathbb{P}(R_+) = \mathbb{P}^3$ has three orbits: the twisted cubic (the cubes $[u^3]$ of linear terms), the tangential surface to the twisted cubic (the products $[u^2 \cdot v]$) and the open orbit, consisting of the products $[u \cdot v \cdot w]$ of pairwise independent linear terms. Therefore, the four $\text{GL}(W)$ -orbits in R_+ are of

dimensions 4, 3, 2 and 0. The four G_3 -orbits in $U_{1,0,0} = R_+ \oplus s(W^* \otimes \delta)$ then have dimensions 6, 5, 4 and 2.

On the other hand we have the following degeneration picture of G_3 -orbits in Alg_3 (see [Ga] for details).



Therefore, only $\overline{[E_3]}$ with $E_3 = k \times k \times k$ can be smooth. It is also easy to assign orbits in R_+ to the degenerations $B = k \times k[x]/(x^2)$ and $A_+(3) = k[x]/(x^3)$. (For more on the algebra $A_+(3)$, see Section 6.)

4.5 Smooth subvarieties of Alg_4

The following proposition completes the proof of Theorem 1.3.

Proposition 4.5.1 *Assume $\text{char}(k) \neq 2$. Let X be a smooth G_4 -invariant subvariety of Alg_4 . Then $X = \overline{[A]}$, where $A = A_0(4)$, $A_1(4)$ or $E_4 = M_2(k)$.*

Proof. In this case, $d = r - 1 = 3$, and Theorem 4.1.3 applies. Thus $\pi(X) = U_{i_1, i_2, i_3}$ for some i_1, i_2 , and $i_3 \in \{0, 1\}$. By Lemma 4.3.1(2), $i_1 = 0$. It remains to investigate the four remaining choices of $i_2, i_3 = 0, 1$. As before, in each case we need to decide whether or not U_{0, i_2, i_3} lies in Alg'_3 .

If $i_2 = i_3 = 0$ then $U_{0,0,0} = \pi([A_0(4)])$. and if $i_2 = 0$ and $i_3 = 1$ then $U_{0,0,1} = \pi(\overline{[A_1(4)]})$; see Lemma 4.3.2. In both cases $U_{0,0, i_3}$ is contained in Alg'_4 .

Next we want to determine whether or not $U_{0,1,0} = R_- \oplus s(W^* \otimes \delta)$, lies in Alg'_4 . We claim that the answer is “yes” and, that, in fact $U = \pi(\overline{[E_4]})$. Indeed, choose the following basis of $E_4 = M_2$:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (26)$$

Using the identity $e_2 \cdot e_3 = e_4 = -e_3 \cdot e_2$, we can easily compute the structure constants c_{ij}^h for M_n in this basis. Combining those c_{ij}^h with $i, j, h \geq 2$ into a tensor, we see that

$$p = e_4^{23} - e_{23}^4 + e_{24}^3 - e_{42}^3 + e_{43}^2 - e_{34}^2 \in [E_4] \subset W^* \otimes W^* \otimes W.$$

Note that p is skew-symmetric (i.e., $s(p) = 0$ in $W^* \otimes W^* \otimes W$) and $c(p) = 0$ in W^* ; see (14) and (15). Thus $p \in R_-$ and consequently, $[E_4] \subset U_{0,1,0}$. On the other hand, since the automorphism group of $M_2(k)$ is 3-dimensional, we have

$$\dim([E_4]) = \dim(G_4) - 3 = 4 \cdot 3 - 3 = 9 = \dim(U_{0,1,0}).$$

Therefore, $[E_4]$ is Zariski dense in $U_{0,1,0}$. In particular, $U_{0,1,0} \subset Alg'_4$, as claimed.

It remains to consider the case $i_2 = i_3 = 1$. Note that this $U_{0,1,1}$ properly contains $\overline{[E_4]} = U_{0,1,0}$. Since E_4 is semi-simple, $\overline{[E_4]}$ is a component of Alg'_4 . This implies that $U_{0,1,1}$ cannot be properly contained in Alg'_4 , thus completing the proof of Proposition 4.5.1. \square

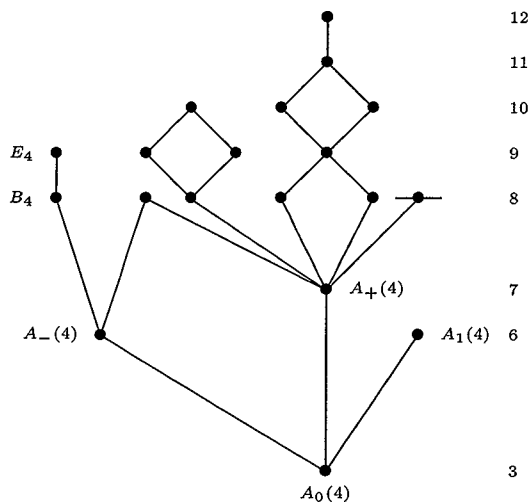
We record the following corollary of the proof of Proposition 4.5.1. Recall that $X_2 = \overline{[M_2(k)]}$.

Corollary 4.5.2 1. $\pi(X_2) = U_{0,1,0} = R_- \oplus s(W^* \otimes \delta)$.

2. (Seshadri [Se78, Thm. 1] or [Se82, p.112]) X_2 is smooth.

Remark 4.5.3 One can also prove Proposition 4.5.1 by a method similar to the one outlined in Remark 4.4.2. If $r = 4$ then $R_- \simeq Sym^2 W$ as $SL(W)$ -module. Thus, as is well known, PGL_3 acts on $\mathbb{P}(R_-) = \mathbb{P}^5$ with 3 orbits, namely, the Veronese surface (the double lines), the chordal variety to it which has dimension 4 (the pairs of lines), and an open orbit (the conics). Hence, there are four G_4 -orbits in $R_- \oplus s(W^* \otimes \delta)$ of dimensions 9, 8, 6 and 3.

On the other hand we have the following degenerations picture of G_4 -orbits in Alg_4 (see [Ga] for more details).



