Generating Graded Central Simple Algebras

Lieven Le Bruyn
(Senior Researcher at the NFWO)

Dept. of Mathematics & Comp. Sci.,
University of Antwerp, UIA, Belgium

November 1996

Report no. 96-19

Division of Pure Mathematics
Department of Mathematics & Computer Science

universitaire
instelling
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Universiteitsplein 1, B-2610 Wilrijk-Antwerpen, BELGIUM

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Lieven Le Bruyn*
Departement Wiskunde UIA
B-2610 Wilrijk (Belgium)
email : lebruyn@wins.uia.ac.be

October 22, 1996

Abstract

In this paper we solve the generator problem for $\mathbb{Z}$-graded central simple algebras. Applications are given to automorphisms of trace rings of generic matrices and to periodic fat point modules.

1 Introduction

If Δ is a simple algebra, finite dimensional over its center K, then it is well known (for example [8, lemma III.1.2]) that Δ can be generated by two elements as K-algebra. In this paper we investigate the analogous question for $\mathbb{Z}$-graded central simple algebras.

Recall that a $\mathbb{Z}$-graded algebra $\Delta = \oplus_{i=-\infty}^{\infty} \Delta_i$ is said to be graded central simple iff Δ has no non-trivial graded ideals. By a graded version of Weddenburn's theorem [7, Thm. I.5.8] we know that

$$\Delta \simeq M_n(D[X, X^{-1}, \phi])(a_1, \ldots, a_n)$$

for some n, a division algebra D, a generator X having degree d and an automorphism ϕ. The numbers a_i can be chosen such that $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n < d$. If R denotes the skew Laurant polynomial algebra $D[X, X^{-1}, \phi]$ graded by the degree of X (that is, $R_{kd} = DX^k$ and $R_i = 0$ otherwise) then

*Senior researcher at the NFWO

the i-th homogenous part of $M_n(D[X,X^{-1},\phi])(a_1,\ldots,a_n)$ is equal to

$$\begin{bmatrix} R_i & R_{i+a_1-a_2} & \cdots & R_{i+a_1-a_n} \\ R_{i+a_2-a_1} & R_i & \cdots & R_{i+a_2-a_n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{i+a_n-a_1} & R_{i+a_n-a_2} & \cdots & R_i \end{bmatrix}$$

We are interested in the case when Δ is a finite module over its center. One verifies easily that this happens if and only if D is finite dimensional over its center L and $\phi \in Aut(D)$ is such that some power becomes an inner automorphism of D, that is, for a minimal m we have $a \in D$ such that $\phi^m(d) = a^{-1}.d.a$. With these notations one verifies that the center becomes

$$Z(M_n(D[X,X^{-1},\phi])(a_1,\ldots,a_n)) = K[T,T^{-1}]$$

where $K = L^\phi$ the invariant field of L and $T = aX^m$ a generator of degree $e = dm$. Furthermore, Δ satisfies all polynomials of $N \times N$ matrices where $N = nim$ with i the index of D, that is, $[D : L] = i^2$.
Obviously, one wonders whether Δ can always be generated by two homogenous elements over its center. However, we have the following

Example 1 *Let $b_1,\ldots,b_n$ be pairwise relatively prime natural numbers and $e = \prod b_i$. Then, one verifies that*

$$\Delta = M_n(K[T,T^{-1}])(b_1,\ldots,b_n)$$

with $deg(T) = e$ cannot be generated by less than n homogenous elements as a $K[T,T^{-1}]$-algebra.

On the other hand, it has been conjectured in [2, Remark p.1697] that, if Δ is generated by Δ_1 as Δ_0-algebra (and hence is a strongly graded algebra as in [7, I.3]), then Δ should be generated by two elements of degree one over its center.
Even in this case the truth is more subtle. A special case of our main theorem can be phrased as follows

Theorem 1 $\Delta = M_n(D[X,X^{-1},\phi])(a_1,\ldots,a_n)$ *with $deg(X) = d$ is generated by Δ_1 as Δ_0-algebra iff*

$$(a_1,\ldots,a_n) = (\underbrace{0,\ldots,0}_{m_1},\underbrace{1,\ldots,1}_{m_2},\ldots,\underbrace{d-1,\ldots,d-1}_{m_d})$$

with all $m_i \geq 1$. Then, Δ is generated by k elements of degree one over its center if and only if

$$m_i \leq k.m_{i\pm 1} \text{ for all } i \text{ mod } d$$

In fact, we will solve the generator problem in full generality. That is, we will give for any $\Delta = M_n(D[X, X^{-1}, \phi])(a_1, \ldots, a_n)$ necessary and sufficient numerical conditions to test whether Δ is generated by k_1 elements of degree d_1, k_2 elements of degree d_2, etc. k_s elements of degree d_s. The proof relies on translating the problem in a certain quiver representation theoretic problem and the algorithmic description of the dimension vectors of simple representations of [4].
Our interest in this problem originated from the following invariant theoretic problem. Consider the space of m-tuples of $n \times n$ matrices $M_n^m = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$. If $A = (A_1, \ldots, A_m) \in M_n^m$ then $PGL_n \times GL_m$ act via $g.A_i = gA_ig^{-1}$ for $g \in PGL_n$ and $a.A_i = \sum a_{ij}A_j$ for $a \in GL_m$.
We call A a generating m-tuple if the matrices A_i generate $M_n(\mathbb{C})$ as a $\mathbb{C}$-algebra and a saturated m-tuple if $(A_1, \ldots, A_{m-1})$ is a generating $m-1$-tuple. For $m \geq 3$ one wonders whether

$$GL_m.Sat_n^m = Gen_n^m$$

where Gen_n^m (resp. Sat_n^m) is the open subvariety of generating (resp. saturated) m-tuples.
If this equality holds one can deduce from the work of Z. Reichstein [9] that any two points in the quotient variety $Q_n^m = M_n^m/PGL_n$ of the same representation type have Zariski isomorphic neighborhoods. Recall that this fact has been proved by Reichstein when $m \geq n+1$. However, we will prove

Theorem 2 *For all $3 \leq m \leq n-1$ we have*

$$GL_m.Sat_n^m \overset{\neq}{\hookrightarrow} Gen_m^n$$

which may be seen as evidence that Reichstein's transitivity result of the automorphism group on the strata cannot be generalized to $m < n$.

