Local Structure of Schelter-Procesi Smooth Orders

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Abstract

In this paper we give the étale local classification of Schelter-Procesi smooth orders in central simple algebras. In particular, we prove that if Δ is a central simple K-algebra of dimension n^2 where K is a field of transcendence degree d, then there are only finitely many étale local classes of smooth orders in Δ. This result is a non-commutative generalization of the fact that a smooth variety is analytically a manifold, whence has only one type of étale local behaviour.

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ABSTRACT. In this paper we give the étale local classification of Schelter-Procesi smooth orders in central simple algebras. In particular, we prove that if Δ is a central simple K-algebra of dimension n^2 where K is a field of trancendence degree d, then there are only finitely many étale local classes of smooth orders in Δ. This result is a non-commutative generalization of the fact that a smooth variety is analytically a manifold, whence has only one type of étale local behaviour.

1. INTRODUCTION

The quest for a suitable notion of a smooth non-commutative variety is still wide open. Up till now most attention has been focussed on algebras having excellent homological properties. However, even in the case when these algebras are finite modules over their centers, a local characterization of regular algebras seems to be out of reach at the moment.

Still, in the case of algebras finite over their centers we have a workable alternative. In [11] W. Schelter introduced and studied smooth algebras by requiring that they have a formal smoothness lifting property with respect to test-objects satisfying the same (or smaller-size) polynomial identities.

Later, C. Procesi gave a slight variation of this definition by restricting to algebras having a trace map satisfying the n-th formal Cayley-Hamilton identity, [9]. He proved that these smooth non-commutative algebras have a nice geometrical characterization. Consider the affine variety X_A of all n-dimensional representations of A (compatible with the trace maps), then A is smooth if and only if X_A is a smooth (commutative) variety.

In case A is a smooth order in a central simple algebra, the quotient variety of X_A under the natural action of PGL_n is the variety corresponding to the center Z of A. That is, to a maximal ideal m of Z corresponds a closed orbit in X_A determining a semi-simple n-dimensional representation M_m of A with decomposition in simple components say

$$M_m = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

where the S_i are distinct simple A-modules of dimension d_i occuring with multiplicity e_i. We then say that m has representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$ with $\sum e_i d_i = n$.

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We want to determine the étale local structure of A near m, that is, we want to determine the algebra structure of

$$A_m^{sh} = A \otimes_Z Z_m^{sh}$$

where Z_m^{sh} is the strict henselization of Z at m. Alternatively, we want to determine the m-adic completion of A.

We succeed to do this by applying the Luna slice theorems in a point x of the smooth variety X_A lying in the closed orbit corresponding to m. The stabilizer group in x is with notations as above $GL(\mathbf{e}) = GL_{e_1} \times \ldots \times GL_{e_r}$ embedded in GL_n using the numbers d_i. The étale local structure of A and Z near m is fully determined by the $GL(\mathbf{e})$-module structure of the normal space N_x to the orbit of x.

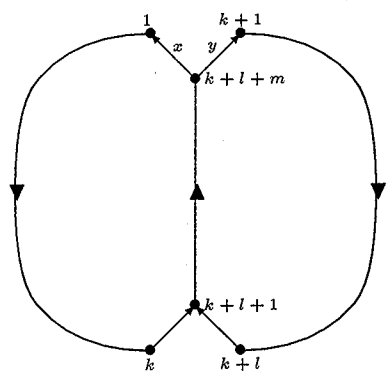
The isotypical decomposition of N_x as $GL(\mathbf{e})$-module can be encoded in a 'local chart' $C = (M, \mathbf{e})$ where the map M is a labeled directed marked graph on r vertices and N_x is the space of representations $R(M, \mathbf{e})$ of M of dimension vector $\mathbf{e} = (e_1, \ldots, e_r)$.

If A is a smooth order in a central simple algebra Δ of dimension n^2 over its center K which is a field of trancendence degree d we must have that $C = (M, \mathbf{e})$ is such that the dimension of the quotient variety $R(M, \mathbf{e})/GL(\mathbf{e})$ is equal to d and that there are simple $\mathbf{e}$-dimensional representations of M. Both conditions can be expressed in terms of the Euler-form of the map M.

We give a method to enumerate all charts that can occur for a given dimension d. Moreover, these local charts not only determine the étale local structure of Z and A in m but also contain enough information to determine the local charts in nearby points (and hence the local structure).

To illustrate our general results, let us give the étale local structure of smooth orders in dimension $d = 2$.

In this case the only local charts $C = (M, \mathbf{e})$ that can occur have as underlying map M a directed graph of the form

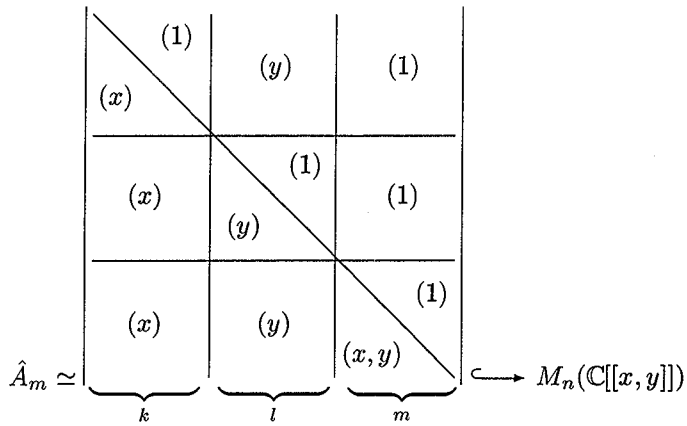


A_{klm}

and dimension vector $\mathbf{e} = (1, \ldots, 1)$. This means that A can only have semi-simple n dimensional representations of type $\tau = (1, d_1; \ldots; 1, d_{k+l+m})$ with $\sum d_i = n$.

From the local chart we can determine the local structure of Z and A in m.

Theorem 1.1. *With notations as before, we have that* $\hat{Z}_m \simeq \mathbb{C}[[x,y]]$ *and*

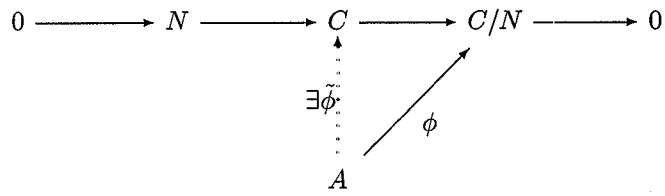


where at place (i,j) *(for every* $1 \leq i,j \leq k+l+m$*) there is a block of dimension* $d_i \times d_j$ *with entries the indicated ideal of* $\mathbb{C}[[x,y]]$.

In particular, we see that a smooth two-dimensional order has a smooth center, its non-Azumaya locus has as its worst singularities normal crossings and A is étale locally split at every m.
In higher dimensions d it is no longer true that the center of smooth orders are smooth nor that a smooth order is étale locally split everywhere.

2. Smooth non-commutative algebras

In this section we recall the definition of smooth algebras satisfying polynomial identities due to W. Schelter [11] as modified by C. Procesi [9] and draw some consequences.
Throughout, we work over an algebraically closed field of characteristic zero which we will denote by $\mathbb{C}$. Recall that a commutative affine algebra A is regular if and only if A satisfies the formally smooth lifting property, see for example [3].
A test-object (C,N) is a commutative algebra C together with an ideal N satisfying $N^k = 0$. The lifting property requires that every diagram



with ϕ an algebra morphism can be completed.
If A is non-commutative one can define A to be smooth provided A has the above lifting property for testobjects (C,N) with C restricted to some suitable category of not necessarely commutative algebras.
A first (rather idea (as pursued for example by J. Cuntz and D. Quillen [1, 2]) is to vary C in the category of all associative algebras. This is too stringent a condition as it is equivalent to Ω_A being a projective $A^e = A \otimes_{\mathbb{C}} A^{op}$-module where Ω_A is the kernel of the multiplication map $\mu : A^e \longrightarrow A$, see [11, lemma 2.3]. Therefore, if

A is a regular commutative algebra it can only be 'smooth' in this sense provided $Kdim\ A \leq 1$.

A more realistic approach is to restrict the class of testobjects C to those algebras having similar commutativity conditions as A. W. Schelter defines a smooth algebra by requiring that if A satisfies the identities of $n \times n$ matrices, then we consider only testobjects C of pi-degree $\leq n$. In this way we recover the commutative characterization of regular algebras as the $n = 1$ case.

Later, Procesi [9] gave a slightly different definition of smoothness which allows a geometric study of these algebras. He considers only algebras A having a linear trace map $tr : A \longrightarrow A$ satisfying the following conditions for all $a, b \in A$

(1) $tr(ab) = tr(ba)$
(2) $tr(a)b = btr(a)$
(3) $tr(tr(a)b) = tr(a)tr(b)$

In particular, the image of tr is a subalgebra of the center of A. We can then define the n-th Cayley-Hamilton polynomial formally. In $\mathbb{Q}[x_1, \ldots, x_n]$ one defines the elementary symmetric functions by the identity

$$\prod(t - x_i) = \sum_{i=0}^{n}(-1)^i \sigma_i t^{d-i}$$

and the power sums functions $\tau_k = \sum x_i^k$. Because $\{\sigma_i\}$ and $\{\tau_i\}$ are generators of the symmetric functions, there are functions with rational coefficients such that

$$\sigma_k = p_k(\tau_1, \ldots, \tau_n)$$

and we define the functions σ_k on A formally as

$$\sigma_k(a) = p_k(tr(a), tr(a^2), \ldots, tr(a^n))$$

and define the n-th Cayley-Hamilton polynomial for A to be

$$\chi_{n,a}(t) = \sum_{i=0}^{n}(-1)^i \sigma_i(a) t^{n-i}$$

Definition 2.1. We say that an algebra A with a trace function tr is an **n-th Cayley-Hamilton algebra** if

(1) For all $a \in A$ we have $\chi_{n,a}(a) = 0$ in A
(2) $tr(1) = n$

With CH_n we will denote the category with objects (A, tr_A) algebras A with a trace function tr_A which are n-th Cayley-Hamilton algebras and morphisms

$$f : (A, tr_A) \longrightarrow (B, tr_B)$$

are algebra morphisms which are trace preserving, that is, the diagram below is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

The archetypical example of an n-th Cayley-Hamilton algebra is an order in a central simple algebra.