Therefore, only $\overline{[E_4]}$ with $E_4 = M_2(k)$ is smooth. (We shall discuss the “minimal” algebras $A_+(4)$ and $A_-(4)$ in Section 6.)

4.6 Lie algebra structures

In this section we prove the analogue of Theorem 1.3 for the variety Lie_d of Lie algebra structures on a d -dimensional vector space W .

The definition of Lie_d is similar to the definition of Alg_r given in Section 1. Let W be a d -dimensional k -vector space. To be consistent with our previous notation, we shall denote a basis of W by e_2, \dots, e_{d+1} . A Lie algebra structure on W is given by a skew-symmetric bilinear form $W \times W \rightarrow W$, i.e., by an element $p = \sum c_{ij}^h e_h^{ij}$ of $\wedge^2 W^* \otimes W$, where

$$[e_i, e_j] = \sum_h c_{ij}^h e_h.$$

Here $c_{ij} = -c_{ji}$ and e_i, e_j, e_k satisfy the Jacobi identity for every $i, j, k = 2, \dots, d+1$. The Jacobi identity translates into a system of polynomial equations on c_{ij}^h which cut out the variety Lie_d in $\wedge^2 W^* \otimes W$.

The transport of structure action of $GL(W)$ on Lie_d is induced from the natural $GL(W)$ -action on $\wedge^2 W^* \otimes W$. It is easy to see that isomorphism classes of d -dimensional Lie algebras are in 1-1 correspondence with $GL(W)$ -orbits in Lie_d . If L is a d -dimensional Lie algebra, we shall denote the corresponding $GL(W)$ -orbit in Lie_d by $[L]$.

Let A be an r -dimensional associative algebra and let W be the linear subspace consisting of all $x \in A$ such that $RTr(x) + LTr(x) = 0$. Then the bracket $[x, y] = x \cdot y - y \cdot x$ defines a Lie algebra structure on W . We shall denote this $r-1$ -dimensional Lie algebra by $L(A)$.

Lemma 4.6.1 *Suppose $\text{char}(k) \nmid 2r$. Then the anti-symmetrization map*

$$a : W^* \otimes W^* \otimes W \rightarrow \wedge^2 W^* \otimes W$$

sends Alg'_r to Lie_{r-1} . Moreover, for every r -dimensional associative algebra A we have $a([A]) = [L(A)]$.

Proof. Let A be an associative algebra and let W be the kernel of $RTr + LTr$, as above. By our assumption on $\text{char}(k)$, $1_A \notin W$; thus we can choose a basis e_1, \dots, e_r of A so that $e_1 = 1_A$ and e_2, \dots, e_r form a basis of W . If $p = \sum_{i,j,h=2}^r c_{ij}^h e_h^{ij} \in Alg'_r$ is the tensor of structure constants in this basis then our definition of $[\cdot, \cdot]$ shows that the structure constants of $L(A)$ in the basis e_2, \dots, e_r are given by $a(p)$. Any other point $p' \in \pi([A])$ is of the form $ug(p)$, where $u \in U$ and $g \in GL(W)$; see (11). By Lemma 2.4.1 we have $a(p') = a(g(p)) = g(a(p)) \in [L(A)]$; see (17). \square

Of particular interest to us will be the d -dimensional Lie algebras

$$L_0(d) = L(A_0(d+1)) \quad \text{and} \quad L_1(d) = [A_1(d+1)].$$

Note that $L_0(d)$ is the trivial Lie algebra with $[x, y] = 0$ and $L_1(d)$ is a "minimal solvable" Lie algebra with $[e_2, e_i] = e_i$ and $[e_i, e_j] = 0$ for all $i, j \geq 3$.

Theorem 4.6.2 *Assume that $d \geq 3$ and $\text{char}(k) \nmid 2(d+1)$. Let X be a smooth closed $GL(W)$ -invariant subvariety of Lie_d . Then*

1. $X = [L_0(d)] = \{0\}$ or

2. $X = \overline{[L_1(d)]}$ or

3. $d = 3$ and $X = \overline{[sl_2(k)]}$.

Proof. Since X is $GL(W)$ -invariant, it is a cone in $\Lambda^2 W^* \otimes W$. Thus X is smooth iff it is a $GL(W)$ invariant subspace of $\Lambda^2 W^* \otimes W$. Under our assumption on $\text{char}(k)$, we can write

$$\Lambda^2 W^* \otimes W = R_- \oplus a(W^* \otimes \delta)$$

a direct sum of irreducible $GL(W)$ -modules; see Sections 4.1 and 4.2. Thus we only have 3 possibilities, namely (i) $X = (0)$, (ii) $X = a(W^* \otimes \delta)$ and (iii) $X = R_-$.

By Lemma 4.6.1 cases (i) and (ii) correspond to $X = [L(A_0(d+1))] = [L_0(d)]$ and $X = [L(A_1(r))] = \overline{[L_1(d)]}$ respectively; see Lemma 4.3.2. Moreover, if $d = 3$ then (iii) implies $X = [L(M_2(k))] = \overline{[sl_2(k)]}$; see Corollary 4.5.2(1).

It therefore, remains to rule our case (iii) when $d \geq 4$, i.e., show that in this case R_- is not contained in Lie_d . Indeed, let $p = e_3^{12} - e_3^{21} + e_2^{34} - e_2^{43}$. Since $c(p) = 0$, we have $p \in R_-$. On the other hand the bilinear form given by p does not satisfy the Jacobi identity, since

$$[[e_1, e_2], e_4] + [[e_2, e_4], e_1] + [[e_4, e_1], e_2] = [e_3, e_4] + [0, e_1] + [0, e_2] = e_2 \neq 0.$$

This completes the proof of the theorem. □

5 Tangent spaces and singularities

Let A be an r -dimensional associative algebra. and let $T(A)$ be the tangent space to $\pi(\overline{[A]})$ at the origin of $W^* \otimes W^* \otimes W$. Note that $T(A)$ is a G_r -invariant subspace of $W^* \otimes W^* \otimes W$. Assume $\text{char}(k) \nmid 2(d^2 - 1)$. Then, by Theorem 4.1.3 $T(A)$ is one of the eight subspaces U_{i_1, i_2, i_3} defined in (20). Thus $T(A)$ can be viewed as a discrete invariant of the algebra A . We shall now give an explicit description of this invariant in terms of the algebra structure of A .

