étale cohomology in non-commutative geometry

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CHAPTER 1

Introduction

EXAMPLE 1.0.1. (The quantum $\mathbb{P}^2$)
Consider the graded algebra A_q on three generators with defining relations

$$\begin{aligned} XY &= qYX \\ YZ &= qZY \\ ZX &= qXZ \end{aligned}$$

where $q \in \mathbb{C}^*$.
If q is a primitive n-th root of unity, then the center of A_q is generated by three elements in degree n, namely X^n, Y^n and Z^n and one in degree 3 namely XYZ.
If n is not divisible by 3 this implies that after localizing at all homogeneous non-zero central elements A_q we obtain the algebra

$$Q^g(A_q) = D[t, t^{-1}]$$

where t is a central element of degree one and D is a division algebra of dimension n^2 over its center which is isomorphic to $\mathbb{C}(x, y)$.
In fact, D is the division ring of fractions of the so called 'quantum-plane'

$$\mathbb{C}_q[u, v] \qquad : \quad uv = qvu \quad u^n = x \quad v^n = y$$

EXAMPLE 1.0.2. (The Sklyanin algebra)
Consider the graded algebra S_τ on three generators with defining relations

$$\begin{aligned} cX^2 + bZY + aYZ &= 0 \\ aZX + cY^2 + bXZ &= 0 \\ bYX + aXY + cZ^2 &= 0 \end{aligned}$$

where $a, b, c \in \mathbb{C}^*$ such that the curve

$$E : (a^3 + b^3 + c^3)xyz = abc(x^3 + y^3 + z^3)$$

is a smooth elliptic curve in $\mathbb{P}^2$.
We have an automorphism on E defined by

$$(x : y : z) \mapsto (acy^2 - b^2xz : bcx^2 - a^2yz : abz^2 - c^2xy)$$

If we choose $(1 : -1 : 0) \in E$ as the origin, then this automorphism is translation by the point $\tau = (a : b : c)$ on E.
If τ is an n-torsion point on E, then the center of S_τ is again generated by three elements of degree n and one of degree 3.
Again, if we localize at all homogeneous non-zero central elements we obtain an algebra

$$Q^g(S_\tau) = D'[t, t^{-1}]$$

with t central of degree one and D' a division algebra of dimension n^2 over its center which is isomorphic to $\mathbb{C}(x, y)$.

Motivated by these examples we say that a graded affine algebra A is a **model** for a central simple $\mathbb{C}(x, y)$-algebra Δ of dimension n^2 if its graded ring of fractions

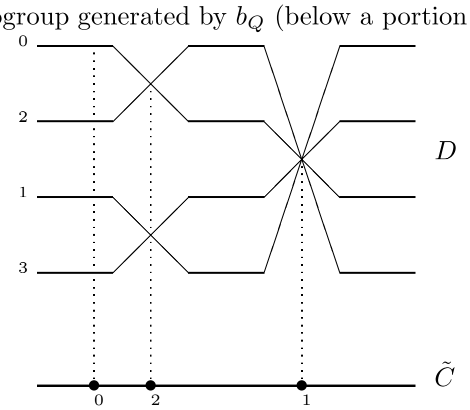
$$Q^g(A) = \Delta[t, t^{-1}]$$

for some central degree one element t.
The first question we like to answer is that of the birational classification : can we describe the $\mathbb{C}(x, y)$-isomorphism classes of central simple algebras Δ of dimension n^2? In particular, can one show that D and D' above are non-isomorphic?

1.1. $\mathbb{Z}_n$-wrinkles on $\mathbb{P}^2$

By a $\mathbb{Z}_n$**-wrinkle on** $\mathbb{P}^2$ we mean the following data-package

- A finite collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of **irreducible curves** in $\mathbb{P}^2$, that is, $C_i = V(F_i)$ for an irreducible form in $\mathbb{C}[X, Y, Z]$ of degree d_i.
- A finite collection $\mathcal{P} = \{P_1, \ldots, P_l\}$ of **points** of $\mathbb{P}^2$ where each P_i is either an intersection point of two or more C_i or a singular point of some C_i.
- For each $P \in \mathcal{P}$ the **branch-data** $b_P = (b_1, \ldots, b_{i_P})$ with $b_i \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\{1, \ldots, i_P\}$ the different branches of $\mathcal{C}$ in P. These numbers must satisfy the admissibility condition
$$\sum b_P = \sum_i b_i = 0 \in \mathbb{Z}_n \text{ for every } P \in \mathcal{P}$$
- for each $C \in \mathcal{C}$ a cyclic $\mathbb{Z}_n$-cover of smooth curves
$$D \longrightarrow\!\!\!\!\!\!\rightarrow \tilde{C}$$
of the desingularization of C which is compatible with the branch-data, that is, if $Q \in \tilde{C}$ corresponds to a C-branch in P, then D is ramified in Q with stabilizer subgroup generated by b_Q (below a portion of a $\mathbb{Z}_4$-wrinkle)



We have a grip on the covers $D \longrightarrow\!\!\!\!\!\!\rightarrow \tilde{C}$ as follows. Let $\{Q_1, \ldots, Q_z\}$ be the points of $\tilde{C}$ where the cover ramifies with branch numbers $\{b_1, \ldots, b_z\}$, then D is determined by a continuous module structure of

$$\pi_1(\tilde{C} - \{Q_1, \ldots, Q_z\}) \text{ on } \phi_D\mathbb{Z}_n$$

where the fundamental group is equal to

$$\langle u_1, v_1, \ldots, u_g, v_g, x_1, \ldots, x_z \rangle / ([u_1, v_1] \ldots [u_g, v_g] x_1 \ldots x_z)$$

with g the genus of $\tilde{C}$ and the action of x_i is determined by b_i.

EXAMPLE 1.1.1. Let us consider the first cases

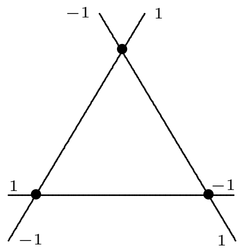
1. If $\tilde{C} = \mathbb{P}^1$ then $g = 0$ and hence $\pi_1(\mathbb{P}^1 - \{Q_1, \ldots, Q_z\}$ is zero if $z \leq 1$ (whence no covers exist) and is $\mathbb{Z}$ if $z = 2$. Hence, there exists a unique cover $D \twoheadrightarrow \mathbb{P}^1$ with branch-data $(1, -1)$ in say $(0, \infty)$ namely with D the normalization of $\mathbb{P}^1$ in $\mathbb{C}(\sqrt[n]{x})$.
2. If $\tilde{C} = E$ an elliptic curve, then $g = 1$. Hence, $\pi_1(C) = \mathbb{Z} \oplus \mathbb{Z}$ and there exist unramified $\mathbb{Z}_n$-covers. They are given by the isogenies
$$E' \twoheadrightarrow E$$
where E' is another elliptic curve and $E = E'/\langle\tau\rangle$ where τ is an n-torsion point on E'.

One can show that any such cover $D \twoheadrightarrow \tilde{C}$ is determined by a function $f \in \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$ which allows us to put a group-structure on the equivalence classes of $\mathbb{Z}_n$-wrinkles where we call a wrinkle trivial provided all coverings $D_i \twoheadrightarrow \hat{C}_i$ are trivial (that is, n copies of $\tilde{C}$).
One of the main results we will prove in these notes is the Artin-Mumford exact sequence for Brauer groups of simply connected surfaces. In the case of $\mathbb{C}(x, y)$ this result can be phrased as

THEOREM 1.1.2. *If Δ is a central simple $\mathbb{C}(x, y)$-algebra of dimension n^2, then Δ determines uniquely a $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$. Conversely, any $\mathbb{Z}_n$-wrinkle on $\mathbb{P}^2$ determines a unique division $\mathbb{C}(x, y)$- algebra whose class in the Brauer group has order n.*

EXAMPLE 1.1.3. Returning to the 'quantum'-algebras defined above

1. The division algebra D with non-commutative model the quantum $\mathbb{P}^2$ algebra A_q is determined by the wrinkle with shadow



   which completely determines the covers.
2. The division algebra D' with non-commutative model the Sklyanin algebra A_τ is determined by the wrinkle where $\mathcal{C} = \{E'\}$ where E' is the elliptic curve in $\mathbb{P}^2$ with unramified cover the isogeny $E \twoheadrightarrow E/ < \tau >= E'$.

In particular, the division algebras D and D' of dimension n^2 over $\mathbb{C}(x, y)$ cannot be isomorphic as $\mathbb{C}(x, y)$-algebras. Or, phrased differently, A_q and S_τ are not projectively birational.
Is there a non-commutative version of Hironaka's resolution of singularities for the algebras for the central simple algebras Δ of dimension n^2 over $\mathbb{C}(x, y)$?
Let A be a model for Δ generated in degree one by elements $a_1, \ldots, a_m$ and defining homogeneous equations
$$f_j(a_1, \ldots, a_n) = 0 \text{ for } 1 \leq j \leq k$$

Consider all solutions to this set of equations with $a_i \in M_n(\mathbb{C})$. They form a subvariety

$$mod_n\ A \hookrightarrow \mathbb{A}^{mn^2}$$

which is homogeneous (that is, a cone).
Observe that if A is commutative (that is, $n = 1$), $mod_n\ A = Max(A)$ and A is a smooth model if the corresponding projective variety $\mathbb{P}(mod_n\ A)$ is smooth. For this reason, we define (actually, the definition has to be modified slightly)

Definition 1.1.4. A model A for Δ is said to be **smooth** iff $proj_n\ A = \mathbb{P}(mod_n\ A) \hookrightarrow \mathbb{P}^{mn^2-1}$ is a smooth (commutative) variety.

Exercise 1.1.5. Are A_q and S_τ smooth models ? Try to compute this in the easiest case $n = 2$, that is, when $q = -1$ (for A_q) and $a = b$ (for S_τ).

We will impose restrictions on the existence of smooth models by computing their étale (or analytic) local structure.
Let A be a model with center C and let S be the projective variety defined by C. We say that $X(c) \hookrightarrow S$ is a **excellent** open subset provided

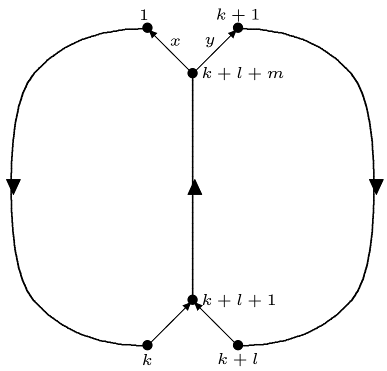
$$A_c^g = B[d, d^{-1}]$$

where d is central of degree one. If $P \in X(c)$ we will denote with m_P the corresponding maximal ideal of the center of B.

1.2. $\mathbb{Z}_n$-loops

By a $\mathbb{Z}_n$-loop we mean the following data :

- A directed graph on $k + l + m \leq n$ vertices of the form

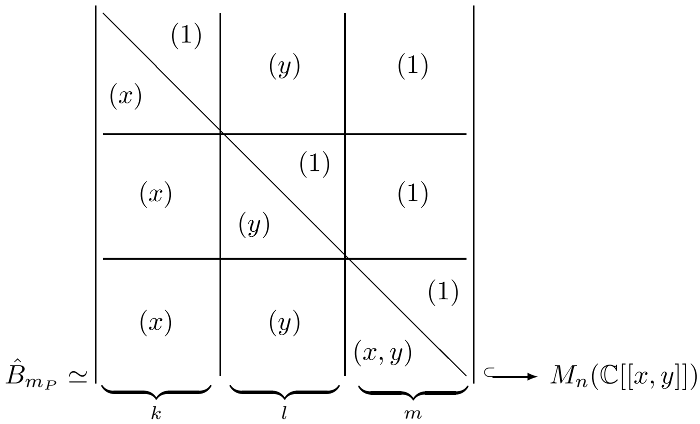


A_{klm}

  where the indicated numbering of vertices will be used later. In this picture we make the natural changes whenever k or l is zero.
- An unordered partition $\mathbf{p} = (p_1, \ldots, p_{k+l+m})$ of n with all $p_i \neq 0$

The second main application of étale machinery we will prove is the local characterization of smooth models (in arbitrary dimension). In the special case under consideration we have

THEOREM 1.2.1. *With notations as before, let A be a model and $X(c) \hookrightarrow S$ an excellent open subset. Then, A is locally on $X(c)$ a smooth model if and only if it assigns to each point $P \in X(c)$ a $\mathbb{Z}_n$-loop, say of type $(A_{klm}, \mathbf{p})$ such that*



where at place (i, j) (for every $1 \leq i, j \leq k + l + m$) there is a block of dimension $p_i \times p_j$ with entries the indicated ideal of $\mathbb{C}[[x, y]]$.

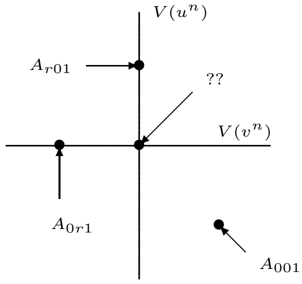
EXAMPLE 1.2.2. Consider the graded algebra A with defining relations

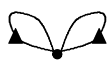
$$\begin{aligned} XY &= qYX \\ XZ &= ZX \\ YZ &= ZY \end{aligned}$$

where q is a primitive n-th root of unity. Then $X(Z)$ is an excellent open subset and

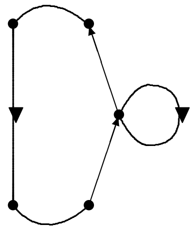
$$A_Z^g = \mathbb{C}_q[u, v][Z, Z^{-1}]$$

where $uv = qvu$. Clearly, $proj_n\ A \mid X(Z) = mod_n\ \mathbb{C}_q[u, v]$. One can verify that on $X(Z)$ the local loops are of the form





A_{001}



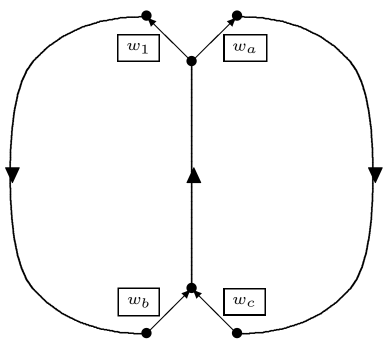
A_{r01}

where there are $r + 1 = n$ vertices in the non-trivial cycle. The corresponding partitions are (n) resp. $(1, \ldots, 1)$.
Observe that there is no $\mathbb{Z}_n$-loop in the origin. This can be seen by observing that the completion at $p = (u^n, v^n)$ is $\mathbb{C}_q[[u, v]]$ which is a division algebra and hence cannot be of the split-form corresponding to a $\mathbb{Z}_n$-loop.

We will show that a central simple $\mathbb{C}(x, y)$-algebra Δ of dimension n^2 has a model which is locally smooth on an excellent cover if and only if all the branch-data in the $\mathbb{Z}_n$- wrinkle on $\mathbb{P}^2$ determining Δ are trivial (that is, zero). Moreover, any Δ has a model with isolated singularities all of which are locally of quantum-plane type.
Hence, in order to construct smooth models in any central simple algebra Δ we have to relax the condition of having an excellent open cover.
Let A be a smooth model with central projective surface $S = Proj\ C$ (which may contain singularities), then locally around $P \in S$ A has the form

$$A_c^g = \ldots \oplus I^{-2}d^{-2} \oplus I^{-1}d^{-1} \oplus B \oplus Id \oplus I^2d^2 \oplus \ldots \hookrightarrow \Delta[d, d^{-1}]$$

with d central of degree one and I an invertible ideal of B.
This time, we will characterize the graded completion of A with respect to the graded maximal ideal m_P^g of C determining P. The underlying combinatorial object is a $\mathbb{Z}_n$-**weighted loop**, that is a $\mathbb{Z}_n$-loop $(A_{klm}, \mathbf{p})$ with



$A_{klm}^{\mathbf{w}}$

- a given **period** $e \in \mathbb{N}_+$

- a **weight** $w_\phi \in \mathbb{Z}/e\mathbb{Z}$ associated to every arrow ϕ in A_{klm}. If we take the sum of the weights along the two cycles we get numbers w_x and w_y.
- a compatible partition $\mathbf{m} = \{m_1 \leq m_2 \leq \ldots \leq m_n < e\}$ of n

THEOREM 1.2.3. *With notations as above, A is a smooth model iff to every point $P \in S$ is associated a weighted $\mathbb{Z}_n$-loop of type, say $(A_{klm}, \mathbf{m}, \mathbf{p}, \mathbf{w}, e)$ such that*

$$\hat{A}^g_{m^g_P} \simeq \left|\begin{array}{c|c|c} {\scriptstyle (x)'} \diagdown {\scriptstyle (1)'} & (y)' & (1)' \\ \hline (x)' & {\scriptstyle (y)'} \diagdown {\scriptstyle (1)'} & (1)' \\ \hline (x)' & (y)' & {\scriptstyle (x,y)'} \diagdown {\scriptstyle (1)'} \end{array}\right| \xhookrightarrow{\simeq^g} M_n(\mathbb{C}[[x,y]][t^e,t^{-e}])(m_1,\ldots,m_n)$$
$$\underbrace{\qquad\quad}_{k}\;\underbrace{\qquad\quad}_{l}\;\underbrace{\qquad\quad}_{m}$$

where at place (i,j) there is a block of dimension $p_i \times p_j$ of form $I.t^{a_{ij}}$ where a_{ij} is the minimal total weight in $\mathbb{Z}_e$ of an oriented path from i to j and I is the intersection of the indicated ideal in $\mathbb{C}[[x,y]][t^e,t^{-e}]$ with the invariant ring $\mathbb{C}[[x,y]]^{\mathbb{Z}_e}[t^e,t^{-e}]$ where the action is given by $x \mapsto \zeta^{w_x}x$ and $y \mapsto \zeta^{w_y}y$ for ζ a primitive e-th root of unity.
The graded matrix-algebra on the right has as its i-th part homogeneous component is defined to be

$$\begin{bmatrix} R_i & R_{i+a_1-a_2} & \ldots & R_{i+a_1-a_n} \\ R_{i+a_2-a_1} & R_i & \ldots & R_{i+a_2-a_n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{i+a_n-a_1} & R_{i+a_n-a_2} & \ldots & R_i \end{bmatrix}.$$

where $R_j = \mathbb{C}[[x,y]]t^j$.

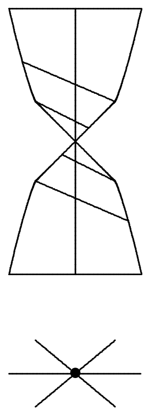
We can now construct smooth models in any Δ by blowing-up the remaining quantum-plane singularities. Let us recall the ringtheoretical interpretation of a blow-up of a point in $\mathbb{A}^2$.

EXAMPLE 1.2.4. Let $\tilde{\mathbb{A}}^2 \twoheadrightarrow \mathbb{A}^2$ be the blow-up of the origin $p = (0,0)$ in $\mathbb{A}^2$. If $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x,y]$, consider the graded algebra

$$R = \mathbb{C}[x,y] \oplus (x,y)t \oplus (x,y)^2t^2 \oplus \ldots \hookrightarrow \mathbb{C}[x,y][t]$$

Then R is generated by two elements in degree zero x, y and two in degree one $X = xt$ and $Y = yt$. The defining (homogeneous) relation of R is $xY - yX$.
Then, $\tilde{\mathbb{A}}^2 = Proj\ R$ and the projection morphism is given by the inclusion (in degree zero) $\mathbb{C}[x,y] \hookrightarrow R$. Geometrically, the blow-up can be viewed as a spiral

staircase covering $\mathbb{A}^2$ by rotating a line through $(0,0)$.



The projection map

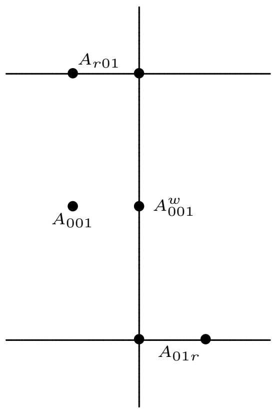
$$\tilde{\mathbb{A}^2} \twoheadrightarrow \mathbb{A}^2 \text{ where } (x,y,X:Y) \mapsto (x,y)$$

is an isomorphism on $\mathbb{A}^2 - (0,0)$ and has a $\mathbb{P}^1$ as fiber over the origin.

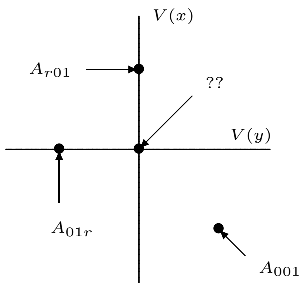
Assume we have a local quantum-plane singularity $B = \mathbb{C}_q[u,v]$ with $u^n = x$ and $v^n = y$. We consider the graded algebra

$$A = B \oplus (u,v)t \oplus (u,v)^2t^2 \oplus \dots \hookrightarrow \mathbb{C}_q(u,v)[t]$$

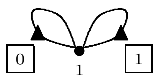
and call it the the non-commutative blow-up of a quantum-plane singularity. We will show that A is a (local) smooth model with corresponding weighted $\mathbb{Z}_n$-loops of type



projecting onto



where $r = n - 1$ and where A^w_{001} is the weighted loop



A is generated by two elements of degree zero u, v and two of degree one $U = ut$ and $V = vt$. $c = U^n = xt^n$ is central and if we localize we obtain

$$A^g_c = \mathbb{C}_q[u, v'][U, U^{-1}, \phi]$$

where $v' = VU^{-1}$ and $\phi(u) = u$, $\phi(v') = qv'$. If we consider a point P in the open set of the 'exceptional fiber' where $v' \neq 0$ (that is, $V \neq 0$), then we can adjoin an n-th root of v'^n to obtain

$$\mathbb{C}_q[u, v'] \otimes_{\mathbb{C}[u^n, v'^n]} \mathbb{C}[u^{\pm n}, v'^{\pm 1}] \simeq M_n(\mathbb{C}[u^{\pm n}, v'^{\pm 1}])$$

whence

$$\hat{A}^g_{m_P} = M_n(\mathbb{C}[[u^{\pm n}, w^{\pm 1}]][U^n, U^{-n}])(0, 1, 2, \ldots, n-1)$$

via the identifications

$$u = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & u^n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \quad v' = \begin{bmatrix} w & 0 & \ldots & 0 \\ 0 & \zeta w & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \zeta^{n-1} w \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ U^n & 0 & 0 & \ldots & 0 \end{bmatrix}$$

and we see that in P, A has weighted $\mathbb{Z}_n$-loop type A^w_{001}.

Acknowledgements These notes are based on a course given for the 'Erasmus Intensive Program Ringtheory' organized by Fred Van Oystaeyen, may 12-22 at the UIA.
Michel Van den Bergh suggested using the coniveau spectral sequence to prove the Artin-Mumford sequence.
As always, my research is supported by a grant from the Foundation for Scientific Research (F.W.O.).

Part 1

Birational classification

CHAPTER 2

The étale site of an algebra

Throughout, A will be a commutative $\mathbb{C}$-algebra. The algebra equivalent of a finite separable field extension is that of an étale morphism.

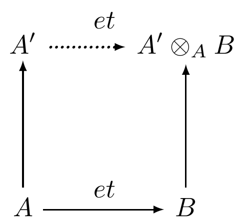
2.1. Etale morphisms

DEFINITION 2.1.1. A finite morphism $A \xrightarrow{f} B$ of commutative $\mathbb{C}$-algebras is said to be **étale** if and only if

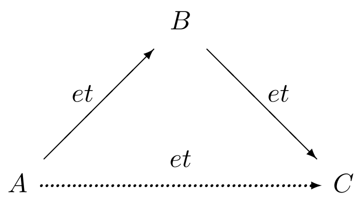
$$B = A[t_1, \ldots, t_k]/(f_1, \ldots, f_k) \text{ such that } det\ (\frac{\partial f_i}{\partial x_j})_{i,j} \in B^*$$

PROPOSITION 2.1.2. *Etale morphisms satisfy 'sorite', that is*

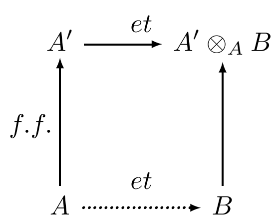
1. *(basechange)*



2. *(composition)*



3. *(descent)*



4. *(morphisms)*

$$\begin{array}{ccc} & A & \\ {}^{et}\swarrow & & \searrow^{et} \\ B & \overset{et}{\underset{A-alg}{\cdots\cdots\rightarrow}} & B' \end{array}$$

Definition 2.1.3. The étale site of A, which we will denote by A_{et} is the category with

- objects : the étale extensions $A \xrightarrow{f} B$ of A
- morphisms : compatible A-algebra morphisms

$$\begin{array}{ccc} & A & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ B_1 & \xrightarrow{\phi} & B_2 \end{array}$$

Observe that by the foregoing proposition all morphisms in A_{et} are étale. We can put on A_{et} a (Grothendieck) topology by defining

- cover : a collection $\mathcal{C} = \{B \xrightarrow{f_i} B_i\}$ in A_{et} such that

$$Spec\ B = \cup_i\ Im\ (Spec\ B_i \xrightarrow{f} Spec\ B\ )$$

2.2. Etale sheaves

An étale presheaf of groups on A_{et} is a functor

$$\mathbf{G} : A_{et} \longrightarrow \textbf{\textit{Groups}}$$

In analogy with usual (pre)sheaf notation we denote for each

- object $B \in A_{et}$: $\Gamma(B, \mathbf{G}) = \mathbf{G}(B)$
- morphism $B \xrightarrow{\phi} C$ in A_{et} : $Res^B_C = \mathbf{G}(\phi) : \mathbf{G}(B) \longrightarrow \mathbf{G}(C)$ and $g \mid C = \mathbf{G}(\phi)(g)$.

A presheaf $\mathbf{G}$ is a sheaf provided for every $B \in A_{et}$ and every cover $\{B \longrightarrow B_i\}$ we have exactness of the equalizer diagram

$$0 \longrightarrow \mathbf{G}(B) \longrightarrow \prod_i \mathbf{G}(B_i) \rightrightarrows \prod_{i,j} \mathbf{G}(B_i \otimes_B B_j)$$

Example 2.2.1. **Constant sheaf :** If G is a group, then

$$\mathbf{G} : A_{et} \longrightarrow \mathbf{Groups} \quad \mathbf{B} \mapsto \mathbf{G}^{\oplus \pi_0(\mathbf{B})}$$

is a sheaf where $\pi_0(B)$ is the number of connected components of $Spec\ B$.

Example 2.2.2. **Multiplicative group $\mathbb{G}_m$:** The functor

$$\mathbb{G}_m : A_{et} \longrightarrow \mathbf{Ab} \quad \mathbf{B} \mapsto \mathbf{B}^*$$

is a sheaf on A_{et}.

A sequence of sheaves of Abelian groups on A_{et} is said to be exact

$$\mathbf{G}' \xrightarrow{f} \mathbf{G} \xrightarrow{g} \mathbf{G}"$$

if for every $B \in A_{et}$ and $s \in \mathbf{G}(B)$ such that $g(s) = 0 \in \mathbf{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in A_{et} and sections $t_i \in \mathbf{G}'(B_i)$ such that $f(t_i) = s \mid B_i$.