2 Reduction to graded matrices

Throughout this section we keep the same notation as above. That is,

$$\Delta = M_n(D[X, X^{-1}, \phi])(a_1, \ldots, a_n)$$

where D is a division algebra of index i with center L, the order of the automorphism ϕ is m in $Aut(D)/Inn(D)$ and the degree of X is d. Then, the center of Δ is the graded field $K[T,T^{-1}]$ with $K = L^{\phi}$ and T an element of degree $e = dm$. Moreover, the index of $Q(\Delta)$ over $K(T)$ is $N = nim$.
For a $2s$-tuple of natural numbers

$$\boldsymbol{g} = (k_1, d_1; \ldots; k_s, d_s)$$

we say that Δ is $\boldsymbol{g}$-generated (over its center $K[T,T^{-1}]$) iff Δ can be generated as $K[T,T^{-1}]$-algebra by k_1 elements of degree d_1, etc. and k_s elements of degree d_s. It is clear that we may assume that all $d_i < e = deg(T)$.
We will reduce the problem of $\boldsymbol{g}$-generateness of Δ to that of $\boldsymbol{g}$-generateness of a certain graded $N \times N$ matrix-algebra over a graded field. The crucial property that we need is the fact that Δ can be split in degree zero, see [1, IV.1.7]. Therefore, if $\overline{K}$ is the algebraic closure of K we know that $\Delta \otimes \overline{K}[T,T^{-1}]$ is a graded matrix algebra. Remains to determine the relevant numbers.

Lemma 1 *With notations as before,*

$$\Delta \otimes \overline{K}[T,T^{-1}] \simeq M_N(\overline{K}[T,T^{-1}])(b_1, \ldots, b_N)$$

where $(b_1, \ldots, b_N)$ *is*

$$(\underbrace{a_1, \ldots, a_1}_{i}, \ldots, \underbrace{a_n, \ldots, a_n}_{i}, \underbrace{a_1 + d, \ldots, a_1 + d}_{i}, \ldots, \underbrace{a_n + d, \ldots, a_n + d}_{i}, \ldots,$$

$$\underbrace{a_1 + (m-1)d, \ldots, a_1 + (m-1)d}_{i}, \ldots, \underbrace{a_n + (m-1)d, \ldots, a_n + (m-1)d}_{i})$$

Proof : Easy by comparing the dimensions of the homogenous components. □

Proposition 1 *Let* $\boldsymbol{g} = (k_1, d_1; \ldots; k_s, d_s)$ *then the following statements are equivalent*

1. $M_n(D[X, X^{-1}, \phi])(a_1, \ldots, a_n)$ *is* $\boldsymbol{g}$*-generated*
2. $M_N(\overline{K}[T,T^{-1}])(b_1, \ldots, b_N)$ *is* $\boldsymbol{g}$*-generated*

Proof : Δ_{d_i} is a finite dimensional K-vectorspace say with basis $b_{i1}, \ldots, b_{iv_i}$. Consider k_i general elements in Δ_{d_i}

$$g_{ij} = \sum_{k=1}^{v_i} \alpha_{ij,k} b_{ik} \text{ with } 1 \leq i \leq k_i$$

and consider all monomials in the elements g_{ij} where $1 \leq i \leq s$ and $1 \leq j \leq k_i$ and order this list with respect to the degree of the elements. Let $\{c_1, \ldots\}$ be this (infinite) list. Now consider the matrix

$$(Tr(c_i.c_j))_{i,j \in \mathbb{N}}$$

where Tr is the reduced trace of Δ with values in $K[T, T^{-1}]$. Any entry of this matrix is a polynomial in the coefficients $\alpha_{ij,k}$ with coefficients in $K[T, T^{-1}]$.

Clearly, the elements g_{ij} generate Δ over $K[T, T^{-1}]$ if and only if the determinant of some $N^2 \times N^2$ minor of the above matrix is non-zero. As these determinants are polynomials in the $\alpha_{ij,k}$ over $K[T, T^{-1}]$ it suffices to show that they are not all formally zero.

As the b_{ij} also form a basis for the homogenous part of degree d_i of $\Delta \otimes \overline{K}[T, T^{-1}]$ we can repeat the above argument to find a necessary and sufficient condition for $M_N(\overline{K}[T, T^{-1}])(b_1, \ldots, b_N)$ to be $\boldsymbol{g}$-generated. If this is the case one of these determinants has a non-zero value for some $\alpha_{ij,k} \in \overline{K}$. But this means that the corresponding polynomial is not formally zero, whence the corresponding $N^2 \times N^2$ minor for Δ has rank N^2 entailing that Δ is $\boldsymbol{g}$-generated. □

3 Reduction to a quiver problem

From now on we will work over the algebraically closed field $\overline{K}$ of characteristic zero and denote it with $\mathbb{C}$. We will slightly change our notation and use the dictionary of the foregoing section to translate the obtained results back to arbitrary graded central simple algebras.

We want to find necessary and sufficient conditions for the graded matrix algebra

$$M = M_N(\mathbb{C}[T, T^{-1}])(b_1, \ldots, b_N)$$

to be $\boldsymbol{g}$-generated where $\boldsymbol{g} = (k_1, d_1; \ldots; k_s, d_s)$ and where $deg\ T = e$. It will be more convenient to denote the N-tuple $(b_1, \ldots, b_N)$ as

$$\boldsymbol{b} = (m_1, e_1; \ldots; m_l, e_l)$$

where $0 \leq e_1 < e_2 < \ldots < e_l < e$ are the distinct numbers occurring as a b_i and m_i is the multiplicity with which they appear. Hence, in particular we have that $\sum m_i = N$.