5.1 Tangent spaces

Lemma 5.1.1 $T(A) = \text{Span}(\pi(\overline{[A]})) = \text{Span}(\pi([A]))$. In particular, $T(A)$ is the smallest of the spaces $U = U_{i_1, i_2, i_3}$, such that $\pi([A]) \subset U$.

Proof. The first equality follows from the fact that $\pi(\overline{[A]})$ is a cone in $W^* \otimes W^* \otimes W$; see Lemma 2.2.2. The second equality follows from $\pi(\overline{[A]}) \subset \text{Span}(\pi([A]))$. □

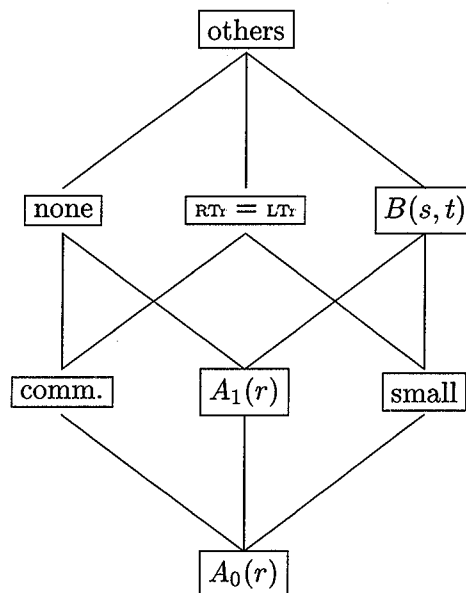
We now classify $T(A)$ in terms of the algebra structure of A . In the sequel $B(s, t)$ will denote the algebra with $\text{Rad}^2 = 0$, whose underlying quiver has two vertices v_1 and v_2 , s arrows from v_1 to v_2 , and t arrows from v_2 to v_1 . Note that the dimension of this algebra is $s + t + 2$ and that $B(0, r - 2) = A_1(r)$. Moreover, since $B(s, t) \simeq B(t, s)$ we shall always assume $s \leq t$.

We call an r -dimensional algebra A *small* provided it is isomorphic to one of the algebras on the following list.

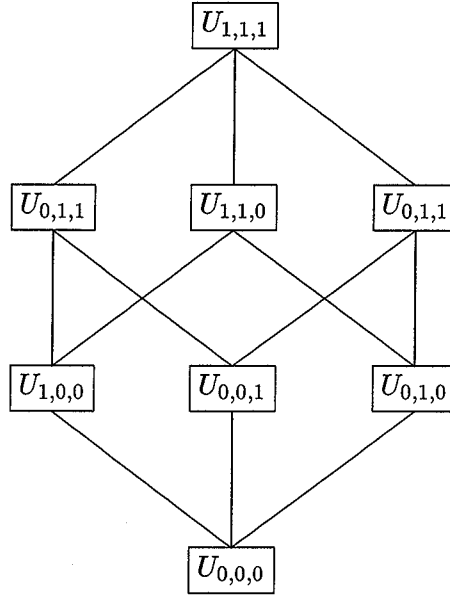
1. $A = k1_A \oplus I$ where $xy = -yx$ for any $x, y \in I$.
2. $A = B(s, s)$ with $2s + 2 = r$.
3. $r = 4$ and $A = M_2(k)$.

With these conventions, we can now state the main theorem of this section.

Theorem 5.1.2 *Assume $\text{char}(k) \nmid 2r(r-2)$. Suppose A belongs to one of the classes of algebras indicated below (and does not belong to a lower class).*



Then the tangent space $T(A)$ to $\overline{\pi([A])}$ at the origin is given by the following diagram.



Proof. We already know that $T(A) = U_{0,0,0}$ (resp. $U_{0,0,1}$) if and only if $A = A_0(r)$ (resp. $A_1(r)$); see Lemma 4.3.2.

Next we note that the subspace $U_{1,1,0}$ is precisely the kernel of $c - c\sigma : W^* \otimes W^* \otimes W \rightarrow W^*$; see Proposition 4.1.1. Thus by Lemma 2.3.3 $\pi([A]) \subset U_{1,1,0}$ if and only if $\text{RTr}_A(w) = \text{LTr}_A(w)$ for every $w \in W$. Since $A = V$ is spanned by W and $e_1 = 1_A$, the later condition is, in turn, equivalent to $\text{RTr}_q(x) = \text{LTr}_q(x)$ for every $x \in A$.

This proves the correspondence between the classes of algebras A and the tangent spaces $T(A)$ given by the four “middle” columns of the diagram. The side columns are covered by Theorems 5.2.3 and 5.3.3, which are proved in the next two sections. \square

Corollary 5.1.3 *Let $n \geq 3$. Then the dimension of the tangent space to $X_n = \overline{[M_n(k)]}$ at the origin of $V^* \otimes V^* \otimes V$ is $n^2(n^2 - 1)(n^2 - 2)$.*

Proof. Theorem 5.1.2 tells us that the tangent space to $\pi(X_n)$ is $U_{1,1,0}$. Since $\pi : X_n \rightarrow \pi(X_n)$ is an isomorphism (see Lemma 2.2.1), the tangent space to X_n at the origin of $V^* \otimes V^* \otimes V$ has the same dimension as $U_{1,1,0}$. This dimension is $d^3 - d$, where $d = \dim(W) = r - 1 = n^2 - 1$, and the corollary follows. \square

We remark that since $\dim(X_n) = (n^2 - 1)^2 < n^2(n^2 - 1)(n^2 - 2)$ (see Lemma 3.2.1), Corollary 5.1.3 may be viewed as a strengthening of Theorem 1.2.