EXAMPLE 2.2.3. **Roots of unity $\boldsymbol{\mu_n}$:** We have a sheaf morphism

$$\mathbb{G}_{\boldsymbol{m}} \xrightarrow{(-)^n} \mathbb{G}_{\boldsymbol{m}}$$

and we denote the kernel with $\boldsymbol{\mu_n}$. As A is a $\mathbb{C}$-algebra we can identify $\boldsymbol{\mu_n}$ with the constant sheaf $\mathbb{Z}_{\mathbf{n}} = \mathbb{Z}/\mathbf{n}\mathbb{Z}$ via the isomorphism $\zeta^i \mapsto i$ after choosing a primitive n-th root of unity $\zeta \in \mathbb{C}$.

LEMMA 2.2.4. *The (Kummer)-sequence of sheaves of Abelian groups*

$$0 \longrightarrow \boldsymbol{\mu_n} \longrightarrow \mathbb{G}_{\boldsymbol{m}} \xrightarrow{(-)^n} \mathbb{G}_{\boldsymbol{m}} \longrightarrow 0$$

is exact on A_{et} (but not necessarily on A_{Zar}).

PROOF. We only need to verify surjectivity. Let $B \in A_{et}$ and $b \in \mathbb{G}_{\boldsymbol{m}}(B) = B^*$. Consider the étale extension $B' = B[t]/(t^n - b)$ of B, then b has an n-th root over in $\mathbb{G}_{\boldsymbol{m}}(B')$. Observe that this n-th root does not have to belong to $\mathbb{G}_{\boldsymbol{m}}(B)$. □

2.3. Derived functors

Before we define cohomology of sheaves on A_{et} let us recall the definition of derived functors. Let $\mathcal{A}$ be an Abelian category. An object I of $\mathcal{A}$ is said to be injective if the functor

$$\mathcal{A} \longrightarrow \mathbf{Ab} \quad M \mapsto Hom_{\mathcal{A}}(M, I)$$

is exact. We say that $\mathcal{A}$ has enough injectives if, for every object M in $\mathcal{A}$, there is a monomorphism $M \hookrightarrow I$ into an injective object.
If $\mathcal{A}$ has enough injectives and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor from $\mathcal{A}$ into a second Abelian category $\mathcal{B}$, then there is an essentially unique sequence of functors

$$R^i\, f : \mathcal{A} \longrightarrow \mathcal{B} \quad i \geq 0$$

called the right derived functors of f having the following properties

- $R^0\, f = f$
- $R^i\, I = 0$ for I injective and $i > 0$
- For every short exact sequence in $\mathcal{A}$

  $$0 \longrightarrow M' \longrightarrow M \longrightarrow M" \longrightarrow 0$$

  there are connecting morphisms $\delta^i : R^i\, f(M") \longrightarrow R^{i+1}\, f(M')$ for $i \geq 0$ such that we have a long exact sequence

$$\ldots \longrightarrow R^i\, f(M) \longrightarrow R^i\, f(M") \xrightarrow{\delta^i} R^{i+1}\, f(M') \longrightarrow R^{i+1}\, f(M) \longrightarrow \ldots$$

- For any morphism $M \longrightarrow N$ there are morphisms $R^i\, f(M) \longrightarrow R^i\, f(N)$ for $i \geq 0$

In order to compute the objects $R^i\ f(M)$ define an object N in $\mathcal{A}$ to be f-acyclic if $R^i\ f(M) = 0$ for all $i > 0$. If we have a resolution of M

$$0 \longrightarrow M \longrightarrow N_0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \ldots$$

by f-acyclic object N_i, then the objects $R^i\ f(M)$ are canonically isomorphic to the cohomology objects of the complex

$$0 \longrightarrow f(N_0) \longrightarrow f(N_1) \longrightarrow f(N_2) \longrightarrow \ldots$$

One can show that all injectives are f-acyclic and hence that derived objects of M can be computed from an injective resolution of M.

2.4. Etale cohomology

Now, let $\mathbf{S}^{ab}(A_{et})$ be the category of all sheaves of Abelian groups on A_{et}. This is an Abelian category having enough injectives whence we can form right derived functors of left exact functors. In particular, consider the global section functor

$$\Gamma : \mathbf{S}^{ab}(A_{et}) \longrightarrow \mathbf{Ab} \quad \mathbf{G} \mapsto \mathbf{G}(\mathbf{A})$$

which is left exact. The right derived functors of Γ will be called the étale cohomology functors and we denote

$$R^i\ \Gamma(\mathbf{G}) = H^i_{et}(A, \mathbf{G})$$

In particular, if we have an exact sequence of sheaves of Abelian groups $0 \longrightarrow \mathbf{G}' \longrightarrow \mathbf{G} \longrightarrow \mathbf{G}" \longrightarrow 0$, then we have a long exact cohomology sequence

$$\ldots \longrightarrow H^i_{et}(A, \mathbf{G}) \longrightarrow H^i_{et}(A, \mathbf{G}") \longrightarrow H^{i+1}_{et}(A, \mathbf{G}') \longrightarrow \ldots$$

If $\mathbf{G}$ is a sheaf of non-Abelian groups (written multiplicatively), we cannot define cohomology groups. Still, one can define a pointed set $H^1_{et}(A, \mathbf{G})$ as follows. Take an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of A and define a 1-cocycle for $\mathcal{C}$ with values in $\mathbf{G}$ to be a family

$$g_{ij} \in \mathbf{G}(A_{ij}) \text{ with } A_{ij} = A_i \otimes_A A_j$$

satisfying the cocycle condition

$$(g_{ij} \mid A_{ijk})(g_{jk} \mid A_{ijk}) = (g_{ik} \mid A_{ijk})$$

where $A_{ijk} = A_i \otimes_A A_j \otimes_A A_k$.
Two cocycles g and g' for $\mathcal{C}$ are said to be cohomologous if there is a family $h_i \in \mathbf{G}(A_i)$ such that for all $i, j \in I$ we have

$$g'_{ij} = (h_i \mid A_{ij}) g_{ij} (h_j \mid A_{ij})^{-1}$$

This is an equivalence relation and the set of cohomology classes is written as $H^1_{et}(\mathcal{C}, \mathbf{G})$. It is a pointed set having as its distinguished element the cohomology class of $g_{ij} = 1 \in \mathbf{G}(A_{ij})$ for all $i, j \in I$.
We then define the non-Abelian first cohomology pointed set as

$$H^1_{et}(A, \mathbf{G}) = \varinjlim\ H^1_{et}(\mathcal{C}, \mathbf{G})$$

where the limit is taken over all étale coverings of A. It coincides with the previous definition in case $\mathbf{G}$ is Abelian.
A sequence $1 \longrightarrow \mathbf{G}' \longrightarrow \mathbf{G} \longrightarrow \mathbf{G}" \longrightarrow 1$ of sheaves of groups on A_{et} is said to be exact if for every $B \in A_{et}$ we have

- $\mathbf{G}'(B) = Ker\ \mathbf{G}(B) \longrightarrow \mathbf{G}"(B)$

- For every $g" \in \mathbf{G}"(B)$ there is a cover $\{B \longrightarrow B_i\}$ in A_{et} and sections $g_i \in \mathbf{G}(B_i)$ such that g_i maps to $g" \mid B$.

PROPOSITION 2.4.1. *For an exact sequence of groups on A_{et}*

$$1 \longrightarrow \mathbf{G}' \longrightarrow \mathbf{G} \longrightarrow \mathbf{G}" \longrightarrow 1$$

there is associated an exact sequence of pointed sets

$$1 \longrightarrow \mathbf{G}'(A) \longrightarrow \mathbf{G}(A) \longrightarrow \mathbf{G}"(A) \xrightarrow{\delta} H^1_{et}(A, \mathbf{G}') \longrightarrow$$

$$\longrightarrow H^1_{et}(A, \mathbf{G}) \longrightarrow H^1_{et}(A, \mathbf{G}") \dashrightarrow H^2_{et}(A, \mathbf{G}')$$

where the last map exists when $\mathbf{G}'$ is contained in the center of $\mathbf{G}$ (and therefore is Abelian whence H^2 is defined).

PROOF. The connecting map δ is defined as follows. Let $g" \in \mathbf{G}"(A)$ and let $\mathcal{C} = \{A \longrightarrow A_i\}$ be an étale covering of A such that there are $g_i \in \mathbf{G}(A_i)$ that map to $g \mid A_i$ under the map $\mathbf{G}(A_i) \longrightarrow \mathbf{G}"(A_i)$. Then, $\delta(g)$ is the class determined by the one cocycle

$$g_{ij} = (g_i \mid A_{ij})^{-1}(g_j \mid A_{ij})$$

with values in $\mathbf{G}'$. The last map can be defined in a similar manner, the other maps are natural and one verifies exactness. □

The main applications of this non-Abelian cohomology for non-commutative algebra is as follows. Let Λ be a not necessarily commutative A-algebra and M an A-module. Consider the sheaves of groups $\mathbf{Aut}(\Lambda)$ resp. $\mathbf{Aut}(M)$ on A_{et} associated to the presheaves

$$B \mapsto Aut_{B-alg}(\Lambda \otimes_A B) \text{ resp. } B \mapsto Aut_{B-mod}(M \otimes_A B)$$

for all $B \in A_{et}$. A twisted form of Λ (resp. M) is an A-algebra Λ' (resp. an A-module M') such that there is an étale cover $\mathcal{C} = \{A \longrightarrow A_i\}$ of A such that there are isomorphisms

$$\Lambda \otimes_A A_i \xrightarrow{\phi_i} \Lambda' \otimes_A A_i \text{ resp. } M \otimes_A A_i \xrightarrow{\psi_i} M' \otimes_A A_i$$

of A_i-algebras (resp. A_i-modules). The set of A-algebra isomorphism classes (resp. A-module isomorphism classes) of twisted forms of Λ (resp. M) is denoted by $Tw_A(\Lambda)$ (resp. $Tw_A(M)$). To a twisted form Λ' one associates a cocycle on $\mathcal{C}$

$$\alpha_{\Lambda'} = \alpha_{ij} = \phi_i^{-1} \circ \phi_j$$

with values in $\mathbf{Aut}(\Lambda)$. Moreover, one verifies that two twisted forms are isomorphic as A-algebras if their cocycles are cohomologous. That is, there is an embedding

$$Tw_A(\Lambda) \hookrightarrow H^1_{et}(A, \mathbf{Aut}(\Lambda)) \text{ and similarly } Tw_A(M) \hookrightarrow H^1_{et}(A, \mathbf{Aut}(M))$$

In favorable situations one can even show bijectivity. In particular, this is the case if the automorphisms group is a smooth affine algebraic group-scheme.

2.5. Stalks and Henselizations

If $\mathfrak{p}$ is a prime ideal of A we will denote with $\mathbf{k}_{\mathfrak{p}}$ the algebraic closure of the field of fractions of $A/\mathfrak{p}$. An étale neighborhood of $\mathfrak{p}$ is an étale extension $B \in A_{et}$ such that the diagram below is commutative

$$\begin{array}{ccc} A & \xrightarrow{nat} & \mathbf{k}_{\mathfrak{p}} \\ {\scriptstyle et}\downarrow & \nearrow & \\ B & & \end{array}$$

The analogue of the localization $A_{\mathfrak{p}}$ for the étale topology is the strict Henselization

$$A_{\mathfrak{p}}^{sh} = \underset{lim}{\longrightarrow} B$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$.

Recall that a local algebra L with maximal ideal m and residue map π : $L \twoheadrightarrow L/m = k$ is said to be Henselian if the following condition holds. Let $f \in L[t]$ be a monic polynomial such that $\pi(f)$ factors as $g_0.h_0$ in $k[t]$, then f factors as $g.h$ with $\pi(g) = g_0$ and $\pi(h) = h_0$. If L is Henselian then tensoring with k induces an equivalence of categories between the étale A-algebras and the étale k-algebras.

An Henselian local algebra is said to be strict Henselian if and only if its residue field is algebraically closed. Thus, a strict Henselian ring has no proper finite étale extensions and can be viewed as a local algebra for the étale topology.

Example 2.5.1. Consider the local algebra of $\mathbb{C}[x_1, \ldots, x_d]$ in the maximal ideal $(x_1, \ldots, x_d)$, then the Henselization and strict Henselization are both equal to

$$\mathbb{C}\{x_1, \ldots, x_d\}$$

the ring of algebraic functions. That is, the subalgebra of $\mathbb{C}[[x_1, \ldots, x_d]]$ of formal power-series consisting of those series $\phi(x_1, \ldots, x_d)$ which are algebraically dependent on the coordiate functions x_i over $\mathbb{C}$. In other words, those ϕ forw which there exists a non-zero polynomial $f(x_i, y) \in \mathbb{C}[x_1, \ldots, x_d, y]$ with $f(x_1, \ldots, x_d, \phi(x_1, \ldots, x_d)) = 0$.

These algebraic functions may be defined implicitly by polynomial equations. Consider a system of equations

$$f_i(x_1, \ldots, x_d; y_1, \ldots, y_m) = 0 \text{ for } f_i \in \mathbb{C}[x_i, y_j] \text{ and } 1 \leq i \leq m$$

Suppose there is a solution in $\mathbb{C}$ with

$$x_i = 0 \text{ and } y_j = y_j^o$$

such that the Jacobian matrix is non-zero

$$det\ (\frac{\partial f_i}{\partial y_j}(0, \ldots, 0; y_1^o, \ldots, y_m^0)) \neq 0$$

Then, the system can be solved uniquely for power series $y_j(x_1, \ldots, x_d)$ with $y_j(0, \ldots, 0) = y_j^o$ by solving inductively for the coefficients of the series. One can show that such implicitly defined series $y_j(x_1, \ldots, x_d)$ are algebraic functions and that, conversely, any algebraic function can be obtained in this way.

If $\mathbf{G}$ is a sheaf on A_{et} and $\mathfrak{p}$ is a prime ideal of A, we define the stalk of $\mathbf{G}$ in $\mathfrak{p}$ to be

$$\mathbf{G}_{\mathfrak{p}} = \varinjlim \mathbf{G}(B)$$

where the limit is taken over all étale neighborhoods of $\mathfrak{p}$. One can verify mono- epi- or isomorphisms of sheaves by checking it in all the stalks.
If A is an affine algebra defined over an algebraically closed field, then it suffices to verify in the maximal ideals of A.

CHAPTER 3

Central simple algebras and cohomology

In this chapter we will use étale cohomology to begin the study of central simple algebras of dimension n^2 over functionfields.

3.1. The étale site of a field

Let K be a field of characteristic zero, choose an algebraic closure $\mathbb{K}$ with absolute Galois group $G_K = Gal(\mathbb{K}/K)$.

LEMMA 3.1.1. *The following are equivalent*

1. $K \longrightarrow A$ *is étale*
2. $A \otimes_K \mathbb{K} \simeq \mathbb{K} \times \ldots \times \mathbb{K}$
3. $A = \prod L_i$ *where* L_i/K *is a finite field extension*

PROOF. Assume (1), then $A = K[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ where f_i have invertible Jacobian matrix. Then $A \otimes \mathbb{K}$ is a smooth algebra (hence reduced) of dimension 0 so (2) holds.
Assume (2), then

$$Hom_{K-alg}(A, \mathbb{K}) \simeq Hom_{\mathbb{K}-alg}(A \otimes \mathbb{K}, \mathbb{K})$$

has $dim_{\mathbb{K}}(A \otimes \mathbb{K})$ elements. On the other hand we have by the Chinese remainder theorem that

$$A/Jac\ A = \prod_i L_i$$

with L_i a finite field extension of K. However,

$$dim_{\mathbb{K}}(A \otimes \mathbb{K}) = \sum_i dim_K(L_i) = dim_K(A/Jac\ A) \leq dim_K(A)$$

and as both ends are equal A is reduced and hence $A = \prod_i L_i$ whence (3).
Assume (3), then each $L_i = K[x_i]/(f_i)$ with $\partial f_i / \partial x_i$ invertible in L_i. But then $A = \prod L_i$ is étale over K whence (1). □

To each finite étale extension $A = \prod L_i$ we can associate the finite set $rts(A) = Hom_{K-alg}(A, \mathbb{K})$ on which the Galois group G_K acts via a finite quotient group. If we write $A = K[t]/(f)$, then $rts(A)$ is the set of roots in $\mathbb{K}$ of the polynomial f with obvious action by G_K. Galois theory, in the interpretation of Grothendieck can now be stated as

PROPOSITION 3.1.2. *The functor*

$$K_{et} \xrightarrow{rts(-)} \textit{finite } G_K - \textit{sets}$$

is an anti-equivalence of categories.

We will now give a similar interpretation of the Abelian sheaves on K_{et}. Let $\mathbf{G}$ be a presheaf on K_{et}. Define

$$M_{\mathbf{G}} = \underrightarrow{lim} \ \mathbf{G}(L)$$

where the limit is taken over all subfields $L \hookrightarrow \mathbb{K}$ that are finite over K. The Galois group G_K acts on $\mathbf{G}(L)$ on the left through its action on L whenever L/K is Galois. Hence, G_K acts an $M_{\mathbf{G}}$ and $M_{\mathbf{G}} = \cup M_{\mathbf{G}}^H$ where H runs through the open subgroups of G_K whence $M_{\mathbf{G}}$ is a continuous G_K-module.
Conversely, given a continuous G_K-module M we can define a presheaf $\mathbf{G}_M$ on K_{et} such that

- $\mathbf{G}_M(L) = M^H$ where $H = G_L = Gal(\mathbb{K}/L)$.
- $\mathbf{G}_M(\prod L_i) = \prod \mathbf{G}_M(L_i)$.

One verifies that $\mathbf{G}_M$ is a sheaf of Abelian groups on K_{et}.

Theorem 3.1.3. *There is an equivalence of categories*

$$\mathbf{S}(K_{et}) \rightleftarrows G_K - \mathbf{mod}$$

induced by the correspondences $\mathbf{G} \mapsto M_{\mathbf{G}}$ *and* $M \mapsto \mathbf{G}_M$*.*

Proof. A G_K-morphism $M \longrightarrow M'$ induces a morphism of sheaves $\mathbf{G}_M \longrightarrow \mathbf{G}_{M'}$. Conversely, if H is an open subgroup of G_K with $L = \mathbb{K}^H$, then if $\mathbf{G} \xrightarrow{\phi} \mathbf{G}'$ is a sheafmorphism, $\phi(L) : \mathbf{G}(L) \longrightarrow \mathbf{G}'(L)$ commutes with the action of G_K by functoriality of ϕ. Therefore, $\underrightarrow{lim} \ \phi(L)$ is a G_K-morphism $M_{\mathbf{G}} \longrightarrow M_{\mathbf{G}'}$.
One verifies easily that $Hom_{G_K}(M, M') \longrightarrow Hom(\mathbf{G}_M, \mathbf{G}_{M'})$ is an isomorphism and that the canonical map $\mathbf{G} \longrightarrow \mathbf{G}_{M_{\mathbf{G}}}$ is an isomorphism. □

In particular, we have that $\mathbf{G}(K) = \mathbf{G}(\mathbb{K})^{G_K}$ for every sheaf $\mathbf{G}$ of Abelian groups on K_{et} and where $\mathbf{G}(\mathbb{K}) = M_{\mathbf{G}}$. Hence, the right derived functors of Γ and $(-)^G$ coincide for Abelian sheaves.
The category $G_K - \mathbf{mod}$ of continuous G_K-modules is Abelian having enough injectives. Therefore, the left exact functor

$$(-)^G : G_K - \mathbf{mod} \longrightarrow \mathbf{Ab}$$

admits right derived functors. They are called the Galois cohomology groups and denoted

$$R^i \ M^G = H^i(G_K, M)$$

Therefore, we have.

Proposition 3.1.4. *For any sheaf of Abelian groups* $\mathbf{G}$ *on* K_{et} *we have a group isomorphism*

$$H^i_{et}(K, \mathbf{G}) \simeq H^i(G_K, \mathbf{G}(\mathbb{K}))$$

Therefore, étale cohomology is a natural extension of Galois cohomology to arbitrary algebras.

3.2. Central simple algebras

The following definition-characterization of central simple algebras is classical

PROPOSITION 3.2.1. *Let A be a finite dimensional K-algebra. The following are equivalent :*

1. *A has no proper twosided ideals and the center of A is K.*
2. *$A_{\mathbb{K}} = A \otimes_K \mathbb{K} \simeq M_n(\mathbb{K})$ for some n.*
3. *$A_L = A \otimes_K L \simeq M_n(L)$ for some n and some finite Galois extension L/K.*
4. *$A \simeq M_k(\Delta)$ for some k where D is a division algebra of dimension l^2 with center K.*

The last part of this result suggests the following definition. Call two central simple algebras A and A' equivalent if and only if $A \simeq M_k(\Delta)$ and $A' \simeq M_l(\Delta)$ with Δ a division algebra. From the second characterization it follows that the tensorproduct of two central simple K-algebras is again central simple. Therefore, we can equip the set of equivalence classes of central simple algebras with a product induced from the tensorproduct. This product has the class $[K]$ as unit element and $[\Delta]^{-1} = [\Delta^{opp}]$, the opposite algebra as $\Delta \otimes_K \Delta^{opp} \simeq End_K(\Delta) = M_{l^2}(K)$. This group is called the **Brauer group** and is denoted $Br(K)$. We will quickly recall its cohomological description, all of which is classical.
GL_r is an affine smooth algebraic group defined over K and is the automorphism group of a vectorspace of dimension r. It defines a sheaf of groups on K_{et} that we will denote by $\mathbf{GL_r}$. Using the general results on twisted forms of the foregoing chapter we have

LEMMA 3.2.2.

$$H^1_{et}(K, \mathbf{GL_r}) = H^1(G_K, GL_r(\mathbb{K})) = 0$$

In particular, we have 'Hilbert's theorem 90'

$$H^1_{et}(K, \mathbb{G}_m) = H^1(G_K, \mathbb{K}^*) = 0$$

PROOF. The cohomology group classifies K-module isomorphism classes of twisted forms of r-dimensional vectorspaces over K. There is just one such class. □

PGL_n is an affine smooth algebraic group defined over K and it is the automorphism group of the K-algebra $M_n(K)$. It defines a sheaf of groups on K_{et} denoted by $\mathbf{PGL_n}$. By the proposition we know that any central simple K-algebra Δ of dimension n^2 is a twisted form of $M_n(K)$. Therefore,

LEMMA 3.2.3. *The pointed set of K-algebra isomorphism classes of central simple algebras of dimension n^2 over K coincides with the cohomology set*

$$H^1_{et}(K, \mathbf{PGL_n}) = H^1(G_K, PGL_n(\mathbb{K}))$$

THEOREM 3.2.4. *There is a natural inclusion*

$$H^1_{et}(K, \mathbf{PGL_n}) \hookrightarrow H^2_{et}(K, \boldsymbol{\mu_n}) = Br_n(K)$$

where $Br_n(K)$ is the n-torsion part of the Brauer group of K. Moreover,

$$Br(K) = H^2_{et}(K, \mathbb{G}_m)$$

is a torsion group.

PROOF. Consider the exact commutative diagram of sheaves of groups on K_{et}

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \boldsymbol{\mu_n} & \longrightarrow & \mathbb{G}_m & \xrightarrow{(-)^n} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{SL_n} & \longrightarrow & \mathbf{GL_n} & \xrightarrow{det} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{PGL_n} & = & \mathbf{PGL_n} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Taking cohomology of the second exact sequence we obtain

$$GL_n(K) \xrightarrow{det} K^* \longrightarrow H^1_{et}(K, \mathbf{SL_n}) \longrightarrow H^1_{et}(K, \mathbf{GL_n})$$

where the first map is surjective and the last term is zero, whence

$$H^1_{et}(K, \mathbf{SL_n}) = 0$$

Taking cohomology of the first vertical exact sequence we get

$$H^1_{et}(K, \mathbf{SL_n}) \longrightarrow H^1_{et}(K, \mathbf{PGL_n}) \longrightarrow H^2_{et}(K, \boldsymbol{\mu_n})$$

from which the first claim follows.
As for the second, taking cohomology of the first exact sequence we get

$$H^1_{et}(K, \mathbb{G}_m) \longrightarrow H^2_{et}(K, \boldsymbol{\mu_n}) \longrightarrow H^2_{et}(K, \mathbb{G}_m) \xrightarrow{n.} H^2_{et}(K, \mathbb{G}_m)$$

By Hilbert 90, the first term vanishes and hence $H^2_{et}(K, \boldsymbol{\mu_n})$ is equal to the n-torsion of the group

$$H^2_{et}(K, \mathbb{G}_m) = H^2(G_K, \mathbb{K}^*) = Br(K)$$

where the last equality follows from the crossed product result. □

So far, the field K was arbitrary. If K is of trancendence degree d, this will put restrictions on the 'size' of the Galois group G_K. In particular this will enable us to show that $H^i(G_K, \boldsymbol{\mu_n}) = 0$ for $i > d$. Before we can prove this we need to refresh our memory on spectral sequences.

3.3. Spectral sequences

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and consider left exact functors

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

Let the functors be such that f maps injectives of $\mathcal{A}$ to g-acyclic objects in $\mathcal{B}$, that is $R^i\ g(f\ I) = 0$ for all $i > 0$. Then, there are connections between the objects

$$R^p\ g(R^q\ f(A)) \text{ and } R^n\ gf(A)$$

for all objects $A \in \mathcal{A}$. These connections can be summarized by giving a spectral sequence

THEOREM 3.3.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories with $\mathcal{A}, \mathcal{B}$ having enough injectives and left exact functors*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

such that f takes injectives to g-acyclics.
Then, for any object $A \in \mathcal{A}$ there is a spectral sequence

$$E_2^{p,q} = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$$

In particular, there is an exact sequence

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow \dots$$

Moreover, if f is an exact functor, then we have

$$R^p\ gf(A) \simeq R^p\ g(f(A))$$

A spectral sequence $E_2^{p,q} \Longrightarrow E^n$ (or $E_1^{p,q} \Longrightarrow E^n$) consists of the following data

1. A family of objects $E_r^{p,q}$ in an Abelian category for $p, q, r \in \mathbb{Z}$ such that $p, q \geq 0$ and $r \geq 2$ (or $r \geq 1$).
2. A family of morphisms in the Abelian category

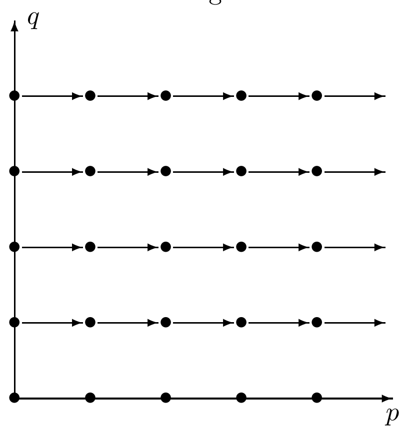
$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

satisfying the complex condition

$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$$

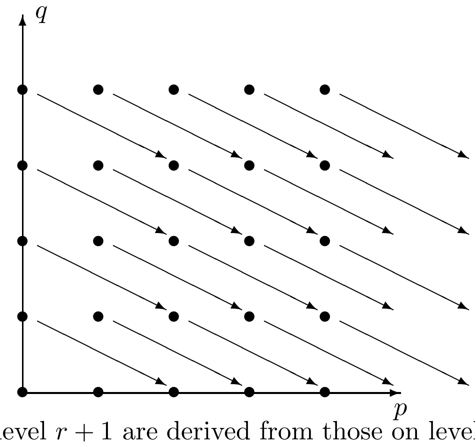
and where we assume that $d_r^{p,q} = 0$ if any of the numbers $p, q, p+r$ or $q-r+1$ is < 1. At level one we have the following

$E_1^{p,q} \quad =$



At level two we have the following

$E_2^{p,q}$ =



3. The objects $E_{r+1}^{p,q}$ on level $r+1$ are derived from those on level r by taking the cohomology objects of the complexes, that is,

$$E_{r+1}^{p} = Ker\ d_r^{p,q}\ /\ Im\ d_r^{p-r,q+r-1}$$

At each place (p,q) this process converges as there is an integer r_0 depending on (p,q) such that for all $r \geq r_0$ we have $d_r^{p.q} = 0 = d_r^{p-r,q+r-1}$. We then define

$$E_\infty^{p,q} = E_{r_0}^{p,q} (= E_{r_0+1}^{p,q} = \ldots)$$

Observe that there are injective maps $E_\infty^{0,q} \hookrightarrow E_2^{0,q}$.