If we denote $R = \mathbb{C}[T, T^{-1}]$ then using our new notation we see that the homogenous part of degree i of $M_N(R)(\boldsymbol{b})$ can be given a block decomposition

$$\left[\begin{array}{c|c|c|c} R_i & R_{i+e_1-e_2} & \cdots & R_{i+e_1-e_l} \\ \hline R_{i+e_2-e_1} & R_i & \cdots & R_{i+e_2-e_l} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline R_{i+e_l-e_1} & R_{i+e_l-e_2} & \cdots & R_i \end{array}\right]$$

where the block at position (k, l) has size $m_k \times m_l$.
This block-decomposition suggests the following quiver-setting. The matrix-skeleton $MSk(\boldsymbol{b})$ for $\boldsymbol{b} = (m_1, e_1; \ldots; m_l, e_l)$ is defined to be the complete labeled directed graph on l vertices where we give the directed arrow

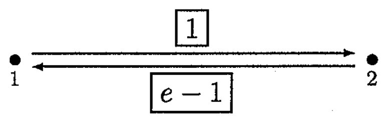
$$\underset{i}{\bullet} \longrightarrow \underset{j}{\bullet} \quad \text{label } e_j - e_i \bmod e$$

This matrix-skeleton encodes the relevant information of the graded matrix algebra $M_N(R)(\boldsymbol{b})$ if we also give the corresponding dimension vector $\boldsymbol{m} = (m_1, \ldots, m_l)$. We have the following observation

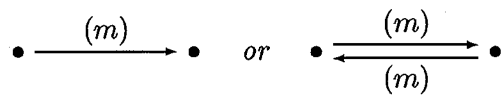
Lemma 2 *All oriented cycles in the matrix-skeleton $MSk(\boldsymbol{b})$ have total label equal to zero in $\mathbb{Z}/e\mathbb{Z}$.*

Given a potential generator datum $\boldsymbol{g} = (k_1, d_1; \ldots; k_s, d_s)$ we will form out of the matrix-skeleton $MSk(\boldsymbol{b})$ a quiver $Q(\boldsymbol{b}, \boldsymbol{g})$ in the following way.
$Q(\boldsymbol{b}, \boldsymbol{g})$ is the quiver on the l vertices (those of $MSk(\boldsymbol{b})$) which has k_i directed arrows for every arrow of label d_i in $MSk(\boldsymbol{b})$.

Example 2 *Consider $M_N(\mathbb{C}[T, T^{-1}])(\underbrace{0, \ldots, 0}_{a}, \underbrace{1, \ldots, 1}_{b})$, then $\boldsymbol{b} = (a, 0; b, 1)$ and the matrix-skeleton $MSk(\boldsymbol{b})$ is the labeled digraph*



If the generator data is $\boldsymbol{g} = (m, 1)$ then the quiver $Q(\boldsymbol{b}, \boldsymbol{g})$ is



according to whether deg $T = e$ is not (resp. is) equal to 2.

If $\boldsymbol{m} = (m_1, \ldots, m_l) \in \mathbb{N}^l$ then the variety of representations of a quiver Q with dimension vector $\boldsymbol{m}$, $Rep(Q, \boldsymbol{m})$ is the vectorspace where we assign to each directed arrow

$$\underset{i}{\bullet} \longrightarrow \underset{j}{\bullet} \text{ the space } M_{m_j \times m_i}(\mathbb{C})$$

that is, if we assign to each vertex i the space $\mathbb{C}^{\oplus m_i}$ then each arrow corresponds to a linear map between the vertex-spaces. Observe that the group $GL(\boldsymbol{m}) = GL_{m_1} \times \ldots \times GL_{m_l}$ has a natural action on $Rep(Q, \boldsymbol{m})$ by basechange in the vertexspaces. Two representations in $Rep(Q, \boldsymbol{m})$ are isomorphic iff they belong to the same $GL(\boldsymbol{m})$-orbit.
If Q is a quiver on l vertices, the Ringel bilinear form R on $\mathbb{Z}^l$ is determined by the matrix with entries

$$R_{ij} = \delta_{ij} - \#\{\underset{i}{\bullet} \longrightarrow \underset{j}{\bullet}\}$$

Not only do we recover the quiver Q from the Ringel form but also a lot of homological information on representations of Q. Let V (resp. W) be a representation of Q with dimension vector α (resp. β), then we have

$$R(\alpha, \beta) = dim_{\mathbb{C}}\ Hom(V, W) - dim_{\mathbb{C}}\ Ext^1(V, W)$$

The Ringel form can also be used to give an algorithmic description of the dimension vectors of the simple representations of Q. Recall that $\widetilde{A_l}$ is the cyclic quiver on l-vertices with one arrow between successive vertices with the cyclic orientation. The following result was proved in [4, Thm. 4]

Theorem 3 *If Q is not equal to $\widetilde{A_l}$, then $\boldsymbol{m} \in \mathbb{N}^l$ is the dimension vector of a simple representation of Q iff and only if*

1. *$supp(\boldsymbol{m})$ is a strongly connected subquiver, that is, if $m_i \neq 0 \neq m_j$ then there is an oriented path from i to j in $supp(\boldsymbol{m})$*
2. *For all $1 \leq i \leq l$ we have the numerical conditions*

$$R(\boldsymbol{m}, \delta_i) \leq 0 \text{ and } R(\delta_i, \boldsymbol{m}) \leq 0$$

where $\delta_i = (\delta_{ij})_j$ is the standard base-vector of $\mathbb{Z}^l$

Finally, recall the stratification result of [4, Thm. 3] which implies that if $\boldsymbol{m}$ is the dimension vector of a simple representation, then $Rep(Q, \boldsymbol{m})$ has a Zariski open subset of simple representations. Moreover, there is an obvious notion of degeneration of representation-types which allows to determine the closures and inclusions of strata, see [4] for more details.