5.2 Alternating algebras

In this section we classify the algebras A with $T(A) = U_{0,1,1}$ and $T(A) = U_{0,1,0}$ under the assumption $\text{char}(k) \nmid 2r$.

We call an r -dimensional algebra A *alternating* if $\pi([A]) \subset U_{0,1,1}$; see (20). The motivation behind the terminology is explained by the following lemma.

Lemma 5.2.1 *A is an alternating algebra if and only if for every $x, y \in A$*

$$xy + yx - l(x)y - l(y)x \in k1_A, \quad (27)$$

where $l(x) = \frac{1}{r}(\text{RTr}(x) + \text{LTr}(x))$.

Proof. Suppose $\pi([A]) \subset U_{0,1,1}$. Translating by an element of $U \subset G_r$, we can find a basis $e_1 = 1_A, e_2, \dots, e_r$ of $A = V$ which gives the (reduced) set of structure constants $q \in R_- \oplus a(W^* \otimes \delta)$. Thus $c(q) = -c\sigma(q)$. By Lemma 2.3.3, this means $\text{RTr}(w) = -\text{LTr}(w)$ and consequently,

$$l(w) = 0 \quad (28)$$

for every $w \in W_0$. Clearly (27) holds if x or $y = e_1 = 1_A$. Thus by linearity it is enough to prove (27) for $x = e_i$ and $y = e_j$, where $i, j \geq 2$. In other words, we only need to show that $e_i e_j + e_j e_i \in k e_1$ for every $i, j \geq 2$; see (28). This, follows from the fact that $q \in R_- \oplus a(W^* \otimes \delta)$ and thus is skew-symmetric (i.e., $s(q) = 0$).

Conversely, suppose (27) holds for every $x, y \in A$. Let W_0 be the vector subspace of $V = A$ given by $l(x) = 0$. Since $e_1 = 1_A \notin W_0$, we see that $\dim(W_0) = r - 1$ and that $e_1 = 1_A$ can be completed to a basis e_1, e_2, \dots, e_r of A with $e_2, \dots, e_r \in W_0$. Thus for every $i, j \geq 2$ we have $e_i \cdot e_j + e_j \cdot e_i \in k1_A$. This means that the tensor $q \in W^* \otimes W^* \otimes W$ given by the structure constants c_{ij}^h in this basis (with $i, j, h \geq 2$) is skew-symmetric in the first two components. In other words, $s(q) = 0$ or, equivalently, $q \in R_- \oplus a(W^* \otimes \delta)$. Consequently, $\pi([A]) \subset U_{0,1,0}$ as claimed. \square

We are now ready to classify the alternating algebras.

Proposition 5.2.2 *Let A be an alternating algebra. Then*

1. $xy = -yx$ for any $x, y \in \text{Rad}(A)$.
2. $\text{SS}(A) = k, k \times k, \text{ or } M_2(k)$.
3. If $\text{SS}(A) = k$, then $A = k1_A \oplus \text{Rad}(A)$ where $xy = -yx$ for all $x, y \in \text{Rad}(A)$ and $\text{Rad}^3(A) = (0)$.
4. If $\text{SS}(A) = k \times k$, then $A = B(u, v)$ with $u + v + 2 = r$.
5. If $\text{SS}(A) = M_2(k)$, then $r = 4$ and $A = M_2(k)$.

Proof. (1): Note that $l(x) = 0$ for every $x \in \text{Rad}(A)$; see Lemma 2.3.1. Thus (27) implies that $xy + yx \in k1_A$ for every $x, y \in \text{Rad}(A)$. That is, $xy + yx \in k1_A \cap \text{Rad}(A) = (0)$, as claimed.

(2): By Lemma 5.2.1 every element of A satisfies a quadratic equation over k . Hence, the same is true of $\text{SS}(A)$. The three semi-simple algebras we listed are the only ones that satisfy this condition.

(3): follows from (1) and (2).

(4): A is the quotient of the path algebra of a quiver Q on two vertices $\{v_1, v_2\}$ having u arrows $v_1 \xrightarrow{a_i} v_2$ and v arrows $v_2 \xrightarrow{b_j} v_1$ by an admissible ideal $I \subset \text{Rad}^2(kQ)$. Since $a_i \cdot b_j + b_j \cdot a_i = 0$ in A , we conclude that $I = \text{Rad}^2(kQ)$, and hence $A = B(u, v)$ with $u + v + 2 = r$.

(5): Write $A = M_2(k) \oplus \text{Rad}(A)$. We want to show that $\text{Rad}(A) = (0)$. Assume the contrary. Choose $a, b \in M_2(k)$ so that $l(a) = l(b) = 0$ and let $z \in \text{Rad}(A)$. Then (27) says that a and b skew-commute with elements of $\text{Rad}(A)$ and thus

$$(ab)z = -a(zb) = -(az)b = b(az) = (ba)z.$$

That is, $[a, b]z = 0$ for every $z \in \text{Rad}(A)$, i.e., $[a, b]$ lies in the kernel of the left multiplication map $\phi : M_2(k) \rightarrow \text{End}(\text{Rad}(A))$. Since $M_2(k)$ is simple, we conclude that $[a, b] = 0$ for every $a, b \in M_2(k)$ with $l(a) = l(b) = 0$. Since $M_2(k)$ is linearly spanned by $\text{Ker}(l) \cap M_2(k)$ and 1_{M_2} , this implies that $[x, y] = 0$ for every $x, y \in M_2(k)$, which is clearly absurd. This contradiction shows that $\text{Rad}(A) = (0)$. \square

Theorem 5.2.3 1. $T(A) = U_{0,1,1}$ if and only if $A = B(u, v)$ with $1 \leq u < v$, $u + v + 2 = r$.

2. $T(A) = U_{0,1,0}$ if and only if A is small and $A \neq A_0(r)$.