4. A family of objects E^n for integers $n \geq 0$ and for each we have a filtration

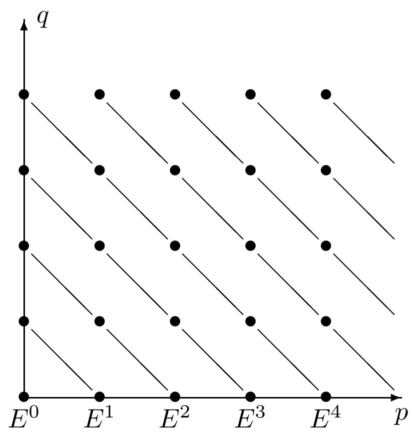
$$0 \subset E_n^n \subset E_{n-1}^n \subset \ldots \subset E_1^n \subset E_0^n = E^n$$

such that the successive quotients are given by

$$E_p^n\ /\ E_{p+1}^n = E_\infty^{p,n-p}$$

That is, the terms $E_\infty^{p,q}$ are the composition terms of the limiting terms E^{p+q}. Pictorially,

$E_\infty^{p,q}$ =



For small n one can make the relation between E^n and the terms $E_2^{p,q}$ explicit. First note that

$$E_2^{0,0} = E_\infty^{0,0} = E^0$$

Also, $E^1_1 = E^{1,0}_\infty = E^{1,0}_2$ and $E^1/E^1_1 = E^{0,1}_\infty = Ker\ d^{0,1}_2$. This gives an exact sequence

$$0 \longrightarrow E^{1,0}_2 \longrightarrow E^1 \longrightarrow E^{0,1}_2 \xrightarrow{d^{0,1}_2} E^{2,0}_2$$

Further, $E^2 \supset E^2_1 \supset E^2_2$ where

$$E^2_2 = E^{2,0}_\infty = E^{2,0}_2 \ /\ Im\ d^{0,1}_2$$

and $E^2_1/E^2_2 = E^{1,1}_\infty = Ker\ d^{1,1}_2$ whence we can extend the above sequence to

$$\ldots \longrightarrow E^{0,1}_2 \xrightarrow{d^{0,1}_2} E^{2,0}_2 \longrightarrow E^2_1 \longrightarrow E^{1,1}_2 \xrightarrow{d^{1,1}_2} E^{3,0}_2$$

as $E^2/E^2_1 = E^{0,2}_\infty \hookrightarrow E^{0,2}_2$ we have that $E^2_1 = Ker\ (E^2 \longrightarrow E^{0,2}_2)$. If we specialize to the spectral sequence $E^{p,q}_2 = R^p\ g(R^q\ f(A)) \Longrightarrow R^n\ gf(A)$ we obtain the exact sequence

$$0 \longrightarrow R^1\ g(f(A)) \longrightarrow R^1\ gf(A) \longrightarrow g(R^1\ f(A)) \longrightarrow R^2\ g(f(A)) \longrightarrow$$

$$\longrightarrow E^2_1 \longrightarrow R^1\ g(R^1\ f(A)) \longrightarrow R^3\ g(f(A))$$

where $E^2_1 = Ker\ (R^2\ gf(A) \longrightarrow g(R^2\ f(A)))$.

3.4. Forced solutions and Tsen fields

Definition 3.4.1. A field K is said to be a $Tsen^d$-field if every homogeneous form of degree deg with coefficients in K and $n > deg^d$ variables has a non-trivial zero in K.

For example, an algebraically closed field $\mathbb{K}$ is a $Tsen^0$-field as any form in n-variables defines a hypersurface in $\mathbb{P}^{n-1}_{\mathbb{K}}$. In fact, algebraic geometry tells us a stronger story

Lemma 3.4.2. *Let $\mathbb{K}$ be algebraically closed. If $f_1, \ldots, f_r$ are forms in n variables over $\mathbb{K}$ and $n > r$, then these forms have a common non-trivial zero in $\mathbb{K}$.*

Proof. Each f_i defines a hypersurface $V(f_i) \hookrightarrow \mathbb{P}^{n-1}_{\mathbb{K}}$. The intersection of r hypersurfaces has dimension $\geq n-1-r$ from which the claim follows. □

We want to extend this fact to higher Tsen-fields. The proof of the following result is technical unenlightening inequality manipulation.

Proposition 3.4.3. *Let K be a $Tsen^d$-field and $f_1, \ldots, f_r$ forms in n variables of degree deg. If $n > rdeg^d$, then they have a non-trivial common zero in K.*

For our purposes the main interest in Tsen-fields comes from :

Theorem 3.4.4. *Let K be of trancendence degree d over an algebraically closed field $\mathbb{C}$, then K is a $Tsen^d$-field.*

Proof. First we claim that the purely trancendental field $\mathbb{C}(t_1, \ldots, t_d)$ is a $Tsen^d$-field. By induction we have to show that if L is $Tsen^k$, then $L(t)$ is $Tsen^{k+1}$. By homogeneity we may assume that $f(x_1, \ldots, x_n)$ is a form of degree deg with coefficients in $L[t]$ and $n > deg^{k+1}$. For fixed s we introduce new variables $y^{(s)}_{ij}$ with $i \leq n$ and $0 \leq j \leq s$ such that

$$x_i = y^{(s)}_{i0} + y^{(s)}_{i1} t + \ldots + y^{(s)}_{is} t^s$$

If r is the maximal degree of the coefficients occurring in f, then we can write

$$f(x_i) = f_0(y_{ij}^{(s)}) + f_1(y_{ij}^{(s)})t + \ldots + f_{deg.s+r}(y_{ij}^{(s)})t^{deg.s+r}$$

where each f_j is a form of degree deg in $n(s+1)$-variables. By the proposition above, these forms have a common zero in L provided

$$n(s+1) > deg^k(ds+r+1) \Longleftrightarrow (n - deg^{i+1})s > deg^i(r+1) - n$$

which can be satisfied by taking s large enough. the common non-trivial zero in L of the f_j, gives a non-trivial zero of f in $L[t]$.
By assumption, K is an algebraic extension of $\mathbb{C}(t_1, \ldots, t_d)$ which by the above argument is $Tsen^d$. As the coefficients of any form over K lie in a finite extension E of $\mathbb{C}(t_1, \ldots, t_d)$ it suffices to prove that E is $Tsen^d$.
Let $f(x_1, \ldots, x_n)$ be a form of degree deg in E with $n > deg^d$. Introduce new variables y_{ij} with

$$x_i = y_{i1}e_1 + \ldots y_{ik}e_k$$

where e_i is a basis of E over $\mathbb{C}(t_1, \ldots, t_d)$. Then,

$$f(x_i) = f_1(y_{ij})e_1 + \ldots + f_k(y_{ij})e_k$$

where the f_i are forms of degree deg in $k.n$ variables over $\mathbb{C}(t_1, \ldots, t_d)$. Because $\mathbb{C}(t_1, \ldots, t_d)$ is $Tsen^d$, these forms have a common zero as $k.n > k.deg^d$. Finding a non-trivial zero of f in E is equivalent to finding a common non-trivial zero to the $f_1, \ldots, f_k$ in $\mathbb{C}(t_1, \ldots, t_d)$, done. □

A direct application of this result is Tsen's theorem :

Theorem 3.4.5. *Let K be the functionfield of a curve C defined over an algebraically closed field. Then, the only central simple K-algebras are $M_n(K)$. That is, $Br(K) = 1$.*

Proof. Assume there exists a central division algebra Δ of dimension n^2 over K. There is a finite Galois extension L/K such that $\Delta \otimes L = M_n(L)$. If $x_1, \ldots, x_{n^2}$ is a K-basis for Δ, then the reduced norm of any $x \in \Delta$,

$$N(x) = det(x \otimes 1)$$

is a form in n^2 variables of degree n. Moreover, as $x \otimes 1$ is invariant under the action of $Gal(L/K)$ the coefficients of this form actually lie in K.
By the main result, K is a $Tsen^1$-field and $N(x)$ has a non-trivial zero whenever $n^2 > n$. As the reduced norm is multiplicative, this contradicts $N(x)N(x^{-1}) = 1$. Hence, $n = 1$ and the only central division algebra is K itself. □

If K is the functionfield of a surface, we also have an immediate application :

Proposition 3.4.6. *Let K be the functionfield of a surface defined over an algebraically closed field. If Δ is a central simple K-algebra of dimension n^2, then the reduced norm map*

$$N : \Delta \longrightarrow K$$

is surjective.

Proof. Let $e_1, \ldots, e_{n^2}$ be a K-basis of Δ and $k \in K$, then

$$N(\sum x_i e_i) - k x_{n^2+1}^n$$

is a form of degree n in n^2+1 variables. Since K is a $Tsen^2$ field, it has a non-trivial solution (x_i^0), but then, $\delta = (\sum x_i^0 e_i) x_{n^2+1}^{-1}$ has reduced norm equal to k. □

3.5. Cohomological dimension and Tate fields

From the cohomological description of the Brauer group it is clear that we need to have some control on the absolute Galois group $G_K = Gal(\mathbb{K}/K)$. In this section we will see that finite trancendence degree forces some cohomology groups to vanish.

Definition 3.5.1. The cohomological dimension of a group G, $cd(G) \leq d$ if and only if $H^r(G, A) = 0$ for all $r > d$ and all torsion modules $A \in G$-mod.

Definition 3.5.2. A field K is said to be a $Tate^d$-field if the absolute Galois group $G_K = Gal(\mathbb{K}/K)$ satisfies $cd(G) \leq d$.

First, we will reduce the condition $cd(G) \leq d$ to a more manageable one. To start, one can show that a profinite group G has $cd(G) \leq d$ if and only if

$$H^{d+1}(G, A) = 0 \text{ for all torsion } G\text{-modules } A$$

Further, as all Galois cohomology groups of profinite groups are torsion, we can decompose the cohomology in its p-primary parts and relate their vanishing to the cohomological dimension of the p-Sylow subgroups G_p of G. This problem can then be verified by computing cohomology of finite simple G_p-modules of p-power order, but for a profinite p-group there is just one such module namely $\mathbb{Z}/p\mathbb{Z}$ with the trivial action.
Combining these facts we have the following manageable criterium on cohomological dimension.

Proposition 3.5.3. *$cd(G) \leq d$ if $H^{d+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for the simple G-modules with trivial action $\mathbb{Z}/p\mathbb{Z}$.*

We will need the following spectral sequence in Galois cohomology

Proposition 3.5.4. *(Hochschild-Serre spectral sequence) If N is a closed normal subgroup of a profinite group G, then*

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Longrightarrow H^n(G, A)$$

holds for every continuous G-module A.

Now, we are in a position to state and prove Tate's theorem

Theorem 3.5.5. *Let K be of trancendence degree d over an algebraically closed field, then K is a $Tate^d$-field.*

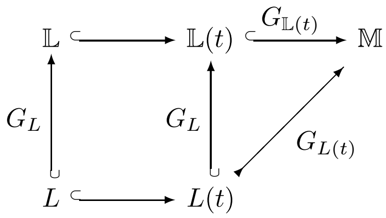
Proof. Let $\mathbb{C}$ denote the algebraically closed basefield, then K is algebraic over $\mathbb{C}(t_1, \ldots, t_d)$ and therefore

$$G_K \hookrightarrow G_{\mathbb{C}(t_1, \ldots, t_d)}$$

Thus, K is $Tate^d$ if $\mathbb{C}(t_1, \ldots, t_d)$ is $Tate^d$. By induction it suffices to prove

$$\text{If } cd(G_L) \leq k \text{ then } cd(G_{L(t)}) \leq k+1$$

Let $\mathbb{L}$ be the algebraic closure of L and $\mathbb{M}$ the algebraic closure of $L(t)$. As $L(t)$ and $\mathbb{L}$ are linearly disjoint over L we have the following diagram of extensions and Galois groups



where $G_{L(t)}/G_{\mathbb{L}(t)} \simeq G_L$.
We claim that $cd(G_{\mathbb{L}(t)}) \leq 1$. Consider the exact sequence of $G_{L(t)}$-modules

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{M}^* \xrightarrow{(-)^p} \mathbb{M}^* \longrightarrow 0$$

where μ_p is the subgroup (of $\mathbb{C}^*$) of p-roots of unity. As $G_{L(t)}$ acts trivially on μ_p it is after a choice of primitive p-th root of one isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Taking cohomology with respect to the subgroup $G_{\mathbb{L}(t)}$ we obtain

$$0 = H^1(G_{\mathbb{L}(t)}, \mathbb{M}^*) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^2(G_{\mathbb{L}(t)}, \mathbb{M}^*) = Br(\mathbb{L}(t))$$

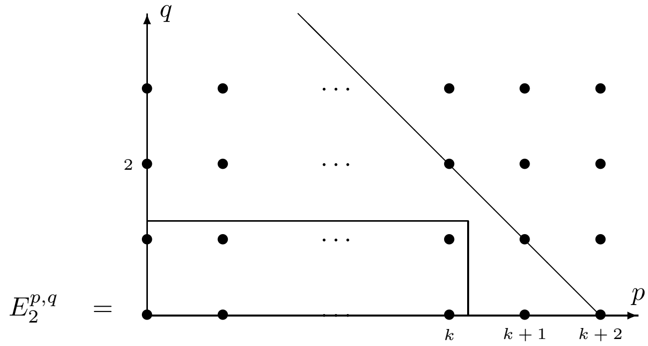
But the last term vanishes by Tsen's theorem as $\mathbb{L}(t)$ is the functionfield of a curve defined over the algebraically closed field $\mathbb{L}$. Therefore, $H^2(G_{\mathbb{L}(t)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for all simple modules $\mathbb{Z}/p\mathbb{Z}$, whence $cd(G_{\mathbb{L}(t)}) \leq 1$.
By the inductive assumption we have $cd(G_L) \leq k$ and now we are going to use exactness of the sequence

$$0 \longrightarrow G_L \longrightarrow G_{L(t)} \longrightarrow G_{\mathbb{L}(t)} \longrightarrow 0$$

to prove that $cd(G_{L(t)}) \leq k+1$. For, let A be a torsion $G_{L(t)}$-module and consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(G_L, H^q(G_{\mathbb{L}(t)}, A)) \Longrightarrow H^n(G_{L(t)}, A)$$

By the restrictions on the cohomological dimensions of G_L and $G_{\mathbb{L}(t)}$ the level two term has following shape



where the only non-zero groups are lying in the lower rectangular region. Therefore, all $E_\infty^{p,q} = 0$ for $p+q > k+1$. Now, all the composition factors of $H^{k+2}(G_{L(t)}, A)$ are lying on the indicated diagonal line and hence are zero. Thus, $H^{k+2}(G_{L(t)}, A) = 0$ for all torsion $G_{L(t)}$-modules A and hence $cd(G_{L(t)}) \leq k+1$.

□

As a consequence we obtain

THEOREM 3.5.6. *If* $\mathbf{A}$ *is a constant sheaf of an Abelian torsion group* A *on* K_{et}*, then*

$$H^i_{et}(K, \mathbf{A}) = 0$$

whenever $i > trdeg_{\mathbb{C}}(K)$.

CHAPTER 4

The Artin-Mumford sequence

In this chapter we will prove the geometric classification by $\mathbb{Z}_n$-wrinkles of central simple algebras over surfaces.

4.1. Leray spectral sequence

Assume we have an algebra morphism $A \xrightarrow{f} A'$ and a sheaf of groups $\mathbf{G}$ on A'_{et}. We define the **direct image** of $\mathbf{G}$ under f to be the sheaf of groups $f_* \ \mathbf{G}$ on A_{et} defined by

$$f_* \ \mathbf{G}(B) = \mathbf{G}(B \otimes_A A')$$

for all $B \in A_{et}$ (recall that $B \otimes_A A' \in A'_{et}$ so the right hand side is well defined). This gives us a left exact functor

$$f_* : \mathbf{S}^{ab}(A'_{et}) \longrightarrow \mathbf{S}^{ab}(A_{et})$$

and therefore we have right derived functors of it $R^i \ f_*$.
If $\mathbf{G}$ is an Abelian sheaf on A'_{et}, then $R^i \ f_*\mathbf{G}$ is a sheaf on A_{et}. One verifies that its stalk in a prime ideal $\mathfrak{p}$ is equal to

$$(R^i \ f_*\mathbf{G})_{\mathfrak{p}} = H^i_{et}(A^{sh}_{\mathfrak{p}} \otimes_A A', \mathbf{G})$$

where the right hand side is the direct limit of cohomology groups taken over all étale neighborhoods of $\mathfrak{p}$.
We can relate cohomology of $\mathbf{G}$ and $f_*\mathbf{G}$ by the following

Theorem 4.1.1. *(Leray spectral sequence) If* $\mathbf{G}$ *is a sheaf of Abelian groups on* A'_{et} *and* $A \xrightarrow{f} A'$ *an algebra morphism, then there is a spectral sequence*

$$E_2^{p,q} = H^p_{et}(A, R^q \ f_*\mathbf{G}) \Longrightarrow H^n_{et}(A, \mathbf{G})$$

In particular, if $R^j \ f_*\mathbf{G} = 0$ *for all* $j > 0$*, then for all* $i \geq 0$ *we have isomorphisms*

$$H^i_{et}(A, f_*\mathbf{G}) \simeq H^i_{et}(A', \mathbf{G})$$

4.2. Cohomology for discrete valuation rings

Consider the setting

$$\begin{array}{ccc} A & \xrightarrow{i} & K \\ \Big\downarrow{\scriptstyle \pi} & & \\ k & & \end{array}$$

where A is a discrete valuation ring in K with residue field $A/m = k$. As always, we will assume that A is a $\mathbb{C}$-algebra. By now we have a grip on the Galois cohomology groups

$$H^i_{et}(K, \boldsymbol{\mu_n}^{\otimes l}) \text{ and } H^i_{et}(k, \boldsymbol{\mu_n}^{\otimes l})$$

and we will use this information to compute the étale cohomology groups

$$H^i_{et}(A, \boldsymbol{\mu_n}^{\otimes l})$$

Here, $\boldsymbol{\mu_n}^{\otimes l} = \underbrace{\boldsymbol{\mu_n} \otimes \ldots \otimes \boldsymbol{\mu_n}}_{l}$ where the tensorproduct is as sheafs of invertible $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$-modules.
We will consider the Leray spectral sequence for i and hence have to compute the derived sheaves of the direct image

Lemma 4.2.1. 1. $R^0\ i_*\boldsymbol{\mu_n}^{\otimes l} \simeq \boldsymbol{\mu_n}^{\otimes l}$ *on* A_{et}.
2. $R^1\ i_*\boldsymbol{\mu_n}^{\otimes l} \simeq \boldsymbol{\mu_n}^{\otimes l-1}$ *concentrated in* m.
3. $R^j\ i_*\boldsymbol{\mu_n}^{\otimes l} \simeq 0$ *whenever* $j \geq 2$.

Proof. The strict Henselizations of A at the two primes $\{0, m\}$ are resp.

$$A^{sh}_0 \simeq \mathbb{K} \text{ and } A^{sh}_m \simeq \mathbf{k}\{t\}$$

where $\mathbb{K}$ (resp. $\mathbf{k}$) is the algebraic closure of K (resp. k). Therefore,

$$(R^j\ i_*\boldsymbol{\mu_n}^{\otimes l})_0 = H^j_{et}(\mathbb{K}, \boldsymbol{\mu_n}^{\otimes l})$$

which is zero for $i \geq 1$ and $\boldsymbol{\mu_n}^{\otimes l}$ for $j = 0$. Further, $A^{sh}_m \otimes_A K$ is the field of fractions of $\mathbf{k}\{t\}$ and hence is of trancendence degree one over the algebraically closed field $\mathbf{k}$, whence

$$(R^j\ i_*\boldsymbol{\mu_n}^{\otimes l})_m = H^j_{et}(L, \boldsymbol{\mu_n}^{\otimes l})$$

which is zero for $j \geq 2$ because L is $Tate$[1].
For the field-tower $K \subset L \subset \mathbb{K}$ we have that $G_L = \hat{\mathbb{Z}} = \underleftarrow{lim}\ \mu_m$ because the only Galois extensions of L are the Kummer extensions obtained by adjoining $\sqrt[m]{t}$. But then,

$$H^1_{et}(L, \boldsymbol{\mu_n}^{\otimes l}) = H^1(\hat{Z}, \boldsymbol{\mu_n}^{\otimes l}(\mathbb{K})) = Hom(\hat{Z}, \boldsymbol{\mu_n}^{\otimes l}(\mathbb{K})) = \boldsymbol{\mu_n}^{\otimes l-1}$$

from which the claims follow. □

Theorem 4.2.2. *We have a long exact sequence*

$$0 \longrightarrow H^1(A, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow H^1(K, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow H^0(k, \boldsymbol{\mu_n}^{\otimes l-1}) \longrightarrow$$

$$H^2(A, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow H^2(K, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow H^1(k, \boldsymbol{\mu_n}^{\otimes l-1}) \longrightarrow \ldots$$

Proof. By the foregoing lemma, the second term of the Leray spectral sequence for $i_*\boldsymbol{\mu_n}^{\otimes l}$ looks like

0	0	0	...
$H^0(k, \boldsymbol{\mu_n}^{\otimes l-1})$	$H^1(k, \boldsymbol{\mu_n}^{\otimes l-1})$	$H^2(k, \boldsymbol{\mu_n}^{\otimes l-1})$	...
$H^0(A, \boldsymbol{\mu_n}^{\otimes l})$	$H^1(A, \boldsymbol{\mu_n}^{\otimes l})$	$H^2(A, \boldsymbol{\mu_n}^{\otimes l})$	...

with connecting morphisms

$$H_{et}^{i-1}(k, \boldsymbol{\mu_n}^{\otimes l-1}) \xrightarrow{\alpha_i} H_{et}^{i+1}(A, \boldsymbol{\mu_n}^{\otimes l})$$

The spectral sequences converges to its limiting term which looks like

0	0	0	...
$Ker\ \alpha_1$	$Ker\ \alpha_2$	$Ker\ \alpha_3$	...
$H^0(A, \boldsymbol{\mu_n}^{\otimes l})$	$H^1(A, \boldsymbol{\mu_n}^{\otimes l})$	$Coker\ \alpha_1$	...

and the Leray sequence yields short exact sequences

$$0 \longrightarrow H_{et}^1(A, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow H_{et}^1(K, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow Ker\ \alpha_1 \longrightarrow 0$$

$$0 \longrightarrow Coker\ \alpha_1 \longrightarrow H_{et}^2(K, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow Ker\ \alpha_2 \longrightarrow 0$$

$$0 \longrightarrow Coker\ \alpha_{i-1} \longrightarrow H_{et}^i(K, \boldsymbol{\mu_n}^{\otimes l}) \longrightarrow Ker\ \alpha_i \longrightarrow 0$$

and gluing these sequences yields the required result.

□

In particular, if A is a discrete valuation ring of K with residue field k we have for each i a connecting morphism

$$H_{et}^i(K, \boldsymbol{\mu_n}^{\otimes l}) \xrightarrow{\partial_{i,A}} H_{et}^{i-1}(k, \boldsymbol{\mu_n}^{\otimes l-1})$$

4.3. Coniveau spectral sequence

Like any other topology, the étale topology can be defined locally on any scheme X. That is, we call a morphism of schemes

$$Y \xrightarrow{f} X$$

an étale extension (resp. cover) if locally f has the form

$$f^a \mid U_i : A_i = \Gamma(U_i, \mathcal{O}_X) \longrightarrow B_i = \Gamma(f^{-1}(U_i), \mathcal{O}_Y)$$

with $A_i \longrightarrow B_i$ an étale extension (resp. cover) of algebras.
Again, we can construct the étale site of X locally and denote it with X_{et}. Presheaves and sheaves of groups on X_{et} are defined similarly and the right derived functors of the left exact global sections functor

$$\Gamma : \mathbf{S}^{ab}(X_{et}) \longrightarrow \mathbf{Ab}$$

will be called the cohomology functors and we denote

$$R^i\ \Gamma(\mathbf{G}) = H^i_{et}(X, \mathbf{G})$$

From now on we restrict to the case when X is a smooth, irreducible projective variety of dimension d over $\mathbb{C}$. In this case, we can initiate the computation of the cohomology groups $H^i_{et}(X, \boldsymbol{\mu_n}^{\otimes l})$ via Galois cohomology of functionfields of subvarieties using the coniveau spectral sequence

Theorem 4.3.1. *Let X be a smooth irreducible variety over $\mathbb{C}$. Let $X^{(p)}$ denote the set of irreducible subvarieties x of X of codimension p with functionfield $\mathbb{C}(x)$, then there exists a coniveau spectral sequence*

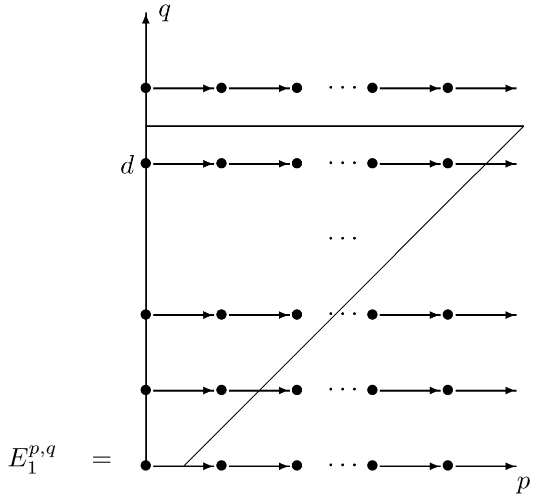
$$E_1^{p.q} = \bigoplus_{x \in X^{(p)}} H^{q-p}_{et}(\mathbb{C}(x), \boldsymbol{\mu_n}^{\otimes l-p}) \Longrightarrow H^{p+q}_{et}(X, \boldsymbol{\mu_n}^{\otimes l})$$

In contrast to the spectral sequences used before, the existence of the coniveau spectral sequence by no means follows from general principles. In it, a lot of heavy machinery on étale cohomology of schemes is encoded. In particular,

- cohomology groups with support of a closed subscheme, see Milne's book "Etale cohomology" pp. 91-94
- cohomological purity and duality, see loc.cit. Chpt VI, §5,6 pp. 241-252

a detailed exposition of which would take us too far afield.
Using the results on cohomological dimension and vanishing of Galois cohomology of $\boldsymbol{\mu_n}^{\otimes k}$ when the index is larger than the trancendence degree, we see that the

coniveau spectral sequence has the following shape



where the only non-zero terms are in the indicated region.
Let us understand the connecting morphisms at the first level, a typical instance of which is

$$\bigoplus_{x\in X^{(p)}} H^i(\mathbb{C}(x), \boldsymbol{\mu_n}^{\oplus l-p}) \longrightarrow \bigoplus_{y\in X^{(p+1)}} H^{i-1}(\mathbb{C}(y), \boldsymbol{\mu_n}^{\oplus l-p-1})$$

and consider one of the closed irreducible subvarieties x of X of codimension p and one of those y of codimension $p+1$. Then, either y is not contained in x in which case the component map

$$H^i(\mathbb{C}(x), \boldsymbol{\mu_n}^{\oplus l-p}) \longrightarrow H^{i-1}(\mathbb{C}(y), \boldsymbol{\mu_n}^{\oplus l-p-1})$$

is the zero map. Or, y is contained in x and hence defines a codimension one subvariety of x. That is, y defines a discrete valuation on $\mathbb{C}(x)$ with residue field $\mathbb{C}(y)$. In this case, the above component map is the connecting morphism defined in the previous section.
In particular, let K be the functionfield of X. Then we can define the unramified cohomology groups

$$F_n^{i,l}(K/\mathbb{C}) = Ker\ H^i(K, \boldsymbol{\mu_n}^{\otimes l}) \xrightarrow{\oplus\partial_{i,A}} \oplus H^{i-1}(k_A, \boldsymbol{\mu_n}^{\otimes l-1})$$

where the sum is taken over all discrete valuation rings A of K (or equivalently, the irreducible codimension one subvarieties of X) with residue field k_A. By definition, this is a (stable) birational invariant of X. In particular, if X is (stably) rational over $\mathbb{C}$, then

$$F_n^{i,l}(K/\mathbb{C}) = 0 \text{ for all } i, l \geq 0$$

4.4. The case of surfaces

In this section S will be a smooth irreducible projective surface.

