Automorphisms and Lie Stacks

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To an endomorphism ϕ of $\mathbb{C}[x_1, \ldots, x_n]$ we associate $co(\phi)$ the dimension of the smallest sub-coalgebra $C(\phi)$ of $\mathbb{C}[\mathbb{G}_a^n]$ containing the $\phi(x_i)$. If the Jacobian of ϕ is invertible, $C(\phi)$ will be a generating set. If $co(\phi)$ is minimal (that is $n+1$), then ϕ is a tame automorphism. We use Lie stacks to construct such tame automorphisms and link their study to the classification problem of local commutative $\mathbb{C}$-algebras of dimension $n+1$.

Automorphisms and Lie Stacks

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Abstract

To an endomorphism ϕ of $\mathbb{C}[x_1, \ldots, x_n]$ we associate $co(\phi)$ the dimension of the smallest sub-coalgebra $C(\phi)$ of $\mathbb{C}[\mathbb{G}_a^n]$ containing the $\phi(x_i)$. If the Jacobian of ϕ is invertible, $C(\phi)$ will be a generating set. If $co(\phi)$ is minimal (that is $n+1$), then ϕ is a tame automorphism. We use Lie stacks to construct such tame automorphisms and link their study to the classification problem of local commutative $\mathbb{C}$-algebras of dimension $n+1$.

1 Introduction

The study of automorphisms of the commutative polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is a fascinating topic with many open problems. For example, if $n \geq 2$ the Jacobian conjecture [2] is still open and so is for $n \geq 3$ the Nagata conjecture on the existence of wild automorphisms, see for example [1]. For a good introduction to these problems we refer the reader to [2], and [8].

For some problems one would like to have a numerical invariant associated to an endo- or automorphism ϕ to allow for inductive arguments. Usually one uses the degree of ϕ which is the maximum of the degrees of the polynomials $\phi_i = \phi(x_i)$ or the total degree which is the sum of $deg(\phi_i)$.

In this note we introduce a finer invariant $co(\phi)$ which depends on the natural underlying coalgebra structure of $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbb{G}_a^n]$, where $\mathbb{G}_a$ is the

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additive group scheme, see for example [9]. Recall that the Hopf structure is induced by the following comultiplication Δ and counit ϵ

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i \quad \text{and} \quad \epsilon(x_i) = 0$$

That is, we take all generators x_i to be primitive elements.
The fundamental theorem of coalgebras (see for example [7]) asserts that any coalgebra is the directed union of finite dimensional sub-coalgebras. In particular, every finite set of elements is contained in a finite dimensional sub-coalgebra.

Definition 1 *The* **co-invariant** *$co(\phi)$ of an algebra endomorphism ϕ of $\mathbb{C}[x_1, \ldots, x_n]$ is defined to be the dimension of the minimal sub-coalgebra $C(\phi)$ of the Hopf algebra $\mathbb{C}[\mathbb{G}_a^n]$ which contains the elements $\phi_i = \phi(x_i)$ for $1 \leq i \leq n$.*

Observe that as the intersection of sub-coalgebras is again a sub-coalgebra, $C(\phi)$ is uniquely determined.
In section two we will show that if ϕ is an endomorphism with invertible Jacobian, then the coalgebra $C(\phi)$ is an algebra generating set of $\mathbb{C}[x_1, \ldots, x_n]$. It is clear from the definitions that $n + 1 \leq c(\phi) \leq \binom{n+d}{n}$ where $d = deg(\phi)$. In section three we show that ϕ is a tame automorphism when $co(\phi)$ is minimal, that is, is equal to $n+1$. On the other hand, the potential wild automorphism proposed by Nagata has $co(\eta) = 22$ which is rather large compared to the theoretical upper bound. Perhaps our invariant can be used to construct large classes of potential wild automorphisms by maximizing the value of $co(\phi)$ with respect to the degree of ϕ.
In the final section we give a method to construct automorphism ϕ with $co(\phi) = n + 1$ and link this to the study of local commutative algebras of dimension $n+1$. In particular we construct an embedding of $N(n)$ the variety of these local algebras in the variety of automorphisms with $co(\phi) = n + 1$. There is a canonical GL_n-action on both varieties and the embedding is GL_n-equivariant. The method is based on the theory of Lie stacks and their enveloping algebras as introduced by the author in [4] and [5]. Whereas this theory originated from the desire to construct non-commutative and non-cocommutative Hopf algebra domains, we apply it here in the easier cocommutative case where we can suffice with enveloping algebras.
In fact, it should be stressed that we could have applied some of the arguments of this paper to the case when $\mathbb{C}[x_1, \ldots, x_n]$ is replaced by the enveloping algebra $U(\mathfrak{g})$ of an n-dimensional Lie algebra. In this case we obtain information on algebra generating sets rather than on automorphisms.