Proof. (1): By Lemma 5.1.1 we need to show $T(A) = U_{0,1,1}$ iff A is alternating but $T(A) \not\subset U_{1,1,0}$ or $U_{0,0,1}$. As we showed in Section 5.1, $T(A) \not\subset U_{1,1,0}$ iff $\text{RTr}_A \neq \text{LTr}_A$. Examining the algebras listed in Proposition 5.2.2, we see that the only ones with this property are $B(u, v)$ with $u \neq v$. On the other hand, $T(A) \subset U_{0,0,1}$ if and only if $A = A_0(r)$ or $A_1(r) = B(0, r - 2)$ (see Lemma 4.3.2), and part (1) follows.

(2): $T(A) = U_{0,1,0}$ iff A alternating, $\text{RTr}_A = \text{LTr}_A$ but $A \neq A_0(r)$. \square

5.3 Quasi-commutative algebras

In this section we classify the algebras A with $T(A) = U_{1,0,1}$ and $T(A) = U_{1,0,0}$ under the assumption $\text{char}(k) \nmid 2(r - 2)$.

We call an r -dimensional algebra A quasi-commutative if $\pi([A]) \subset U_{1,0,1}$.

Lemma 5.3.1 *A is quasi-commutative if and only if for every $x, y \in A$*

$$xy - yx + m(x)y - m(y)x \in k1_A$$

where $m(x) = \frac{1}{r-2}(RTr(x) - LTr(x))$.

Proof. Let $q \in \pi([A])$. Then by our assumption q lies in $U_{1,0,1}$. Denote the $(W^* \otimes \delta) \oplus (\delta \otimes W^*)$ -component of q by

$$\sum_{\alpha, \beta \geq 2} c_\alpha e_\beta^{\alpha\beta} + d_\alpha e_\beta^{\beta\alpha}.$$

Note that the identity we want to verify is linear in both x and y . Moreover, it holds trivially if x or $y = 1_A$. Thus we only need to prove it for $x = e_i, y = e_j$ as i, j range from 2 to r . A direct calculation shows that

$$e_i \cdot e_j - e_j \cdot e_i = (c_{ij}^1 - c_{ji}^1)e_1 + (c_i - d_i)e_j + (d_j - c_j)e_i. \quad (29)$$

On the other hand,

$$c(q) = \sum_{\alpha=2}^r ((r-1)d_\alpha + c_\alpha)e^\alpha, \quad c\sigma(q) = \sum_{\alpha=2}^r ((r-1)c_\alpha + d_\alpha)e^\alpha$$

and thus by Lemma 2.3.3 $RTr(e_i) = (r-1)d_i + c_i$ and $LTr(e_i) = (r-1)c_i + d_i$. Hence, $c_i - d_i = m(e_i)$, and the desired identity follows from (29). \square

Proposition 5.3.2 *Let A be a quasi-commutative r -dimensional algebra. Then, either A is commutative or $A \simeq A_1(r)$.*

Proof. We claim that A is basic, i.e., $SS(A)$ is commutative. Assume $M_n(k) \triangleleft SS(A)$ with $n \geq 2$ and take $x = E_{1n}, y = E_{n1}$ where E_{ij} is the elementary matrix whose (i, j) -entry is 1 and all other entries are 0. Then, $[x, y], x, y$ and 1 cannot be k -dependent contradicting the above lemma. This proves the claim.

Assume that A is not commutative. Then by Lemma 5.3.1 there is an $x \in A$ with $m(x) \neq 0$. Indeed, otherwise $[x, y] \in k1_A$ for all x, y , and taking traces on both sides implies that A must be commutative. As $LTr(y) = RTr(y) = 0$ for $y \in Rad A$ (see Lemma 2.3.1), we may assume that $x \in SS(A) = k^{\oplus s}$.

If A is not commutative, we claim that its quiver must have precisely two vertices v_1 and v_2 . Indeed, as $m(1) = 0$, the number of vertices must be greater than one. On the other hand, if $m(x) \neq 0$, then 1, x and x' are k -dependent for any $x' \in SS(A)$; see Lemma 5.3.1. Hence, $s \leq 2$, as claimed.

Therefore, A is a quotient of the path algebra of Q having u arrows $v_1 \xrightarrow{a_i} v_2$ and v arrows $v_2 \xrightarrow{b_j} v_1$ modulo an admissible ideal. Assume $m(f_1) = \alpha \neq 0$ where f_1 is the idempotent corresponding to v_1 . Then

$$-a_i = a_i e_i - e_i a_i = \alpha a_i + \beta_i 1_A$$

implies $\alpha = -1$. But then, $b_j = b_j f_1 - f_1 b_j = -b_j$, a contradiction, unless there are no arrows from v_2 to v_1 . Consequently, $A \simeq A_0(r)$, as claimed. \square

We can now prove the main theorem of this section.

Theorem 5.3.3 1. *There are no algebras with $T(A) = U_{1,0,1}$.*

2. *$T(A) = U_{1,0,0}$ if and only if A is commutative and $A \neq A_0(r)$.*

Proof. In view of Lemmas 5.1.1 and 5.3.1, we only need to show that $\pi([A]) \subset U_{1,0,0}$ iff A is commutative.

Assume A is commutative. Then its (reduced) tensor of structure constants $p \in W^* \otimes W^* \otimes W$ is symmetric (i.e., $a(p) = 0$) for any choice of basis. In other words, $\pi([A]) \subset \text{Ker}(a) = U_{1,0,0}$. Conversely, if $\pi([A]) \subset U_{1,0,0}$ then $[x, y] \in k1_A$ for all $x, y \in A$. Taking traces on both sides we conclude that $[x, y] = 0$, i.e., A is commutative. \square

6 Minimal deformations and minimal singularities

6.1 Proof of Theorem 1.4

We begin with the following lemma.