Example 2.2. Let K be the functionfield of a normal variety X and let Δ be a central simple K-algebra of dimension n^2 over K. By classical theory of Brauer groups we know that

$$\Delta \otimes_K L \simeq M_n(L)$$

for some finite Galois extension L of K with Galois group G.
On $M_n(L)$ we have the usual trace map Tr and we can define for $\delta \in \Delta$, $tr(\delta) = Tr(\delta \otimes 1)$. As the image is invariant under the Galois group it follows that $tr(\delta) \in K$ and it is called the reduced trace map on Δ.
Let R be the coordinate ring of an affine open set of X and let A be an R-order in Δ. That is, A is a finitely generated R-submodule of Δ such that $A.K = \Delta$. As R is integrally closed it follows that $tr(a) \in R$ for all $a \in A$ and as we are in characteristic zero $tr(A) = R$. Clearly, A equipped with the reduced trace is an n-th Cayley-Hamilton algebra.

One can study n-th Cayley-Hamilton algebras via (commutative) algebraic geometry and geometric invariant theory.
Let $(A, tr_A) \in CH_n$ be m-generated, that is, there are elements $a_1, \ldots, a_m \in A$ such that the subalgebra in CH_n generated by them is equal to A (note that this is weaker than A being generated as algebra by m elements). Consider

$$X_A = \{\phi : (A, tr_A) \longrightarrow (M_n(\mathbb{C}), Tr) \text{ in } CH_n\}$$

By taking the images $\phi(a_i) \in M_n(\mathbb{C})$ for $1 \leq i \leq m$ it is clear that X_A is a closed subvariety of the affine space $M_n(\mathbb{C})^{\oplus m}$. There is a natural action of PGL_n on $M_n(\mathbb{C})^{\oplus m}$ by simultanous conjugation. Clearly, X_A is a PGL_n-stable closed subvariety of $M_n(\mathbb{C})^{\oplus m}$. If we denote by $CH_n^{(m)}$ the subcategory of CH_n consisting of algebras which are trace generated by m elements we have

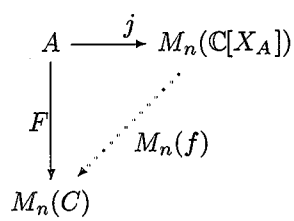
Theorem 2.3 (Procesi, [9]). *The functor*

$$CH_n^{(m)} \longrightarrow PGL_n\text{-closed subvarieties of } M_n(\mathbb{C})^{\oplus m}$$

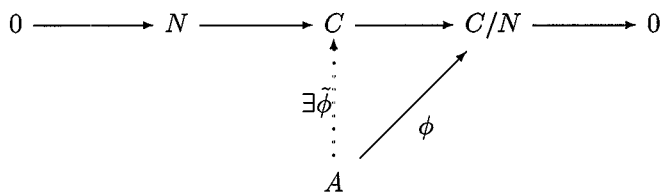
assigning X_A to $A \in CH_n^{(m)}$ has a left inverse. This inverse assigns to a PGL_n-closed subvariety X the ring of PGL_n-equivariant maps $X \longrightarrow M_n(\mathbb{C})$, or equivalently, the ring of concomitants

$$M_n(\mathbb{C}[X])^{PGL_n}$$

This means that we can recover $A \in CH_n^{(m)}$ from the affine PGL_n-variety X_A as $A \simeq M_n(\mathbb{C}[X_A])^{PGL_n}$. The embedding $j_A : A \hookrightarrow M_n(\mathbb{C}[X_A])$ has the following universal property. Let C be a commutative algebra and $F : A \longrightarrow M_n(C)$ a morphism in CH_n (with the usual trace map on $M_n(C)$) then there is a uniquely determined morphism $f : \mathbb{C}[X_A] \longrightarrow C$ making the diagram below commutative



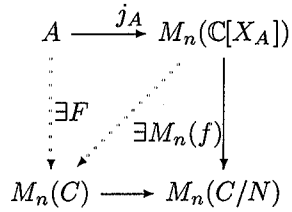
Definition 2.4. An affine algebra (A, tr_A) in CH_n is said to be **smooth** if and only if for every test-object (C, N) where $(C, tr_C) \in CH_n$, N a nilpotent ideal (invariant under the trace map such that also $(C/N, \overline{tr_C}) \in CH_n$) and every morphism $\phi : (A, tr_A) \longrightarrow (C/N, \overline{tr_C})$ in CH_n the diagram below can be completed in CH_n



Procesi gave the following geometric characterization of smooth algebras

Proposition 2.5 (Procesi [9]). *Equivalent are*
(1) *A is a smooth algebra in CH_n*
(2) *X_A is a smooth (commutative) variety*

Proof. (1) $\Rightarrow$ (2) : Let (C, N) be a commutative test-object for $\mathbb{C}[X_A]$. We have to lift the map $\mathbb{C}[X_A] \longrightarrow C/N$ to C. By smoothness in CH_n of A we can complete with F the diagram



but then by the universal property of j_A there is a uniquely determined map $f : \mathbb{C}[X_A] \longrightarrow C$ which is the required lift.

The reverse implication makes essential use of the Reynolds operator in invariant theory, see [9]. □

From now on we will restrict attention to prime smooth algebras A, or equivalently, to the case when X_A is an irreducible smooth variety. In this case we can show that A has to be an order over a normal domain. In fact, we have the following geometric interpretation of the center Z of A.

Lemma 2.6. *The center Z of A is the coordinate ring of the quotient variety X_A/PGL_n and is a Cohen-Macaulay normal domain.*

Proof. By the characterization of smooth algebras in CH_n, the module varety X_A is a smooth affine PGL_n-variety. The coordinate ring of the quotient variety X_A/PGL_n is by definition the ring of invariants $\mathbb{C}[X_A]^{PGL_n}$ which by [9] is equal to $tr(A) = Z$. The ringtheoretical properties of Z follow from the fact that $\mathbb{C}[X_A]$ is regular and the Hochster-Roberts result. □

Let us recall a few standard facts on the quotient map

$$\pi : X_A \longrightarrow X_A/PGL_n = Y_A$$

where Y_A is the affine variety corresponding to the center Z. From invariant theory we know that points of Y parametrize closed PGL_n- orbits in X_A.

By the Artin-Voigt theorem (see for example [4]) we know that closed orbits in X_A correspond to isomorphism classes of semi-simple n-dimensional representations of A. Moreover, as A is an order in a central simple algebra of dimension n^2 over K, there is a Zariski open subset $Y_{Az} \hookrightarrow Y_A$ the points of which correspond to isoclasses of simple n-dimensional representations of A. We call Y_{Az} the **Azumaya locus** of A. Equivalently, if m is the maximal ideal of Z corresponding to a point $y \in Y$, then $y \in Y_{Az}$ if and only if $A/mA \simeq M_n(\mathbb{C})$.
For an arbitrary point $y \in Y$ with corresponding maximal ideal m of Z we have

$$(A/mA)/rad \simeq M_{d_1}(\mathbb{C})^{\oplus e_1} \oplus \ldots \oplus M_{d_r}(\mathbb{C})^{\oplus e_r}$$

with $\sum d_i e_i = n$. Here, the factors correspond to the simple components of the semi-simple representation determined by y, these components have dimension d_i and occur with multiplicity e_i.
With $\tau(m)$ or $\tau(y)$ we will denote the **representation type** of this semi-simple n-dimensional representation, that is the numerical data $(e_1, d_1; \ldots; e_r, d_r)$. With $Y(\tau)$ we denote the subset of Y consisting of points of representation type τ. In particular, $Y_{Az} = Y(1, n)$.
There is a natural order relation on the representation types given by degenerations of the corresponding representations. This order is induced by the following two operations

$$(\ldots; e, d; \ldots) \geq (\ldots; e, d'; e, d"; \ldots) \text{ where } d = d' + d"$$

$$(\ldots; e', d; e", d; \ldots) \geq (\ldots; e' + f, d; e" - f, d; \ldots)$$

This combinatorial order relation relates to the stabilizer subgroups of closed orbits. If $\tau = (e_1, d_1; \ldots; e_r, d_r)$ then one verifies that the stabilizer subgroup of GL_n in the semi-simple representation of type τ is equal to

$$GL(\tau) = GL_{e_1} \times \ldots \times GL_{e_r}$$

embedded via

$$\begin{bmatrix} GL_{e_1} \otimes 1_{d_1} & & \\ & \ddots & \\ & & GL_{e_r} \otimes 1_{d_r} \end{bmatrix} \hookrightarrow GL_n$$

Then $\tau \leq \tau'$ if and only if $GL(\tau')$ is conjugated to a subgroup of $GL(\tau)$.

3. Etale local structure

If X is a commutative smooth variety of dimension d and x a point of X then there is only one type of étale local behaviour at x, namely

$$\mathcal{O}_x^{sh} \simeq \mathbb{C}\{x_1, \ldots, x_d\}$$

the strict Henselization of the local ring in x is the ring of algebraic functions on d variables.
In this section we will prove an analogous result for smooth orders in CH_n. We will show in the next section that for fixed n and d (the Krull dimension of the center) there are only finitely many types of étale behaviour.
From now on we fix a smooth algebra (A, tr) in CH_n which we assume to be an order in a central simple K-algebra Δ of dimension n^2. K is the field of fractions of the center Z of A.