Lemma 3 *If* $\Delta = M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$, *where* $\boldsymbol{b} = (m_1, e_1; \ldots; m_l, e_l)$ *and* $\boldsymbol{g} = (k_1, d_1; \ldots; k_s, d_s)$ *then if we denote* $\boldsymbol{m} = (m_1, \ldots, m_l)$, *there is a natural identification*

$$\phi : Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m}) \longrightarrow \Delta_{d_1}^{\oplus k_1} \oplus \ldots \oplus \Delta_{d_s}^{\oplus k_s}$$

Proof : It follows from the block-description of Δ that an element $\delta \in \Delta_{d_i}$ has only non-zero entries in the block at place (k,l) iff $d_i + e_k - e_l$ is a multiple of e, that is, when $d_i = e_l - e_k \mod e$. This block has size $m_k \times m_l$. Hence, δ is fully determined by taking in $Q(\boldsymbol{b},\boldsymbol{g})$ one arrow between the vertices k and l whenever the corresponding arrow in $MSk(\boldsymbol{b})$ has label d_i. The lemma follows by superposition. □

We are now in a position to state and prove the main theorem :

Theorem 4 *With notations as above, the following statements are equivalent*

1. $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ *is* $\boldsymbol{g}$*-generated*
2. $\boldsymbol{m}$ *is the dimension vector of a simple representation of* $Q(\boldsymbol{b},\boldsymbol{g})$

Proof :
(1) $\Longrightarrow$ (2) : Every element δ in Δ_j is a matrix whose entries are all of the form αT^k for $\alpha \in \mathbb{C}$ and some $k \in \mathbb{Z}$. With δ_λ we will denote the matrix in $M_N(\mathbb{C})$ obtained from δ after setting T equal to λ. If $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ is $\boldsymbol{g}$-generated there are elements $(\delta(k)) \in \Delta_{d_1}^{\oplus k_1} \oplus \ldots \oplus \Delta_{d_s}^{\oplus k_s}$ which generate $M_N(\mathbb{C}[T,T^{-1})(\boldsymbol{b})$ over $\mathbb{C}[T,T^{-1}]$. But then, for a generic specialization $T \mapsto \lambda$ the matrices $\delta(k)_\lambda \in M_N(\mathbb{C})$ will generate $M_N(\mathbb{C})$ as $\mathbb{C}$-algebra. These matrices correspond to a simple representation of $Q(\boldsymbol{b},\boldsymbol{g})$ with dimension vector $\boldsymbol{m}$ under the identification of the previous lemma.
(2) $\Longrightarrow$ (1) : This part will be proved by induction on $\boldsymbol{m}$ and parallels the proof of [4, Thm. 4]. We will sketch only the main ideas. If $k = \sum k_i$ we denote the map

$$Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m}) \xrightarrow{\phi} \Delta_{d_1}^{\oplus k_1} \oplus \ldots \oplus \Delta_{d_s}^{\oplus k_s} \xrightarrow{T \mapsto \lambda} M_N(\mathbb{C})^{\oplus k}$$

by ϕ_λ. The set of $V \in Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m})$ such that $\phi_\lambda(V)$ is a generating k-tuple of $N \times N$ matrices is Zariski open for all $\lambda \neq 0$.
If all $m_i = 1$ and $\boldsymbol{m}$ is the dimension vector of a simple representation of $Q(\boldsymbol{b},\boldsymbol{g})$ then one can use the strongly connectedness and the fact that the total label of any oriented cycle is a multiple of e to produce the primitive

matrix-idempotents e_{ii} from a simple representation in $Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m})$. using these idempotents one can then generate $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$.
Hence we may assume that there is a vertex i with m_i maximal and ≥ 2 and that the result holds for all dimension vectors $\boldsymbol{f} < \boldsymbol{m}$. In particular we can consider the vector

$$\boldsymbol{m'} = (m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_l)$$

As in [4, Thm 4] one can easily reduce to the case that i is a good vertex (that is, there is no direct successor (resp. predecessor) j of i with $m_j = m_i$ and j is a prism (resp. focus) vertex). In this case, one verifies easily that $\boldsymbol{m'}$ is again the dimension vector of a simple representation of $Q(\boldsymbol{b},\boldsymbol{g})$ and by induction we may assume that $M_{N-1}(\mathbb{C}[T,T^{-1}])(\boldsymbol{b'})$ is $\boldsymbol{g}$-generated with $\boldsymbol{b'} = (m_1, e_1; \ldots; m_i - 1, e_i; \ldots; m_l, e_l)$. Consider the non-empty Zariski open subset U' of $Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m'})$ such that the maps ϕ_λ to $M_{N-1}(\mathbb{C})^{\oplus k}$ give generating tuples for $\lambda \neq 0$.
As $R(\boldsymbol{m'},\delta_i) < 0$ and $R(\delta_i,\boldsymbol{m'}) < 0$ we know that for any $V' \in U'$ we have

$$Ext^1(V', S_i) \neq 0 \neq Ext^1(S_i, V')$$

for S_i the one-dimensional simple representation concentrated in vertex i. Now consider the open subvariety U of $Rep(Q(\boldsymbol{b},\boldsymbol{g}),\boldsymbol{m})$ of representations V such that $V' = V \mid \boldsymbol{m'}$ lies in U'. Consider a point in U and consider the subalgebra of $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ generated by $\phi(V)$. As $\phi(V')$ generates $M_{N-1}(\mathbb{C}[T,T^{-1}])(\boldsymbol{b'})$ it contains an homogenous element (of degree a multiple of e) with $N-1$ distinct eigenvalues. There exists an open set of V with $V \mid \boldsymbol{m'} = V'$ such that the corresponding element $C(T)$ in $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ has N distinct eigenvalues. By the block-form of $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ we know that the (finitely many) eigenspaces of $\phi_\lambda(V)$ are concentrated in the vertex-spaces. As U contains an open subset consisting of simple representations we may assume that V is a simple representation. Hence, for each of these finite number of eigenspaces there is a M_z among the components of $\phi_\lambda(V)$ which does not leave this subspace invariant. But then $C(\lambda)$ and a linear combination of the M_z generate $M_N(\mathbb{C})$ and this for a dense set of $\lambda \neq 0$.
Hence, let Γ be the $\mathbb{C}[T,T^{-1}]$-subalgebra of $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ generated by the homogenous elements $\phi(V)$. By the above argument Γ must be a graded prime ring with center the graded field $\mathbb{C}[T,T^{-1}]$. But then, Γ is a graded central simple algebra and must be equal to $M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ finishing the proof. □

4 Some consequences

In view of the main theorem and the numerical condition of theorem 3 to determine the dimension vectors of semi-simple representations we have a complete solution to the generator problem for graded matrix algebras and hence by the descent results of section 2 also for graded central simple algebras. In this section we draw some immediate consequences.