Definition 4.4.1. *S is called simply connected if every étale cover $Y \longrightarrow S$ is trivial, that is, Y is isomorphic to a finite disjoint union of copies of S.*

The first term of the coniveau spectral sequence of S has following shape

0	0	0	0	$\dots$
$H^2(\mathbb{C}(S), \boldsymbol{\mu_n})$	$\oplus_C\ H^1(\mathbb{C}(C), \mathbb{Z}_n)$	$\oplus_P \boldsymbol{\mu_n}^{-1}$	0	$\dots$
$H^1(\mathbb{C}(S), \boldsymbol{\mu_n})$	$\oplus_C \mathbb{Z}_n$	0	0	$\dots$
$\boldsymbol{\mu_n}$	0	0	0	$\dots$

where C runs over all irreducible curves on S and P over all points of S.

LEMMA 4.4.2. *For any smooth S we have $H^1(\mathbb{C}(S), \boldsymbol{\mu_n}) \twoheadrightarrow \oplus_C\ \mathbb{Z}_n$. If S is simply connected, $H^1_{et}(S, \boldsymbol{\mu_n}) = 0$.*

PROOF. Using the Kummer sequence $1 \longrightarrow \boldsymbol{\mu_n} \longrightarrow \mathbb{G}_m \xrightarrow{(-)} \mathbb{G}_m \longrightarrow 1$ and Hilbert 90 we obtain that

$$H^1_{et}(\mathbb{C}(S), \boldsymbol{\mu_n}) = \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$$

The first claim follows from the exact diagram describing divisors of rational functions

$$\begin{array}{ccccccccc}
 & & \mu_n & \simeq & \mu_n & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{C}(S)^* & \xrightarrow{div} & \oplus_C \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle (-)^n} & & \downarrow {\scriptstyle n.} & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathbb{C}(S)^* & \xrightarrow{div} & \oplus_C \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \oplus_C \mathbb{Z}_n & \simeq & \oplus_C \mathbb{Z}_n & &
\end{array}$$

By the coniveau spectral sequence we have that $H^1_{et}(S, \boldsymbol{\mu_n})$ is equal to the kernel of the morphism

$$H^1_{et}(\mathbb{C}(S), \boldsymbol{\mu_n}) \xrightarrow{\gamma} \oplus_C\ \mathbb{Z}_n$$

and in particular, $H^1(S, \boldsymbol{\mu_n}) \hookrightarrow H^1(\mathbb{C}(S), \boldsymbol{\mu_n})$.
As for the second claim, an element in $H^1(S, \boldsymbol{\mu_n})$ determines a cyclic extension $L = \mathbb{C}(S)\sqrt[n]{f}$ with $f \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$ such that in each fieldcomponent L_i of L

there is an étale cover $T_i \longrightarrow S$ with $\mathbb{C}(T_i) = L_i$. By assumption no non-trivial étale covers exist whence $f = 1 \in \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$. □

In we invoke another major tool in étale cohomology of schemes, Poincaré duality, we obtain the following information on the cohomology groups for S.

PROPOSITION 4.4.3. *(Poincaré duality for S) If S is simply connected, then*

1. $H^0_{et}(S, \boldsymbol{\mu_n}) = \boldsymbol{\mu_n}$
2. $H^1_{et}(S, \boldsymbol{\mu_n}) = 0$
3. $H^3_{et}(S, \boldsymbol{\mu_n}) = 0$
4. $H^4_{et}(S, \boldsymbol{\mu_n}) = \boldsymbol{\mu_n}^{-1}$

PROOF. The third claim follows from the second as both groups are dual to each other. The last claim follows from the fact that for any smooth irreducible projective variety X of dimension d one has that

$$H^{2d}_{et}(X, \boldsymbol{\mu_n}) \simeq \boldsymbol{\mu_n}^{\otimes 1-d}$$

□

We are now in a position to state and prove the important

THEOREM 4.4.4. *(Artin-Mumford exact sequence) If S is a simply connected smooth projective surface, then the sequence*

$$0 \longrightarrow Br_n(S) \longrightarrow Br_n(\mathbb{C}(S)) \longrightarrow \oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow$$

$$\longrightarrow \oplus_P \boldsymbol{\mu_n}^{-1} \longrightarrow \boldsymbol{\mu_n}^{-1} \longrightarrow 0$$

is exact.

PROOF. The top complex in the first term of the coniveau spectral sequence for S was

$$H^2(\mathbb{C}(S), \boldsymbol{\mu_n}) \xrightarrow{\alpha} \oplus_C H^1(\mathbb{C}(C), \mathbb{Z}_n) \xrightarrow{\beta} \oplus_P \boldsymbol{\mu_n}$$

The second term of the spectral sequence (which is also the limiting term) has the following form

0	0	0	0	...
$Ker\ \alpha$	$Ker\ \beta/Im\ \alpha$	$Coker\ \beta$	0	...
$Ker\ \gamma$	$Coker\ \gamma$	0	0	...
$\boldsymbol{\mu_n}$	0	0	0	...

By the foregoing lemma we know that $Coker\ \gamma = 0$. By Poincare duality we know that $Ker\ \beta = Im\ \alpha$ and $Coker\ \beta = \boldsymbol{\mu_n}^{-1}$. Hence, the top complex was exact in its middle term and can be extended to an exact sequence

$$0 \longrightarrow H^2(S, \boldsymbol{\mu_n}) \longrightarrow H^2(\mathbb{C}(S), \boldsymbol{\mu_n}) \longrightarrow \oplus_C H^1(\mathbb{C}(C), \mathbb{Z}_n) \longrightarrow$$

$$\oplus_P \boldsymbol{\mu_n}^{-1} \longrightarrow \boldsymbol{\mu_n}^{-1} \longrightarrow 0$$

As $\mathbb{Z}_n \simeq \boldsymbol{\mu_n}$ the third term is equal to $\oplus_C \mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$ by the argument given before and the second term we remember to be $Br_n(\mathbb{C}(S)$. The identification of $Br_n(S)$ with $H^2(S, \boldsymbol{\mu_n})$ will be explained in the next section. □

Some immediate consequences can be drawn from this :

- For a smooth simply connected surface S, $Br_n(S)$ is a birational invariant (it is the birational invariant $F_n^{2,1}(\mathbb{C}(S)/\mathbb{C})$ of the foregoing section.
- In particular, if $S = \mathbb{P}^2$ we have that $Br_n(\mathbb{P}^2) = 0$ and we obtain the description of $Br_n(\mathbb{C}(x,y))$ by $\mathbb{Z}_n$-wrinkles as

$$0 \longrightarrow Br_n\ \mathbb{C}(x,y) \longrightarrow \oplus_C\ \mathbb{C}(C)^*/\mathbb{C}(C)^{*n} \longrightarrow \oplus_P\ \boldsymbol{\mu_n}^{-1} \longrightarrow \boldsymbol{\mu_n} \longrightarrow 0$$

Exercise 4.4.5. If S is not necessarily simply connected, show that any class in $Br(\mathbb{C}(S))_n$ determines a $\mathbb{Z}_n$-wrinkle.

Exercise 4.4.6. If X is a smooth irreducible rational projective variety of dimension d, show that the obstruction to classifying $Br(\mathbb{C}(X))_n$ by $\mathbb{Z}_n$-wrinkles is given by $H^3_{et}(X, \boldsymbol{\mu_n})$.

4.5. Interpretation via maximal orders

In this section we will give a ringtheoretical interpretation of the maps in the Artin-Mumford sequence. Observe that nearly all maps are those of the top complex of the first term in the coniveau spectral sequence for S. We gave an explicit description of them using discrete valuation rings. The statements below follow from this description.

Let us consider a discrete valuation ring A with field of fractions K and residue field k. Let Δ be a central simple K-algebra of dimension n^2.

Definition 4.5.1. An A-subalgebra Λ of Δ will be called an A-**order** if it is a free A-module of rank n^2 with $\Lambda.K = \Delta$. An A-order is said to be maximal if it is not properly contained in any other order.

In order to study maximal orders in Δ (they will turn out to be all conjugated), we consider the completion $\hat{A}$ with respect to the m-adic filtration where $m = At$ with t a uniformizing parameter of A. $\hat{K}$ will denote the field of fractions of $\hat{A}$ and $\hat{\Delta} = \Delta \otimes_K \hat{K}$.

Because $\hat{\Delta}$ is a central simple $\hat{K}$-algebra of dimension n^2 it is of the form

$$\hat{\Delta} = M_t(D)$$

where D is a division algebra with center $\hat{K}$ of dimension s^2 and hence $n = s.t$. We call t the capacity of Δ at A.

In D we can construct a unique maximal $\hat{A}$-order Γ, namely the integral closure of $\hat{A}$ in D. We can view Γ as a discrete valuation ring extending the valuation v defined by A on K. If $v : \hat{K} \longrightarrow \mathbb{Z}$, then this extended valuation

$$w : D \longrightarrow n^{-2}\mathbb{Z} \text{ is defined as } w(a) = (\hat{K}(a) : \hat{K})^{-1} v(N_{\hat{K}(a)/\hat{K}}(a))$$

for every $a \in D$ where $\hat{K}(a)$ is the subfield generated by a and N is the norm map of fields.

The image of w is a sugroup of the form $e^{-1}\mathbb{Z} \hookrightarrow n^{-2}.\mathbb{Z}$. The number $e = e(D/\hat{K})$ is called the ramification index of D over $\hat{K}$. We can use it to normalize the valuation w to

$$v_D : D \longrightarrow \mathbb{Z} \text{ defined by } v_D(a) = \frac{e}{n^2} v(N_{D/\hat{K}}(a))$$

With these conventions we have that $v_D(t) = e$.
The maximal order Γ is then the subalgebra of all elements $a \in D$ with $v_D(a) \geq 0$. It has a unique maximal ideal generated by a prime element T and we have that $\overline{\Gamma} = \Gamma/T\Gamma$ is a division algebra finite dimensional over $\hat{A}/t\hat{A} = k$ (but not necessarily having k as its center).
The inertial degree of D over $\hat{K}$ is defined to be the number $f = f(D/\hat{K}) = (\overline{\Gamma} : k)$ and one shows that

$$s^2 = e.f \text{ and } e \mid s \text{ whence } s \mid f$$

After this detour, we can now take $\Lambda = M_t(\Gamma)$ as a maximal $\hat{A}$-order in $\hat{\Delta}$. One shows that all other maximal $\hat{A}$-orders are conjugated to Λ. Λ has a unique maximal ideal M with $\overline{\Lambda} = M_t(\overline{\Gamma})$.

Definition 4.5.2. With notations as above, we call the numbers $e = e(D/\hat{K})$, $f = f(D/\hat{K})$ and t resp. the ramificaton, inertia and capacity of the central simple algebra Δ at A. If $e = 1$ we call Λ an Azumaya algebra over A, or equivalently, if $\Lambda/t\Lambda$ is a central simple k-algebra of dimension n^2.

Now let us consider the case of a discrete valuation ring A in K such that the residue field k is $Tsen^1$. The center of the division algebra $\overline{\Gamma}$ is a finite dimensional field extension of k and hence is also $Tsen^1$ whence has trivial Brauer group and therefore must coincide with $\overline{\Gamma}$. Hence,

$$\overline{\Gamma} = k(\overline{a})$$

a commutative field, for some $a \in \Gamma$. But then, $f \leq s$ and we have $e = f = s$ and $k(\overline{a})$ is a cyclic degree s field extension of k.
Because $s \mid n$, the cyclic extension $k(\overline{a})$ determines an element of $H^1_{et}(k, \mathbb{Z}_n)$.

Definition 4.5.3. Let Z be a normal domain with field of fractions K and let Δ be a central simple K-algebra of dimension n^2. A Z-order B is a subalgebra which is a finitely generated Z-module. It is called maximal if it is not properly contained in any other order. One can show that B is a maximal Z-order if and only if $\Lambda = B_p$ is a maximal order over the discrete valuation ring $A = Z_p$ for every height one prime ideal p of Z.

Return to the situation of an irreducible smooth projective surface S. If Δ is a central simple $\mathbb{C}(S)$-algebra of dimension n^2, we define a maximal order as a sheaf $\mathcal{B}$ of $\mathcal{O}_S$-orders in Δ which for an open affine cover $U_i \hookrightarrow S$ is such that

$$B_i = \Gamma(U_i, \mathcal{B}) \text{ is a maximal } Z_i = \Gamma(U_i, \mathcal{O}_S) \text{ order in } \Delta$$

Any irreducible curve C on S defines a discrete valuation ring on $\mathbb{C}(S)$ with residue field $\mathbb{C}(C)$ which is $Tsen^1$. Hence, the above argument can be applied to obtain from $\mathcal{B}$ a cyclic extension of $\mathbb{C}(C)$, that is, an element of $\mathbb{C}(C)^*/\mathbb{C}(C)^{*n}$.

Definition 4.5.4. We call the union of those curves such that $\mathcal{B}$ determines a non-trivial cyclic extension of $\mathbb{C}(C)$ the **ramification divisor** of Δ (or of $\mathcal{B}$).

The map in the Artin-Mumford exact sequence

$$Br_n(\mathbb{C}(S)) \longrightarrow \bigoplus_C H^1_{et}(\mathbb{C}(C), \boldsymbol{\mu_n})$$

assigns to the class of Δ the cyclic extensions introduced above.

Definition 4.5.5. An S-Azumaya algebra (of index n) is a sheaf of maximal orders in a central simple $\mathbb{C}(S)$-algebra Δ of dimension n^2 such that it is Azumaya at each curve C, that is, such that $[\Delta]$ lies in the kernel of the above map.

One can show that if $\mathcal{B}$ and $\mathcal{B}'$ are S-Azumaya algebras of index n resp. n', then $\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{B}'$ is an Azumaya algebra of index $n.n'$. We call an Azumaya algebra trivial if it is of the form $End(\mathcal{P})$ where $\mathcal{P}$ is a vectorbundle over S. The equivalence classes of S-Azumayay algebras can be given a group-structure called the Brauer-group of the surface S.

Part 2

Non-commutative smooth models

CHAPTER 5

Restricted smooth models

Assume K is a field of trancendence degree d over $\mathbb{C}$. By Hironaka's result on resolution of singularities we know that K has a smooth model. Ringtheoretically, this means that there is a positively graded affine $\mathbb{C}$-domain

$$A = \mathbb{C}[x_0, \ldots, x_m]/(f_1, \ldots, f_k)$$

generated in degree one by the x_i and where the f_i are homogeneous polynomials such that

1. A is a **model** for K. That is, if we localize at the multiplicative system of non-zero homogeneous elements of A we obtain the graded field
$$Q^g(A) = K[t, t^{-1}]$$
with t of degree one.
2. A is a **smooth** model for K if $X = Proj\ A$ is smooth. That is, consider the zero set
$$Proj\ A = V(f_1, \ldots, f_k) \hookrightarrow \mathbb{P}^m$$
then at each point $p \in V(f_1, \ldots, f_k)$, the kernel of the linear map
$$(\frac{\partial f_i}{\partial x_j})(p) : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^k$$
will be of dimension $d + 1$.

In this chapter we will consider a non-commutative analogous situation, where the role of K is replaced by a central simple K-algebra Δ of dimension n^2.

5.1. Cayley-Hamilton algebras

We fix a field K of trancendence degree d, a central simple K-algebra Δ of dimension n^2 and a connected graded algebra

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus A_3 \oplus \ldots$$

which is affine and generated in degree one, that is,

$$A_1 = \mathbb{C}a_1 + \ldots + \mathbb{C}a_m$$

and there is an epimorphism

$$\mathbb{C}\langle x_1, \ldots, x_m \rangle \twoheadrightarrow A$$

mapping x_i to a_i. Moreover, we will assume that A is a finite module over its center C which is itself a graded algebra. We will always assume that C is a normal (that is, integrally closed) domain.

DEFINITION 5.1.1. The graded algebra A is said to be a **model** for Δ if A is prime and if we localize at the Ore-set of non-zero central homogeneous elements we obtain

$$Q^g(A) = \Delta[t, t^{-1}]$$

where t is a central element of degree one. In particular, C will be a model for K though not necessarily generated in degree one.

We recall the definition of the reduced trace map $tr : A \longrightarrow C$. As $\Delta[t, t^{-1} \otimes_K \mathbb{K} \simeq M_n(\mathbb{K}[t, t^{-1})$ we can define for any $a \in A$ its reduced trace $tr(a) = Tr(a \otimes 1) \in \mathbb{K}[t, t^{-1}]$. As Tr is compatible with the Galois action and $a \otimes 1$ is invariant under G_K, it follows that $tr(a) \in K[t, t^{-1}]$. Moreover, as C is integrally closed in $K[t, t^{-1}]$ and a is integral over C, it follows that $tr(a) \in C$. Moreover, as we are in characteristic zero we have that $tr(A) = C$. Remark that tr is a homogeneous linear map. A with its reduced trace map is a special instance of a Cayley-Hamilton algebra.
Let A be an arbitrary $\mathbb{C}$-algebra having a linear trace map $tr : A \longrightarrow A$ satisfying the following conditions for all $a, b \in A$

1. $tr(ab) = tr(ba)$
2. $tr(a)b = btr(a)$
3. $tr(tr(a)b) = tr(a)tr(b)$

In particular, the image of tr is a subalgebra of the center of A. We can then define the n-th Cayley-Hamilton polynomial formally. In $\mathbb{Q}[x_1, \ldots, x_n]$ one defines the elementary symmetric functions by the identity

$$\prod(t - x_i) = \sum_{i=0}^{n} (-1)^i \sigma_i t^{d-i}$$

and the power sums functions and $\{\tau_i\}$ are generators of the symmetric functions, there are functions with rational coefficients such that

$$\sigma_k = p_k(\tau_1, \ldots, \tau_n)$$

and we define the functions σ_k on A formally as

$$\sigma_k(a) = p_k(tr(a), tr(a^2), \ldots, tr(a^n))$$

and define the n-th Cayley-Hamilton polynomial for A to be

$$\chi_{n,a}(t) = \sum_{i=0}^{n} (-1)^i \sigma_i(a) t^{n-i}$$

DEFINITION 5.1.2. We say that an algebra A with a trace function tr is an n-th Cayley-Hamilton algebra if

1. For all $a \in A$ we have $\chi_{n,a}(a) = 0$ in A
2. $tr(1) = n$

With CH_n we will denote the category with objects (A, tr_A) algebras A with a trace function tr_A which are n-th Cayley-Hamilton algebras and morphisms $f : (A, tr_A) \longrightarrow (B, tr_B)$ are algebra morphisms which are trace preserving, that is, the diagram below is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle tr_A}\downarrow & & \downarrow{\scriptstyle tr_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

As we have seen above, orders in central simple algebras are the archetypical examples of Cayley-Hamilton algebras. One can study Cayley-Hamilton algebras using algebraic geometry and invariant theory.

Let $(A, tr_A) \in CH_n$ be m-generated, that is, there are elements $a_1, \ldots, a_m \in A$ such that the subalgebra in CH_n generated by them is equal to A (note that this is weaker than A being generated as algebra by m elements). Consider

$$mod_n\ A = \{\phi : (A, tr_A) \longrightarrow (M_n(\mathbb{C}), Tr) \text{ in } CH_n\}$$

the set of n-dimensional trace preserving representations of A. By taking the images $\phi(a_i) \in M_n(\mathbb{C})$ for $1 \leq i \leq m$ it is clear that $mod_n\ A$ is a closed subvariety of the affine space $M_n(\mathbb{C})^{\oplus m}$.

There is a natural action of PGL_n on $M_n(\mathbb{C})^{\oplus m}$ by simultaneous conjugation. Clearly, $mod_n\ A$ is a PGL_n-stable closed subvariety of $M_n(\mathbb{C})^{\oplus m}$. The PGL_n-orbits correspond to isomorphism classes of representations.

If we denote by $CH_n^{(m)}$ the subcategory of CH_n consisting of algebras which are trace generated by m elements we have the following important result due to C. Procesi

Theorem 5.1.3 (Procesi). *The functor*

$$CH_n^{(m)} \longrightarrow PGL_n - \text{ closed subvarieties of } M_n(\mathbb{C})^{\oplus m}$$

assigning $mod_n\ A$ to $A \in CH_n^{(m)}$ has a left inverse.

This inverse assigns to a PGL_n-closed subvariety X the ring of PGL_n-equivariant maps $X \longrightarrow M_n(\mathbb{C})$, or equivalently, the ring of concomitants

$$M_n(\mathbb{C}[X])^{PGL_n}$$

This means that we can recover $A \in CH_n^{(m)}$ from the affine PGL_n-variety $mod_n\ A$ as $A \simeq M_n(\mathbb{C}[mod_n\ A])^{PGL_n}$.

The embedding $j_A : A \hookrightarrow M_n(\mathbb{C}[X_A])$ has the following universal property. Let C be a commutative algebra and $F : A \longrightarrow M_n(C)$ a morphism in CH_n (with the usual trace map on $M_n(C)$) then there is a uniquely determined morphism $f : \mathbb{C}[X_A] \longrightarrow C$ making the diagram below commutative

$$\begin{array}{ccc} A & \xrightarrow{j} & M_n(\mathbb{C}[X_A]) \\ {\scriptstyle F}\downarrow & \swarrow{\scriptstyle M_n(f)} & \\ M_n(C) & & \end{array}$$

After this short excursion to Cayley-Hamilton algebras let us return to the situation at hand, that is, the connected graded algebra A is a model for Δ. In this case,

$$mod_n\ A \hookrightarrow M_n(\mathbb{C})^{\oplus m}$$

is a cone because all the defining relations of A are homogeneous. Alternatively, there is a $\mathbb{C}^*$-action on $mod_n\ A$ which commutes with the PGL_n-action. Using Procesi's result we have

LEMMA 5.1.4. *Let A be a model for Δ, then $mod_n\ A$ has a natural action by $PGL_n \times \mathbb{C}^*$. Moreover, we recover the graded algebra A from the action on $mod_n\ A$.*

PROOF. By Procesi's theorem we recover the algebra A as the ring of PGL_n-equivariant maps $mod_n\ A \longrightarrow M_n(\mathbb{C})$. The $\mathbb{C}^*$-action defines the gradation on A. An element $f \in A_k$ iff the diagram

$$\begin{array}{ccc} mod_n\ A & \xrightarrow{f} & M_n(\mathbb{C}) \\ \lambda. \downarrow & & \downarrow \lambda^k. \\ mod_n\ A & \xrightarrow{f} & M_n(\mathbb{C}) \end{array}$$

commutes where the vertical map on the left is action by $\lambda \in \mathbb{C}^*$ on $mod_n\ A$ and on the right left multiplication by λ^k. □

As $mod_n\ A$ is a cone we can define its projective space

$$proj_n\ A = \mathbb{P}(mod_n\ A) \hookrightarrow \mathbb{P}(M_n(\mathbb{C})^{\oplus m}) = \mathbb{P}^{mn^2-1}$$

which has an induced PGL_n-action. We would like to call A smooth whenever $proj_n\ A$ is a smooth variety. However, we have to be careful about representations having the zero representation in the closure of its orbit.

DEFINITION 5.1.5. The semi-stable points $proj_n^{ss}\ A$ of $proj_n\ A$ are those determined by a representation $A \longrightarrow M_n(\mathbb{C})$ on which a central homogeneous element of A does not vanish.

DEFINITION 5.1.6. A model A for Δ is said to be a smooth model if and only if $proj_n^{ss}\ A$ is a smooth variety.

EXERCISE 5.1.7. If A is commutative, verify that $proj_1^{ss}\ A = Proj\ A$. Hence the above definition generalizes the classical one.

5.2. Module varieties

LEMMA 5.2.1. *Let A be a model for Δ and $0 \neq c \in C$ homogeneous. The localization at the Ore-set $\{1, c.c^2.\ldots\}$ has the form*

$$A_c^g = \ldots \oplus I^{-2} \oplus I^{-1} \oplus B \oplus I \oplus I^2 \oplus \ldots$$

where I is an invertible ideal of B, that is, for $I^{-1} = \{\delta \in \Delta \mid I.\delta \subset B\}$ we have $I.I^{-1} = B$.

PROOF. Let $deg(c) = u$ and write $c = \sum d_i a_i$ with $deg(d_i) = u - 1$, then

$$1 = \sum (c^{-1} d_i).a_i \in (A_c^g)_{-1}.(A_c^g)_1$$

whence A_c^g is strongly graded and the last claim follows from the structure result of strongly graded algebras and $Q^g(A) = \Delta[t, t^{-1}]$ with t central. □

A particularly interesting case is when $I = Bc'$ with c' central.

DEFINITION 5.2.2. A smooth model A of Δ is said to be **restricted** if one can cover $proj_n^{ss}\ A$ with affine open sets $X(c)$ where $c \in C$ homogeneous such that

$$A_c^g = B[d, d^{-1}]$$

and d central of degree one.