Let $y \in Y(\tau)$ and corresponding maximal ideal $m \lhd Z$, by the étale local type of A at y we mean the structure of the algebra

$$A_m^{sh} = A \otimes_Z Z_m^{sh}$$

The next result was proved in [11] by a different method.

Proposition 3.1. *An Azumaya algebra A is smooth in CH_n if and only if its center Z is smooth.*

Proof. As A is an Azumaya algebra, X_A is a principal PGL_n-bundle over Y_A. Clearly, a principal bundle is smooth if and only if the base-space is smooth. □

Lemma 3.2. *If $y \in Y_{Az}$ then $A_m^{sh} \simeq M_n(\mathbb{C}\{x_1, \ldots, x_d\})$.*

Proof. If $y \in Y_{Az}$ then $A \otimes_Z Z_m$ is an Azumaya algebra and hence split by an étale extension. By the foregoing proposition, y is a smooth point of Y and the result follows. □

In general, let m be the maximal ideal of Z corresponding to $y \in Y(\tau)$. Fix a point $x \in X_A$ in the closed orbit corresponding to y. The semi-simple A-module M_x has decomposition into simple components

$$M_x = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

where $dim(S_i) = d_i$ and the stabilizer GL_x of the GL_n-action on X_A is equal to $GL(\tau)$. With Orb_x we will denote the orbit $(P)GL_n.x$ of x in X_A.
We have GL_n-equivariant closed embeddings

$$Orb_x \hookrightarrow X_A \hookrightarrow M_n(\mathbb{C})^{\oplus m}$$

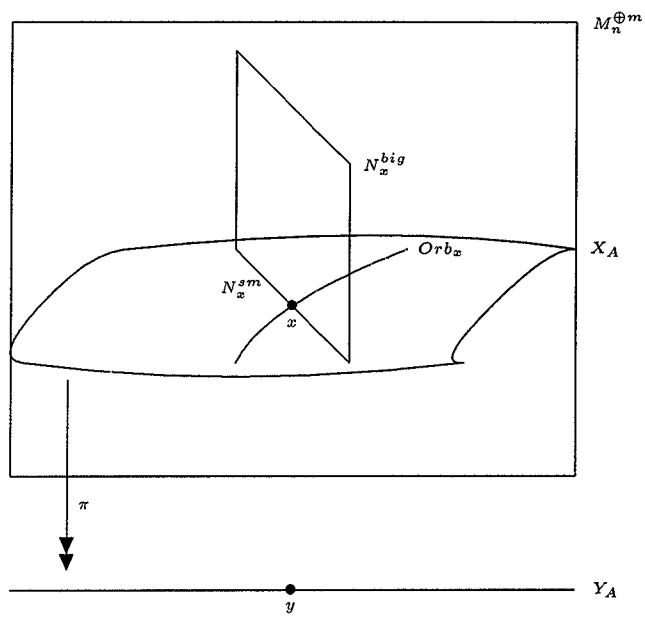
and corresponding embeddings of the tangent spaces in x

$$T_x\ Orb_x \hookrightarrow T_x\ X_A \hookrightarrow T_x\ M_n(\mathbb{C})^{\oplus m}$$

which are embeddings as $GL(\tau)$-modules and hence by reductivity of $GL(\tau)$ they are direct factors.
Therefore, we have for the normal spaces to the orbit in X_A resp. $M_n(\mathbb{C})^{\oplus m}$ that

$$N_x^{sm} = \frac{T_x\ X_A}{T_x\ Orb_x} \lhd N_x^{big} = \frac{T_x\ M_n(\mathbb{C})^{\oplus m}}{T_x\ Orb_x}$$

as $GL(\tau)$-modules. That is, we have the following picture



Before we compute these $GL(\tau)$-modules, let us explain the relevance to our problem.
This is an application of the Luna slice theorem in invariant theory (see [8] or [12]) adapted to the situation of interest to us.
In general, if H is a reductive subgroup of G acting on an affine variety Z then one defines an H-action on $G \times Z$ via the map

$$h.(g,z) = (gh^{-1}, h.z)$$

The corresponding quotient is called the associated fiber bundle

$$G \times^H Z = (G \times Z)/H$$

and it acquires a G-action via multiplication on the left in the first component. One can show that the corresponding quotient satisfies

$$(G \times^H Z)/G \simeq Z/H$$

Theorem 3.3 (Luna slice theorem). *Let x be a smooth point of X_A of representation type τ. Then, there exists a locally closed affine smooth subvariety $S_x \hookrightarrow X_A$ containing x, which is stable under the action of $GL(\tau)$ satisfying the following properties*

- *The map $GL_n \times S_x \longrightarrow X_A$ obtained by $(g,s) \mapsto g.s$ induces a GL_n-equivariant étale map*

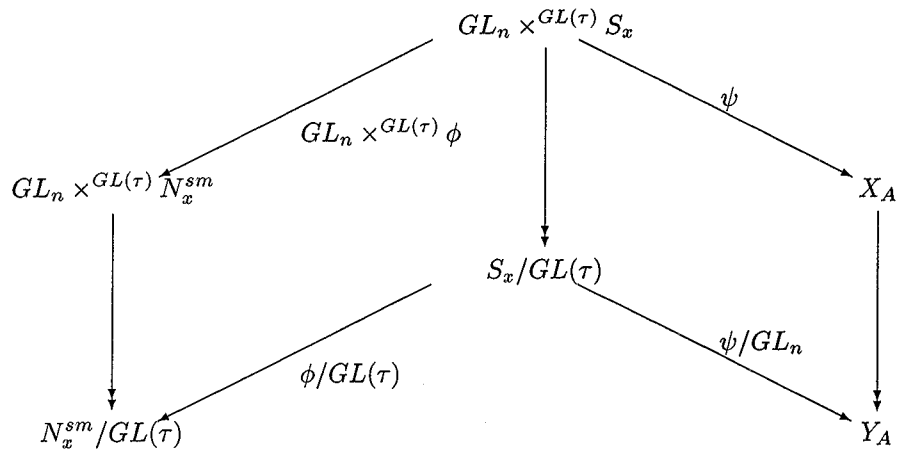
$$\psi : GL_n \times^{GL(\tau)} S_x \longrightarrow X_A$$

with affine image. Moreover the induced quotient map

$$\psi/GL_n : (GL_n \times^{GL(\tau)} S_x)/GL_n = S_x/GL(\tau) \longrightarrow X_A/GL_n = Y_A$$

is also étale.

- *There is a $GL(\tau)$-equivariant map*
$$\phi : S_x \longrightarrow N_x^{sm} = T_x\ S_x$$
such that $\phi(x) = 0$ and with affine image. The induced quotient map
$$\phi/GL(\tau) : S_x/GL(\tau) \longrightarrow N_x^{sm}/GL(\tau)$$
is also étale.
- *The above maps induce the following commutative diagram*



where the vertical maps are the quotient maps, all diagonal maps are étale and the upper ones are GL_n-equivariant.

Hence, the GL_n-local structure of X_A in a neighborhood of x is the same as that of $GL_n \times^{GL(\tau)} N_x^{sm}$ in a neighborhood of $\overline{(1_n, 0)}$. Similarly, the local structure of Y_A in a neighborhood of m is the same as that of $N_x^{sm}/GL(\tau)$ in a neighborhood of $\overline{0}$. Therefore, we have

Theorem 3.4. *Let A be a smooth order in CH_n with center Z.*
Let m be a maximal ideal of Z corresponding to a point $x \in mod_n\ B$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$.
Let p denote the maximal ideal of $\mathbb{C}[N_x^{small}/GL(\tau)]$ corresponding to the point $\overline{0}$.
Then,
(1) $Z_m^{sh} \simeq \mathbb{C}[N_x^{sm}/GL(\tau)]_p^{sh}$
(2) $A_m^{sh} \simeq (M_n(\mathbb{C}[GL_n \times^{GL(\tau)} N_x^{sm}])^{GL_n})_p^{sh}$

Hence, we know the étale local structure of Z and A in m if we know the $GL(\tau)$-module structure of N_x^{sm}.

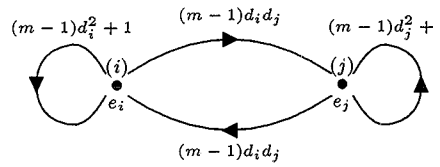
Since we know the embedding $GL(\tau) \hookrightarrow GL_n$ and the action of GL_n on $M_n(\mathbb{C})^{\oplus m}$ (by simultaneous conjugation) we know the structure of $T_x\ M_n(\mathbb{C})^{\oplus m} = M_n(\mathbb{C})^{\oplus m}$ as $GL(\tau)$-module. Further, the exact sequence
$$0 \longrightarrow Lie\ GL(\tau) \longrightarrow Lie\ GL_n \longrightarrow T_x\ Orb_x \longrightarrow 0$$
allows us to determine the $GL(\tau)$-module structure of $T_x\ Orb_x$ and consequently that of $N_x^{big} = T_x\ M_n(\mathbb{C})^{\oplus m}/T_x\ Orb_x$.
Once we know an isotypical decomposition of N_x^{big}, taking a direct subsum we obtain all possibilities for N_x^{sm}. Of course, later on, we will have to verify which of these theoretical possibilities actually occur for a smooth order A in CH_n.