Lemma 4 *$M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ can only be $\boldsymbol{g}$-generated if $Q(\boldsymbol{b},\boldsymbol{g})$ is a strongly connected quiver.*

Proof : If $\boldsymbol{m}$ is the dimension vector of a simple representation of $Q(\boldsymbol{b},\boldsymbol{g})$ then its support which is $\{1,\ldots,l\}$ has to be a strongly connected (sub)quiver by theorem 3. □

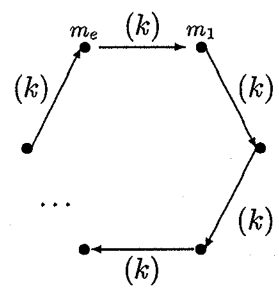
We will now concentrate on the special (but important) case of matrix-algebras generated in degree one.

Proposition 2 *$M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})$ is generated in degree one if and only if $\boldsymbol{b} = (m_1, 0, m_2, 1, \ldots, m_e, e-1)$ with all $m_i \geq 1$. In fact, it can be generated by k elements of degree one if and only if $m_i \leq k m_{i \pm 1 \ mod\ e}$.*

Proof : Let us denote $\boldsymbol{b} = (m_1, e_1, \ldots, m_l, e_l)$ and $\boldsymbol{g} = (k, 1)$ for $k = dim\ M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})_1$, then the only arrows in $Q(\boldsymbol{b},\boldsymbol{g})$ are those from vertex i to vertex j when $e_j - e_i$ is equal to one or $1-e$. As $e_1 < e_2 < \ldots < e_l < e$ this means that there are only arrows in $Q(\boldsymbol{b},\boldsymbol{g})$ between two consecutive vertices if $e_{i+1} = e_i + 1\ mod\ e$. Hence, $Q(\boldsymbol{b},\boldsymbol{g})$ can only be strongly connected if $e_1 = 0, e_2 = 1$ etc and $l = e$ and $e_l = e-1$. In this case

$$k = dim\ M_N(\mathbb{C}[T,T^{-1}])(\boldsymbol{b})_1 = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} m_i . m_{i+1}$$

and the quiver $Q(\boldsymbol{b},\boldsymbol{g})$ and dimension-vector $\boldsymbol{m}$ can be visualized as



The Ringel form of $Q(\boldsymbol{b},\boldsymbol{g})$ is given by the $e \times e$ matrix

$$\begin{bmatrix} 1 & -k & & & & \\ & 1 & -k & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & -k \\ -k & & & & & 1 \end{bmatrix}$$

and $\boldsymbol{m} = (m_1, \ldots, m_e)$ is the dimension vector of a simple representation of $Q(\boldsymbol{b},\boldsymbol{g})$ by theorem 3 if and only if

$$\forall i : m_i \leq k.m_{i \pm 1 \ mod \ e}$$

from which the result follows. □

By descent we can now prove theorem 1 of the introduction

Theorem 1 *$\Delta = M_n(D[X, X^{-1}, \phi])(a_1, \ldots, a_n)$ with $deg(X) = d$ is generated by Δ_1 as Δ_0-algebra iff*

$$(a_1, \ldots, a_n) = (\underbrace{0, \ldots, 0}_{m_1}, \underbrace{1, \ldots, 1}_{m_2}, \ldots, \underbrace{d-1, \ldots, d-1}_{m_d})$$

with all $m_i \geq 1$. Then, Δ is generated by k elements of degree one over its center if and only if

$$m_i \leq k.m_{i \pm 1} \text{ for all } i \text{ mod } d$$

Proof : If $\Delta = M_n(D[X, X^{-1}, \phi])(\boldsymbol{a})$ with

$$\boldsymbol{a} = (m_1, 0; m_2, 1; \ldots; m_d, d-1)$$

and if the degree of ϕ in $Aut(D)/Inn(D)$ is m and the index of D is i we know from lemma 1 that

$$\Delta \otimes \mathbb{C}[T, T^{-1}] = M_N(\mathbb{C}[T, T^{-1}])(\boldsymbol{b})$$

where $\mathbb{C}$ is the algebraic closure of the center K of D and $deg(T) = md = e$ and $\boldsymbol{b}$ is equal to

$$(im_1, 0; \ldots; im_d, d-1; im_1, d; \ldots; im_d, 2d-1; \ldots; im_1, (m-1)d; \ldots; im_d, e-1)$$

and the numerical condition of the previous proposition applied to this case is that

$$im_i \leq kim_{i \pm 1 \ mod \ d}$$

from which the statement follows. □

It is now easy to find counter-examples to the conjecture of [2, Remark p. 1697]

Example 3 *Let deg $X = 2$ and consider the graded central simple algebra*

$$\Delta = M_{k+1}(D[X, X^{-1}, \phi])(1, 0; k, 1)$$

Then, Δ can be generated by k elements of degree one but not by $k-1$ elements of degree one.

5 Automorphisms of trace rings of generic matrices

In this section we give an application to invariant theory. Again, we will work over an algebraically closed field of characteristic zero and denote it with $\mathbb{C}$.
The group PGL_n acts on the space of m-tuples of $n \times n$ matrices $M_n^m = M_n(\mathbb{C})^{\oplus m}$ by simultaneous conjugation. Let Q_n^m be the algebraic quotient variety M_n^m/PGL_n for this action, that is, the coordinate ring $\mathbb{C}[Q_n^m]$ is the ring of polynomial invariant functions $\mathbb{C}[M_n^m]^{PGL_n}$.
A point $\zeta \in Q_n^m$ can be lifted to a (unique up to simultaneous conjugation) m-tuple of matrices $x_\zeta = (x_1, \ldots, x_m) \in M_n^m$ such that the representation

$$\phi(x_\zeta) : \mathbb{C}\langle u_1, \ldots, u_m\rangle \longrightarrow M_n(\mathbb{C}) \quad u_i \mapsto x_i$$

