Degenerations of Matrices and Rationality

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ABSTRACT. In this note we study the irreducible component X_n of the variety of n^2-dimensional algebras determined by the orbit $\mathcal{O}_n$ of $M_n(\mathbb{C})$. In particular we prove that $X_n - \mathcal{O}_n$ has components of codimension one and that smoothness of X_n implies stable rationality of PGL_n-quotients.

DEGENERATIONS OF MATRICES AND RATIONALITY

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ABSTRACT. In this note we study the irreducible component X_n of the variety of n^2-dimensional algebras determined by the orbit $\mathcal{O}_n$ of $M_n(\mathbb{C})$. In particular we prove that $X_n - \mathcal{O}_n$ has components of codimension one and that smoothness of X_n implies stable rationality of PGL_n-quotients.

1. THE ALGEBRA VARIETY Alg_r

In this section we will show that any smooth irreducible component of the variety of all algebra structures of dimension r, Alg_r, has to be isomorphic to an affine space.

Our definition of Alg_r below differs slightly from that given by P. Gabriel [4] in that we fix the identity element. Hence we use the conventions as given in [9] or [7].

Any associative algebra structure on an r-dimensional vectorspace $V = \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_r$ with e_1 the identity element is determined by its r^3-tuple of structure constants c_{ij}^k where

$$e_i.e_j = \sum_{k=1}^{r} c_{ij}^k e_k$$

Clearly, they must express the fact that e_1 is the identity element, that is, $e_1.e_i = e_i = e_i.e_1$ whence

$$c_{1i}^k = \delta_i^k = c_{i1}^k$$

and that the multiplication is associative, that is, for all $i,j,k \geq 2$ we have $(e_i.e_j).e_k = e_i.(e_j.e_k)$ whence

$$(1.1) \qquad \sum_{l=1}^{r} (\ c_{ij}^l c_{lk}^m - c_{jk}^l c_{il}^m \ ) = 0$$

for all $m \geq 1$.

We denote by f_{ijkm} the left hand side of equation 1.1 where we substitute all ocurences of $c_{1u}^v = c_{u1}^v = \delta_u^v$ and replace all remaining c_{uv}^w by an indeterminate x_{uv}^w. Then, the variety Alg_r of all associative algebra structures on V with identity element e_1 is an affine variety with coordinate ring

$$\mathbb{C}[Alg_r] = \mathbb{C}[x_{uv}^1, x_{uv}^w \mid u,v,w \geq 2]/(f_{ijkm} \mid i,j,k \geq 2, m \geq 1)$$

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Let G be the subgroup of GL_r consisting of those automorphisms fixing e_1, that is,

$$G = \begin{bmatrix} \begin{array}{c|ccc} 1 & a_2 & \dots & a_r \\ \hline 0 & & & \\ \vdots & & GL_{r-1} & \\ 0 & & & \end{array} \end{bmatrix}$$

If $A \in Alg_r$ with multiplication $._A$ and $g \in G$, then we define a new algebra structure $._{g.A}$ on V by defining for all $v, v' \in V$

$$v ._{g.A} v' = g^{-1}(\ g(v) ._A g(v')\ )$$

The G-orbits in Alg_r correspond to the $\mathbb{C}$-algebra isomorphism classes and the stabilizer subgroup $Stab_G(A)$ is the group $Aut(A)$ of $\mathbb{C}$-algebra automorphisms of A.

Lemma 1.1. *$\mathbb{C}[Alg_r]$ is a positively graded algebra by defining*

$$deg\ x^1_{uv} = 2 \text{ and } deg\ x^w_{uv} = 1 \text{ for all } u, v, w \geq 2$$

Proof. We have to verify that all the relations f_{ijkm} are homogeneous. Now, f_{ijkm} is equal to

$$\underbrace{x^1_{ij}\delta^m_k - x^1_{jk}\delta^m_i}_{A} + \underbrace{\sum_{l=2}^{r} (x^l_{ij}x^m_{lk} - x^l_{jk}x^m_{il})}_{B}$$

A can only be non-zero in case $m \geq 2$. In this case, A and B are both of degree two and so f_{ijkm} is homogeneous. If however $m = 1$, then $f_{ijkm} = B$ and is homogeneous of degree three. □

Remark 1.2. If $r \geq 3$ we can even prove that $\mathbb{C}[Alg_r]$ is generated in degree one. For we take $2 \leq m = k \neq i$ in equation 1.1 to obtain that

$$x^1_{ij} = \sum_{l=2}^{r} (x^l_{jk}x^k_{il} - x^l_{ij}x^k_{lk})$$

The above lemma has a remarkable consequence.