LEMMA 5.2.3. *If A is a restricted smooth model for Δ, then locally for the defining cover we have*

$$proj_n^{ss}\ A \mid X(c) = mod_n\ B$$

PROOF. A trace preserving n-dimensional representation

$$\phi : A_c^g = B[d, d^{-1}] \longrightarrow M_n(\mathbb{C})$$

is determined by $\phi \mid B \in mod_n\ B$ and $\phi(d) = \lambda I_n$ for some $\lambda \in \mathbb{C}^*$. The corresponding point in $proj_n\ A$ is hence fully determined by $\phi \mid B$. ☐

Therefore, the local study of restricted smooth models reduces to that of affine algebras B with normal center $Z(B)$ such that $mod_n\ B$ is a smooth affine variety. We can give a ringtheoretical interpretation of this condition. In analogy with the infinitesimal lifting property of smooth commutative algebras we define

DEFINITION 5.2.4. An affine algebra (B, tr_B) in CH_n is said to be smooth if and only if for every test-object (C, N) where $(C, tr_C) \in CH_n$, N a nilpotent ideal (invariant under the trace map such that also $(C/N, \overline{tr_C}) \in CH_n$) and every morphism $\phi : (B, tr_B) \longrightarrow (C/N, \overline{tr_C})$ in CH_n the diagram below can be completed in CH_n

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & C/N & \longrightarrow & 0 \\ & & & & \uparrow \exists \tilde{\phi} & \nearrow \phi & & & \\ & & & & B & & & & \end{array}$$

Using the universal property of $j_B : B \hookrightarrow M_n(\mathbb{C}[mod_n\ B])$ recalled in the previous section we then have

PROPOSITION 5.2.5 (Procesi). *Equivalent are*

1. *B is a smooth algebra in CH_n*
2. *$mod_n\ B$ is a smooth variety*

PROOF. (1) $\Rightarrow$ (2) : Let (C, N) be a commutative test-object for $\mathbb{C}[mod_n\ B]$. We have to lift the map $\mathbb{C}[mod_n\ B] \longrightarrow C/N$ to C. By smoothness in CH_n of B we can complete with F the diagram

$$\begin{array}{ccc} B & \xrightarrow{j_B} & M_n(\mathbb{C}[mod_n\ B]) \\ \vdots\ \exists F & \swarrow \exists M_n(f) & \downarrow \\ M_n(C) & \longrightarrow & M_n(C/N) \end{array}$$

but then by the universal property of j_B there is a uniquely determined map $f : \mathbb{C}[mod_n\ B] \longrightarrow C$ which is the required lift.

The reverse implication makes essential use of the Reynolds operator in invariant theory. □

We will make a few comments about the **module varieties** $mod_n\ B$ for an arbitrary affine $\mathbb{C}$-algebra B in CH_n. Let $b_1, \ldots, b_s$ be generators of B, then

$$mod_n\ B = \{x : B \longrightarrow M_n(\mathbb{C})\} \hookrightarrow M_n(\mathbb{C})^{\oplus s} = \mathbb{A}^{sn^2}$$

via $x \mapsto (x(b_1), \ldots, x(b_s))$. A point x determines an n-dimensional B-module $M_x = \mathbb{C}^n_x$ via $b_i.m = x(b_i)m$ for all $m \in \mathbb{C}^n$. There is a natural action of GL_n on $mod_n\ B$ via conjugation in $M_n(\mathbb{C})$ and x, x' lie in the same orbit if and only if $M_x \simeq M_{x'}$ as B-module.

Definition 5.2.6. With notations as above we denote

1. The orbit $GL_n.x$ of x by Orb_x.
2. The stabilizer subgroup $Stab_x(GL_n) = \{g \in GL_n \mid g.x = x\}$ by GL_x.

Lemma 5.2.7. *For any $x \in mod_n\ B$ we have a canonical isomorphism*

$$GL_x \simeq Aut_{B-mod}(M_x)$$

Proof. Consider a B-module isomorphism $g : M_x \longrightarrow M_x$ determined by $g \in GL_n$. Then, $g(b_i.m) = b_i.g(m)$ and hence

$$g.x(b_i).m = x(b_i).g.m \text{ for all } m \in M_x = \mathbb{C}^n_x \text{ and } 1 \leq i \leq s$$

But then, $g.x.g^{-1} = x$ and $g \in GL_x$. □

A natural question is the correlation between algebraic properties of the B-module M_x and geometric properties of the orbit $Orb_x \hookrightarrow mod_n\ B$.

Definition 5.2.8. A filtration F on a finite dimensional B-module M is a sequence of submodules

$$0 = M_t \subset \ldots \subset M_1 \subset M_0 = M$$

and the associated graded B-module is defined by

$$gr_F(M) = \oplus_{i=1}^t M_{i-1}/M_i$$

By the Jordan-Hölder theorem we know that M has a filtration with all composition factors $S_i = M_{i-1}/M_i$ simple B-modules.

Lemma 5.2.9. *Let $x, x' \in mod_n\ B$. Equivalent are*

1. *There is a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that $lim_{t\to 0}\lambda(t).x = x'$.*
2. *There is a filtration F on M_x such that $gr_F(M_x) \simeq M_{x'}$*

Proof. $(1) \Rightarrow (2)$: We consider the weightdecomposition of $M_x = V$

$$V = \oplus_i V_i \text{ where } V_i = \{v \in V \mid \lambda(t).v = t^i.v \text{ for all } t \in \mathbb{C}^*\} \text{ for } i \in \mathbb{Z}$$

and we consider $M_j = \oplus_{i>j} M_i$. We claim that the M_j define a filtration on M_x with associated graded module $M_{x'}$.
Consider the canonical inclusion and projection maps

$$V_i \overset{\iota_i}{\hookrightarrow} V = \oplus V_i \overset{\pi_i}{\twoheadrightarrow} V_i$$

For $b \in B$ the action $x(b) = \phi = (\phi_{ij}) \in End(V)$ where $\phi_{ij} = \pi_j \circ \phi \circ \iota_i : V_i \longrightarrow V_j$. We have

$$\begin{array}{ccc} V_i & \xrightarrow{\phi_{ij}} & V_j \\ \uparrow \lambda(t)^{-1} & & \downarrow \lambda(t) \\ V_i & \xrightarrow{t^{j-i}\phi_{ij}} & V_j \end{array}$$

and as $lim_{t \rightarrow 0} \lambda(t).x(b) = x'(b)$ exists we have

- $\phi_{ij} = 0$ whenever $j < i$ and hence M_k is a B-submodule of M_x
- $lim_{t \rightarrow 0}(\lambda(t).x(b))_{ij} = (x'(b))_{ij} = 0$ for $i < j$
- $(x'(b))_{ii} = (x(b))_{ii}$ for all i

Therefore $x'(b)$ is the diagonal matrix (ϕ_{ii}) and the claim follows.
$(2) \Rightarrow (1)$: Consider a filtration F

$$0 = M_t \subset \ldots \subset M_1 \subset M_0 = M_x$$

then there exist subspaces V_i of $V = M_x$ such that $M_j = \oplus_{i=j}^t V_i$ and $V = \oplus_{i=0}^t V_i$. If we then define an action $\lambda(t) \mid V_i = t^i.I_n$ this satisfies the requirements. □

THEOREM 5.2.10 (Artin-Voigt).
1. *The closed GL_n-orbits in $mod_n\ B$ are precisely the isomorphism classes of semi-simple B-modules of dimension n.*
2. *The Zariski closure $\overline{Orb_x}$ contains a unique closed orbit determined by the direct sum of the composition factors of M_x.*

PROOF. Let $x \in mod_n\ B$, consider a Jordan-Hölder filtration on M_x with associated graded $gr(M_x)$ a semi-simple B-module of dimension n. Denote x_{ss} the corresponding point of $mod_n\ B$.
(1) : If Orb_x is closed, then by the foregoing lemma we have

$$Orb_{x_{ss}} \subset \overline{Orb_x} = Orb_x$$

and thus, $gr(M_x) \simeq M_x$ whence M_x is semi-simple.
Conversely, assume M_x is semi-simple and let $y \in \overline{Orb_x}$. By the Hilbert-Mumfrd criterium in invariant theory, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow GL_n$ such that

$$lim_{t \rightarrow 0} \lambda(t).x \in Orb_y$$

Again by the foregoing lemma this implies that there is a filtration F on M_x such that $gr_F(M_x) \simeq M_y$. However, as M_x is semi-simple $gr_F(M_x) \simeq M_x$ and thus $M_x \simeq M_y$ and hence Orb_x is closed.
(2) : Uniqueness follows from the Jordan-Hölder theorem. □

In general, if X is an affine variety with an action by a reductive group G, then the closed orbits are parameterized by th e points of an affine variety X/G, the quotient variety. Its coordinate ring is

$$\mathbb{C}[X/G] = \mathbb{C}[X]^G$$

the ring of G-invariant polynomial functions on X. Moreover, the natural inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ defines the quotient map

$$\pi : X \twoheadrightarrow X/G$$

and for each $\zeta \in X/G$ the fiber $\pi^{-1}(\zeta)$ contains a unique closed orbit (that of minimal dimension).
Restricting to the case of interest to us we see that the set $iso_n^{ss}\ B$ of isoclasses of semi-simple n-dimensional representations of B has the structure of an affine variety with coordinate ring

$$\mathbb{C}[iso_n^{ss}\ B] = \mathbb{C}[mod_n\ B]^{GL_n}$$

and we have a ringtheoretical interpretation of this quotient variety namely

$$\mathbb{C}[iso_n^{ss}\ B] = Z(B)$$

If m is a maximal ideal of the center $Z(B)$, then one determines the associated semi-simple representation of B by taking the quotient of B/mB by its Jacobson radical. In conclusion, we have

THEOREM 5.2.11 (Procesi). *If B is an affine algebra in CH_n, then the module variety $mod_n\ B$ together with its natural GL_n-action determines*

1. *B as the ring of equivariant maps $mod_n\ B \longrightarrow M-n(\mathbb{C})$.*
2. *The center Z of B as the coordinate ring of the quotient variety.*

5.3. Etale local structure

If X is a commutative smooth variety of dimension d and x a point of X then there is only one type of étale local behavior at x, namely

$$\mathcal{O}_x^{sh} \simeq \mathbb{C}\{x_1, \ldots, x_d\}$$

the strict Henselization of the local ring in x is the ring of algebraic functions on d variables.
In this section we will prove an analogous result for restricted smooth models of a central simple algebra Δ of dimension n^2 over a field K of trancendence degree d. We will show that for given n and d there are only finitely many types of étale local behavior.
Hence, fix a restricted smooth model A and consider a cover of $proj_n^{ss}\ A$ by affine open sets determined by $X(c)$ where $c \in C$ is homogeneous and

$$A_c^g \simeq B[d, d^{-1}]$$

with d central of degree one. Let Z be the center of B, then Z has field of fractions K. Let $m \triangleleft Z$ be a maximal ideal. We want to study the structure of the algebra

$$B_m^{sh} = B \otimes_Z Z_m^{sh}$$

From the foregoing section we recall that m determines a unique closed orbit Orb_x in $mod_n\ B \overset{open}{\hookrightarrow} proj_n^{ss}\ A$ with M_x a semi-simple B-module of dimension n. Consider the decomposition of M_x in simple components

$$M_x = S_1^{\oplus e_1} \oplus \ldots \oplus S_r^{\oplus e_r}$$

with S_i a simple B-module of dimension d_i. Then, $\sum e_i d_i = n$ and the corresponding point $x \in mod_n\ B$ is given by the trace preserving morphism

$$x : B \twoheadrightarrow \overline{B} = B/mB \twoheadrightarrow \overline{B}/rad(\overline{B}) = M_{d_1}(\mathbb{C})^{\oplus e_1} \oplus \ldots \oplus M_{d_r}(\mathbb{C})^{\oplus e_r}$$

We say that x (or m) is a point of representation type

$$\tau(x) = \tau(m) = (e_1, d_1; \ldots; e_r, d_r)$$

The stabilizer subgroup GL_x is then verified to be

$$GL_{e_1} \times \ldots \times GL_{e_r}$$

embedded in GL_n via

$$\begin{bmatrix} GL_{e_1} \otimes 1_{d_1} & & \\ & \ddots & \\ & & GL_{e_r} \otimes 1_{d_r} \end{bmatrix} \hookrightarrow GL_n$$

We remark that GL_x depends only on its representation type τ and we will denote by $GL(\tau)$ the group $\times GL_{e_i}$ embedded in GL_n as above.
It will turn out that in order to describe B^{sh}_m we have to be able to understand the normal space in x to the orbit Orb_x as a module over the stabilizer subgroup $GL_x = GL(\tau)$.
We have GL_n-equivariant closed embeddings

$$Orb_x \hookrightarrow mod_n\ A \hookrightarrow M_n(\mathbb{C})^{\oplus m}$$

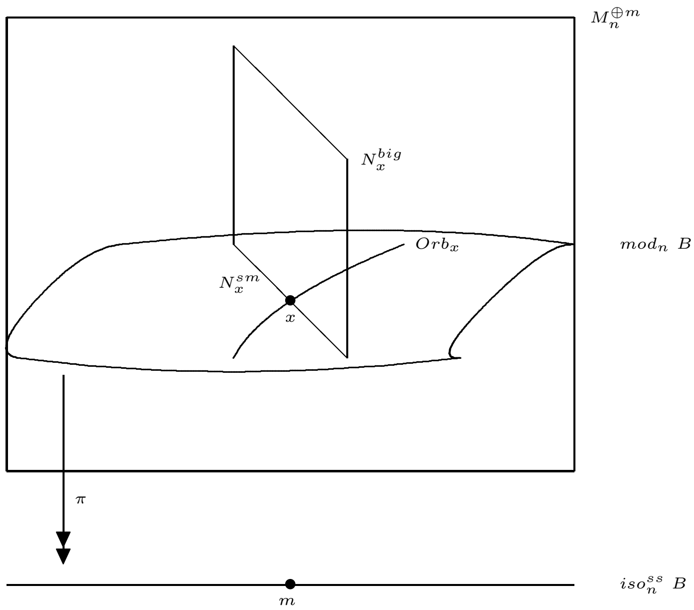
if $b_1, \ldots, b_m$ are generators of B. We have embeddings of the respective tangent spaces in x

$$T_x\ Orb_x \hookrightarrow T_x\ mod_n\ B \hookrightarrow T_x\ M_n(\mathbb{C})^{\oplus m}$$

which are embeddings as $GL(\tau)$-modules and hence by reductivity of $GL(\tau)$ they are direct factors.
Therefore, we have for the normal spaces to the orbit in $mod_n\ B$ resp. $M_n(\mathbb{C})^{\oplus m}$ that

$$N^{sm}_x = \frac{T_x\ X_A}{T_x\ Orb_x} \lhd N^{big}_x = \frac{T_x\ M_n(\mathbb{C})^{\oplus m}}{T_x\ Orb_x}$$

as $GL(\tau)$-modules. That is, we have the following picture



Before we compute these $GL(\tau)$-modules, let us explain the relevance to our problem.

This is an application of the **Luna slice theorem** in invariant theory adapted to the situation of interest to us. In general, if H is a reductive subgroup of G acting on an affine variety Z then one defines an H-action on $G \times Z$ via the map

$$h.(g,z) = (gh^{-1}, h.z)$$

The corresponding quotient is called the associated fiber bundle

$$G \times^H Z = (G \times Z)/H$$

and it acquires a G-action via multiplication on the left in the first component. One can show that the corresponding quotient satisfies

$$(G \times^H Z)/G \simeq Z/H$$

THEOREM 5.3.1 (Luna slice theorem). *Let x be a smooth point of $mod_n\ B$ of representation type τ. Then, there exists a locally closed affine smooth subvariety $S_x \hookrightarrow mod_n\ B$ containing x, which is stable under the action of $GL(\tau)$ satisfying the following properties*

- *The map $GL_n \times S_x \longrightarrow mod_n\ B$ obtained by $(g,s) \mapsto g.s$ induces a GL_n-equivariant étale map*

  $$\psi : GL_n \times^{GL(\tau)} S_x \longrightarrow mod_n\ A$$

  with affine image. Moreover the induced quotient map

  $$\psi/GL_n : (GL_n \times^{GL(\tau)} S_x)/GL_n = S_x/GL(\tau) \longrightarrow mod_n\ B/GL_n = iso_n^{ss}\ B$$

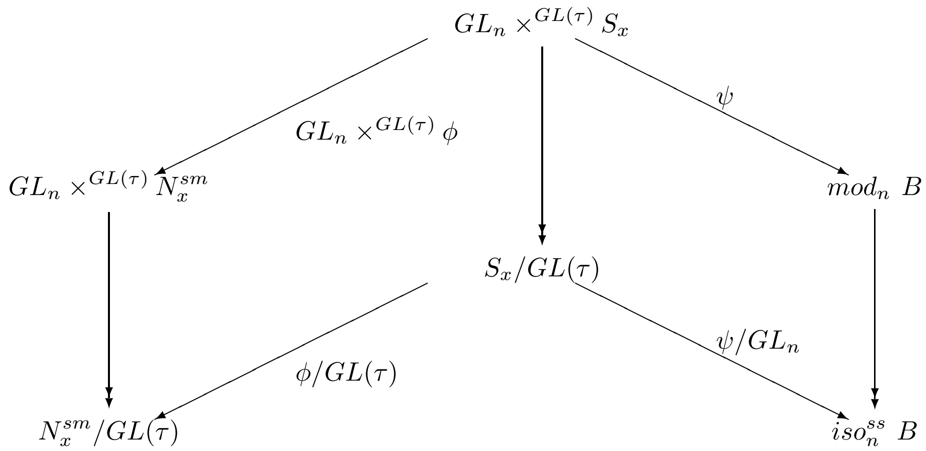
  is also étale.
- *There is a $GL(\tau)$-equivariant map*

  $$\phi : S_x \longrightarrow N_x^{sm} = T_x\ S_x$$

  such that $\phi(x) = 0$ and with affine image. The induced quotient map

  $$\phi/GL(\tau) : S_x/GL(\tau) \longrightarrow N_x^{sm}/GL(\tau)$$

  is also étale.
- *The above maps induce the following commutative diagram*



  where the vertical maps are the quotient maps, all diagonal maps are étale and the upper ones are GL_n-equivariant.

Hence, the GL_n-local structure of $mod_n\ B$ in a neighborhood of x is the same as that of $GL_n \times^{GL(\tau)} N_x^{sm}$ in a neighborhood of $\overline{(1_n, 0)}$. Similarly, the local structure of $iso_n^{ss}\ B$ in a neighborhood of m is the same as that of $N_x^{sm}/GL(\tau)$ in a neighborhood of $\overline{0}$. Therefore, we have

THEOREM 5.3.2. *Let A be a restricted smooth model of Δ and consider an open cover $proj_n^{ss}\ A \mid X(c) = mod_n\ B$ where B has center Z.*
Let m be a maximal ideal of Z corresponding to a point $x \in mod_n\ B$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$.
Let p denote the maximal ideal of $\mathbb{C}[N_x^{small}/GL(\tau)]$ corresponding to the point $\overline{0}$. Then,

1. $Z_m^{sh} \simeq \mathbb{C}[N_x^{sm}/GL(\tau)]_p^{sh}$
2. $B_m^{sh} \simeq (M_n(\mathbb{C}[GL_n \times^{GL(\tau)} N_x^{sm}])^{GL_n})_p^{sh}$

Hence, we know the étale local structure of Z and B in m if we know the $GL(\tau)$-module structure of N_x^{sm}.

Since we know the embedding $GL(\tau) \hookrightarrow GL_n$ and the action of GL_n on $M_n(\mathbb{C})^{\oplus m}$ (by simultaneous conjugation) we know the structure of $T_x\ M_n(\mathbb{C})^{\oplus m} = M_n(\mathbb{C})^{\oplus}$ as $GL(\tau)$-module. Further, the exact sequence

$$0 \longrightarrow Lie\ GL(\tau) \longrightarrow Lie\ GL_n \longrightarrow T_x\ Orb_x \longrightarrow 0$$

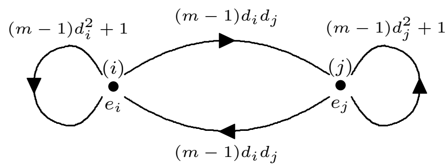
allows us to determine the $GL(\tau)$-module structure of $T_x\ Orb_x$ and consequently that of $N_x^{big} = T_x\ M_n(\mathbb{C})^{\oplus m}/T_x\ Orb_x$.
Once we know an isotypical decomposition of N_x^{big}, taking a direct subsum we obtain all possibilities for N_x^{sm}. Of course, later on, we will have to verify which of these theoretical possibilities actually occur from a restricted smooth model.
Rather than writing down decompositions of $N_x^{sm} \lhd N_x^{big}$ in simple $GL(\tau)$-modules we prefer to represent this information by a 'local chart'. We use the following dictionary

- a **loop** at vertex (i) corresponds with the $GL(\tau)$-module $M_{e_i}(\mathbb{C})$ on which GL_{e_i} acts by conjugation and the other factors act trivially.
- an **arrow** from vertex (i) to vertex (j) corresponds to the $GL(\tau)$-module $M_{e_i \times e_j}(\mathbb{C})$ on which $GL_{e_i} \times GL_{e_j}$ act via $g.m = g_i m g_j^{-1}$ and the other factors act trivially.
- a **marked loop** at vertex (i) corresponds to the simple $GL(\tau)$-module $M_{e_i}^0(\mathbb{C})$, that is, trace zero matrices with action of GL_{e_i} by conjugation and trivial action by the other components.
- the label of a loop or arrow indicates the multiplicity of the corresponding representation.

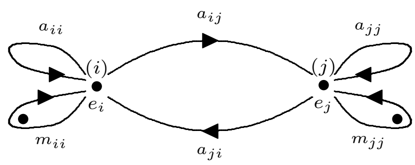
LEMMA 5.3.3. *With conventions as above and x a point of representation type τ we have*

1. *The $GL(\tau) = GL_{e_1} \times \ldots \times GL_{e_r}$-module structure of N_x^{big} can be represented by the local chart on r vertices such that the subchart on any two vertices*

$1 \leq i, j \leq r$ *is of the form*



2. *The* $GL(\tau)$*-module structure of* N_x^{sm} *can be represented by a local chart on* r *vertices such that the subgraph on any two vertices* $1 \leq i, j \leq r$ *is of the form*



where $a_{ij} \leq (m-1)d_i d_j$ *and* $a_{ii} + m_{ii} \leq (m-1)d_i^2 + 1$ *for all* $1 \leq i, j \leq r$*.*

Proof. (2) follows from (1) by observing that $M_{e_i \times e_j}(\mathbb{C})$ is a simple $GL(\tau)$-module and that the isotypical decomposition of $M_{e_i}(\mathbb{C}) = M_{e_i}^0(\mathbb{C}) \oplus \mathbb{C}_{triv}$ where $\mathbb{C}_{triv}$ is the trivial one-dimensional $GL(\tau)$-module. $\square$

5.4. Classifying local charts

A local chart $C = (M, \mathbf{e})$ consists of two data : the underlying 'map' M that is, the marked labeled directed graph and the 'dimension-vector' $\mathbf{e} = (e_1, \ldots, e_r)$. If we specify $\mathbf{e}$ we obtain a $GL(\boldsymbol{e}) = \times GL_{e_i}$-module $R(M, \mathbf{e})$ any vector of which we call a representation of the map M of dimension $\mathbf{e}$. That is $v \in R(M, \mathbf{e})$ assigns to each

- arrow from (i) to (j) a matrix in $M_{e_i \times e_j}(\mathbb{C})$
- unmarked loop in (i) a matrix in $M_{e_i}(\mathbb{C})$
- marked loop in (i) a trace zero matrix in $M_{e_i}^0(\mathbb{C})$

A morphism from a representation $v \in R(M, \mathbf{e})$ to a representation $w \in R(M, \mathbf{f})$ is an r-tuple of linear maps $\psi = (\psi_1, \ldots, \psi_r) \in \oplus_i M_{f_i \times e_i}(\mathbb{C})$ such that every diagram

$$\begin{array}{ccc} \mathbb{C}^{e_i} & \xrightarrow{v} & \mathbb{C}^{e_j} \\ \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_j} \\ \mathbb{C}^{f_i} & \xrightarrow{v} & \mathbb{C}^{f_j} \end{array}$$

is commutative where the horizontal maps are either arrows or (marked) loops in M.
Having morphisms, the notions of sub- , quotient- and simple-representation are obvious as are direct sums of representations of M. If we view the $GL(\mathbf{e})$-module $R(M, \mathbf{e})$ as an affine space on which $GL(\mathbf{e})$ acts, then orbits correspond precisely to isomorphism classes of representations.

Lemma 5.4.1. *The local chart $C = (M, \mathbf{e})$ of a restricted smooth model must be such that $R(M, \mathbf{e})$ contains simple representations of M.*

Proof. Consider a point $x \in mod_n\ B$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$ with $N_x^{sm} = R(M, \mathbf{e})$. By the Luna slice theorem we have étale GL_n-equivariant maps

$$GL_n \times^{GL(\tau)} N_x^{sm} \xleftarrow{et} GL_n \times^{GL(\tau)} S_x \xrightarrow{et} mod_n\ B$$

As B is a prime order, we have that any Zariski neighborhood of x in $mod_n\ B$ contains simple orbits, that is, closed orbits with stabilizer $\mathbb{C}^*$. Because the maps above are GL_n-equivariant and étale every Zariski neighborhood of $\overline{(1_n, 0)}$ contains a closed GL_n-orbits with stabilizer $\mathbb{C}^*$. By the correspondence of orbits in fiber bundles there must be closed $GL(\tau)$-orbits in $N_x^{sm} = R(M, \mathbf{e})$ with stabilizer $\mathbb{C}^*$. By a version of the Artin-Voigt theorem for representations of the map M closed orbits correspond to semi-simple representations of M. If the stabilizer of such a representation is $\mathbb{C}^*$ then it must be simple. □

Hence, we have to determine which dimension vectors can arise from simple representations of the map M. We define the **Euler-form** of M as the bilinear map

$$\chi_M : \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$$

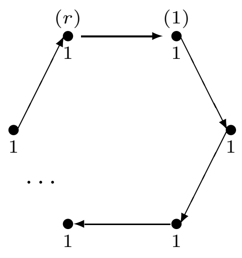
determined by the matrix $\chi_M = (\chi_{ij})$ with entries

$$\chi_{ij} = -a_{ij} \text{ and } \chi_{ii} = 1 - a_{ii} - m_{ii}$$

where a_ij is the number of arrows from (i) to (j) in M and a_{ii} resp. m_{ii} are the number of (resp. marked) loops at (i).

Proposition 5.4.2. *$\mathbf{e} = (e_1, \ldots, e_r)$ is the dimension-vector of a simple representation of the map M if and only if one of the following situations occurs*

1. *$M = \tilde{A}_r$ the extended Dynkin diagram with cyclic orientation and $\mathbf{e} = (1, \ldots, 1)$.*



2. *$M \neq \tilde{A}_r$. Then, M has to be strongly connected (that is, any two vertices can be connected by a directed path) and if $\delta_i = (\delta_{1i}, \ldots, \delta_{ri})$ are a standard basis of $\mathbb{Z}^r$ we must have*

$$\chi_M(\mathbf{e}, \delta_i) \leq 0 \text{ and } \chi(\delta_i, \mathbf{e}) \leq 0$$

for all $1 \leq i \leq r$.