Lemma 6.1.1 *Let A is an r -dimensional k -algebra with the property that $1_A, a, b$, and ab are k -linearly dependent for every $a, b \in A$. Assume $r \geq 4$. Then $A = A_0(r)$ or $A = A_1(r)$.*

Proof. Rewriting the condition on A in terms of structure constants, we see that for any choice of basis $e_1 = 1_A, e_2, \dots, e_r$ of A , we have $c_{ij}^h = 0$, whenever i, j , and h are distinct integers ≥ 2 . Thus

$$\pi([A]) \subset U = \text{Span}\{e_j^{ii}, e_i^{ij}, e_i^{ji} : i, j = 2, \dots, r\}$$

and $T_0(\pi([A])) = \text{Span}(\pi([A]))$ is contained in U ; see Lemma 5.1.1. On the other hand, $T_0(\pi([A]))$ is a G_r -invariant subspace of $W^* \otimes W^* \otimes W$ and thus is necessarily of the form U_{i_1, i_2, i_3} for some $i_1, i_2, i_3 \in \{0, 1\}$. Since neither R_+ , nor R_- is contained in U (to see this, consider the tensors $e_4^{23} \pm e_4^{32} \in R_{\pm}$), we have $T_0(\pi([A])) = U_{0,0,0}$ or $U_{0,0,1}$. In the former case $A = A_0(1)$, in the latter case $A = A_1(r)$; see Lemma 4.3.2. \square

We are now ready to prove the main result of this section. Note that it is, in fact, stronger than Theorem 1.4.

Proposition 6.1.2 *Suppose $r \geq 3$ and A is an r -dimensional k -algebra which is not isomorphic to $A_0(r)$ or $A_1(r)$. Then A degenerates to $A_+(r)$ or $A_-(r)$. In other words, $A_+(r) \leq A$ or $A_-(r) \leq A$ (and possibly both) in the degeneration order defined in Section 1.*

Proof. By Lemma 6.1.1 there exist $a, b \in A$ such that $1_A, a, b$ and ab are linearly independent. In other words, A has a basis e_1, e_2, \dots, e_r with $e_1 = 1_A, e_2 = a, e_3 = b,$ and $e_4 = ab$. Let $W = \text{Span}\{e_2, \dots, e_r\}$ and let $f_i = t^{\alpha_i} e_i$, where $\alpha_2 = \alpha_3 = 2, \alpha_4 = 4,$ and $\alpha_m = 3$ for all $m \geq 5$. Then in the basis $\{1_A, f_2, \dots, f_r\}$, the multiplication rules become

$$f_i \cdot f_j = \sum_h t^{\alpha_i + \alpha_j - \alpha_h} c_{ij}^h f_h.$$

By our choice of α_i , the exponent $\alpha_i + \alpha_j - \alpha_h$ is 0 if $h = 4, i, j \in \{2, 3\}$, and positive for all other choices of i, j, h . Thus letting $t \rightarrow 0$, we see that A degenerates to an algebra A' with $f_i \cdot f_j = 0$ for all $(i, j) \neq (2, 2), (2, 3), (3, 2),$ and $(3, 3)$. Moreover, $f_2 \cdot f_3 = f_4 \neq 0$. Thus

$$A' = k \cdot 1 \oplus I \oplus J$$

with $J^2 = 0$ and $IJ = JI = 0$. Moreover, $k \cdot 1 \oplus I$ is a non-trivial 4-dimensional algebra which is local with $\text{Rad}^2 \neq 0$ and $\text{Rad}^3 = 0$. The degeneration picture of four dimensional algebras shows that $k \cdot 1 \oplus I$ degenerates to either $A_+(4)$ or $A_-(4)$; see Remark 4.5.3. Thus A' degenerates to $A_{\pm}(4) \oplus J = A_{\pm}(r)$, and the proposition follows. \square

Remark 6.1.3 The orbits $\pi([A])$, where $A = A_0(r), A_+(r)$, and $A_-(r)$ have, a natural representation-theoretic interpretation. Recall that for every irreducible SL_n -module U , there is a unique closed orbit of minimal dimension in $\mathbb{P}(U)$ corresponding to the highest weight vector; see, e.g., [PV, Sect. 1] or [FH, §23.3]. If U is an irreducible GL_n -module then this orbit in $\mathbb{P}(U)$ lifts to a unique GL_n -orbit in U , which we shall denote by $m(U)$. We claim that

1. $\pi[A_1(r)] = m(W^* \otimes \delta) \oplus U_{0,0,0},$
2. $\pi[A_+(r)] = m(R_+) \oplus U_{0,0,0},$
3. $\pi[A_-(r)] = m(R_-) \oplus U_{0,0,0}.$

Indeed, (1) follows from Lemma 4.3.2. To prove (2) and (3) recall that in the course of the proof of Proposition 4.1.1 we saw that $R_+ = S_{(3,2^{d-2},0)}$ and $R_- = S_{(2,1^{d-3},0^2)}$ with highest weights, respectively

$$\lambda_+ = 3L_1 + 2L_2 + \dots + 2L_{d-1} \quad \text{and} \quad \lambda_- = 2L_1 + L_2 + \dots + L_{d-2}.$$

(Here we are using the notational conventions of [FH].) The highest weight vectors associated to these weights are, respectively,

$$w_+ = e_2^{r,r} \in R_+ \quad \text{and} \quad w_- = e_2^{r-1,r} - e_2^{r,r-1} \in R_-.$$

Here $r = d + 1$ and $W = \text{Span}\{e_2, \dots, e_r\}$. Now it is easy to see that w_+ and w_- give rise, respectively, to the algebras $A_+(r)$ and $A_-(r)$, defined in (6) and (7), where $x_i = e_{i+1}$ and $y_i = e_{r+2-i}$ for $i = 2, \dots, d$.

6.2 Minimal singularities

Consider an action of an algebraic group G on a variety X . The degeneration partial order on the set G -of orbits in X is defined as follows: $O_1 \leq O_2$ if $O_1 \subset \overline{O_2}$. Moreover, O_1 is a minimal degeneration of O_2 if $O_1 \leq O \leq O_2$ is only possible for $O = O_1$ or $O = O_2$. A pointed variety (\overline{O}, p) is called a minimal singularity if O is a G -orbit and Gp is a minimal degeneration of O . Minimal singularities have been extensively studied in recent years; for details we refer the reader to [B] and [KP].