of the free algebra on m letters is semi-simple, that is, the x_i generate a semi-simple subalgebra

$$M(x_\zeta) = M_{e_1}(\mathbb{C})^{\oplus d_1} \oplus \ldots \oplus M_{e_r}(\mathbb{C})^{\oplus d_r} \hookrightarrow M_n(\mathbb{C})$$

with $\sum e_i d_i = n$. We say that ζ or x_ζ has representation type

$$\tau(\zeta) = \tau(x_\zeta) = (d_1, e_1; \ldots; d_r, e_r)$$

and denote by $Q_n^m(\tau)$ the set of all points of Q_n^m of representation type τ.
For more details we refer to [3] where it was shown (among other things) that any two points of the same representation type have étale (or analytic) isomorphic neighborhoods.
In [9] and [10] Z. Reichstein studied the analogous (but much harder) problem for the Zariski topology. By constructing PGL_n-equivariant automorphisms on M_n^m he was able to show (at least if m is large enough) that the automorphism group acts transitively on the strata and hence

Theorem 5 (Reichstein [9]) *Any two points of $Q^n_m(\tau)$ have isomorphic Zariski neighborhoods if $m \geq n+1$.*

This result raises the obvious question whether there can be different orbits under the automorphism group for small values of m. For $m \geq 3$ it follows from Reichstein's strategy that the hearth of the problem consists of points of representation type $(1,n)$ (that is, those corresponding to simple representations) which form a Zariski open and dense set in Q^m_n. We will study here $Q^m_n(1,n)$ under affine automorphisms, that is, automorphisms of Q^m_n induced from those on $\mathbb{C}\langle u_1, \dots, u_m\rangle$ of the form

$$u_i \mapsto \sum a_{ij} u_j$$

with $a = (a_{ij}) \in GL_m(\mathbb{C})$. Note that, the action of affine automorphisms gives a GL_m-action on M^m_n commuting with the PGL_n-action.
An m-tuple $x = (x_1, \dots, x_m) \in M^m_n$ is said to be generating if x determines a simple representation, that is, belongs to $\pi^{-1}(Q^m_n(1,n)) = Gen^m_n$. It is clear that Gen^m_n is a $PGL_n \times GL_m$-stable non-empty Zariski-open subset when $m \geq 2$.
Gen^m_n contains a Zariski open subset Sat^m_n consisting of the saturated m-tuples, that is, those x such that $x_1, \dots, x_{m-1}$ already generate $M_n(\mathbb{C})$. Clearly, Sat^m_n is a PGL_n-stable non-empty Zariski open subset whenever $m \geq 3$.
As Sat^m_n is not stable under the GL_m-action, we wonder whether any generating m-tuple can be mapped by an affine automorphism to a saturated m-tuple, or equivalently, for which $m \geq 3$ do we have

$$GL_m.Sat^m_n = Gen^m_n \ (?)$$

The relevance of this question comes from the following result which an be proved by mimicking the arguments in [9] and [10].

Proposition 3 *If m is such that $GL_m.Sat^m_n = Gen^m_n$, then for every representation type τ we have that any two points in $Q^m_n(\tau)$ have isomorphic Zariski neighborhoods.*

However, we will prove the following result mentioned in the introduction

Theorem 2 *For all $3 \leq m \leq n-1$*

$$GL_m.Sat^m_n \overset{\neq}{\hookrightarrow} Gen^m_n$$

Proof : First we will give a procedure to associate to any m-tuple $x = (x_1, \ldots, x_m) \in Gen_n^m$ a graded matrix-algebra.
Equip $M_n(\mathbb{C}[t])$ with the usual gradation and consider the $\mathbb{C}$-subalgebra A_x generated by the homogenous elements

$$tx_i \in M_n(\mathbb{C}[t])$$

Clearly, A_x is a graded algebra and we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}\langle u_1, \ldots, u_m \rangle & \xrightarrow{\phi(x)} & M_n(\mathbb{C}) \\ \downarrow & & \uparrow \, ./(t-1) \\ \mathbb{G}_n^m & \xrightarrow[\Phi(x)]{} & A_x \end{array}$$

where $\mathbb{G}_n^m$ is the ring of m generic $n \times n$ matrices, that is, the subalgebra of $M_n(\mathbb{C}[v_{ij}(k)])$ generated by the generic matrices $V_k = (v_{ij}(k))$. $\mathbb{G}_n^m$ is a graded algebra generated in degree one by giving $deg(v_{ij}(k)) = 1$ and the map $\Phi(x)$ determined by sending V_k to tx_k is gradation preserving.
Because $\phi(tx)$ is a simple representation it follows from the Artin-Procesi theorem that there exists an homogenous central polynomial whose evaluation at A_x is non-zero. That is, there exists a $c = t^f \in Z(A_x)$ for some f. But then, the graded localization at c is a graded field, hence of the form

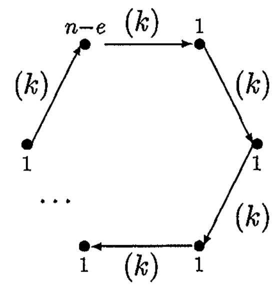
$$Q_c^g(Z(A_x)) = \mathbb{C}[t^e, t^{-e}]$$

for some e and as any specialization $Q_c^g(A_x)/(t - \lambda) = M_{\hat{n}}(\mathbb{C})$, $Q_c^g(A_x)$ is a graded Azumaya algebra over the graded field $\mathbb{C}[t^e, t^{-e}]$ and hence by [1] of the form

$$Q_c^g(A_x) \simeq M_n(\mathbb{C}[t^e, t^{-e}])(m_1, 0; m_2, 1; \ldots; m_e, e-1)$$

where the determination of $\boldsymbol{b} = (m_1, e_1; \ldots; m_l, e_l)$ follows from the fact that A_x and hence $Q_c^g(A_x)$ is generated in degree one. More precisely, $Q_c^g(A_x)$ is generated as $\mathbb{C}[t^e, t^{-e}]$-algebra by the m-elements tx_i.
Next, we will use our generator results to construct $x \in Gen_n^m - GL_m.Sat_n^m$ with $m = n - e$. Consider the situation $\boldsymbol{b} = (1, 0; 1, 1; \ldots; 1, e-1; n-e, e)$

with corresponding quiver-setting



Then we see that this $\boldsymbol{m}$ is a dimension vector of a simple representation iff $k \geq n - e$.
Hence, if $m = n - e$ we can take x the m-tuple of matrices corresponding to a simple representation of the quiver. Therefore, $x \in Gen_n^m$. Further, any point of the orbit $GL_m.x$ is again a representation of this quiver. If some $GL_m.x$ would lie in Sat_n^m then this would mean that the quiver with $k = m - 1$ arrows between the vertices would have a simple representation of dimension vector $\boldsymbol{m}$. Quod non. □