PROOF. We will only prove necessity of the conditions in (2). Sufficiency follows from a degeneration argument and induction.
Let $v \in R(M, \mathbf{e})$ be a simple representation (that is, contains no proper subrepresentations) and let $v(\phi)$ denote the linear map determined by the arrow, loop or marked loop ϕ.
Assume M is not strongly connected, then we can divide M into maximal strongly connected submaps $M_1, \ldots, M_z$ say. The direction of all arrows between two such components must be all the same by maximality. Hence, there is a component M_i having no arrows to other components. Now, define a proper subrepresentation w of v with dimension-vector $\mathbf{f} = \delta_M.\mathbf{e}$ by $w(\phi) = v(\phi)$ if ϕ is a map in M_i and $w(\phi) = 0$ otherwise. Hence, M must be strongly connected.
For each (i) we have $\chi_M(\delta_i, \mathbf{e}) = e_i - \sum_{(i) \xrightarrow{\phi} (j)} e_j$ Hence, if $\chi_M(\delta_i, \boldsymbol{e}) > k$ then the natural morphism

$$\bigoplus_{(i) \xrightarrow{\phi} (j)} v(\phi) : \mathbb{C}^{e_i} \longrightarrow \bigoplus_{(i) \xrightarrow{\phi} (j)} \mathbb{C}^{e_j}$$

has a non-trivial kernel K of dimension $k > 0$ and determines a proper subrepresentation of v of dimension-vector $\mathbf{f} = (\delta_i j.k)_j$.
Similarly, if $\chi_M(\mathbf{e}, \delta_i) = e_i - \sum_{(j) \xrightarrow{\phi} (i)} e_j > 0$ then the image of the natural morphism

$$\bigoplus_{(j) \xrightarrow{\phi} (i)} v(\phi) : \bigoplus_{(i) \xrightarrow{\phi} (j)} \mathbb{C}^{e_j} \longrightarrow \mathbb{C}^{e_i}$$

is a proper subspace of $\mathbb{C}^{e_i}$ of dimension $k < e_i$ and hence determines a proper subrepresentation of v with dimension-vector $\mathbf{e} + (k - e_i)\delta_i$. $\square$

PROPOSITION 5.4.3. *The local chart $C = (M, \mathbf{e})$ of a restricted smooth model for Δ a central simple K-algebra with $trdeg_{\mathbb{C}}(K) = d$ must be such that*

$$1 - \chi_M(\mathbf{e}, \mathbf{e}) - \sum_i m_{ii} = d$$

PROOF. Consider a point $x \in mod_n\ B$ of representation type $\tau = (e_1, d_1; \ldots; e_r, d_r)$ with $N_x^{sm} = R(M, \mathbf{e})$. By the Luna slice theorem we have étale maps

$$N_x^{sm}/GL(\tau) \xleftarrow{et} S_x/GL(\tau) \xrightarrow{et} iso_n^{ss}\ B$$

Because $\mathbb{C}[iso_n^{ss}\ B] = Z$ with functionfield K we have that $iso_n^{ss}\ B$ and hence $N_x^{sm}/GL(\tau)$ must be of dimension d.
By definition of the Euler-form of M we have that

$$\chi(\mathbf{e}, \mathbf{e}) = -\sum_{i \neq j} e_i e_j a_{ij} + \sum_i e_i^2 (1 - a_{ii} - m_{ii})$$

On the other hand we have the following dimensions

$$dim\ R(M, \mathbf{e}) = \sum_{i \neq j} e_i e_j a_{ij} + \sum_i e_i^2 (a_{ii} + m_{ii}) - \sum_i m_{ii}$$

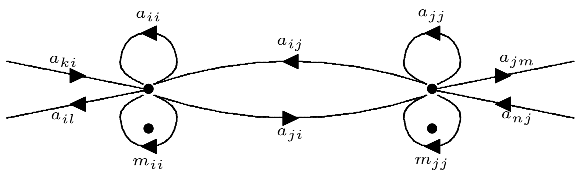
$$dim\ GL(\mathbf{e}) = \sum_i e_i^2$$

As $\mathbf{e}$ is the dimension vector of a simple representation we know that the orbits in general position in $R(M,\mathbf{e})$ are closed and have stabilizer $\mathbb{C}^*$. Therefore, the dimension of the quotient variety $R(M,\mathbf{e})/GL(\mathbf{e}) = N_x^{sm}/GL(\tau)$ is equal to

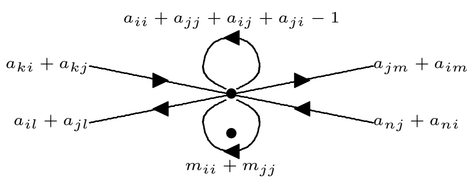
$$dim\ R(M,\mathbf{e}) - dim\ GL(\mathbf{e}) + 1$$

and plugging in the above information we see that this is equal to $1 - \chi(\mathbf{e},\mathbf{e}) - \sum_i m_{ii}$. □

If we want to study the local structure of restricted smooth models for central simple algebras over a field of trancendence degree d, we have to compile a list of admissible charts. We will give the first steps in such a classification.
The basic idea that we use is to shrink a chart to its simplest form and classify these simplest forms for given d. By **shrinking** we mean the following process. Assume $\mathbf{e}$ is the dimension vector of a simple representation of M and let (i) and (j) be two connected vertices with $e_i = e_j = e$. That is we have locally the following situation



We will use one of the arrows connecting (i) with (j) to identify the two vertices. That is, we form the shrinked chart $C^s = (M^s, \mathbf{e^s})$ where M^s is a map on $r-1$ vertices $\{(1),\ldots,\hat{(i)},\ldots,(r)\}$ and $\mathbf{e^s}$ is the dimension vector with (i) removed. That is, locally round z the shrinked chart has the form



That is, in M^s we have for all $k,l \neq j$ that $a^s_{kl} = a_{kl}$. Moreover, the number of arrows and (marked) loops connected to j are determined as follows

- $a^s_{jk} = a_{ik} + a_{jk}$
- $a^s_{kj} = a_{ki} + a_{kj}$
- $a^s_{jj} = a_{ii} + a_{jj} + a_{ij} + a_{ji} - 1$
- $m^s_{jj} = m_{ii} + m_{jj}$

Lemma 5.4.4. $\mathbf{e}$ *is the dimension vector of a simple representation of* M *if and only if* $\mathbf{e^s}$ *is the dimension vector of a simple representation of* M^s. *Moreover,*

$$dim\ R(M, \mathbf{e})/GL(\mathbf{e}) = dim\ R(M^s, \mathbf{e^s})/GL(\mathbf{e^s})$$

Proof. Fix an arrow ϕ connecting (i) and (j). As $e_i = e_j = e$ there is a Zariski open subset $U \hookrightarrow R(M, \mathbf{e})$ of points v such that $v(\phi)$ is invertible. By basechange in either (i) or (j) we can find a point w in its orbit such that $w(\phi) = I_e$. If we think of $w(\phi)$ as identifying $\mathbb{C}^{e_i}$ with $\mathbb{C}^{e_j}$ we can view the remaining maps of w as a representation in $R(M^s, \mathbf{e^s})$ and denote it with w^s. the map $U \longrightarrow R(M^s, \mathbf{e^s})$ is well-defined and maps $GL(\mathbf{e})$-orbits to $GL(\mathbf{e^s})$-orbits.
Conversely, given a representation $w' \in R(M^s, \mathbf{e^s})$ we can uniquely determine a representation $w \in U$ mapping to w'.
Both claims follow immediately from this observation. □

It is clear that any chart can uniquely be reduced to its simplest form, which has the property that no connecting vertices can have same dimension. Also note that the shrinking process has a not necessarily unique converse operation which we will call **splitting of a vertex**.

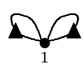
Proposition 5.4.5. *Let* $\mathbf{e}$ *be the dimension vector of a simple representation of* M *and let* $d = dim\ R(M, \mathbf{e})/GL(\mathbf{e})$. *If* $e = max\ e_i$, *then* $d \geq e + 1$.

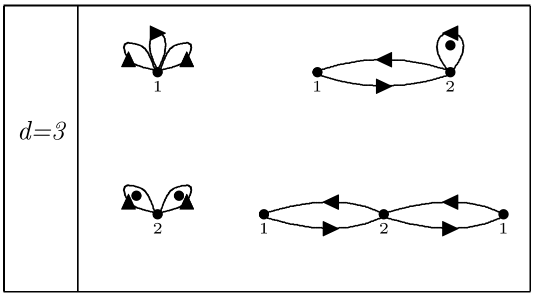
Proof. Exercise! First reduce the chart to its simplest form and compute the incoming and outgoing contributions in a vertex to the dimension of the quotient-variety. □

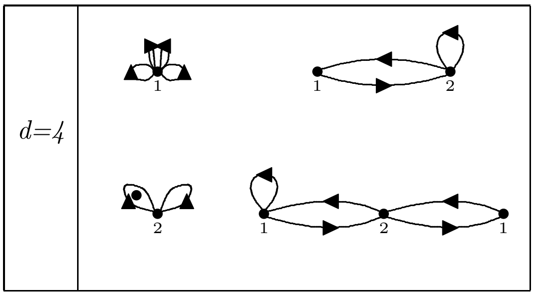
Definition 5.4.6. Two charts $C = (M, \mathbf{e})$ and $C' = (M, \mathbf{e})$ are said to be equivalent if their corresponding $GL(\mathbf{e})$-modules are isomorphic.

Example 5.4.7. The charts below are equivalent

Theorem 5.4.8. *The local charts occurring for a restricted smooth model for central simple algebras over a field of trancendence degree d can be shrinked to one of the following equivalence classes of charts*

d=1	
d=2	





5.5. Reading the local chart

Knowing which local charts can occur, we will now investigate what information can be derived from the local chart.
We will fix the following situation : A is a restricted smooth model of Δ and we consider on open subvariety $proj_m^{ss}\ A \mid X(c) = mod_n\ B$ where B has center Z. m will be a maximal ideal of Z corresponding to the closed orbit $GL_n.x \hookrightarrow mod_n\ B$ where x has representation type $\tau = (e_1, d_1; \dots; e_r, d_r)$.
We have $N_x^{sm} = R(M, \mathbf{e})$ as $GL(\tau) = \times GL_{e_i}$- module. The local structure of Z near m is determined by that of $\mathbb{C}[N_x^{sm}/GL_\tau]$ near the zero representation, so we better have an interpretation of this ring

PROPOSITION 5.5.1. *$\mathbb{C}[N_x^{sm}/GL(\tau)]$ is generated by traces along oriented cycles in the chart $C = (M, \mathbf{e})$.*

That is, for every arrow ϕ (resp. loop or marked loop) from (i) to (j) we take a generic rectangular matrix

$$M_\phi = \begin{bmatrix} x_{11}(\phi) & \dots & \dots & x_{1,e_j}(\phi) \\ \vdots & & & \vdots \\ x_{e_i1}(\phi) & \dots & \dots & x_{e_ie_j}(\phi) \end{bmatrix}$$

(resp. a generic square matrix or generic trace zero matrix).
If $cyc = \phi_k \circ \dots \circ \phi_2 \circ \phi 1$ is an oriented cycle in the map M, then we compute the following matrix

$$M_{cyc} = M_{\phi_k}. \dots .M_{\phi_2}.M_{\phi_1}$$

over $\mathbb{C}[x_{kl}(\phi)] = \mathbb{C}[R(M, \mathbf{e})]$. If the starting vertex of ϕ_1 is (i), then this is a square $e_i \times e_i$ matrix and we can consider its trace

$$Tr(M_{cyc}) \in \mathbb{C}[R(M, \mathbf{e})]$$

and one verifies easily that this polynomial is invariant under the action of GL_τ. Slightly harder to prove is that these functions actually generate

$$\mathbb{C}[R(M,\mathbf{e})]^{GL_\tau} = \mathbb{C}[N_x^{sm}/GL_\tau]$$

The essential ingredient in this proof is the fact that the polynomial invariants of tuples of matrices under simultaneous conjugation are generated by traces of products of generic matrices.
In fact, one can even bound the length of the oriented cycles to be considered by $(\sum_i e_i)^2$.
Next, let us consider the étale local structure of B near m. By the results proved before, we have to control for this the ring of GL_n-equivariant maps

$$GL_n \times^{GL_\tau} R(M,\mathbf{e}) \longrightarrow M_n(\mathbb{C})$$

on which the multiplication is given by that in target space $M_n(\mathbb{C})$.

Proposition 5.5.2. *The ring of GL_n-equivariant maps is Morita equivalent to the ring of GL_τ-equivariant maps*

$$R(M,\mathbf{e}) \longrightarrow M_{\sum e_i}(\mathbb{C})$$

where for any two vertices (i) and (j) the GL_τ-equivariant maps

$$R(M,\mathbf{e}) \longrightarrow Hom(\mathbb{C}^{\oplus e_i}, \mathbb{C}^{\oplus e_j})$$

are generated as a module over $\mathbb{C}[N_x^{sm}/GL(\tau)]$ by the paths in the map M starting from (i) and ending in (j).

Again, if $path = \phi_k \circ \ldots \circ \phi_1$ is such a path, then the corresponding module element is M_{path}. Again, this result follows from a minor adaptation to existing results on invariants and concomitants of representations of quivers proved by C. Procesi and myself.
Apart from allowing us to compute the local structure of Z and B near m, the local chart $N_x^{sm} = R(M,\mathbf{e})$ also allows us to describe the local charts in nearby points and the dimensions of subvarieties of points having a specific local chart.
The points ζ in the quotient variety $N_x^{sm}/GL(\tau) = R(M,\mathbf{e})/GL(\mathbf{e})$ are in one-to-one correspondence with the isomorphism classes of semi-simple representations of the map M of dimension vector $\mathbf{e}$.
If V_ζ is a representative in the closed orbit corresponding to ζ then we can decompose V_ζ into its simple representations

$$V_\zeta = W_1^{\oplus m_1} \oplus \ldots \oplus W_k^{\oplus m_k}$$

where W_i is a simple representation of the map M of dimension vector $\mathbf{b_i}$ and occurring in V_ζ with multiplicity m_i.
Extending previous terminology we will say that V_ζ (or ζ) is of representation type $\sigma = (m_1, \mathbf{b_1}; \ldots; m_k, \mathbf{b_k})$.
As we have a combinatorial description of all simple dimension vectors for M we can determine which representation types can occur for a given $\mathbf{e}$.
With V_σ we will denote the set of all points $\zeta \in N_x^{sm}/GL_\tau$ of representation type σ.

Proposition 5.5.3. *$\{V_\sigma : \sigma$ a representation type for $\mathbf{e}\}$ is a finite stratification of the quotient variety $N_x^{sm}/GL(\tau)$ into locally closed irreducible smooth subvarieties.*

Moreover, the dimension of the stratum V_σ determined by $\sigma = (m_1, \mathbf{b_1}; \ldots; m_k, \mathbf{b_k})$ is equal to

$$\sum_{j=1}^{k}(1 - \chi_M(\mathbf{b_j}, \mathbf{b_j})) - \sum_{i=1}^{r} m_{ii}$$

PROOF. According to the Luna slice results we have to verify that the representation type determines the stabilizer subgroup of a point in the closed orbit up to conjugation in $GL(\mathbf{e})$.
Let $\mathbf{b_i} = (b_{i1}, \ldots, b_{ir})$ and denote $b_i = \sum_j b_{ij}$. We choose a basis in $\oplus_i \mathbb{C}^{\oplus e_i}$ in the following way : the first $m_1 b_1$ vectors give a basis for the simple components of type W_1, the next $e_2 b_2$ vectors give a basis for the simple components of type W_2 and so on.
If $m = \sum a_i$, the subring of $M_m(\mathbb{C})$ generated by the representation V_ζ expressed in this basis is

$$\begin{bmatrix} M_{b_1}(\mathbb{C}) \otimes I_{e_1} & & \\ & \ddots & \\ & & M_{b_k}(\mathbb{C}) \otimes I_{e_k} \end{bmatrix}$$

Therefore, the stabilizer GL_V in $GL(\mathbf{e})$ of V_ζ is the group of units of the centralizer of this ring and is therefore equal to $GL_{m_1} \times \ldots \times GL_{m_k}$ which is embedded in $GL(\mathbf{e})$ with respect to the chosen basis as

$$\begin{bmatrix} GL_{m_1}(\mathbb{C} \otimes I_{b_1}) & & \\ & \ddots & \\ & & GL_{m_k}(\mathbb{C} \otimes I_{b_k}) \end{bmatrix}$$

It is easy to see that the conjugacy class of GL_V depends only on the representation type τ.
Finally, we have seen before that the dimension of the variety of isoclasses of simple representations of M of dimension vector $\mathbf{b_j}$ is equal to $1 - \chi(\mathbf{b_j}, \mathbf{b_j}) - \sum_i m_{ii}$ from which the claim about the dimension of the stratum follows. □

Given two representations types σ and σ', the stratum $V_{\sigma'}$ lies in the closure of the stratum V_σ if and only if the stabilizer subgroup GL_σ is conjugated to a subgroup of GL_σ in $GL(\mathbf{e})$. Again, mimicking similar results for representations of quivers we can give a combinatorial solution to this problem.
Two representation types

$$\sigma = (m_1, \mathbf{b_1}; \ldots; m_k, \mathbf{b_k}) \text{ and } \sigma' = (m'_1, \mathbf{b_1}'; \ldots; m'_{k'}, \mathbf{b_{k'}}')$$

are said to be direct successors $\sigma < \sigma'$ if and only if either

- (splitting one simple type) $k' = k + 1$ and for all but one $1 \le i \le k$ we have $(m_i, \mathbf{b_i}) = (m'_j, \mathbf{b_j}')$ for a uniquely determined j and for the remaining i we have corresponding to it $(m_i, \mathbf{b_u}'; m_i, \mathbf{b_v}')$ with $\mathbf{b_i} = \mathbf{b_u}' + \mathbf{b_v}'$.
- (combining two simple types) $k' = k - 1$ and for all but one $1 \le i \le k'$ we have $(m'_i, \mathbf{b_i}') = (m_j, \mathbf{b_j})$ for a uniquely determined j and for the remaining i we have corresponding to it $(m_u, \mathbf{b_i}'; m_v, \mathbf{b_i}')$ with $m_u + m_v = m'_i$

The direct successor relation $<$ induces an ordering which we will denote with $\ll$.

PROPOSITION 5.5.4. *The stratum $V_{\sigma'}$ lies in the closure of the stratum V_σ if and only if $\sigma \ll \sigma'$.*

Finally, we want to understand the étale local structure of the quotient variety $N_x^{sm}/GL(\tau)$ in a neighborhood of a point $\zeta \in V_\sigma$. This again is an application of the Luna slice results.
So, let V be a semi-simple representation of M corresponding to $\zeta \in V_\sigma$ with stabilizer subgroup $GL_\sigma = GL_{m_1} \times \ldots GL_{m_k}$. We have to investigate the GL_σ-module structure of the normal space to the orbit of V.
The tangentspace to the $GL(\mathbf{e})$ orbit of V is equal to the image of the natural linear map

$$Lie\ GL(\mathbf{e}) \longrightarrow R(M, \mathbf{e})$$

sending an element $y \in Lie\ GL(\mathbf{e})$ to the representation determined by the commutator $[y, V] = y.V - V.y \in M_m(\mathbb{C})$ where as above $m = \sum e_i$ and all embeddings are with respect to the choice of basis we introduced in the proof of proposition.
The kernel of the above map is the centralizer of the subalgebra of $M_m(\mathbb{C})$ generated by the representation V, that is, the algebra

$$C_V = \begin{bmatrix} M_{m_1}(\mathbb{C} \otimes I_{b_1}) & & \\ & \ddots & \\ & & M_{m_r}(\mathbb{C} \otimes I_{b_r}) \end{bmatrix}$$

We thus have an exact sequence of GL_σ-modules

$$0 \longrightarrow C_V \longrightarrow Lie\ GL(\mathbf{e}) \longrightarrow T_V\ Orb_V \longrightarrow 0$$

where the action of GL_σ is given by conjugation in $M_m(\mathbb{C})$ via the embedding given before.
A typical element $\gamma \in GL_\sigma = GL_{m_1} \times \ldots \times GL_{m_k}$ will be written as $(\gamma_1, \ldots, \gamma_k)$ and we will express the actions in terms of the γ_i.
C_V as GL_σ-module consists of

- one m_1^2-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action

$$\vdots$$

- one m_k^2-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action

Moreover, using our notation $\mathbf{b_i} = (b_{i1}, \ldots, b_{ir})$ we have that $Lie\ GL(\mathbf{e})$ as GL_σ-module consists of

- $\sum_{j=1}^k b_{1j}^2$ times the m_1^2-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action

$$\vdots$$

- $\sum_{j=1}^k b_{kj}^2$ times the m_k^2-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action
- $\sum_{j=1}^k b_{1j}b_{2j}$ times the $m_1 \times m_2$-dimensional representation with $\gamma_1^{-1}.\gamma_2$-action

$$\vdots$$

- $\sum_{j=1}^k b_{kj}b_{k-1j}$ times the $m_k \times m_{k-1}$ -dimensional representation with $\gamma_k^{-1}.\gamma_{k-1}$-action

Hence, we know the GL_σ-module structure of $T_V\ Orb_V$. Next, we have to determine the GL_σ-module structure of $R(M, \mathbf{e})$. For each arrow ϕ with start vertex (i) and (distinct) end vertex (j) there are

- $b_{1i}b_{1j}$ times the $m_1 \times m_1$-dimensional representation with $\gamma_1^{-1}.\gamma_1$-action

- $b_{1i}b_{2j}$ times the $m_1 \times m_2$-dimensional representation with $\gamma_1^{-1}.\gamma_2$-action

$$\vdots$$

- $b_{ki}b_{kj}$ times the $m_k \times m_k$-dimensional representation with $\gamma_k^{-1}.\gamma_k$-action

For each unmarked loop in (i) we have the same decomposition as above replacing all occurrences of j with i. For a marked loop in (i) we have to replace the terms of dimension $m_i \times m_i$ by

- b_{li}^2 times the $m_i^2 - 1$-dimensional representation of trace zero matrices with $\gamma_l^{-1}.\gamma_l$-action.

We now have all the information on the GL_σ-module structure on the normal space to the orbit using the (split) exact sequence of GL_σ-modules

$$0 \longrightarrow T_V\ Orb_V \longrightarrow R(M, \mathbf{e}) \longrightarrow N_V \longrightarrow 0$$

and we obtain

PROPOSITION 5.5.5. *The étale local structure of $N_x^{sm}/GL(\tau)$ near $\zeta \in V_\sigma$ is determined by a local chart $C_\sigma = (M_\sigma, \mathbf{e}_\sigma)$ where M_σ has k vertices $\{(1), \ldots, (k)\}$ and there are*

- $-\chi_M(\mathbf{b_i}, \mathbf{b_j})$ *directed arrows from (i) to (j) when $i \neq j$*
- $1 - \chi_1(\mathbf{b_i}, \mathbf{b_i})$ *unmarked loops in (i)*
- $-\chi_2(\mathbf{b_i}, \mathbf{b_i})$ *marked loops in (i)*

where $\chi_1 = (\delta_{ij} - a_{ij})_{i,j}$ and $\chi_2 = (-\delta_{ij} m_{ii})_{i,j}$ and $\chi_M = \chi_1 + \chi_2$. Moreover, the dimension vector $\mathbf{e}_\sigma = (m_1, \ldots, m_k)$.

CHAPTER 6

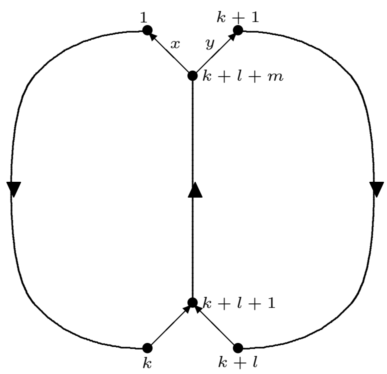
Non-commutative surfaces

In this chapter we apply the results proved before in the special case of surfaces, that is $d = 2$.

6.1. Local characterization

If $d = 2$ we will give an alternative proof of the classification of local charts as $\mathbb{Z}_n$-loops.

PROPOSITION 6.1.1. *For restricted smooth models of central simple algebras over the functionfield K of surfaces the local charts $C = (M, \mathbf{e})$ are such that $\mathbf{e} = (1, \dots, 1)$ and the map M has the following form :*



A_{klm}

where the indicated numbering of vertices and labeling of arrows will be used later. In this picture we make the obvious changes whenever k or l are zero.

PROOF. The strongly connected map M must contain more than one oriented cycle and hence contains a submap of the indicated type (possibly degenerated). It is easy to verify that for A_{klm}, $\mathbf{f} = (1, \dots, 1)$ is the dimension vector of a simple representation.
If M contains additional vertices $\{s = k + l + m + 1, \dots, r\}$ and/or the dimension vector $\mathbf{e} = (e_1, \dots, e_r) \neq \mathbf{f}$, there exist semi-simple representations in $R(M, \mathbf{e})$ with dimension-vector decomposition

$$(\underbrace{1, \dots, 1}_{k+l+m}, 0, \dots, 0) \oplus \delta_1^{e_1 - 1} \oplus \dots \oplus \delta_{s-1}^{e_{s-1} - 1} \oplus \delta_s^{e_s} \oplus \dots \oplus \delta_r^{\oplus e_r}$$

As $dim\ R(A_{klm}, \mathbf{f})/GL(\mathbf{f})$ is equal to 2 there is a two-dimensional family of such semi-simple representations. Hence, they cannot be properly semi-simple as their locus must be of dimension $< d = 2$. Therefore, $M = A_{klm}$ and $\mathbf{e} = \mathbf{f}$. □

Let A be a restricted smooth model for Δ, a central simple K-algebra of dimension n^2. Locally, A is of the form

$$A_c^g = B[d, d^{-1}]$$

with d central of degree one. Let m be a maximal ideal of $Z = Z(B)$ corresponding to a semi-simple n-dimensional B-module M_x. By the above characterization we know that M_x must have a decomposition

$$M_x = S_1 \oplus \ldots \oplus S_r$$

where S_i is a simple B-module of dimension d_i and all components are distinct. That is, $n = \sum_i d_i$ and the embedding of $GL(\mathbf{e}) = \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{r}$ in GL_n is given via

$$(\lambda_1, \ldots, \lambda_r) \mapsto diag(\underbrace{\lambda_1, \ldots, \lambda_1}_{d_1}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{d_r})$$

We want to describe the étale local structure of B near m, that is, the ring $B_m^{sh} = B \otimes_Z Z_m^{sh}$. In order to do this we have to compute the rings of invariants and concomitants of the local chart near the zero representation.

Proposition 6.1.2. *Using the labeling of vertices and arrows in the chart A_{klm} given above we have*

1. *The ring of polynomial invariants is equal to*

$$\mathbb{C}[R(A_{klm}, \epsilon)/GL(\epsilon)] = \mathbb{C}[x, y]$$

2. *The rings of GL_n-equivariant maps*

$$M_n(GL_n \times^{GL(\epsilon)} \mathbb{C}[R(A_{klm}, \epsilon])^{GL_n}$$

is isomorphic to the subring of $M_n(\mathbb{C}[x, y])$ with block decomposition

k	l	m
(x) ╲ (1)	(y)	(1)
(x)	(y) ╲ (1)	(1)
(x)	(y)	(x, y) ╲ (1)

where at place (i, j) (for every $1 \le i, j \le r$) there is a block of dimension $d_i \times d_j$ with entries the indicated ideal of $\mathbb{C}[x, y]$.

PROOF. By basechange in the vertices we see that all non-zero maps in a minimal oriented cycle can be taken to be the identity map except for one. If we define these remaining maps x and y then the traces along oriented cycles in the chart are of the form $x^i y^j$. The result about the equivariant maps follows from computing the $\mathbb{C}[x,y]$-module of paths from a vertex (i) to vertex (j) and applying the general results of the previous chapter. □

Using the Luna slice theorem, we obtain the required étale local classification

THEOREM 6.1.3. *With notations as before, we have*

1. $Z_m^{sh} \simeq \mathbb{C}\{x,y\}$
2. B_m^{sh} *is isomorphic to the subring of* $M_n(\mathbb{C}\{x,y\})$ *with the above block decomposition.*

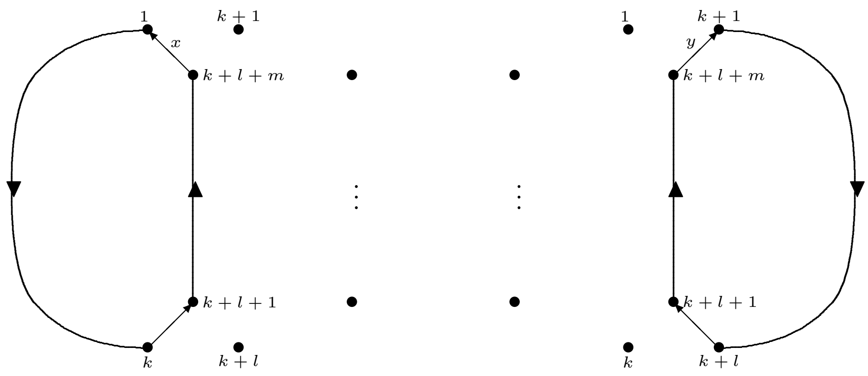
DEFINITION 6.1.4. A Z-order B in a central simple K-algebra Δ of dimension n^2 is said to be **étale locally split** in a maximal ideal m of Z iff B_m^{sh} has ring of fractions $M_n(K \otimes_Z Z_m^{sh})$.