Rather than writing down decompositions of $N_x^{sm} \triangleleft N_x^{big}$ in simple $GL(\tau)$-modules we prefer to represent this information by a **'local chart'**. We use the following dictionary

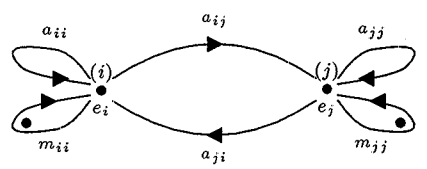
- a **loop** at vertex (i) corresponds with the $GL(\tau)$-module $M_{e_i}(\mathbb{C})$ on which GL_{e_i} acts by conjugation and the other factors act trivially.
- an **arrow** from vertex (i) to vertex (j) corresponds to the $GL(\tau)$-module $M_{e_i \times e_j}(\mathbb{C})$ on which $GL_{e_i} \times GL_{e_j}$ act via $g.m = g_i m g_j^{-1}$ and the other factors act trivially.
- a **marked loop** at vertex (i) corresponds to the simple $GL(\tau)$-module $M_{e_i}^0(\mathbb{C})$, that is, trace zero matrices with action of GL_{e_i} by conjugation and trivial action by the other components.
- the label of a loop or arrow indicates the multiplicity of the corresponding representation.

Lemma 3.5. *With conventions as above and x a point of representation type τ we have*

(1) *The $GL(\tau) = GL_{e_1} \times \ldots \times GL_{e_r}$-module structure of N_x^{big} can be represented by the local chart on r vertices such that the subchart on any two vertices $1 \leq i, j \leq r$ is of the form*



(2) *The $GL(\tau)$-module structure of N_x^{sm} can be represented by a local chart on r vertices such that the subgraph on any two vertices $1 \leq i, j \leq r$ is of the form*



where $a_{ij} \leq (m-1)d_id_j$ and $a_{ii} + m_{ii} \leq (m-1)d_i^2 + 1$ for all $1 \leq i, j \leq r$.

Proof. (2) follows from (1) by observing that $M_{e_i \times e_j}(\mathbb{C})$ is a simple $GL(\tau)$-module and that the isotypical decomposition of $M_{e_i}(\mathbb{C}) = M_{e_i}^0(\mathbb{C}) \oplus \mathbb{C}_{triv}$ where $\mathbb{C}_{triv}$ is the trivial one-dimensional $GL(\tau)$-module. The isotypical decomposition of N_x^{big} has been proved in [5]. □

4. Classifying local charts

A local chart $C = (M, \mathbf{e})$ consists of two data : the underlying 'map' M that is, the marked labeled directed graph and the 'dimension-vector' $\mathbf{e} = (e_1, \ldots, e_r)$. If we specify $\mathbf{e}$ we obtain a $GL(\mathbf{e}) = \times GL_{e_i}$-module $R(M, \mathbf{e})$ any vector of which we call a representation of the map M of dimension $\mathbf{e}$. That is $v \in R(M, \mathbf{e})$ assigns to each

- arrow from (i) to (j) a matrix in $M_{e_i \times e_j}(\mathbb{C})$
- unmarked loop in (i) a matrix in $M_{e_i}(\mathbb{C})$
- marked loop in (i) a trace zero matrix in $M^0_{e_i}(\mathbb{C})$

A morphism from a representation $v \in R(M, \mathbf{e})$ to a representation $w \in R(M, \mathbf{f})$ is an r-tuple of linear maps $\psi = (\psi_1, \ldots, \psi_r) \in \oplus_i M_{f_i \times e_i}(\mathbb{C})$ such that every diagram

$$\begin{array}{ccc} \mathbb{C}^{e_i} & \xrightarrow{v} & \mathbb{C}^{e_j} \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_j} \\ \mathbb{C}^{f_i} & \xrightarrow{v} & \mathbb{C}^{f_j} \end{array}$$

is commutative where the horizontal maps are either arrows or (marked) loops in M.

Having morphisms, the notions of sub-, quotient- and simple-representation are obvious as are direct sums of representations of M. If we view the $GL(\mathbf{e})$-module $R(M, \mathbf{e})$ as an affine space on which $GL(\mathbf{e})$ acts, then orbits correspond precisely to isomorphism classes of representations.

Lemma 4.1. *The local chart $C = (M, \mathbf{e})$ of a smooth order A in CH_n at a point x must be such that $R(M, \mathbf{e})$ contains simple representations of M.*

Proof. Consider a point $x \in X_A$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$ with $N_x^{sm} = R(M, \mathbf{e})$. By the Luna slice theorem we have étale GL_n-equivariant maps

$$GL_n \times^{GL(\tau)} N_x^{sm} \xleftarrow{et} GL_n \times^{GL(\tau)} S_x \xrightarrow{et} X_A$$

As A is a prime order, we have that any Zariski neighborhood of x in X_A contains simple orbits, that is, closed orbits with stabilizer $\mathbb{C}^*$. Because the maps above are GL_n-equivariant and étale every Zariski neighborhood of $\overline{(1_n, 0)}$ contains a closed GL_n-orbits with stabilizer $\mathbb{C}^*$. By the correspondence of orbits in fiber bundles there must be closed $GL(\tau)$-orbits in $N_x^{sm} = R(M, \mathbf{e})$ with stabilizer $\mathbb{C}^*$. By a version of the Artin-Voigt theorem for representations of the map M closed orbits correspond to semi-simple representations of M. If the stabilizer of such a representation is $\mathbb{C}^*$ then it must be simple. □

Hence, we have to determine which dimension vectors can arise from simple representations of the map M. We define the **Euler-form** of M as the bilinear map

$$\chi_M : \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$$

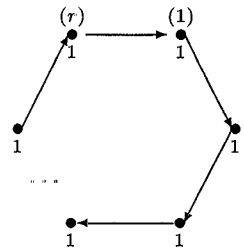
determined by the matrix $\chi_M = (\chi_{ij})$ with entries

$$\chi_{ij} = -a_{ij} \text{ and } \chi_{ii} = 1 - a_{ii} - m_{ii}$$

where a_{ij} is the number of arrows from (i) to (j) in M and a_{ii} resp. m_{ii} are the number of (resp. marked) loops at (i).

Proposition 4.2. $\mathbf{e} = (e_1, \ldots, e_r)$ *is the dimension-vector of a simple representation of the map* M *if and only if one of the following situations occurs*

(1) $M = \tilde{A}_r$ *the extended Dynkin diagram with cyclic orientation and* $\mathbf{e} = (1, \ldots, 1)$.



(2) $M \neq \tilde{A}_r$. *Then,* M *has to be strongly connected (that is, any two vertices can be connected by a directed path) and if* $\delta_i = (\delta_{1i}, \ldots, \delta_{ri})$ *are a standard basis of* $\mathbb{Z}^r$ *we must have*

$$\chi_M(\mathbf{e}, \delta_i) \leq 0 \text{ and } \chi(\delta_i, \mathbf{e}) \leq 0$$

for all $1 \leq i \leq r$.

Proof. We will only prove necessity of the conditions in (2). Sufficiency follows from a degeneration argument and induction as in [6].

Let $v \in R(M, \mathbf{e})$ be a simple representation (that is, contains no proper subrepresentations) and let $v(\phi)$ denote the linear map determined by the arrow, loop or marked loop ϕ.

Assume M is not strongly connected, then we can divide M into maximal strongly connected submaps $M_1, \ldots, M_z$ say. The direction of all arrows between two such components must be all the same by maximality. Hence, there is a component M_i having no arrows to other components. Now, define a proper subrepresentation w of v with dimension-vector $\mathbf{f} = \delta_M \mathbf{e}$ by $w(\phi) = v(\phi)$ if ϕ is a map in M_i and $w(\phi) = 0$ otherwise. Hence, M must be strongly connected.

For each (i) we have $\chi_M(\delta_i, \mathbf{e}) = e_i - \sum_{(i) \xrightarrow{\phi} (j)} e_j$ Hence, if $\chi_M(\delta_i, \mathbf{e}) > k$ then the natural morphism

$$\bigoplus_{(i) \xrightarrow{\phi} (j)} v(\phi) : \mathbb{C}^{e_i} \longrightarrow \bigoplus_{(i) \xrightarrow{\phi} (j)} \mathbb{C}^{e_j}$$

has a non-trivial kernel K of dimension $k > 0$ and determines a proper subrepresentation of v of dimension-vector $\mathbf{f} = (\delta_{ij} k)_j$.

Similarly, if $\chi_M(\mathbf{e}, \delta_i) = e_i - \sum_{(j) \xrightarrow{\phi} (i)} e_j > 0$ then the image of the natural morphism

$$\bigoplus_{(j) \xrightarrow{\phi} (i)} v(\phi) : \bigoplus_{(i) \xrightarrow{\phi} (j)} \mathbb{C}^{e_j} \longrightarrow \mathbb{C}^{e_i}$$

is a proper subspace of $\mathbb{C}^{e_i}$ of dimension $k < e_i$ and hence determines a proper subrepresentation of v with dimension-vector $\mathbf{e} + (k - e_i)\delta_i$. □

Proposition 4.3. *The local chart $C = (M, \mathbf{e})$ of a smooth model A in CH_n having a center Z of (Krull) dimension d must be such that*

$$1 - \chi_M(\mathbf{e}, \mathbf{e}) - \sum_i m_{ii} = d$$

Proof. Consider a point $x \in X_A$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$ with $N_x^{sm} = R(M, \mathbf{e})$. By the Luna slice theorem we have étale maps

$$N_x^{sm}/GL(\tau) \xleftarrow{et} S_x/GL(\tau) \xrightarrow{et} Y_A$$

Because $\mathbb{C}[Y_A] = Z$ we have that Y_A and hence $N_x^{sm}/GL(\tau)$ must be of dimension d.
By definition of the Euler-form of M we have that

$$\chi(\mathbf{e}, \mathbf{e}) = -\sum_{i \neq j} e_i e_j a_{ij} + \sum_i e_i^2 (1 - a_{ii} - m_{ii})$$