The idea underlying the above proof is the following. From [6] we know that in the Hesselink stratification of the nullcone $Null_n^m$ of the PGL_n-action on M_n^m there appear for each $m \leq n-1$ non-empty strata which were still empty in $Null_n^{m-1}$. By taking the associated cone $C(x)$ of a simple representation x one obtains a subvariety of $Null_n^m$ of dimension $n^2 - 1$. Clearly, it should make a difference whether $C(x)$ hits (or does not) one of this new strata. The construction of A_x is a ringtheoretical version of taking the cone over the orbit, $\overline{PGL_n \times \mathbb{C}^*.x}$ and $Q_x^g(A_x)$ is the algebra corresponding to $\overline{PGL.x \times \mathbb{C}^*.x} - C(x)$.

6 Periodic fat points

In this section we will give an application to the study of the *Proj* of graded algebras and in particular to the determination of the types of periodic fat points which can occur.
The setting will be the following : let A be a connected positively graded affine $\mathbb{C}$-algebra which is $\boldsymbol{g} = (k_1, d_1; \ldots ; k_s, d_s)$ generated and let $m = \sum k_i$. We consider $Rep_n\ A$ the variety of n-dimensional representations of A. Clearly, $Rep_n\ A$ is a PGL_n-stable closed subvariety of M_n^m. Moreover, the gradation of A endows this variety with an additional $\mathbb{C}^*$-action. Consider

the one-dimensional torus

$$\mathbb{C}^* \hookrightarrow GL_m \qquad \text{via } t \mapsto \begin{bmatrix} t^{d_1} & & \\ & \ddots & \\ & & t^{d_s} \end{bmatrix}$$

then $Rep_n\ A$ is $PGL_n \times \mathbb{C}^*$-stable.
It is natural to define the n-th approximation of $Proj\ A$ to be the orbit-space

$$proj_n\ A = Orb(Rep_n\ A, PGL_n \times \mathbb{C}^*)$$

as for commutative A, $proj_1\ A = Proj(A)$. Observe that there is only one closed orbit in $Rep_n\ A$ for this $PGL_n \times \mathbb{C}^*$- action, namely the trivial representation, so the usual affine quotient variety will not be useful in this case.
Now, assume that A has an n-dimensional simple representation then the set of all irreducible n-dimensional representations

$$Irr_n\ A \xhookrightarrow{\text{open}} Rep_n\ A$$

is a Zariski open subset which is clearly $PGL_n \times \mathbb{C}^*$-stable. As a first approximation to the orbit space $proj_n\ A$ one can study the orbit-space $irr_n\ A$ of orbits in $Irr_n\ A$. As the stabilizer of any point in $Irr_n\ A$ is of the form $\mathbf{1} \times \boldsymbol{\mu_e}$ for some e, all orbits have dimension n^2 and hence are closed in $Irr_n\ A$. Therefore, if we cover $Irr_n\ A$ by affine $PGL_n \times \mathbb{C}^*$- stable subvarieties we can construct $irr_n\ A$ locally by studying the corresponding affine quotient-varieties. A natural way to do this is to consider the special affine open sets in M_n^m determined by a homogenous (with respect to the by the torus induced gradation on $\mathbb{C}\langle u_1, \ldots, u_m\rangle$ or on $\mathbb{G}_n^m$) central polynomial. In this way we get a scheme-structure on $irr_n\ A$.
The orbits in $Irr_n\ A$ have the following module-theoretic interpretation. Recall that a fat point module of A is an equivalence class in $Proj\ A$, a representant F of which is a graded (left) A-module which is 1-critical with respect to Gelfand-Kirillov dimension. Recall that fat point modules are simple objects in $Proj\ A$ (which is the quotient category of $gr\ A$ the category of graded left A-modules by the Serre subcategory of torsion A-modules).
We will say that a fat point with representing module F is periodic of multiplicity n if F has a simple quotient of dimension n. The reason for this terminology is that we can choose F such that the Hilbert series has rational expression

$$\mathcal{H}(F,t) = \frac{m_1 t^{e_1} + \ldots + m_l t^{e_l}}{(1-t^2)}$$

We will say that the fat point has period e, multiplicity $\sum m_i$ and type $\boldsymbol{b} = (m_1, e_1; \ldots; m_l, e_l)$. The fact on the Hilbert series will follow from the proof of the following result. It is based on similar results in [5] and [11].

Proposition 4 *With notations as before, there is a one-to-one correspondence between*

1. *$PGL_n \times \mathbb{C}^*$-orbits in $Irr_n\ A$*
2. *Periodic multiplicity n fat point A-modules*

Proof :
(2) $\Longrightarrow$ (1) : Take a representant F of the fat point with n-dimensional simple quotient determined by the matrix m-tuple $x = (x_1, \ldots, x_m) \in Irr_n\ A$. The orbits corresponding to F is $PGL_n \times \mathbb{C}^*.x$.