Proposition 1.3. *Let X be an irreducible component of Alg_r of dimension d. If X is smooth, then*

$$X \simeq \mathbb{A}^d$$

Proof. Irreducible components of Alg_r correspond to minimal prime ideals of $\mathbb{C}[Alg_r]$. As $\mathbb{C}[Alg_r]$ is graded, these prime ideals are graded whence $\mathbb{C}[X]$ is positively graded.
If X is smooth, $\mathbb{C}[X]$ is a positively graded affine regular algebra whence isomorphic to $\mathbb{C}[x_1, \dots, x_d]$. □

For small dimensions r many irreducible components are indeed smooth. However, invariant-theoretic intuition tells us that most components will contain singularities if r increases.

Example 1.4. $Alg_2 \simeq \mathbb{A}^2$ with open orbit determined by $\mathbb{C} \times \mathbb{C}$. The orbit of $\mathbb{C}[x]/(x^2)$ defines a line pair.
Alg_3 has two irreducible components, one the closure of the orbit of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ which is isomorphic to $\mathbb{A}^6$, the other the closure of the orbit of

$$\begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix}$$

which is isomorphic to $\mathbb{A}^3$.
As we will see below, the closure of the orbit of $M_2(\mathbb{C})$ in Alg_4 is an irreducible component isomorphic to $\mathbb{A}^9$.

2. Degenerations of $M_n(\mathbb{C})$

From now on we restrict attention to the case when $r = n^2$. As $M_n(\mathbb{C})$ is a simple algebra, its orbit $\mathcal{O}_n$ is open in Alg_{n^2} (see for example [4, Cor. 2.6]) and hence determines an irreducible component

$$\overline{\mathcal{O}_n} = X_n \overset{open}{\hookrightarrow} Alg_{n^2}$$

The dimension of the base-change group G is equal to $n^2(n^2-1)$ and the stabilizer subgroup of $M_n(\mathbb{C})$ is its automorphism group which is PGL_n. Therefore

$$\mathcal{O}_n \simeq G/PGL_n \text{ and } dim\ X_n = (n^2-1)^2$$

Definition 2.1. An n^2-dimensional $\mathbb{C}$-algebra A is said to be a **degeneration** of $M_n(\mathbb{C})$ iff $A \in X_n - \mathcal{O}_n$.

Example 2.2. X_2 is made of the orbits of 4 algebras with degeneration picture

$$M_2(\mathbb{C}) > \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix} > \wedge^2(\mathbb{C}^2) > \mathbb{C}[x,y,z]/(x,y,z)^2$$

which have orbit dimensions resp. 9, 8, 6 and 3. C.S. Seshadri has shown [9, p.112] that $X_2 \simeq \mathbb{A}^9$.

For $n \geq 3$ surprisingly little is known about X_n. In fact, the only reference known to me is a paper due to F. Flanigan [3] studying the easier degenerations (those having enough idempotents). Still, X_n plays a crucial role not only in the study of (projective) orders in central simple algebras but also in the desingularization of moduli spaces of vector bundles on smooth curves.
From the foregoing section we recall that

$$\mathbb{C}[X_n] = \mathbb{C}[x_{ij}^1, x_{ij}^k \mid 2 \leq i,j,k \leq n^2]/(I_n)$$

and we can express the x_{ij}^1 as quadratic expressions in the x_{ij}^k. We define the **generic order** $\mathbb{G}_n$ to be the $\mathbb{C}[X_n]$-algebra

$$\mathbb{G}_n = \mathbb{C}[X_n]e_1 \oplus \ldots \oplus \mathbb{C}[X_n]e_{n^2}$$

with multiplication defined by

$$e_i.e_j = \sum_{k=1}^{n^2} x_{ij}^k e_k$$

By the relations on the x_{ij}^k it follows that $\mathbb{G}_n$ is an associative algebra with unit element and is a free module of rank n^2 over $\mathbb{C}[X_n]$. In fact,