On the other hand we have the following dimensions

$$dim\ R(M, \mathbf{e}) = \sum_{i \neq j} e_i e_j a_{ij} + \sum_i e_i^2 (a_{ii} + m_{ii}) - \sum_i m_{ii}$$

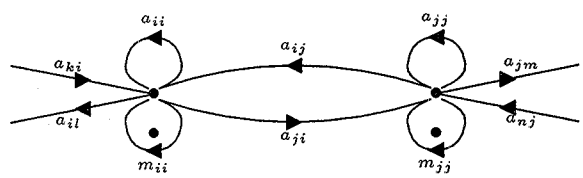
$$dim\ GL(\mathbf{e}) = \sum_i e_i^2$$

As $\mathbf{e}$ is the dimension vector of a simple representation we know that the orbits in general position in $R(M, \mathbf{e})$ are closed and have stabilizer $\mathbb{C}^*$. Therefore, the dimension of the quotient variety $R(M, \mathbf{e})/GL(\mathbf{e}) = N_x^{sm}/GL(\tau)$ is equal to

$$dim\ R(M, \mathbf{e}) - dim\ GL(\mathbf{e}) + 1$$

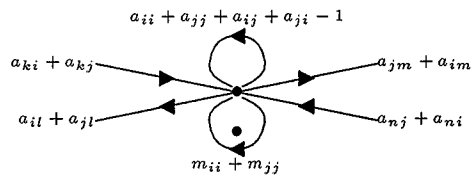
and plugging in the above information we see that this is equal to $1 - \chi(\mathbf{e}, \mathbf{e}) - \sum_i m_{ii}$. □

If we want to study the local structure of smooth orders A in CH_n having a center of dimension d, we have to compile a list of admissible charts. We will give the first steps in such a classification.
The basic idea that we use is to shrink a chart to its simplest form and classify these simplest forms for given d. By **shrinking** we mean the following process. Assume $\mathbf{e}$ is the dimension vector of a simple representation of M and let (i) and (j) be two connected vertices with $e_i = e_j = e$. That is we have locally the following situation



We will use one of the arrows connecting (i) with (j) to identify the two vertices. That is, we form the shrinked chart $C^s = (M^s, \mathbf{e}^s)$ where M^s is a map on $r - 1$

vertices $\{(1), \ldots, \hat{(i)}, \ldots, (r)\}$ and $\mathbf{e}^s$ is the dimension vector with (i) removed. That is, locally round z the shrinked chart has the form



That is, in M^s we have for all $k, l \neq j$ that $a^s_{kl} = a_{kl}$. Moreover, the number of arrows and (marked) loops connected to j are determined as follows

- $a^s_{jk} = a_{ik} + a_{jk}$
- $a^s_{kj} = a_{ki} + a_{kj}$
- $a^s_{jj} = a_{ii} + a_{jj} + a_{ij} + a_{ji} - 1$
- $m^s_{jj} = m_{ii} + m_{jj}$

Lemma 4.4. $\mathbf{e}$ *is the dimension vector of a simple representation of* M *if and only if* $\mathbf{e}^s$ *is the dimension vector of a simple representation of* M^s. *Moreover,*

$$dim\ R(M, \mathbf{e})/GL(\mathbf{e}) = dim\ R(M^s, \mathbf{e}^s)/GL(\mathbf{e}^s)$$

Proof. Fix an arrow ϕ connecting (i) and (j). As $e_i = e_j = e$ there is a Zariski open subset $U \hookrightarrow R(M, \mathbf{e})$ of points v such that $v(\phi)$ is invertible. By basechange in either (i) or (j) we can find a point w in its orbit such that $w(\phi) = I_e$.
If we think of $w(\phi)$ as identifying $\mathbb{C}^{e_i}$ with $\mathbb{C}^{e_j}$ we can view the remaining maps of w as a representation in $R(M^s, \mathbf{e}^s)$ and denote it with w^s. the map $U \longrightarrow R(M^s, \mathbf{e}^s)$ is well-defined and maps $GL(\mathbf{e})$-orbits to $GL(\mathbf{e}^s)$-orbits.
Conversely, given a representation $w' \in R(M^s, \mathbf{e}^s)$ we can uniquely determine a representation $w \in U$ mapping to w'.
Both claims follow immediately from this observation. □

It is clear that any chart can uniquely be reduced to its simplest form, which has the property that no connecting vertices can have the same dimension. Also note that the shrinking process has a not necessarily unique converse operation which we will call **splitting of a vertex**.

Proposition 4.5. *Let* $\mathbf{e}$ *be the dimension vector of a simple representation of* M *and let* $d = dim\ R(M, \mathbf{e})/GL(\mathbf{e})$. *If* $e = max\ e_i$, *then* $d \geq e + 1$.

Proof. By the above lemma we may assume that $C = (M, \mathbf{e})$ is in its simplest form, that is, no two connecting vertices have same dimension. Let $\chi_M = I_r - (g_{ij})_{i,j}$ where g_{ij} is the number of arrows (or loops marked or unmarked) from (i) to (j). We can then rewrite

$$-\chi_M(\mathbf{e}, \mathbf{e}) = \sum_i e_i(\sum_j g_{ij}e_j - e_i) = \sum_i e_i(\sum_j g_{ji}e_j - e_i)$$

and observe that the terms between brackets are positive by the requirement that $\mathbf{e}$ is the dimension vector of a simple representation. For any vertex (k) we will call

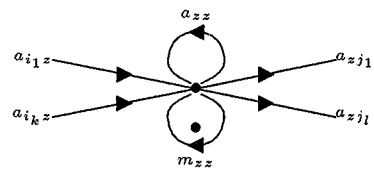
$$e_k(\sum_{l\neq k} a_{lk}e_l + (a_{kk}+m_{kk}-1)e_k) - m_{kk}$$

$$e_k(\sum_{l\neq k} a_{kl}e_l + (a_{kk}+m_{kk}-1)e_k) - m_{kk}$$

the incoming (resp. outgoing) contribution of (k) to $dim\ R(M,\mathbf{e})/GL(\mathbf{e})$.
Observe that the in(out) contributions of vertices having no marked loops is always ≥ 0 by the restrictions on $\mathbf{e}$. Further, if there are marked loops at a vertex (i) then the in(out) contribution of that vertex is ≥ 1. For, it is

$$e_i((a_{ii}+m_{ii}-1)e_i + \sum_j a_{ij}e_j) - m_{ii} \geq a_{ii}+m_{ii}+1-m_{ii} \geq 1$$

(note that $e_i \neq e_j$).
Consider a vertex (z) where the maximum dimesionvector e is obtained. We have locally the following situation



The out-contribution of (z) is equal to

$$e((a_{zz}+m_{zz}-1)e + \sum_u a_{zj_u}e_{j_u}) - m_{zz}$$

If $m_{zz} \geq 2$ this expression is $\geq e^2(m_{zz}-1)+2-m_{zz} \geq e$ as $e \geq 2$. Hence, the dimension is at least $a+1$.
If $m_{zz}=1$ then the in-contribution of (z) is $\geq e-1$ and the in-contributions of j_u are $\geq e-1$, so the total dimension is $\geq 2e-1 \geq e+1$ if $e \geq 2$.
Finally, if $m_{zz}=0$ then the out-contribution of (z) is

$$\geq e(\sum_u a_{zj_u}e_{j_u} - e)$$

which is $\geq e$ whenever $\sum_u a_{zj_u}e_{j_u} \neq a$. If this happens to be an equality, then the in-contribution at j_u is

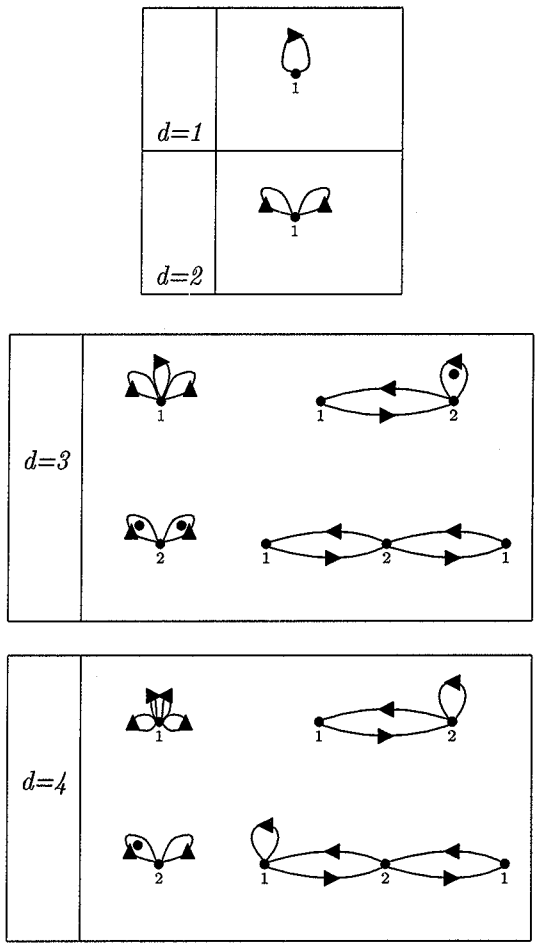
$$\geq e_{j_u}((m_{j_uj_u}-1)e_{j_u}+e) - m_{j_uj_u} \geq e-1$$

but there are at least two j_u as none of the $e_{j_u} = e$, so again the dimension $\geq 2e-1 \geq e+1$. □

Definition 4.6. Two charts $C=(M,\mathbf{e})$ and $C'=(M,\mathbf{e})$ are said to be **equivalent** if their corresponding $GL(\mathbf{e})$-modules are isomorphic.