From the étale local description of Z and B and étale descent we deduce

PROPOSITION 6.1.5. *If A is a restricted smooth model for a central simple K-algebra Δ of dimension n^2 and if $A_c^g = B[d,d^{-1}]$ with d central of degree one. Then,*

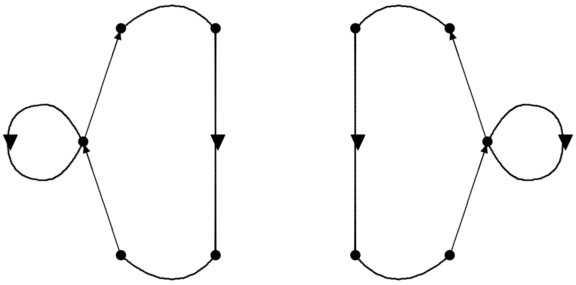
1. *The center $Z = Z(B)$ is smooth.*
2. *The non-Azumaya locus of B, $ram_B = iso_n^{ss}\ B - iso_n^s\ B$ consists at worst out of isolated (possibly embedded) points and a reduced divisor whose worst singularities are normal crossings.*
3. *B is étale locally split at every point $m \in iso_n^{ss}\ B$.*

PROOF. (1) and (3) are immediate from the foregoing theorem. As for (2) we have to proper semi-simple representations of $R(A_{klm}, \mathbf{e})$.
Their decomposition into simple representations can be depicted by one of the following two situations



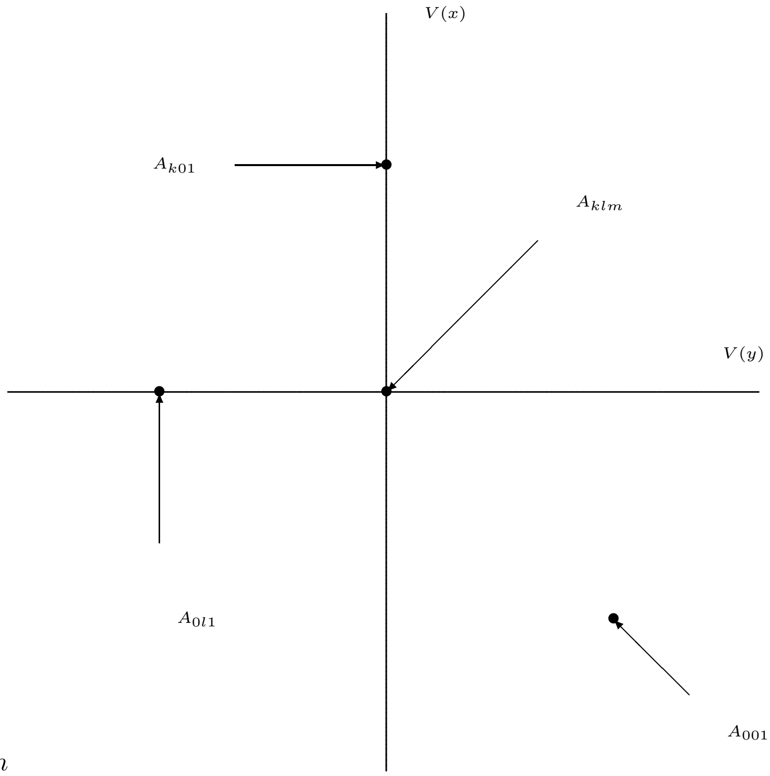
$V(y)$ $V(x)$

where the trace along the indicated oriented cycle is non-zero. By the general results of the foregoing chapter we can compute the local charts of $iso_n^{ss}\ B$ near such a point. They are resp. of the following types



A_{0l1} A_{k01}

and we have the following local picture of the structure of $iso_n^{ss}\ B$ near m.



from which the statement follows (taking care of possible degenerate cases, for example, an isolated point occurs for local charts of type A_{00m} with $m \geq 2$). □

LEMMA 6.1.6. *With notations as before, B is a projective module over Z if and only if all local charts are of type A_{kl1}. In particular, if a local chart is of type A_{klm} with $m \geq 2$, then gldim $B = \infty$.*

PROOF. As the center is smooth, projectivity and reflexivity as Z-module are equivalent. Observe that B_m^{sh} is reflexive only if no block of type (x, y) occurs, that is, iff $m = 1$. The last statement follows from the fact that an order of finite global dimension with smooth center has to be projective. □

6.2. Central blow-ups

We fix a field K of trancendence degree 2 and a central simple K-algebra of dimension n^2. In this section we begin our construction of a smooth model for Δ. We will assume throughout that all (commutative) smooth models for K are simply connected.

Choose a smooth projective surface X with functionfield K and fix an embedding $X \hookrightarrow \mathbb{P}^z$, or equivalently, a representation of the homogeneous coordinate ring $C = \mathbb{C}[u_0, \ldots, u_z]/(f_1, \ldots, f_v)$. We can cover $X = Proj\ C$ with affine open subsets $X(c)$ such that

$$C_c^g = Z[d, d^{-1}]$$

with $d \in C$ of degree one.

From the Artin-Mumford exact sequence we recall that Δ is determined by a $\mathbb{Z}_n$-wrinkle on X, that is, we are given

- A divisor $D \hookrightarrow X$ and a list of its irreducible components C_i which are irreducible curves on X
- The list of singular points $p_j \in X$ on D
- For each branch B_k of D at p_i a number $n_{i,k} \in \mathbb{Z}_n$ such that $\sum_k n_{i,k} = 0$.

and we recall that we have a ringtheoretical interpretation of D as the non-Azumaya (or ramification) locus of a maximal order in Δ on X. That is, locally on $X(c)$ we have a Z-maximal order B in Δ with $ram_B = D \mid X(c)$.

We will investigate if we can change X, C, Z and B such that $mod_n\ B$ is a smooth variety.

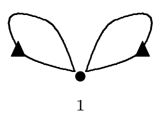
Definition 6.2.1. If m is a maximal ideal of Z we say that B is smooth in m if $mod_n\ B$ is smooth in a point x corresponding to the semi-simple n-dimensional B-module determined by m.

Proposition 6.2.2. *If $m \in iso_n^{ss}\ B$ is a non-singular point of D or if m does not lie on D, then B is smooth at m.*

Proof. If m does not lie on D, then it determines a simple n-dimensional B-module and hence B_m is an Azumaya algebra over Z_m. As Azumaya algebras are split by an étale extension we have

$$B_m^{sh} = B \otimes_Z Z_m^{sh} \simeq M_n(Z_m^{sh})$$

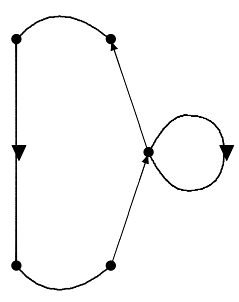
which is the ring corresponding to the local chart of type A_{001}



and therefore B is smooth at m.
Let m be a nonsingular point of the ramification divisor D. Consider the pointed spectrum $Spec\ Z_m - \{m\}$. The only prime ideals are of height one (the curves passing through m) and hence this is a Dedekind scheme. Moreover, B determines a sheaf of maximal orders on this Dedekind scheme. Hence B_m^{sh} determines a sheaf of hereditary orders on the pointed scheme $Spec\ \mathbb{C}\{x,y\} - (x,y)$ and we can choose the variables such that x is a local parameter determining D near m.
From the characterization theorem of hereditary orders over discrete valuation rings we know the structure of $(B_m^{sh})_p$ at every height one prime of Z_m^{sh}. As B and hence B_m^{sh} is a reflexive (even projective) module, this information suffices to determine B_m^{sh}.
One can prove that B_m^{sh} must be of isomorphic to an algebra of the form

$$\begin{bmatrix} M_{d_1}(\mathbb{C}\{x,y\}) & M_{d_1 \times d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_1 \times d_r}(\mathbb{C}\{x,y\}) \\ M_{d_2 \times d_1}(x\mathbb{C}\{x,y\}) & M_{d_2}(\mathbb{C}\{x,y\}) & \ldots & M_{d_2 \times d_r}(\mathbb{C}\{x,y\}) \\ \vdots & \vdots & \ddots & \vdots \\ M_{d_r \times d_1}(x\mathbb{C}\{x,y\}) & M_{d_r \times d_2}(x\mathbb{C}\{x,y\}) & \ldots & M_{d_r}(\mathbb{C}\{x,y\}) \end{bmatrix}$$

for $\sum d_i = n$ (as a matter of fact, as we started out from a maximal order B one can show that the d_i are all equal). Anyway, this is the algebra corresponding to a local chart of type A_{r01}



A_{r01}

and so we have that B is smooth in m. □

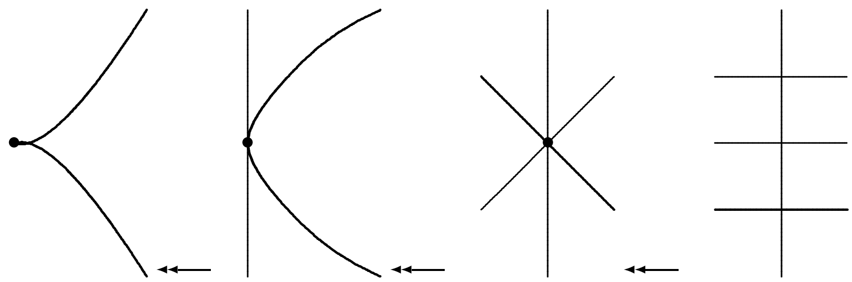
In conclusion, a maximal order on a smooth surface can have only isolated 'singularities' in the singularities of the divisor D. We claim that we may assume that all the singularities of D are normal crossings.
Recall the classical result on commutative surfaces

THEOREM 6.2.3 (Embedded resolution of curves in surfaces). *Let D be any curve on the surface X. Then, there exists a finite sequence of blow-ups*

$$X' = X_s \longrightarrow X_{s-1} \longrightarrow \dots \longrightarrow X_0 = X$$

and, if $f : X' \twoheadrightarrow X$ is their composition, then the total inverse image $f^{-1}(D)$ is a divisor with normal crossings.

EXAMPLE 6.2.4. Consider the cusp $D : y^2 = x^3$ in $\mathbb{P}^2$, then we need three blow-ups to get $f^{-1}(D)$ with normal crossings



In order to apply this result, we need to understand how the ramification divisor D of Δ changes if we blow up a singular point p of it.

LEMMA 6.2.5. *Let $\tilde{X} \twoheadrightarrow X$ be the blow-up of X at a singular point p of D, the ramification divisor of Δ on X. Let $\tilde{D}$ be the strict transform f D and E the exceptional line on $\tilde{X}$. Let D' be the ramification divisor of Δ on the smooth model $\tilde{X}$ of K. Then,*

1. *Assume the local branch data at p distribute in an admissible way on $\tilde{D}$, that is,*

$$\sum_{i \text{ at } q} n_{i,p} = 0 \text{ for all } q \in E \cap \tilde{D}$$

where the sum is taken only over the branches at q. Then,

$$D' = \tilde{D}$$

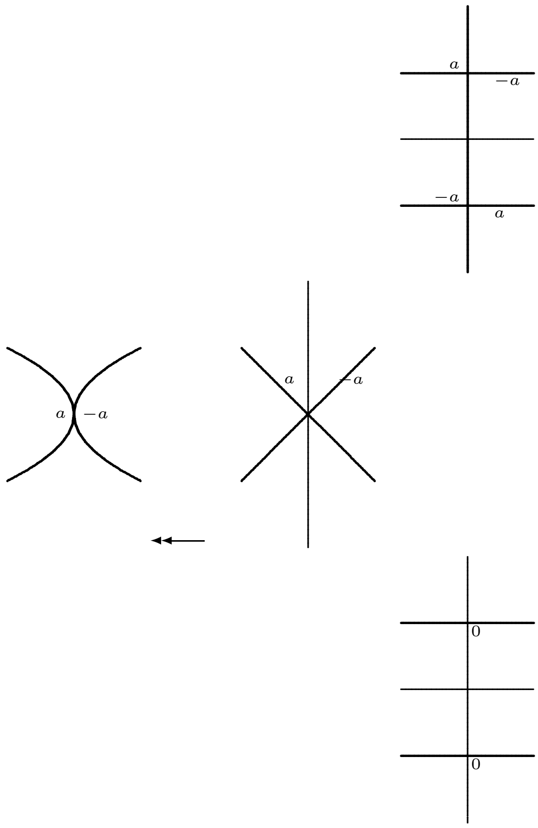
2. *Assume the local branch data at p does not distribute in an admissible way, then*

$$D' = \tilde{D} \cup E$$

PROOF. Clearly, $\tilde{D} \hookrightarrow D' \hookrightarrow \tilde{D} \cup E$. By the Artin-Mumford sequence applied to X' we know that the branch data of D' must add up to zero at all points q of $\tilde{D} \cap E$.
(1) : Assume $E \subset D'$. Then, the E-branch number at q must be zero for all $q \in \tilde{D} \cap E$. But there are no non-trivial étale covers of $\mathbb{P}^1 = E$ so $ram(\Delta)$ gives the trivial element in $H^1_{et}(\mathbb{C}(E), \boldsymbol{\mu_n})$, a contradiction. Hence $D' = \tilde{D}$.
(2) : If at some $q \in \tilde{D} \cap E$ the branch numbers do not add up to zero, the only remedy is to include E in the ramification divisor and let the E-branch number be such that the total sum is zero in $\mathbb{Z}_n$. □

EXAMPLE 6.2.6. Consider the sequence of blow-ups below, where the thick curves indicate the ramification divisor.

After the first blow up we obtain a ramification divisor with normal crossings. Note that the exceptional line is not part of the ramification divisor as the branch-data is admissible.
If we blow up the crossing, the resulting picture depends on whether a is zero or not. If $a = 0$ then the exceptional line is not part of the ramification divisor and hence we can separate the branches.
If $a \neq 0$ then the exceptional line has to become part of the ramification divisor as otherwise the branch data would not be compatible in two points, in contradiction with the Artin-Mumford exact sequence.



6.3. Smooth models

Before we can apply the foregoing to the construction of smooth models we have to make a local computation.
Consider the ring of algebraic functions in two variables $\mathbb{C}\{x, y\}$ and let $X = Spec\ \mathbb{C}\{x, y\}$. There is only one codimension two subvariety $m = (x, y)$.
Let us compute the coniveau spectral sequence for X. If K is the field of fractions of $\mathbb{C}\{x, y\}$ and if we denote with k_p the field of fractions of $\mathbb{C}\{x, y\}/p$ where p is a

height one prime, we have as its first term

0	0	0	0	...
$H^2(K, \boldsymbol{\mu_n})$	$\oplus_p H^1(k_p, \mathbb{Z}_n)$	$\boldsymbol{\mu_n}^{-1}$	0	...
$H^1(K, \boldsymbol{\mu_n})$	$\oplus_p \mathbb{Z}_n$	0	0	...
$\boldsymbol{\mu_n}$	0	0	0	...

As $\mathbb{C}\{x,y\}$ is a unique factorization domain, as before we see that the map

$$H^1_{et}(K, \boldsymbol{\mu_n}) = K^*/(K^*)^n \xrightarrow{\gamma} \oplus_p \mathbb{Z}_n$$

is surjective.
Moreover, all fields k_p are isomorphic to the field of fractions of $\mathbb{C}\{z\}$ whose only cyclic extensions are given by adjoining a root of z and hence they are all ramified in m. Therefore, the component maps

$$\mathbb{Z}_n = H^1_{et}(k_L, \mathbb{Z}_n) \xrightarrow{\beta_L} \mu^{-1}$$

are isomorphisms.
Therefore, the second (and limiting) term of the spectral sequence has the form

0	0	0	0	...
$Ker\ \alpha$	$Ker\ \beta / Im\ \alpha$	0	0	...
$Ker\ \gamma$	0	0	0	...
$\boldsymbol{\mu_n}$	0	0	0	...

Finally, we use the fact that $\mathbb{C}\{x,y\}$ is strict Henselian and hence has no proper étale extensions. But then,

$$H^i_{et}(X, \boldsymbol{\mu_n}) = 0 \text{ for } i \geq 1$$

and substituting this information in the spectral sequence we obtain that the top sequence of the coniveau spectral sequence

$$0 \longrightarrow Br_n\ K \xrightarrow{\alpha} \oplus_L \mathbb{Z}_n \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

is exact.

From this sequence we immediately obtain the following

LEMMA 6.3.1. 1. *Let* $U = X - V(x)$*, then* $Br_n\ U = 0$

2. *Let* $U = X - V(xy)$*, then* $Br_n\ U = \mathbb{Z}_n$ *with generator the quantum-plane algebra*

$$\mathbb{C}_\zeta[u,v] \text{ with } uv = \zeta vu$$

where ζ *is a primitive* n*-th root of one*

We can now state and prove our first result on the existence of smooth models for central simple algebras Δ over functionfields of trancendence degree two.

THEOREM 6.3.2. *Let* S *be a (simply connected) smooth projective surface and* Δ *a central simple* $\mathbb{C}(S)$*-algebra of dimension* n^2*. Then, a restricted smooth model for* Δ *exists if and only if*

$$ram\ [\Delta] \in Ker\ (\bigoplus_C H^1_{et}(\mathbb{C}(C), \mathbb{Z}_n) \longrightarrow \bigoplus_p \boldsymbol{\mu_n}^{-1})$$

PROOF. By the Artin-Mumford sequence, Δ is determined by a $\mathbb{Z}_n$-wrinkle on S. The shadow D of this $\mathbb{Z}_n$-wrinkle is the ramification divisor of any maximal $\mathcal{O}_S$-order in Δ.

The singular points of D can be divided in two finite subsets

- P_{nr} where the branch-data are trivial
- P_r where some of the branch numbers are non-zero

By the foregoing section, we can consider a sequence of blow-ups

$$S' \xrightarrow{\pi} S$$

such that, when D' denotes the ramification divisor of a maximal $\mathcal{O}_{S'}$-order in Δ we have

- D' has at worst normal crossings as singularities
- $\pi(D'_{sing}) = P_r$

For the last fact we use (a) that we can separate the branches of the ramification divisor at a crossing where the branch-data are trivial and (b) that the exceptional line is part of the ramification of the blow-up if the branch-data is non-trivial.

In particular, if $ram\ [\Delta]$ lies in the kernel, D' is a finite disjoint union of smooth curves on S'. In this case, any maximal $\mathcal{O}_{S'}$-order in Δ is locally smooth at every point of S by the result of the previous section.

Conversely, if $ram\ [\Delta]$ does not belong to the kernel, there is a singular point m on D' where the branch-data are non-trivial. If Λ is locally at m any maximal order over S' in Δ, then one can use above lemma to show that Λ cannot be étale locally split in m, that is, the ring of fractions of Λ^{sh}_m is not a full $n \times n$ matrixalgebra.

If there were a restricted smooth order A in Δ which is B locally at m, then B has to be étale locally split at m. However, $B \hookrightarrow \Lambda$ for some maximal order Λ, this contradicts the foregoing. □

Another way to phrase the foregoing result is

PROPOSITION 6.3.3. *If* Δ *is a central simple* K*-algebra of dimension* n^2*, with* $trdeg_{\mathbb{C}}\ K = 2$*. Then, there is a smooth model* S *of* K *such that any maximal* $\mathcal{O}_S$*-order in* Δ *has at worst isolated singularities which are étale locally of quantum-plane type.*

If we want to construct (unrestricted) smooth models in any Δ we have to find a way to resolve quantum-plane singularities.

CHAPTER 7

Non-commutative blow-ups

In this chapter we will modify the local-chart setting in order to study smooth models in the unrestricted sense. As some of the arguments are analogous to these of the restricted case, we will only mention the required changes and leave the details as rather interesting exercises. In compensation we will consider an example in great detail.

7.1. Equivariant desingularization

We have seen that a central simple algebra Δ over a surface with non trivial branch-data cannot have a restricted smooth model.
This contradicts the following (too optimistic) approach to construct such smooth models. Consider a smooth model X for the center K of Δ and consider a maximal order over X in Δ. Locally we have the situation

$$A_c^g = B[d, d^{-1}]$$

and let us assume that

$$mod_n\ B = proj_n^{ss}\ A \mid X(c)$$

has singularities. Because the set of singular points $sing$ is a closed GL_n-stable subvariety we can consider the blow-up with center $sing$. The GL_n-action extends to one on the blow-up.
If we iterate this process we will end up with a smooth variety $\tilde{mod}_n$ with a GL_n-action such that the projection map

$$\tilde{mod}_n \longrightarrow\!\!\!\!\!\rightarrow mod_n\ B$$

is GL_n-equivariant.
Again, we can cover the semi-stable points of the variety $\tilde{mod}_n$ by affine GL_n-stable open subvarieties $X(m)$ and in view of the close connection between affine GL_n-varieties and Cayley-Hamilton algebras, we expect that these open subvarieties are module varieties themselves

$$\tilde{mod}_n^{\ ss} \mid X(m) = mod_n\ B_{(m)}$$

If this were the case, then the sheaf of orders $B_{(m)}$ would give us a restricted smooth model of Δ.
We know that this strategy has to fail. To see clearly where the argument breaks down let us compute an example.

Example 7.1.1. Let us consider the quantum-space example when $q = -1$. In this case, the central simple algebra is the quaternion division algebra

$$\Delta = \begin{pmatrix} x, y \\ \mathbb{C}(x, y) \end{pmatrix}$$

and consider a sheaf of maximal orders over $\mathbb{P}^2$ in Δ with affine section

$$B = \mathbb{C}\langle u, v\rangle/(uv + vu)$$

with $u^2 = x$ and $v^2 = y$.
In particular, u and v have reduced trace zero and the defining relation $uv+vu = 0$ can be reformulated as $tr(uv) = 0$.
Consider the module variety $mod_2\ B$. A point of $mod_2\ B$ is determined by a pair of matrices

$$x = \left(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \quad , \quad \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \right)$$

such that the trace of the product is zero. That is,

$$mod_2\ B = V(2x_1x_4 + x_2x_6 + x_3x_5) \hookrightarrow \mathbb{A}^6 = Spec\ \mathbb{C}[x_1, \ldots, x_6]$$

with action of GL_2 given by simultaneous conjugation.
The quotient-variety under this action $iso_2^{ss}\ B$ is isomorphic to $\mathbb{A}^2$ and the quotient map is given by taking the determinants

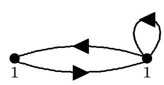
$$mod_2\ B \xrightarrow{\pi} iso_2^{ss}\ B = \mathbb{A}^2$$

$$x \mapsto (x_1^2 + x_2x_3, x_4^2 + x_5x_6)$$

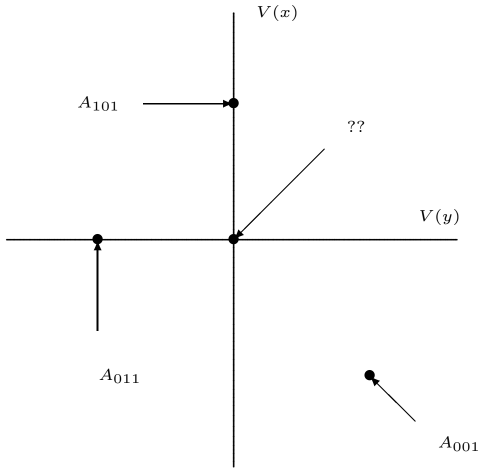
and it is easy to find a representative in the closed orbit determined by a point $m = (\lambda, \mu) \in \mathbb{A}^2$, namely

$$x_{\lambda,\mu} = \left(\begin{bmatrix} i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} \end{bmatrix} \quad , \quad \begin{bmatrix} 0 & \sqrt{\mu} \\ -\sqrt{\mu} & 0 \end{bmatrix} \right)$$

We see that the corresponding 2-dimensional B-module M_x is simple whenever $\lambda\mu \neq 0$, is semi-simple with distinct one-dimensional components if only one of the two is non-zero and has a one-dimensional component occurring with multiplicity two in case $\lambda = \mu = 0$. That is, the ramification divisor of Δ on $\mathbb{A}^2$ has the form as depicted below.
Because B is a maximal order in Δ, we know that it must be smooth in all regular points of $D = V(xy)$ and in fact we can compute that the local charts in these points have are of the form

$A_{101} = A_{011} =$ 

However, for $x_{0,0}$ the semi-simple module $M_x = \mathbb{C}_{triv}^{\oplus 2}$ has equal components which we know cannot happen for smooth algebras over surfaces.
In fact, one verifies that $mod_2\ B$ has an isolated singularity in the origin $p = (0, 0, 0, 0, 0, 0)$.



Now, consider the blow up of p in $\mathbb{A}^6$. We obtain the variety $\tilde{\mathbb{A}^6} \hookrightarrow \mathbb{A}^6 \times \mathbb{P}^5$ which is

$$\tilde{\mathbb{A}^6} = V(x_i X_j - x_j X_i)$$

where X_i are the projective parameters of $\mathbb{P}^5$. The strict transform of $mod_2\ B$ is then the subvariety

$$\tilde{mod_2} = V(x_i X_j - x_j X_i, 2X_1X_4 + X_2X_6 + X_3X_5) \hookrightarrow \mathbb{A}^6 \times \mathbb{P}^5$$

which is a smooth variety with GL_2-action induced by simultaneous conjugation on the four 2×2-matrices

$$\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} \begin{bmatrix} x_4 & x_5 \\ x_6 & -x_4 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{bmatrix} \begin{bmatrix} X_4 & X_5 \\ X_6 & -X_4 \end{bmatrix} \right)$$

As the projection map $\tilde{mod_2} \twoheadrightarrow mod_2\ B$ is a GL_2- isomorphism outside the fiber over p we only need to investigate the (semi-stable) points lying over p. They form the smooth quadric

$$Q = Proj\ V(2X_1X_4 + X_2X_6 + X_3X_5) \hookrightarrow \mathbb{P}^5$$

on which GL_2 acts with quotient-variety

$$Q^{ss}/GL_2 = Proj\ \mathbb{C}[X_1^2 + X_2X_3, X_1^2 + X_5X_6]$$

Therefore

$$\tilde{mod_2}^{ss}/GL_2 \simeq \tilde{\mathbb{A}^2}$$

the blow up of $\mathbb{A}^2$ at the point $(0,0)$.
To a point $(1 : \mu) \in \mathbb{P}^1$ in the exceptional fiber corresponds the closed GL_2-orbit with representative

$$x = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & \sqrt{\mu} \\ -\sqrt{\mu} & 0 \end{bmatrix} \right)$$

One verifies that the stabilizer of this point is

$$Stab(x) \simeq \mu_2 = \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rangle \hookrightarrow PGL_2$$

Let us assume we can cover $\tilde{mod_2}^{ss}$ by GL_2-stable affine open subvarieties such that

$$\tilde{mod_2}^{ss} \mid X(m) = mod_2\ B_{(m)}$$

and consider an open containing the orbit of x.
As the orbit is closed it must correspond to a semi-simple 2-dimensional $B_{(m)}$-module. As the stabilizer is zero-dimensional in PGL_2 this semi-simple must in fact be simple. But then, the PGL_2-stabilizer must be trivial, a contradiction.
Hence, GL_n-equivariant desingularizations of module varieties cannot always be covered by module varieties.