(1) $\Longrightarrow$ (2) : Let $x = (x_1, \ldots, x_m) \in Irr_n\ A$ be a representant of the orbit. The kernel of the corresponding morphism $A \twoheadrightarrow M_n(\mathbb{C})$ is a maximal ideal and consider the maximal graded ideal contained in it. It is easy to verify that this is the kernel of the graded morphism

$$\phi_x : A \twoheadrightarrow A_x$$

where A_x is the graded subalgebra of $M_n(\mathbb{C}[t])$ (endowed with the natural gradation) generated as $\mathbb{C}$-algebra by the elements $t^{d_1}.x_1, \ldots, t^{d_s}.x_m$. Precisely as in the foregoing section one can show that the center of A_x is non-trivial and that the graded ring of quotients of A_x is a graded central simple algebra and hence of the form

$$A \twoheadrightarrow A_x \hookrightarrow Q^g\ A_x = M_n(\mathbb{C}[t^e, t^{-e}])(m_1, e_1; \ldots; m_l, e_l)$$

for certain numbers e, m_i and e_i such that $e_i < e$ and $\sum m_i = n$. The period e can be recovered from the action as

$$\mathbf{1} \times \boldsymbol{\mu}_e = Stab_{PGL_n \times \mathbb{C}^*}(x)$$

Denote the graded field $\mathbb{C}[t^e, t^{-e}]$ by R, then we can view the right hand side as the graded endomorphism ring of the graded R-module

$$V = R(e_1)^{\oplus m_1} \oplus \ldots \oplus R(e_l)^{\oplus m_l}$$

where $R(k)$ denotes the shifted graded module, that is, $R(k)_i = R_{k+i}$. Observe that the graded algebra morphism $A \longrightarrow END\ V$ makes V into a graded A-module. The fat A module corresponding to x is represented by the graded A-module $F = V_{\geq 0}$ which has Hilbert series

$$\mathcal{H}(V,t) = \frac{m_1 t^{e_1} + \ldots + m_l t^{e_l}}{(1-t^e)}$$

It is easy t verify that these two mappings are inverse to each other. □

Our main theorem imposes restrictions on the types of periodic multiplicity n fat points which can arise in *Proj* A.

Theorem 6 *Let A be a connected graded algebra generated by elements of degree $\boldsymbol{g} = (k_1, d_1; \ldots ; k_s, d_s)$. Then, A can have a periodic multiplicity n fat point module F of type $\boldsymbol{b} = (m_1, e_1; \ldots ; m_l, e_l)$ only if*

- $n = \sum m_i$
- $\boldsymbol{m} = (m_1, \ldots, m_l)$ *is the dimension vector of a simple representation of* $Q(\boldsymbol{b}, \boldsymbol{g})$

Clearly, the defining equation of A may impose further restrictions on the types of periodic fat points that can occur. For example, it was shown in [5] that if F is a periodic fat point of A which is generated in degree one and is the quotient of an Auslander regular algebra, then the only types that can occur for A are (m, e).
However, in the generic case when A is $\mathbb{C}\langle u_1, \ldots, u_m \rangle$ or $\mathbb{G}_n^m$ the above restrictions are the only ones and one can describe the scheme irr_n A rather explicitly. It would be interesting to generalize the results of [2] (where the case was treated when all the variables are given degree one). We leave this as a suggestion for further research.
We will end this paper with one application to the *Proj* of generic matrices when we give the generic matrices V_k degree one. As we indicated above, we can cover the scheme irr_n $\mathbb{G}_n^m$ by affine varietes which are determined by graded localizations Q_c^g $\mathbb{G}_n^m$ where c is an homogenous central polynomial for $n \times n$ matrices. As $\mathbb{G}_n^m$ is generated in degree one, we know that Q_c^g $\mathbb{G}_n^m$ is a strongly graded ring. Therefore, one wonders whether it can be reduced to the form $\Lambda[x, x^{-1}, \phi]$ if we localize further and whether irr_n $\mathbb{G}_n^m$ can be covered by such special strongly graded algebras.
In [2] it was shown that this is always the case if $n = 2$ and cannot be so for $n > 2$ and m large enough ($\geq 2n-2$). In fact, the reason for stating the conjecture [2, Remark p. 1697] was the believe that one could take $m = 2$ in this result. Even if the conjecture fails to be true, we will show that the consequence is still valid.

Proposition 5 *For $n > 2$ one cannot cover $irr_n\ \mathbb{G}_n^m$ with special strongly graded algebras of the form*

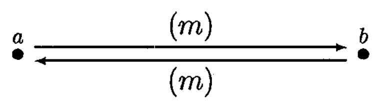
$$\Lambda[x, x^{-1}, \phi]$$

where $deg(x) = 1$ and ϕ is an automorphism of the degree zero part Λ.

Proof : Consider the graded matrix algebra

$$M_n(\mathbb{C}[T, T^{-1}])(a, 0; b, 1)$$

with T of degree two and $a = b = k$ if $n = 2k + 1$ and $a = k$, $b = k - 1$ if $n = 2k$. The corresponding quiver situation is



and one verifies that (a, b) is the dimension vector of a simple representation for all $m \geq 2$. Hence, by our main result, the graded matrix algebra can be generated by m elements of degree one $w_1, \ldots, w_m$. The map

$$\psi : \mathbb{G}_n^m \longrightarrow M_n(\mathbb{C}[T, T^{-1}])(a, 0; b, 1)$$

defined by $\psi(V_k) = w_k$ is graded and we have that $\psi(\mathbb{G}_n^m)\mathbb{C}[T, T^{-1}] = M_n(\mathbb{C}[T, T^{-1}])(a, 0; b, 1)$. That is, ψ is a central extension. Therefore, $P = Ker\ \psi$ is a graded prime ideal of $\mathbb{G}_n^m$ of p.i.-degree n. Therefore, the graded localization at P or $p = P \cap Z(\mathbb{G}_n^m)$ is a graded Azumaya algebra $Q_P^g\ \mathbb{G}_n^m$ and we have

$$Q_P^g\ \mathbb{G}_n^m / p\ Q_P^g\ \mathbb{G}_n^m \simeq M_n(\mathbb{C}[T, T^{-1}])(a, 0; b, 1)$$

One verifies that the degree one part of $M_n(\mathbb{C}[T, T^{-1}])(a, 0; b, 1)$ contains no regular elements (the rank of every element in $\leq n - 1$ by the choice of a and b), therefore $Q_P^g\ \mathbb{G}_n^m$ cannot be of the form $\Lambda[x, x^{-1}, \phi]$. □

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