Example 4.7. The charts below are equivalent

Theorem 4.8. *The local charts of a smooth order A in CH_n with center Z of Krull dimension d can be shrinked to one of the following equivalence classes of charts*



It is clear that one can easily compile a list of equivalence classes of simplified charts for any given dimension d. A direct consequence is

Proposition 4.9. *For fixed n and d there are only finitely many types of étale local behaviour.*

Proof. By the foregoing reduction there are for fixed d only finitely many equivalence classes of local charts shrinked to their simplest form. As $\sum e_i \leq n$, we can only apply splitting of vertices finitely many times. □

5. Reading the local chart

Knowing which local charts can occur, we will now investigate what information can be derived from the local chart.
We will fix the following situation : A is a smooth order in CH_n having a center Z of Krull dimension d, m will be a maximal ideal of Z corresponding to the closed orbit $GL_n.x \hookrightarrow X_A$ where x has representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$.
We have $N_x^{sm} = R(M, \mathbf{e})$ as $GL(\tau) = \times GL_{e_i}$- module. The local structure of Z near m is determined by that of $\mathbb{C}[N_x^{sm}/GL_\tau]$ near the zero representation, so we better have an interpretation of this ring

Proposition 5.1. *$\mathbb{C}[N_x^{sm}/GL(\tau)]$ is generated by traces along oriented cycles in the chart $C = (M, \mathbf{e})$.*

That is, for every arrow ϕ (resp. loop or marked loop) from (i) to (j) we take a generic rectangular matrix

$$M_\phi = \begin{bmatrix} x_{11}(\phi) & \ldots & \ldots & x_{1,e_j}(\phi) \\ \vdots & & & \vdots \\ x_{e_i 1}(\phi) & \ldots & \ldots & x_{e_i e_j}(\phi) \end{bmatrix}$$

(resp. a generic square matrix or generic trace zero matrix).
If $cyc = \phi_k \circ \ldots \circ \phi_2 \circ \phi 1$ is an oriented cycle in the map M, then we compute the following matrix

$$M_{cyc} = M_{\phi_k} \ldots M_{\phi_2} M_{\phi_1}$$

over $\mathbb{C}[x_{kl}(\phi)] = \mathbb{C}[R(M, \mathbf{e})]$. If the starting vertex of ϕ_1 is (i), then this is a square $e_i \times e_i$ matrix and we can consider its trace

$$Tr(M_{cyc}) \in \mathbb{C}[R(M, \mathbf{e})]$$

and one verifies easily that this polynomial is invariant under the action of GL_τ.
Slightly harder to prove is that these functions actually generate

$$\mathbb{C}[R(M, \mathbf{e})]^{GL_\tau} = \mathbb{C}[N_x^{sm}/GL_\tau]$$

The essential ingredient in this proof is the fact that the polynomial invariants of tuples of matrices under simultaneous conjugation are generated by traces of products of generic matrices.
In fact, one can even bound the length of the oriented cycles to be considered by $(\sum_i e_i)^2$, see [6] for more details.
Next, let us consider the étale local structure of A near m. By the results proved before, we have to control for this the ring of GL_n-equivariant maps

$$GL_n \times^{GL_\tau} R(M, \mathbf{e}) \longrightarrow M_n(\mathbb{C})$$

on which the multiplication is given by that in target space $M_n(\mathbb{C})$.

Proposition 5.2. *The ring of GL_n-equivariant maps is Morita equivalent to the ring of GL_τ-equivariant maps*

$$R(M, \mathbf{e}) \longrightarrow M_{\sum e_i}(\mathbb{C})$$

where for any two vertices (i) and (j) the GL_τ-equivariant maps

$$R(M, \mathbf{e}) \longrightarrow Hom(\mathbb{C}^{\oplus e_i}, \mathbb{C}^{\oplus e_j})$$

are generated as a module over $\mathbb{C}[N_x^{sm}/GL(\tau)]$ *by the paths in the map* M *starting from* (i) *and ending in* (j).

Again, if $path = \phi_k \circ \ldots \circ \phi_1$ is such a path, then the corresponding module element is M_{path}. This result follows from a minor adaptation to existing results on invariants and concomitants of representations of quivers proved by C. Procesi and myself, see [6].

Apart from allowing us to compute the local structure of Z and B near m, the local chart $N_x^{sm} = R(M, \mathbf{e})$ also allows us to describe the local charts in nearby points and the dimensions of subvarieties of points having a specific local chart.

The points ζ in the quotient variety $N_x^{sm}/GL(\tau) = R(M, \mathbf{e})/GL(\mathbf{e})$ are in one-to-one correspondence with the isomorphism classes of semi-simple representations of the map M of dimension vector $\mathbf{e}$.

If V_ζ is a representative in the closed orbit corresponding to ζ then we can decompose V_ζ into its simple representations

$$V_\zeta = W_1^{\oplus m_1} \oplus \ldots \oplus W_k^{\oplus m_k}$$

where W_i is a simple representation of the map M of dimension vector $\mathbf{b_i}$ and occurring in V_ζ with multiplicity m_i.

Extending previous terminology we will say that V_ζ (or ζ) is of representation type $\sigma = (m_1, \mathbf{b_1}; \ldots; m_k, \mathbf{b_k})$.

As we have a combinatorial description of all simple dimension vectors for M we can determine which representation types can occur for a given $\mathbf{e}$.

With V_σ we will denote the set of all points $\zeta \in N_x^{sm}/GL_\tau$ of representation type σ.

Proposition 5.3. $\{V_\sigma : \sigma$ *a representation type for* $\mathbf{e}\}$ *is a finite stratification of the quotient variety* $N_x^{sm}/GL(\tau)$ *into locally closed irreducible smooth subvarieties. Moreover, the dimension of the stratum* V_σ *determined by* $\sigma = (m_1, \mathbf{b_1}; \ldots; m_k, \mathbf{b_k})$ *is equal to*

$$\sum_{j=1}^{k}(1 - \chi_M(\mathbf{b_j}, \mathbf{b_j})) - \sum_{i=1}^{r} m_{ii}$$

Proof. According to the Luna slice results we have to verify that the representation type determines the stabilizer subgroup of a point in the closed orbit up to conjugation in $GL(\mathbf{e})$.

Let $\mathbf{b_i} = (b_{i1}, \ldots, b_{ir})$ and denote $b_i = \sum_j b_{ij}$. We choose a basis in $\oplus_i \mathbb{C}^{\oplus e_i}$ in the following way : the first $m_1 b_1$ vectors give a basis for the simple components of type W_1, the next $e_2 b_2$ vectors give a basis for the simple components of type W_2 and so on.

If $m = \sum a_i$, the subring of $M_m(\mathbb{C})$ generated by the representation V_ζ expressed in this basis is

$$\begin{bmatrix} M_{b_1}(\mathbb{C}) \otimes I_{e_1} & & \\ & \ddots & \\ & & M_{b_k}(\mathbb{C}) \otimes I_{e_k} \end{bmatrix}$$

Therefore, the stabilizer GL_V in $GL(\mathbf{e})$ of V_ζ is the group of units of the centralizer of this ring and is therefore equal to $GL_{m_1} \times \ldots \times GL_{m_k}$ which is embedded in

$GL(\mathbf{e})$ with respect to the chosen basis as

$$\begin{bmatrix} GL_{m_1}(\mathbb{C}\otimes I_{b_1}) & & \\ & \ddots & \\ & & GL_{m_k}(\mathbb{C}\otimes I_{b_k}) \end{bmatrix}$$

It is easy to see that the conjugacy class of GL_V depends only on the representation type τ.
Finally, we have seen before that the dimension of the variety of isoclasses of simple representations of M of dimension vector $\mathbf{b_j}$ is equal to $1-\chi(\mathbf{b_j},\mathbf{b_j})-\sum_i m_{ii}$ from which the claim about the dimension of the stratum follows. □

Given two representations types σ and σ', the stratum $V_{\sigma'}$ lies in the closure of the stratum V_σ if and only if the stabilizer subgroup GL_σ is conjugated to a subgroup of GL_σ in $GL(\mathbf{e})$. Again, mimicking similar results for representations of quivers we can give a combinatorial solution to this problem.
Two representation types

$$\sigma=(m_1,\mathbf{b_1};\ldots;m_k,\mathbf{b_k}) \text{ and } \sigma'=(m_1',\mathbf{b_1}';\ldots;m_{k'}',\mathbf{b_{k'}}')$$

are said to be direct successors $\sigma<\sigma'$ if and only if either

- (splitting one simple type) $k'=k+1$ and for all but one $1\leq i\leq k$ we have $(m_i,\mathbf{b_i})=(m_j',\mathbf{b_j}')$ for a uniquely determined j and for the remaining i we have corresponding to it $(m_i,\mathbf{b_u}';m_i,\mathbf{b_v}')$ with $\mathbf{b_i}=\mathbf{b_u}'+\mathbf{b_v}'$.
- (combining two simple types) $k'=k-1$ and for all but one $1\leq i\leq k'$ we have $(m_i',\mathbf{b_i}')=(m_j,\mathbf{b_j})$ for a uniquely determined j and for the remaining i we have corresponding to it $(m_u,\mathbf{b_i}';m_v,\mathbf{b_i}')$ with $m_u+m_v=m_i'$

The direct successor relation $<$ induces an ordering which we will denote with $\ll$.