For this reason, we have to consider unrestricted smooth models if we want to construct smooth models in every central simple algebra over a surface.

7.2. Graded semi-simple modules

If the connected graded algebra A is an (unrestricted) smooth model for Δ, then as before we want to describe the local structure of $proj_n^{ss}\ A$ in the neighborhood of a point x in a closed GL_n-orbit.
By definition, x determines a $\mathbb{C}^*$-family of points $x_\lambda \in mod_n\ A$. In fact under the canonical (quotient-morphism under the $\mathbb{C}^*$-action)

$$\psi : mod_n\ A \longrightarrow\!\!\!\!\!\rightarrow proj_n\ A$$

we have that $\psi^{-1}(GL_n.x)$ is the $GL_n \times \mathbb{C}^*$-orbit of any of the x_λ.

LEMMA 7.2.1. *With notations as above we have that following statements are equivalent*

1. *$GL_n.x$ is a closed orbit in $proj_n^{ss}\ A$*
2. *$GL_n \times \mathbb{C}^*.x_\lambda$ is a closed orbit in $mod_n^{ss}\ A$*
3. *x_λ determines a semi-simple n-dimensional A-module*

We will now investigate the ringtheoretical interpretation of an $x \in proj_n^{ss}\ A$ having a closed orbit.
Assume first that one $x_\lambda \in iso_n^s\ A$ is a simple n-dimensional representation of A, then all x_μ are similar. For, if the matrices $x_\lambda = (m_1, \ldots, m_k)$ generate $M_n(\mathbb{C})$ as $\mathbb{C}$-algebra, then so do the matrices $(tm_1, \ldots, tm_k)$ for any $t \in \mathbb{C}^*$.
the kernel of the epimorphism determined by x_λ :

$$\phi_{x_\lambda} : A \longrightarrow\!\!\!\!\!\rightarrow M_n(\mathbb{C})$$

is a maximal ideal M of A and we consider the maximal graded ideal M_g contained in it. It is easy to verify that this is the kernel of the graded morphism

$$\phi_x : A \longrightarrow\!\!\!\!\!\rightarrow A_x$$

where A_x is the graded subalgebra of $M_n(\mathbb{C}[t])$ (endowed with the natural gradation) generated as $\mathbb{C}$-algebra by the elements

$$(tm_1, \ldots, tm_k) \in M_n(\mathbb{C}[t])$$

LEMMA 7.2.2. *With notations as above we have*

1. *The center of A_x is a non-trivial $\mathbb{C}$-subalgebra of $\mathbb{C}[t]$.*
2. *The graded ring of fractions $Q^g(A_x)$ is a graded central simple algebra, that is, contains no proper graded twosided ideals.*

PROOF. (1) : If a matrix $c(t) \in M_n(\mathbb{C}[t])$ is central in A_x, then $c(\mu) \in M_n(\mathbb{C})$ is central in $A_x/(t-\mu) = M_n(\mathbb{C})$ whence all non-diagonal entries of $c(t)$ are divisible by $t-\mu$. As μ is arbitrary it follows that $c(t)$ is a diagonal matrix, whence $Z(A_x) \hookrightarrow \mathbb{C}[t]$ and is a graded subalgebra. Because $x \in proj_n^{ss}\ A$ there is a homogeneous central element $c \in A$, say of degree f not vanishing on A_x. But then, the image of c in A_x is of the form $\sigma.t^f$ for some $\sigma \in \mathbb{C}^*$ and $f \in \mathbb{N}_+$, whence $Z(A_x)$ is strictly greater than $\mathbb{C}$.
(2) : Let $c = t^f \in Z(A_x)$, then the graded localization at c is a graded field and hence of the form

$$Z(A_x)^g_c = \mathbb{C}[t^e, t^{-e}]$$

for some $e \in \mathbb{N}_+$. Moreover, as any specialization

$$(A_x)^g_c/(t^e - \mu) = M_n(\mathbb{C})$$

we have that $(A_x)^g_c$ is a graded Azumaya algebra over a graded field and hence a graded central simple algebra. □

Combining Tsen's theorem with the characterization of graded simple algebras we obtain

PROPOSITION 7.2.3. *Let $x \in proj_n^{ss}\ A$ corresponding to a simple n-dimensional A-module, then*

$$A \twoheadrightarrow A_x \hookrightarrow Q^g(A_x) = M_n(\mathbb{C}[t^e, t^{-e}])(a_1, \ldots, a_n)$$

for some natural numbers $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n < e$ where the i-th homogeneous part of the graded matrix-ring is defined to be

$$\begin{bmatrix} R_i & R_{i+a_1-a_2} & \ldots & R_{i+a_1-a_n} \\ R_{i+a_2-a_1} & R_i & \ldots & R_{i+a_2-a_n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{i+a_n-a_1} & R_{i+a_n-a_2} & \ldots & R_i \end{bmatrix}.$$

with $R_i = \mathbb{C}[t^e, t^{-e}]_i$. In fact, if A is generated by k elements of degree one, then the numbers a_i are of the form

$$(a_1, \ldots, a_n) = (\underbrace{0, \ldots, 0}_{m_1}, \underbrace{1, \ldots, 1}_{m_2}, \ldots, \underbrace{e-1, \ldots, e-1}_{m_e})$$

with all $m_i \geq 1$ and satisfying the inequalities

$$m_i \leq k.m_{i\pm 1} \textit{ for all } i \bmod e$$

This result allows us to assign numerical invariants to x

DEFINITION 7.2.4. If $x \in proj_n^{ss}\ A$ lies in the image of $iso_n^s\ A$ we say that x is a **graded simple A-module of size** n For such x we have

$$Q^g(A_x) = M_n(\mathbb{C}[t^e, t^{-e}])(\underbrace{0, \ldots, 0}_{m_1}, \underbrace{1, \ldots, 1}_{m_2}, \ldots, \underbrace{e-1, \ldots, e-1}_{m_e})$$

Then, we say that the **period** of x is e and that x is of **matrix-type** $(m_1, \ldots, m_e)$.

LEMMA 7.2.5. *Let $x \in proj_n^{ss}\ A$ be graded simple with period e and matrix-type $(m_1, \ldots, m_e)$ and let $x_1 \in mod_n\ A$ be a simple n-dimensional A-module representing x. Then, the $GL_n \times \mathbb{C}^*$-stabilizer of x_1 is isomorphic to group $\mathbb{C}^* \times \mu_{\mathbf{e}}$ where the cyclic group $\mu_{\mathbf{e}}$ has generator*

$$(g_\zeta, \zeta) \in GL_n \times \mathbb{C}^*$$

where ζ is a primitive e-th root of 1 and

$$g_\zeta = diag(\underbrace{1, \ldots, 1}_{m_1}, \underbrace{\zeta, \ldots, \zeta}_{m_2}, \ldots, \underbrace{\zeta^{e-1}, \ldots, \zeta^{e-1}}_{m_e})$$

PROOF. By the assumptions we can find a point x' in the orbit such that the corresponding k-tuple of $n \times n$-matrices x'_1 have block-decompositions

$$\begin{bmatrix} 0 & z_1 & 0 & \ldots & 0 \\ 0 & 0 & z_2 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \ldots & z_{e-1} \\ z_e & 0 & 0 & \ldots & 0 \end{bmatrix}$$

where z_i is an $m_i \times m_{i+1}$-matrix. The claim follows from this description. □

By a graded version of the Jordan-Hölder theorem we have that closed GL_n-orbits in $proj_n^{ss}\ A$ correspond to graded semi-simple A-modules of size n, that is,

$$M_x = S_1^{\oplus f_1} \oplus \ldots \oplus S_r^{\oplus f_r}$$

where S_i is a graded simple A-module of size s_i, period e_i, matrix-type $(m_{i1}, \ldots, m_{ie_i})$ and occurring with multiplicity f_i.

DEFINITION 7.2.6. The **graded representation type** of x is the collection of numerical data

- the underlying representation type $(s_1, f_1; \ldots; s_r, f_r)$
- the periods $(e_1, \ldots, e_r)$
- the matrix types $(m_{i1}, \ldots, m_{ie_i})$

Precisely as in the case of graded simple modules considered above we have

PROPOSITION 7.2.7. *Let x determine a graded semi-simple A-module of size n with representation-type given by the data*

$$(s_1, f_1; \ldots; s_r, f_r) \quad (e_1, \ldots, e_r) \quad (m_{i1}, \ldots, m_{ie_i})$$

Let $x_1 \in mod_n\ A$ a corresponding semi-simple n-dimensional representation of type $\tau = (s_1, f_1; \ldots; s_r, f_r)$. Then, the $GL_n \times \mathbb{C}^*$-stabilizer of x_1 is equal to

$$Stab(x_1) = GL(\tau) \times \mu_e$$

where $e = gcd(e_1, \ldots, e_r)$. If $e_i = e.c_i$ then a generator of the cyclic component is given by

$$(g_\zeta, \zeta) \in GL_n \times \mathbb{C}^*$$

where ζ is a primitive e-th root of 1 and

$$g_\zeta = \bigoplus_i diag(\underbrace{1^{c_i}, \ldots, 1^{c_i}}_{m_{i1} f_i}, \underbrace{\zeta_i^{c_i}, \ldots, \zeta_i^{c_i}}_{m_{i2} f_i}, \ldots, \underbrace{\zeta_i^{c_i.(e_i-1)}, \ldots, \zeta_i^{c_i.(e_i-1)}}_{m_{ie_i} f_i})$$

where ζ_i is a primitive e_i-th root of 1.

7.3. Local charts revisited

Let A be a smooth model in a central simple K-algebra of dimension n^2. Let $x \in proj_n^{ss}\ A$ a graded semi-simple A-module of size n determining a closed GL_n-orbit and let x_1 a preimage in $mod_n\ A$.
The closed $GL_n \times \mathbb{C}^*$-orbit of x_1 determines a graded maximal ideal m in the center of A and we want to study the graded étale local structure of the $\mathbb{Z}$-graded algebra A_m^g. That is, we want to study the limit of $A_m^g \otimes D$ where D is a $\mathbb{Z}$-graded étale extension of C_m^g. We will denote this limit with $A_m^{g,sh}$.
Let us assume that the graded representation type of x is determined by the numerical data

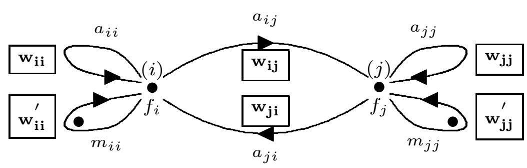
- type $\tau = (s_1, f_1; \ldots; s_r, f_r)$
- periods $(e_1, \ldots, e_r)$
- matrix-types $(m_{i1}, \ldots, m_{ie_i})$

In order to determine the algebra $A_m^{g,sh}$ we will apply the Luna slice theorem in the smooth point $x_1 \in mod_n\ A$, considered as a $G = GL_n \times \mathbb{C}^*$-variety.
From the previous section we recall that the stabilizer subgroup in x_1 (or, by abuse of notation in x) is equal to

$$G_x = GL(\tau) \times \mu_{\mathbf{e}} \hookrightarrow GL_n \times \mathbb{C}^*$$

where $e = gcd(e_1, \ldots, e_r)$.
With N_x^{sm} we will denote the normal space to the $GL_n \times \mathbb{C}^*$- orbit in x_1. By a similar argument as before we obtain

Proposition 7.3.1. *N_x^{sm} is as $GL(\tau) \times \mu_{\mathbf{e}}$-module isomorphic to the representation space of a* **weighted** *local chart $C_w = (M, \mathbf{f}, \mathbf{w})$. Here, $M_w = (M, \mathbf{w})$ is a weighted map on r vertices such that the subgraph on any two vertices $1 \le i, j \le r$ is of the form*



where $\mathbf{w_{kl}} = (w_{kl}^{(1)}, \ldots, w_{kl}^{(a_{kl})})$ and $\mathbf{w'_{kk}} = (w_{kk}^{'(1)}, \ldots, w_{kk}^{'(m_{kk})})$ are series of numbers from $\mathbb{Z}/e\mathbb{Z}$. We use the following dictionary

- *a* **weighted loop** *at vertex (i) with weight m corresponds with the $GL(\tau) \times \mu_{\mathbf{e}}$-module $M_{e_i}(\mathbb{C})$ on which GL_{e_i} acts by conjugation and the other factors of $GL(\tau)$ act trivially, which is a $\mu_{\mathbf{e}}$ eigenspace with eigenvector ζ^m.*
- *a* **weighted arrow** *from vertex (i) to vertex (j) with weight m corresponds to the $GL(\tau) \times \mu_{\mathbf{e}}$-module $M_{e_i \times e_j}(\mathbb{C})$ on which $GL_{e_i} \times GL_{e_j}$ act via $g.m = g_i m g_j^{-1}$ and the other factors of $GL(\tau)$ act trivially, which is a $\mu_{\mathbf{e}}$ eigenspace with eigenvector ζ^m.*

- *a* **weighted marked loop** *at vertex* (i) *corresponds to the simple* $GL(\tau) \times \mu_{\mathbf{e}}$*-module* $M^0_{e_i}(\mathbb{C})$*, that is, trace zero matrices with action of* GL_{e_i} *by conjugation and trivial action by the other components of* $GL(\tau)$*, which is a* $\mu_{\mathbf{e}}$ *eigenspace with eigenvector* ζ^m*.*
- *the label of a loop or arrow indicates the multiplicity of the corresponding representation.*

The classification of underlying charts (that is, forgetting the weights) which can arise in a given dimension d is the same as given before.
Having determined the G_x-module structure of N^{sm}_x, the Luna slice theorem asserts the following

THEOREM 7.3.2. *The ring of* GL_n*-equivariant maps from the fiber bundle*

$$F_x = (GL_n \times \mathbb{C}^*) \times^{GL(\tau)\times\mu_{\mathbf{e}}} N^{sm}_x \longrightarrow M_n(\mathbb{C})$$

to $M_n(\mathbb{C})$ *is an algebra* Λ*. The* $\mathbb{C}^*$*-action on* F_x *induces a* $\mathbb{Z}$*-gradation on* Λ*.*
The center of Λ *is the* $\mathbb{Z}$*-graded ring*

$$R = \mathbb{C}[\mathbb{C}^* \times^{\mu_{\mathbf{e}}} (N^{sm}_x/GL(\tau))]$$

and if we denote by p *the graded maximal ideal of* R *corresponding to the zero representation in* $N^{sm}_x/GL(\tau)$*, we have*

1. $C^{g,sh}_m \simeq R^{g,sh}_p$
2. $A^{g,sh}_m \simeq \Lambda^{g,sh}_m$

The above result gives us a way to calculate the graded étale local structure of A^g_m. In particular, we have

PROPOSITION 7.3.3. *The étale local structure of* $proj^{ss}_n\ A$ *in a neighborhood of the closed orbit of* x *is given by*

$$((GL_n \times \mathbb{C}^*) \times^{GL(\tau)\times\mu_{\mathbf{e}}} N^{sm}_x)/\mathbb{C}^*$$

in a neighborhood of the orbit corresponding to the zero representation in N^{sm}_x*.*

Assume we are in the restricted case treated before, that is, when $\mu_{\mathbf{e}} = 1$. Then, the above theorem asserts that

$$A^{g,sh}_m \simeq B[t, t^{-1}]$$

where B is the strict Henselization of the ring of GL_n-equivariant maps

$$GL_n \times^{GL(\tau)} N^{sm}_x \longrightarrow M_n(\mathbb{C})$$

at the zero representation. Hence, we recover our previous results.

7.4. Smooth models revisited

In this section we apply the foregoing to the construction of smooth non-commutative surfaces.
By taking a suitable smooth model S, such that the ramification divisor of the central simple algebra Δ has only normal crossings and such that the branches with trivial branch-data are separated, we have constructed a restricted model having isolated singularities which are étale -locally of quantum-plane type.
That is, we may assume that locally our maximal order B is of the form

$$B = \mathbb{C}_q[u, v] \text{ with } uv = qvu$$

where q is a primitive n-root of unity. We want to resolve the remaining singularities in $mod_n\ B$ lying over the origin in $iso_n^{ss}\ B$ with is $\mathbb{A}^2$ determined by the center of B, $Z = \mathbb{C}[x,y]$ where $x = u^n$ and $y = v^n$.
We will achieve this by blowing up the closed orbit of the trivial representation in $mod_2\ B$. Let us recall the ringtheoretical interpretation of the blow-up of a point in $\mathbb{A}^2$

EXAMPLE 7.4.1. Let $\tilde{\mathbb{A}}^2 \twoheadrightarrow \mathbb{A}^2$ be the blow-up of the origin $p = (0,0)$ in $\mathbb{A}^2$. If $\mathbb{C}[\mathbb{A}^2] = \mathbb{C}[x,y]$, consider the graded algebra

$$R = \mathbb{C}[x,y] \oplus (x,y)t \oplus (x,y)^2t^2 \oplus \ldots \hookrightarrow \mathbb{C}[x,y][t]$$

Then R is generated by two elements in degree zero x, y and two in degree one $X = xt$ and $Y = yt$. The defining (homogeneous) relation of R is $xY - yX$.
Then, $\tilde{\mathbb{A}}^2 = Proj\ R$ and the projection morphism is given by the inclusion (in degree zero) $\mathbb{C}[x,y] \hookrightarrow R$.

DEFINITION 7.4.2. Let B be an affine $\mathbb{C}$-order in Δ and M the kernel of a semi-simple k-dimensional representation $B \longrightarrow M_k(\mathbb{C})$ occurring as a direct factor of a semi-simple n-dimensional representation x. Then, the **non-commutative blow-up** of B at x is $proj_n^{ss}\ A$ where A is the graded algebra

$$A = B \oplus Mt \oplus M^2t^2 \oplus \ldots \hookrightarrow \Delta[t]$$

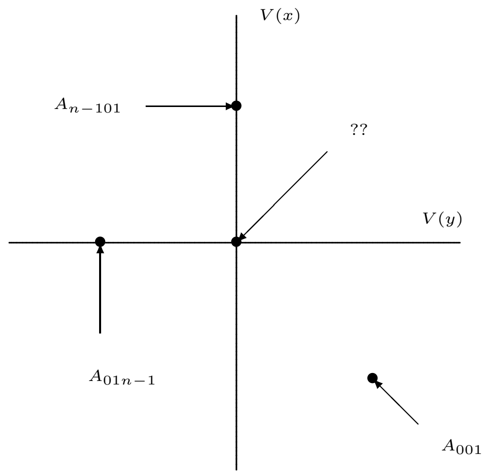
and the projection map

$$proj_n^{ss}\ A \longrightarrow mod_n\ B$$

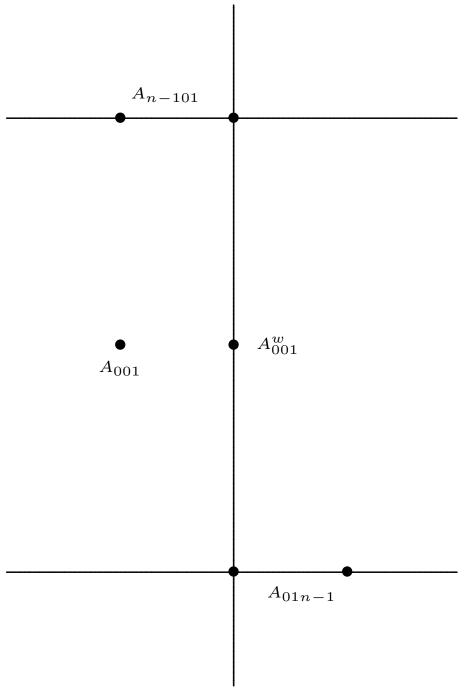
is given by the inclusion in degree zero $B \hookrightarrow A$.

We can resolve the remaining singularities by non-commutative blow-ups.

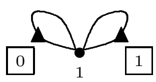
THEOREM 7.4.3. *If A is the blow-up of $B = \mathbb{C}_q[u,v]$ corresponding to the trivial (one-dimensional) representation, then $proj_n^{ss}\ A$ is a smooth variety.*
The local structure of B is summarized in the picture



$proj_n^{ss}$ A has a $\mathbb{P}^1$ of closed orbits lying over the singularity of B and the local structure is summarized in the picture



where A_{001}^w is the weighted local chart



We will give the proof of this theorem in the example considered before.

Example 7.4.4. Consider the quantum-plane with $q = -1$. That is,

$$B = \mathbb{C}\langle u, v\rangle/(uv + vu)$$

is a maximal order in the quaternion algebra $\Delta = (x, y)$ with center $Z = \mathbb{C}[x, y]$ where $x = u^2$ and $y = v^2$.
We have seen that $mod_2\ B$ has an isolated singularity in the closed orbit corresponding to the 2-dimensional semi-simple representation $M_x = \mathbb{C}_{triv}^{\oplus 2}$. The kernel of the trivial representation is $M = (u, v)$. Hence, the graded algebra A defining the non-commutative blow-up is

$$A = B \oplus (u, v)t \oplus (u, v)^2 t^2 \oplus \ldots \hookrightarrow B[t]$$

A is generated by two elements u, v of degree zero and two of degree one U, V satisfying the following defining relations

$$uv + vu = UV + VU = uV + Vu = vU + Uv = uV + vU = uU - Uu = vV - Vv = 0$$

and u^2, v^2, U^2 and V^2 are homogeneous central elements.

If $p = (x, y) \in \mathbb{A}^2$ which is not the origin, then $A_p = B_p[t, t^{-1}]$ whence there is a unique closed orbit in $proj_2^{ss}\ A$ lying over p and in those we know already that $proj_2^{ss}\ A$ is smooth.
Observe that it suffices to verify that $proj_2^{ss}\ A$ is smooth in the closed orbits as the singularities are a GL_2-closed subvariety.
We only have to investigate the closed orbits over $p = (0, 0)$. For those, either U^2 or V^2 must be invertible. Let us consider the former case , then

$$A' = A^g_{U^2} = \mathbb{C}_{-1}[u, w][U, U^{-1}, \sigma]$$

where $w = VU^{-1}$ (observe that $v = uVU^{-1}$). The defining relations of A' are

$$uw + wu = wU + Uw = uU - Uu = 0$$

Lemma 7.4.5. *$mod_2\ A'$ is smooth in the closed orbits lying over $(0, 0) \in \mathbb{A}^2 = iso_2^{ss}\ B$. Representatives of these orbits are of one of following types :*

1. *type I (simple) : the images of (u, w, U) are*
$$\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \quad \begin{bmatrix}a & 0\\0 & -a\end{bmatrix} \quad \begin{bmatrix}0 & b\\b & 0\end{bmatrix}\right)$$
2. *type II (semi-simple) : the images of (u, w, U) are*
$$\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \quad \begin{bmatrix}a & 0\\0 & -a\end{bmatrix}\right)$$

where $a, b \in \mathbb{C}^*$.

Proof. $mod_2\ A$ is of dimension 6 so we have to verify that the tangent spaces to $mod_2\ A$ in the indicated points are 6-dimensional. In order to compute these tangentspaces we consider

$$u' = \phi(u) + \epsilon \begin{bmatrix}\alpha_u & \beta_u\\ \gamma_u & -\alpha_u\end{bmatrix}$$

and similarly w' and U'. We then have to compute the conditions necessary to have that

$$u'w' + w'u' = w'U' + U'w' = u'U' - U'u' = 0$$

if ϵ in infinitesimal, that is, $\epsilon^2 = 0$.
These conditions easily calculated to be

$$\text{type I} : \alpha_u = 0 \quad \beta_u = \gamma_u \text{ and } \alpha_U = -\frac{b}{2a}(\beta_w + \gamma_w)$$

$$\text{type II} : \beta_u = \gamma_u = \alpha_w = 0$$

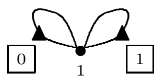
finishing the proof. □

Next, we have to calculate the weighted local charts for these orbits. We have the following stabilizer subgroups in $GL_2 \times \mathbb{C}^*$

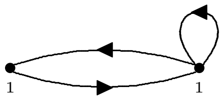
- In type I points, the stabilizer $G_x = \mathbb{C}^* \times \boldsymbol{\mu_2}$ where the generator of $\boldsymbol{\mu_2}$ is
$$\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, -1\right)$$
- In type II points, the stabilizer $G_x = (\mathbb{C}^* \times \mathbb{C}^*) \times 1$ with embedding
$$\left(\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}, 1\right)$$

LEMMA 7.4.6. *The weighted local charts are as follows.*

1. *In type I points they have the form*



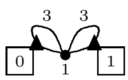
2. *In type II points they have the form*



PROOF. Let us first consider the tangent-spaces calculated above as a module over the stabilizer group.
In type I : GL_2 acts via conjugation and $\mathbb{C}^*$ via degree. Hence, the image of a vector in the tangentspace under the generator of $\mu_{\mathbf{2}}$ is given by computing the entries of

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\cdot\left(\begin{bmatrix}0 & \beta_u\\\beta_u & 0\end{bmatrix}\quad\begin{bmatrix}\alpha_w & \beta_w\\\gamma_w & -\alpha_w\end{bmatrix}\quad-\begin{bmatrix}\alpha_U & \beta_U\\\gamma_U & -\alpha_U\end{bmatrix}\right)\cdot\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

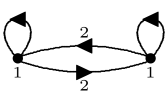
and hence the tangentspace as $\mathbb{C}^*\times\mu_{\mathbf{2}}$-module is represented by the weighted chart



Similarly, in type II points we have to consider the action of $\mathbb{C}^*\times\mathbb{C}^*\times 1$ on a tangentvector which is given by

$$\begin{bmatrix}\lambda & 0\\0 & \mu\end{bmatrix}\cdot\left(\begin{bmatrix}\alpha_u & 0\\0 & -\alpha_u\end{bmatrix}\quad\begin{bmatrix}0 & \beta_w\\\gamma_w & 0\end{bmatrix}\quad-\begin{bmatrix}\alpha_U & \beta_U\\\gamma_U & -\alpha_U\end{bmatrix}\right)\cdot\begin{bmatrix}\lambda^{-1} & 0\\0 & \mu^{-1}\end{bmatrix}$$

which can be represented by the chart



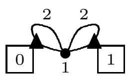
Next, we have to compute the sub-modules corresponding to the tangent space to the $GL_2\times\mathbb{C}^*$-orbit. In order to do this we have to determine the image of the Lie algebra

$$Lie(GL_2\times\mathbb{C}^*) = \left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\quad, t\right)$$

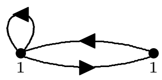
acting via

$$(1+\epsilon t)^{deg}.(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\epsilon\begin{bmatrix}a & b\\ c & d\end{bmatrix}).Z.(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\epsilon\begin{bmatrix}a & b\\ c & d\end{bmatrix})$$

where Z is the image of u resp. w, U. This computation gives that the tangentspace to the orbit as G_x module can be represented by a weighted chart which is for type I points of the form



and for type II points of the form



from which the result follows. □