In this section we consider the case $X = \text{Alg}_r$ and $G = G_r$. Since the G_r -orbits in Alg_r are in 1-1 correspondence with r -dimensional k -algebras (up to isomorphism), the degeneration order is, in fact, defined on the set of algebras. That is, $B \leq A$ if $[B] \subset [\overline{A}]$, as we stated in the Introduction. In this section we shall focus on the minimal singularities in Alg_r of the form $([\overline{A}], 0)$, where 0 is the origin of $V^* \otimes V^* \otimes V$. By Theorem 1.4, $A = A_1(r)$, $A_+(r)$, or $A_-(r)$.

Before we proceed to the main result of this section, we recall the definition of smooth equivalence. Two pointed varieties (X, x) and (Y, y) are smoothly equivalent if there exist smooth morphisms $\lambda : (Z, z) \rightarrow (X, x)$ and $\rho : (Z, z) \rightarrow (Y, y)$ (of pointed varieties). Note that for a smooth morphism $(Z, z) \rightarrow (X, x)$, we have

$$\hat{O}_z \simeq \hat{O}_x[[t_1, \dots, t_i]],$$

where \hat{O}_x is the completion of the local ring of X at x and \hat{O}_z the completion of the local ring of Z at z and $i = \dim Z - \dim X$; see e.g., [KP, Sect. 2.1]. Therefore, the integer $\dim T_x(X) - \dim X$ is an invariant of smooth equivalence.

Proposition 6.2.1 *Suppose $r \geq 4$. Then no two of the minimal singularities $([\overline{A}], 0)$, with $A = A_1(r)$, $A_+(r)$, $A_-(r)$, are smoothly equivalent.*

Proof. By Theorem 1.3 the orbit closure of $A_1(r)$ is smooth at the origin, where as those of $A_+(r)$ and $A_-(r)$ are not. Hence, we only have to show that

$$\dim T_0([\overline{A_+(r)}]) - \dim [\overline{A_+(r)}] \neq \dim T_0([\overline{A_-(r)}]) - \dim [\overline{A_-(r)}]$$

The dimensions of the tangent spaces are given by

$$\dim T_0([\overline{A_+(r)}]) = d + \dim R_+ = d + \frac{d(d-1)(d+2)}{2}$$

and

$$\dim T_0([\overline{A_-(r)}]) = d + \dim R_- = d + \frac{d(d+1)(d-2)}{2}.$$

The dimensions of the orbits are computed in Lemma 6.2.2 below. Combining the formulas we obtain

$$\dim T_0([\overline{A_+(r)}]) - \dim [\overline{A_+(r)}] = \frac{1}{2}(d^3 + d^2 - 6d + 4)$$

and

$$\dim T_0(\overline{[A_-(r)]}) - \dim \overline{[A_-(r)]} = \frac{1}{2}(d^3 - d^2 - 7d + 12),$$

which are unequal for every integer d . □

Lemma 6.2.2 1. $\dim \text{Aut}(A_+(r)) = d^2 - 2d + 2$ for any $d \geq 2$.

2. $\dim \text{Aut}(A_-(r)) = d^2 - 3d + 6$ for any $d \geq 3$.

Proof. We give two proofs of this lemma.

Proof I. (1): The elements $1, x_2, \dots, x_{d-1}$, and x_2^2 form a k -basis $A_+(r)$. An automorphism of $A_+(r)$ is given by

$$\begin{aligned} x_2 &\mapsto a_2x_2 + a_3x_3 \dots + a_dx_d + c_2x_2^2 \\ x_3 &\mapsto b_{33}x_3 + \dots + b_{3d}x_d + c_3x_2^2 \\ &\vdots \\ x_d &\mapsto b_{d3}x_3 + \dots + b_{dd}x_d + c_dx_2^2, \end{aligned}$$

where the a_i, b_{ij} , and c_i are arbitrary elements of k , except that $a_2 \neq 0$ and the matrix $(b_{ij} : i, j = 3, \dots, d)$ is non-singular. Thus we have embedded $\text{Aut}(A_+)$ as a Zariski open subset in an affine space \mathbb{A}^N , where N is the total number of parameters a_i, b_{ij} , and c_i . That is $N = (d-1) + (d-2)^2 + (d-1)$, and part (1) follows.

(2): We use similar reasoning. Clearly $1, x_2, \dots, x_{d-1}$ and x_2x_3 form a k -basis of $A_-(r)$. Moreover, an automorphism of $A_-(r)$ is given by

$$\begin{aligned} x_2 &\mapsto a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots + a_{2d}x_d + c_2x_2x_3 \\ x_3 &\mapsto a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + \dots + a_{3,d}x_d + c_3x_2x_3 \\ x_4 &\mapsto b_{44}x_4 + \dots + b_{4,d}x_d + c_{4d}x_2x_3 \\ &\vdots \\ x_d &\mapsto b_{d,4}x_4 + \dots + b_{d,d}x_d + c_{d,2}x_2x_3, \end{aligned}$$

where the $a_{ij}, b_{ij} \in k$ are constrained only by the inequalities

$$\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \neq 0 \text{ and } \det(b_{ij}) \neq 0.$$

Counting parameters, we see that

$$\dim \text{Aut}(A_-(r)) = 2(d-1) + (d-3)^2 + (d-1) = d^2 - 3d + 6,$$

as claimed.

Proof II. By Remark 6.1.3

$$\dim \overline{[A_{\pm}(r)]} = d^2 + d - \dim P(\lambda_{\pm})$$

where $P(\lambda_{\pm})$ is the parabolic subgroup corresponding to the highest weight λ_{\pm} . Therefore, we only need to determine $P(\lambda_{\pm})$. Since this is quite standard (see, e.g., [FH, p. 388]), we shall only outline the argument and leave out the details.