Proposition 5.4. *The stratum $V_{\sigma'}$ lies in the closure of the stratum V_σ if and only if $\sigma\ll\sigma'$.*

Finally, we want to understand the étale local structure of the quotient variety $N_x^{sm}/GL(\tau)$ in a neighborhood of a point $\zeta\in V_\sigma$. This again is an application of the Luna slice results.
So, let V be a semi-simple representation of M corresponding to $\zeta\in V_\sigma$ with stabilizer subgroup $GL_\sigma=GL_{m_1}\times\ldots GL_{m_k}$. We have to investigate the GL_σ-module structure of the normal space to the orbit of V.
The tangentspace to the $GL(\mathbf{e})$ orbit of V is equal to the image of the natural linear map

$$Lie\ GL(\mathbf{e})\longrightarrow R(M,\mathbf{e})$$

sending an element $y\in Lie\ GL(\mathbf{e})$ to the representation determined by the commutator $[y,V]=y.V-V.y\in M_m(\mathbb{C})$ where as above $m=\sum e_i$ and all embeddings are with respect to the choice of basis we introduced in the proof of proposition.
The kernel of the above map is the centralizer of the subalgebra of $M_m(\mathbb{C})$ generated by the representation V, that is, the algebra

$$C_V=\begin{bmatrix} M_{m_1}(\mathbb{C}\otimes I_{b_1}) & & \\ & \ddots & \\ & & M_{m_r}(\mathbb{C}\otimes I_{b_r}) \end{bmatrix}$$

We thus have an exact sequence of GL_σ-modules

$$0 \longrightarrow C_V \longrightarrow Lie\ GL(\mathbf{e}) \longrightarrow T_V\ Orb_V \longrightarrow 0$$

where the action of GL_σ is given by conjugation in $M_m(\mathbb{C})$ via the embedding given before.
A typical element $\gamma \in GL_\sigma = GL_{m_1} \times \ldots \times GL_{m_k}$ will be written as $(\gamma_1, \ldots, \gamma_k)$ and we will express the actions in terms of the γ_i.
C_V as GL_σ-module consists of

- one m_1^2-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action

$\vdots$

- one m_k^2-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action

Moreover, using our notation $\mathbf{b_i} = (b_{i1}, \ldots, b_{ir})$ we have that $Lie\ GL(\mathbf{e})$ as GL_σ-module consists of

- $\sum_{j=1}^k b_{1j}^2$ times the m_1^2-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action

$\vdots$

- $\sum_{j=1}^k b_{kj}^2$ times the m_k^2-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action
- $\sum_{j=1}^k b_{1j}b_{2j}$ times the $m_1 \times m_2$-dimensional representation with $\gamma_1^{-1}.\gamma_2$-action

$\vdots$

- $\sum_{j=1}^k b_{kj}b_{k-1j}$ times the $m_k \times m_{k-1}$ -dimensional representation with $\gamma_k^{-1}.\gamma_{k-1}$-action

Hence, we know the GL_σ-module structure of $T_V\ Orb_V$. Next, we have to determine the GL_σ-module structure of $R(M, \mathbf{e})$. There are

- $b_{1i}b_{1j}$ times the $m_1 \times m_1$-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action
- $b_{1i}b_{2j}$ times the $m_1 \times m_2$-dimensional representation with $\gamma_1^{-1}.\gamma_2$-action

$\vdots$

- $b_{ki}b_{kj}$ times the $m_k \times m_k$-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action

For each unmarked loop in (i) we have the same decomposition as above replacing all occurrences of j with i. For a marked loop in (i) we have to replace the terms of dimension $m_i \times m_i$ by

- b_{li}^2 times the $m_i^2 - 1$-dimensional representation of trace zero matrices with $\gamma_l^{-1}.\gamma_l$-action.

We now have all the information on the GL_σ-module structure on the normal space to the orbit using the (split) exact sequence of GL_σ-modules

$$0 \longrightarrow T_V\ Orb_V \longrightarrow R(M, \mathbf{e}) \longrightarrow N_V \longrightarrow 0$$

and we obtain

Proposition 5.5. *The étale local structure of $N_x^{sm}/GL(\tau)$ near $\zeta \in V_\sigma$ is determined by a local chart $C_\sigma = (M_\sigma, \mathbf{e}_\sigma)$ where M_σ has k vertices $\{(1), \ldots, (k)\}$ and there are*

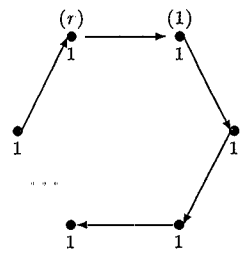
- $-\chi_M(\mathbf{b_i}, \mathbf{b_j})$ *directed arrows from (i) to (j) when $i \neq j$*
- $1 - \chi_1(\mathbf{b_i}, \mathbf{b_i})$ *unmarked loops in (i)*
- $-\chi_2(\mathbf{b_i}, \mathbf{b_i})$ *marked loops in (i)*

where $\chi_1 = (\delta_{ij} - a_{ij})_{i,j}$ *and* $\chi_2 = (-\delta_{ij}m_{ii})_{i,j}$ *and* $\chi_M = \chi_1 + \chi_2$. *Moreover, the dimension vector* $\mathbf{e}_\sigma = (m_1, \ldots, m_k)$.

6. Low dimensional cases

In this section we give an alternative proof of the local classification of smooth orders in CH_n in dimension one and two and draw some conclusions.
In [11] W. Schelter proved that in dimension one smooth orders are hereditary. We will show that this also follows immediately from our local description. The result below also follows from reversing the shrinking process of local charts and the classification given before.

Proposition 6.1. *The local charts* $C = (M, \mathbf{e})$ *for a smooth order* A *in* CH_n *with center of dimension one are such that* $\mathbf{e} = (1, \ldots, 1)$ *and the map* M *has the form* $\tilde{A}_r$



Proof. Let $(M, \mathbf{e})$ be a local chart of A on r vertices. As M is strongly connected there exist oriented cycles in M. Fix, one such cycle (say of length s) and number the vertices of M such that the first s vertices are those making up the cycle. Now, assume the dimension vector is $\mathbf{e} = (e_1, \ldots, e_r)$. Then, there is a semi-simple representation in $R(M, \mathbf{e})$ with composition

$$(\underbrace{1, \ldots, 1}_{s}, \underbrace{0, \ldots, 0}_{r-s}) \oplus \delta_1^{\oplus e_1 - 1} \oplus \ldots \delta_s^{\oplus e_s - 1} \oplus \delta_{s+1}^{\oplus e_{s+1}} \oplus \ldots \oplus \delta_r^{\oplus e_r}$$

where δ_i are the standard basis of $\mathbb{Z}^r$ or equivalently the dimension vector of the simple representation concentrated at vertex (i).
There is a one-parameter family of isoclasses of such semi-simple representations. As $dim\ R(M, \mathbf{e})/GL(\mathbf{e}) = 1$, and $\mathbf{e}$ is the dimension vector of a simpe representation, this can only happen if the above semi-simple representation is actually simple. Hence, M is one cycle (that is, $r = s$) and all $e_i = 1$. □

In particular, the only representation-types that can occur re of the form $\tau = (1, d_1, \ldots, 1, d_r)$ and $GL(\tau) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ embedded in GL_n via

$$(\lambda_1, \ldots, \lambda_r) \mapsto diag(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

One verifies that

$$\mathbb{C}[R(M, \mathbf{e})/GL(\mathbf{e})] \simeq \mathbb{C}[x]$$

where x is the trace along the cycle. Further the ring of equivariant maps

$$M_n(\mathbb{C}[GL_n \times^{GL(\mathbf{e})} R(M, \mathbf{e})])^{GL_n}$$

has the following block decomposition

$$\left[\begin{array}{c|c|c|c} M_{d_1}(\mathbb{C}[x]) & M_{d_1\times d_2}(\mathbb{C}[x]) & \cdots & M_{d_1\times d_r}(\mathbb{C}[x]) \\ \hline M_{d_2\times d_1}(x\mathbb{C}[x]) & M_{d_2}(\mathbb{C}[x]) & \cdots & M_{d_2\times d_r}(\mathbb{C}[x]) \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_{d_r\times d_1}(x\mathbb{C}[x]) & M_{d_r\times d_2}(x\mathbb{C}[x]) & \cdots & M_{d_r}(\mathbb{C}[x]) \end{array}\right]$$

From this local information we deduce

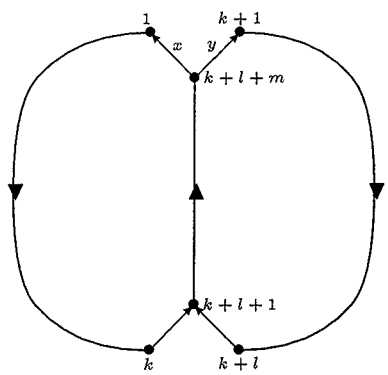
Theorem 6.2. *Equivalent are*

- *A is a smooth order in CH_n with one-dimensional center*
- *A is an hereditary order over a Dedekind domain*

Proof. From the local description of a smooth order given before we deduce that the center must be smooth, hence a Dedekind domain and that the smooth order A must be locally hereditary. Conversely, from the local description of hereditary orders as given in [10, Thm. 39.14] we deduce that hereditary orders are locally smooth. □

In dimension two we can also give an alternative proof of our local characterization of smooth orders.

Proposition 6.3. *For a smooth order A in CH_n with two-dimensional center the local charts $C = (M, \mathbf{e})$ are such that $\mathbf{e} = (1, \ldots, 1)$ and the map M has the following form :*



A_{klm}

where the indicated numbering of vertices and labeling of arrows will be used later. In this picture we make the obvious changes whenever k or l are zero.