Lemma 2.3. *If we define deg $e_1 = 0$ and deg $e_i = 1$ for $i \geq 2$, then $\mathbb{G}_n$ is a positively graded algebra generated in degree one. Moreover, $\mathbb{G}_n$ is an order in a central simple algebra of dimension n^2 over its center which is $\mathbb{C}(X_n)$.*

Proof. We may take $e_2, \ldots, e_{n^2}$ as spanning the trace zero elements of $\mathbb{G}_n$, whence $x_{ij}^1 = tr(e_i.e_j)$. Let d be the determinant of the symmetric $n^2 \times n^2$ matrix (x_{ij}^1), then $d \neq 0$ in $\mathbb{C}[X_n]$ and if we localize at it we obtain by the Artin-Procesi result that $Q_d(\mathbb{G}_n)$ is an Azumaya algebra of rank n^2 over the domain $Q_d(\mathbb{C}[X_n])$ from which the remaining claims follow. □

The relevance of X_n for the study of orders follows from

Proposition 2.4. *Let Y be a variety and Δ a central simple algebra of dimension n^2 over the functionfield $\mathbb{C}(Y)$. Let $\mathcal{A}$ be an $\mathcal{O}_Y$-order in Δ which is locally free of rank n^2. Then, for each point $y \in Y$ there is a Zariski open subvariety U and a morphism*

$$U \xrightarrow{\phi} X_n$$

such that there exists an isomorphism of $\mathcal{O}_U$-algebras

$$\mathcal{A} \mid U \simeq \phi^*(\mathbb{G}_n)$$

Proof. Remembering the Artin-Procesi result this is a reformulation of [9, p.111]. □

Conversely, one can use ringtheory to obtain some information on X_n. Recall that in X_2 there is an orbit of codimension one. In general we have

Theorem 2.5. *$X_n - \mathcal{O}_n$ contains components of codimension one.*

Proof. Let $\widetilde{X_n} \xrightarrow{\pi} X_n$ be the normalization map of X_n. Then $\mathbb{G} = \pi^*(\mathbb{G}_n)$ is an order which is projective of rank n^2 over its center $\mathbb{C}[\widetilde{X_n}]$. Consider the non-Azumaya locus $ram(\mathbb{G})$ of $\mathbb{G}$. Assume that $ram(\mathbb{G})$ does not contain components of codimension one, then $\mathbb{G}$ is a reflexive Azumaya algebra over $\mathbb{C}[\widetilde{X_n}]$. Moreover, as $\mathbb{G}$ is projective, it must be an Azumaya algebra, see for example [5]. But then, $\mathbb{G}/\mathbb{G}m_x \simeq M_n(\mathbb{C})$ for every point $x \in \widetilde{X_n}$. But then, this would also hold for $x \in X_n$ wrt. $\mathbb{G}_n$ whence $X_n = \mathcal{O}_n$. This is clearly absurd as there are degenerations of matrices (for example $\mathbb{C}[x_2, \ldots, x_{n^2}]/(x_i)^2$).
Therefore, $ram(\mathbb{G})$ contains components of codimension one and their images under π give codimension one components of $X_n - \mathcal{O}_n$. □

In the next section we will see that there is a close relation between $\mathbb{G}_n$ and the trace ring of $n^2 - 1$ generic $n \times n$ matrices $\mathbb{T}_{n^2-1,n}$. The results above show that the ringtheoretical properties of both rings are quite different.
Degenerations of matrices are also relevant to the recent attempts to classify graded algebras of Gelfand-Kirillov dimension two. In [1] M. Artin and T. Stafford proved that graded connected domains of $GKdim$ two which are generated in degree one are twisted homogeneous coordinate rings. This raises the obvious question to generalize this result to prime algebras.
The following result shows that classifying degenerations of matrices is a subproblem of this project.