Suppose the i -th node in the Dynkin diagram A_{d-1} corresponds to the simple root $L_i - L_{i+1}$. Then for $d \geq 5$ we can depict the set $\Sigma(\lambda_{\pm})$ of simple roots which are perpendicular to the weight λ_{\pm} by marked nodes in the Dynkin diagram:

$$\begin{aligned} \Sigma(\lambda_+) &= \circ \bullet \bullet \dots \bullet \bullet \circ \\ \Sigma(\lambda_-) &= \circ \bullet \bullet \dots \bullet \circ \bullet \end{aligned}$$

For any subset Σ of the simple roots there is a parabolic subgroup $P(\Sigma)$ whose Lie algebra is of the form

$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{sl}(W)_{\alpha}.$$

Here $T(\Sigma)$ is the set of roots which can be written as sums of negatives of the roots in Σ , see [FH, p. 385]. This allows us to prove Lemma 6.2.2 for $d \geq 5$.

For $d = 3$ the relevant sets are $\Sigma(\lambda_+) = \emptyset$ and $\Sigma(\lambda_-) = \{L_2 - L_3\}$; for $d = 4$, they are $\Sigma(\lambda_+) = \{L_2 - L_3\}$ and $\Sigma(\lambda_-) = \{L_3 - L_4\}$. Repeating the above argument, we complete the proof of the lemma in these two cases. \square

References

- [AS] M. Artin and J.T. Stafford, *Noncommutative graded domains with quadratic growth*, Invent. Math. **122** (1995) 231-276
- [BIB] Ch. Bessenrodt and L. Le Bruyn, *Stable rationality of certain PGL_n -quotients*, Invent. Math. **104** (1991) 179-199
- [B] K. Bongartz, *Minimal singularities for representations of Dynkin quivers*, Comment. Math. Helvetici **69** (1994), 575-611.
- [DS] Th. Dana-Picard, M. Schaps, *A computer-assisted project: classification of algebras*, in "Computational algebraic geometry and commutative algebra" (Cortona 1991), pp. 71-83, Sympos. Math. XXIV, Cambridge University Press, 1993.
- [D] I. V. Dolgachev, *Rationality of fields of invariants*, in Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, vol. 46, part 2, AMS, 1987, 3-16.
- [Do] S. Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993) 39-60
- [Er] K. Erdmann, *Symmetric groups and quasi-hereditary algebras* in "Finite dimensional algebras and related topics" ed. V. Dlab, L.L. Scott, NATO-ASI C-424 (1992) 123-161

- [F68] F. J. Flanigan, *Algebraic geography: varieties of structure constants*, Pacific J. Math. **27** (1968), no. 1, 71-79.
- [F74] F. J. Flanigan, *Which algebras deform into a total matrix algebra?*, J. Algebra **29** (1974), 103-112.
- [FH] W. Fulton, J. Harris, *Representation theory, a first course*, Graduate Texts in Mathematics 129, Springer-Verlag, 1991.
- [Ga] P. Gabriel, *Finite representation type is open*, in Springer Lecture Notes in Math. **488** (1974), 132-155.
- [Ge] M. Gerstenhaber, *On the deformation of rings and algebras*, Annals of Math. **79** (1964), 59-103.
- [FGG] A. Freedman, R. N. Gupta, R. M. Guralnick, *Shirshov's theorem and representations of semi-groups*, Pacific Journal of Mathematics, to appear.
- [Ha] D. Happel, *Deformations of five dimensional algebras with unit*, in "Ring Theory" ed. F. Van Oystaeyen, Lect. Notes in Pure and Applied Mathematics, Marcel Dekker (1979), 459-494.
- [K] H. Kraft, *Geometric methods in representation theory*, in "Representations of Algebras", Springer Lecture Notes in Math. **944** (1980), 180-258.
- [KP] H. Kraft, C. Procesi, *Minimal singularities in GL_n* , Invent. Math. **62** (1981), 503-515.
- [LB] L. Le Bruyn, *Centers of generic division algebras, the rationality problem 1965-1990*, Israel Journal of Math. **76** (1991), 97-111.
- [IBP] L. Le Bruyn, C. Procesi, *Etale local structure of matrix invariants and comitants*, in "Algebraic Groups", Utrecht 1986, Lecture Notes in Math. **1271** (1987), 143-175.
- [IBT] L. Le Bruyn, Y. Teranishi, *Matrix invariants and complete intersections*, Glasgow Math. J. **32** (1990), 227-229.
- [M79] G. Mazzola, *The algebraic and geometric classification of associative algebras of dimension five*, Manuscripta Math. **27** (1979), no. 1, 81-101.
- [M82] G. Mazzola, *How to count the number of irreducible components of the scheme of finite-dimensional algebra structures*, J. Algebra **78**(1982), no. 2, 292-302.
- [N] Yu. A. Neretin, *An estimate for the number of parameters defining an n -dimensional algebra*, Izvestiya Acad. Nauk SSSR, Ser. Mat. **51** (1987), no. 2, 306-318. English Translation in Math. USSR Izvestiya, Vol. 30 (1988), no. 2.

- [OV] A. L. Onishchik, E. B. Vinberg, *Lie Groups and Algebraic Groups*, Springer-Verlag, 1990.
- [PV] V. L. Popov, E. B. Vinberg, *On a class of quasi-homogeneous affine varieties*, Math. USSR, Izvestiya **6** (1972), 743-758.
- [P] C. Procesi, *Non-commutative affine rings*, Atti Acc. Naz. Lincei, S. VIII, v. VIII, fo. 6 (1967), 239-255.
- [Re] Z. Reichstein, *On automorphisms of matrix invariants induced from the trace ring*, Linear Algebra and Its Applications, **193** (1993), 51-74.
- [Ro] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York, 1980.
- [Shaf] I. R. Shafarvich, *Basic Algebraic Geometry*, Springer-Verlag, 1974.
- [Schaps] M. Schaps, *Moduli of commutative and non-commutative finite covers*, Israel J. Math. **58** (1987) 67-102
- [Se78] C. S. Seshadri, *Desingularization of moduli varieties of vector bundles on curves*, in "Proceedings of the International Symposium on Algebraic Geometry" (Kyoto University, 1977), edited by M. Nagata, Tokyo, 1978, pp. 155-184.
- [Se82] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **92** (1982), Société Mathématique de France.