Proof. The strongly connected map M must contain more than one oriented cycle and hence contains a submap of the indicated type (possibly degenerated). It is

easy to verify that for A_{klm}, $\mathbf{f} = (1, \ldots, 1)$ is the dimension vector of a simple representation.
If M contains additional vertices $\{s = k+l+m+1, \ldots, r\}$ and/or the dimension vector $\mathbf{e} = (e_1, \ldots, e_r) \neq \mathbf{f}$, there exist semi-simple representations in $R(M, \mathbf{e})$ with dimension-vector decomposition

$$(\underbrace{1, \ldots, 1}_{k+l+m}, 0, \ldots, 0) \oplus \delta_1^{e_1 - 1} \oplus \ldots \oplus \delta_{s-1}^{e_{s-1}-1} \oplus \delta_s^{e_s} \oplus \ldots \oplus \delta_r^{\oplus e_r}$$

As *dim* $R(A_{klm}, \mathbf{f})/GL(\mathbf{f})$ is equal to 2 there is a two-dimensional family of such semi-simple representations. Hence, they cannot be properly semi-simple as their locus must be of dimension $< d = 2$. Therefore, $M = A_{klm}$ and $\mathbf{e} = \mathbf{f}$. □

Let m be a maximal ideal of the center Z of A corresponding to a semi-simple n-dimensional A-module M_x. By the above characterization we know that M_x must have a decomposition

$$M_x = S_1 \oplus \ldots \oplus S_r$$

where S_i is a simple B-module of dimension d_i and all components are distinct. That is, $n = \sum_i d_i$ and the embedding of $GL(\mathbf{e}) = \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{r}$ in GL_n is given via

$$(\lambda_1, \ldots, \lambda_r) \mapsto diag(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

We want to describe the étale local structure of A near m, that is, the ring $A_m^{sh} = A \otimes_Z Z_m^{sh}$. In order to do this we have to compute the rings of invariants and concomitants of the local chart near the zero representation.

Proposition 6.4. *Using the labeling of vertices and arrows in the chart A_{klm} given above we have*

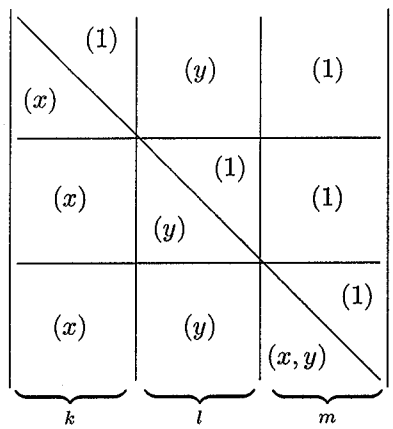
(1) *The ring of polynomial invariants is equal to*

$$\mathbb{C}[R(A_{klm}, \epsilon)/GL(\mathbf{e})] = \mathbb{C}[x, y]$$

(2) *The rings of GL_n-equivariant maps*

$$M_n(GL_n \times^{GL(\mathbf{e})} \mathbb{C}[R(A_{klm}, \epsilon])^{GL_n}$$

is isomorphic to the subring of $M_n(\mathbb{C}[x, y])$ with block decomposition



where at place (i,j) (for every $1 \leq i,j \leq r$) there is a block of dimension $d_i \times d_j$ with entries the indicated ideal of $\mathbb{C}[x,y]$.

Proof. By basechange in the vertices we see that all non-zero maps in a minimal oriented cycle can be taken to be the identity map except for one. If we define these remaining maps x and y then the traces along oriented cycles in the chart are of the form $x^i y^j$. The result about the equivariant maps follows from computing the $\mathbb{C}[x,y]$-module of paths from a vertex (i) to vertex (j) and applying the general results of the previous section. □

Using the Luna slice theorem, we obtain the required étale local classification

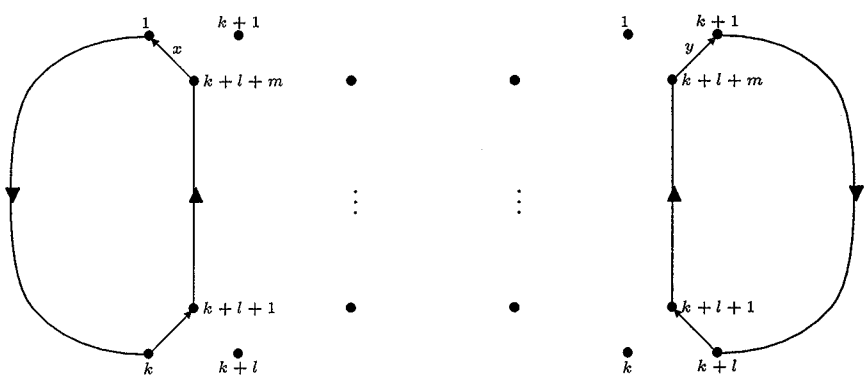
Theorem 6.5. *With notations as before, we have*
(1) $Z_m^{sh} \simeq \mathbb{C}\{x,y\}$
(2) A_m^{sh} *is isomorphic to the subring of* $M_n(\mathbb{C}\{x,y\})$ *with the above block decomposition.*

Definition 6.6. A Z-order A in a central simple algebra Δ of dimension n^2 is said to be **étale locally split** in a maximal ideal m of Z iff B_m^{sh} has ring of fractions $M_n(K \otimes_Z Z_m^{sh})$.

From the étale local description of Z and B and étale descent we deduce

Proposition 6.7. *If A is a smooth order with two-dimensional center, then*
(1) *The center Z is smooth.*
(2) *The non-Azumaya locus of A, $ram_A = Y_A - Y_{Az}$ consists at worst out of isolated (possibly embedded) points and a reduced divisor whose worst singularities are normal crossings.*
(3) *A is étale locally split at every point $m \in Y_A$.*

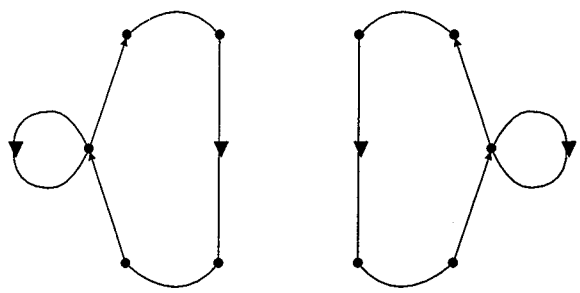
Proof. (1) and (3) are immediate from the foregoing theorem. As for (2) we have to study the local charts in proper semi-simple representations of $R(A_{klm}, \mathbf{e})$. Their decomposition into simple representations can be depicted by one of the following two situations



$V(y)$ $V(x)$

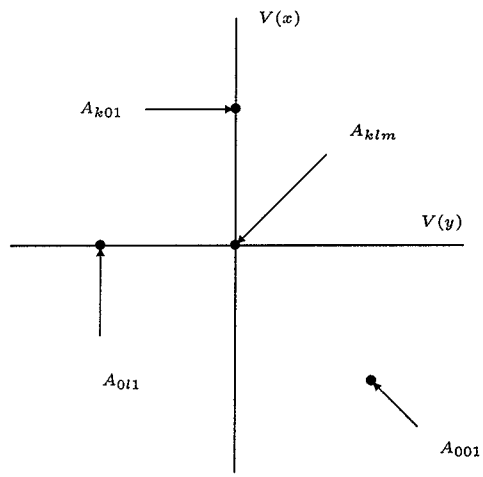
where the trace along the indicated oriented cycle is non-zero. By the general results of the foregoing section we can compute the local charts of A near such a

point. They are resp. of the following types



A_{0l1} A_{k01}

and we have the following local picture of the structure of Y_A near m.



from which the statement follows (taking care of possible degenerate cases, for example, an isolated point occurs for local charts of type A_{00m} with $m \geq 2$). □

We see that already in dimension two, smooth orders do not have to have finite global dimension.

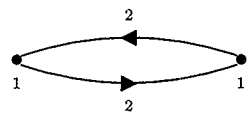
Lemma 6.8. *With notations as before, A is a projective module over Z if and only if all local charts are of type A_{kl1}. In particular, if a local chart is of type A_{klm} with $m \geq 2$, then gldim $A = \infty$.*

Proof. As the center is smooth, projectivity and reflexivity as Z-module are equivalent. Observe that A_m^{sh} is reflexive only if no block of type (x, y) occurs, that is, iff $m = 1$. The last statement follows from the fact that an order of finite global dimension with smooth center has to be projective. □

From dimension $d = 3$ on some new phenomena occur. Rather than giving a full classification, we will give examples of the differences.

Lemma 6.9. *In dimension $d \geq 3$ the center Z of a smooth order A in CH_n no longer has to be smooth.*

Proof. Consider the local chart of the following form



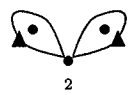
The center of the corresponding ring of equivariant maps is equal to

$$\mathbb{C}[x,y,z,v]/(xy-zv)$$

and hence is not smooth in the origin. In arbitrary dimension $d \geq 3$ this example survives by adding loops in one of the vertex spaces. □

Lemma 6.10. *In dimension $d \geq 3$ a smooth algebra A does not have to be étale locally split everywhere.*

Proof. Consider the following local chart



Then one verifies that the ring of equivariant maps is equal to

$$Cliff_{\mathbb{C}[x,y,z]} \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

the Clifford algebra of a non-degenerate quadratic form. This algebra cannot be split by an étale extension in the origin. □

More generally, we can determine the **local index** at a point m. We have

$$Q(A_m^{sh}) \simeq M_k(D)$$

for a division algebra of dimension l^2 over its center. We call l the local index at m.

Proposition 6.11. *Let A be a smooth algebra in CH_n and m a point with local chart $C = (M, \mathbf{e})$ where $\mathbf{e} = (e_1, \ldots, e_r)$. Then, the local index of A at m is equal to $gcd(e_i)$.*

Proof. This is an adaptation of the results of [7]. □

Proposition 6.12. *If A is a smooth algebra in CH_n with center of dimension d, then A can be split by étale extensions up to parts of index $\leq \sqrt{d+1}$.*

Proof. Follows from the result above and the fact that the minimal dimension of a quotient variety of a local chart with $gcd(e_i) = e$ is $e^2 - 1$, corresponding to the two marked loop map and dimension vector $\mathbf{e} = (e)$. □

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