Proposition 2.6. *To every degeneration of $M_n(\mathbb{C})$ one can associate a graded connected prime algebra generated in degree one of Gelfand Kirillov dimension two. The associated central projective curve may even be taken to be rational.*

Proof. Let A be a degeneration of $M_n(\mathbb{C})$. As $\mathbb{C}(G)$ is rational, the orbit $\mathcal{O}_n$ as well as its closure X_n is a unirational variety. We claim that one can connect any point x of $\mathcal{O}_n$ with A along an affine line. This follows from Hironaka's result on resolution of singularities which allows one to reduce to the case of a unirational variety X which is obtained from a projective space by blowing up a number of times. For such X we can connect any two points along an affine line.
The resulting map $\mathbb{A}^1 \xrightarrow{\phi} X_n$ with say $\phi(0) = A$ and $\phi(1) = x$ determines a prime order $\phi^*(\mathbb{G}_n)$ which is projective of rank n^2 over $\mathbb{C}[\mathbb{A}^1]$. Equip $\phi^*(\mathbb{G}_n)$ with the generator filtration (as $\mathbb{C}$-algebra), then the Rees algebra with respect to this filtration has the claimed properties. □

One can generalize the above by studying the closure of the open orbit of $M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$ in Alg_{n^2r}. Then, one can study prime algebras which are projective modules of rank n^2r over a central subring. This generalization not only involves the variety X_n but also the study of the irreducible component in Alg_r determined by the open orbit of $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$ which is of fundamental importance in the investigation of covers in algebraic geometry. We refer the reader to [6] for some of the remarkable properties of this component in the variety of all commutative $\mathbb{C}$-algebras of dimension r.

3. A rationality problem

The variety X_n is also of interest in the study of vector-bundles on curves. The moduli variety $U(n, d)$ of semi-stable vectorbundles of rank n and degree d over a smooth curve of genus g usually has singularities. In [9, Partie V] C.S. Seshadri constructs a potential desingularization $N_{n,d}$ of $U(n, d)$ using parabolic structures. Further, it is proved [9, p.126] that $N_{n,d}$ is smooth (and hence is a proper desingularization) whenever X_n is smooth. This raises the problem

Question 3.1. For which n is X_n a smooth variety ?

In this section we will show that a positive solution to this problem has implications to the rationality problem of PGL_n-quotients.
Let H be an affine linear reductive group acting almost freely on a finite dimensional $\mathbb{C}$-vectorspace V, that is, the stabilizer subgroup of a generic point is trivial. One of the main open problems in invariant theory is to determine for which groups H the field of rational invariants $\mathbb{C}(V)^H$ is stably rational. By the 'no-name' lemma this property is known to be independent of the choice of the vectorspace V with almost free action.
In particular, for $H = PGL_n$ this problem is still wide open. At the moment we only have a positive solution provided n divides $420 = 2^2.3.5.7$ by [2]. In ringtheory this problem arises in the study of the center of Amitsur's generic division algebras $UD(m, n)$. This center can be viewed as the functionfield of the quotient variety $M_n(\mathbb{C})^{\oplus m}//PGL_n$ of the action of PGL_n by simultanous conjugation on m-tuples of $n \times n$ matrices. Observe that this action is almost free whenever $m \geq 2$.
The relevance of Seshadri's question to this problem is given by the following

Theorem 3.2. *If X_n is smooth, then $\mathbb{C}(V)^{PGL_n}$ is stably rational for every vectorspace V admitting an almost free PGL_n-action.*

Proof. The open orbit $\mathcal{O}_n$ of $M_n(\mathbb{C})$ in X_n is isomorphic to the homogeneous space G/PGL_n. Now, PGL_n embeds in G via the embedding $PGL_n \hookrightarrow GL_{n^2-1}$ given by the action of PGL_n by simultanous conjugation on linearly independent n^2-1-tuples of trace zero $n \times n$ matrices. Therefore,

$$G/PGL_n \simeq \mathbb{A}^{n^2-1} \times GL_{n^2-1}/PGL_n$$

where the second factor is an open subvariety of the quotient variety

$$GL_{n^2-1}/PGL_n \xhookrightarrow{open} M_n^0(\mathbb{C})^{\oplus n^2-1}//PGL_n$$

(where $M_n^0(\mathbb{C})$ are the trace zero matrices).
Therefore, $\mathbb{C}(M_n^0(\mathbb{C})^{\oplus n^2-1})^{PGL_n}$ is stably birational to $\mathbb{C}(X_n)$. If X_n is smooth we have seen that $X_n \simeq \mathbb{A}^{(n^2-1)^2}$ and so $\mathbb{C}(X_n)$ is rational.
The statement now follows from the no-name lemma using the fact that PGL_n acts almost freely on $M_n^0(\mathbb{C})^{\oplus n^2-1}$. □

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