2 A Jacobi-type result

Throughout this section, ϕ will be an algebra endo-morphism of $\mathbb{C}[x_1, \ldots, x_n]$. The Jacobian conjecture asserts that ϕ is an automorphism if and only if

$$J(\phi) = (\frac{\partial \phi_i}{\partial x_j})_{i,j} \in M_n(\mathbb{C}[x_1, \ldots, x_n])$$

is invertible.
Clearly, if ϕ is an automorphism, the sub-coalgebra $C(\phi)$ introduced before must be a generating st for the algebra $\mathbb{C}[\mathbb{G}_a^n]$. In this section we will prove

Theorem 1 *Let ϕ be an endo-morphism of $\mathbb{C}[x_1, \ldots, x_n]$ with $J(\phi)$ invertible. Then, $C(\phi)$ is an algebra generating set of $\mathbb{C}[x_1, \ldots, x_n]$.*

First, we will give a criterium for a sub-coalgebra C of $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbb{G}_a^n]$ to be a generating set. As $\mathbb{G}_a^n$ is a connected unipotent group, $\mathbb{C}[\mathbb{G}_a^n]$ is a pointed irreducible Hopf algebra, that is, $\mathbb{C}.1$ is the unique minimal sub-coalgebra which is contained in any sub-coalgebra. Hence, any sub-coalgebra C is also pointed irreducible.
As such, C comes equipped with a natural exhaustive filtration, the coradical filtration

$$C_0 = \mathbb{C}1 \subset C_1 \subset \ldots \subset C_m = C$$

where for each $c \in C_{i+1}$ we have

$$\Delta'(c) = \Delta(c) - c \otimes 1 - 1 \otimes c \in C_i \otimes C_i$$

In particular, $C_1 = \mathbb{C}.1 + P(C)$ where $P(C)$ is the set of primitive elements of C.
For $p \in \mathbb{C}[x_1, \ldots, x_n]$ we will denote with $lin(p)$ the degree ≤ 1 part of f in the natural filtration on $\mathbb{C}[x_1, \ldots, x_n]$ by giving $deg(x_i) = 1$. Observe that this filtration coincides with the coradical filtration on $\mathbb{C}[\mathbb{G}_a^n]$.

Proposition 1 *Let $C = \mathbb{C}c_1 + \ldots \mathbb{C}c_l$ be a sub-coalgebra of $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[\mathbb{G}_a^n]$. Then, C is generating if and only if $\sum \mathbb{C}\ lin(c_i) = \mathbb{C}1 + \mathbb{C}x_1 + \ldots \mathbb{C}x_n$.*

Proof : We may assume the basis-elements c_i to be ordered with respect to the coradical filtration. As

$$\mathbb{C}c_1 + \ldots + \mathbb{C}c_{i_1} = C_1 = P(C) + \mathbb{C}.1 \hookrightarrow \mathbb{C}[x_1, \ldots, x_n]_{\leq 1}$$

it is clear that these c_j with $j \leq i_1$ have only linear terms and that $P(C_1) = \mathbb{C}c_2 + \ldots + \mathbb{C}c_{i_1}$.

Because every C_k is a pointed irreducible sub-coalgebra, it generates an algebra which is a pointed irreducible cocommutative sub-bialgebra of $\mathbb{C}[\mathbb{G}_a^n]$. By the structure result of cocommutative Hopf algebras these are (in this case commutative) enveloping algebras on the primitive elements. In particular, the algebra generated by C_1 is $\mathbb{C}[S_1]$ where $S_1 = \sum_{j \leq i_1} \mathbb{C}\ lin(c_j)$. Assume by induction we have proved that

$$\mathbb{C}\langle C_i \rangle = \mathbb{C}[S_i]$$

where S_i is the span of all $lin(c_k)$ where $c_k \in C_i$. Now, take $c_l \in C_{i+1}$, then by definition of the coradical filtration we have that

$$\Delta'(c_l) \in C_i \otimes C_i \hookrightarrow \mathbb{C}[S_i] \otimes \mathbb{C}[S_i]$$

As we know that $\Delta'(c_l) \in \mathbb{C}[\mathbb{G}_a^n] \otimes \mathbb{C}[\mathbb{G}_a^n]$ we can use the PBW-basis to show that there must be an element $m \in \mathbb{C}[S_i]$ such that

$$\Delta'(c_l) = \Delta'(m)$$

As Δ' is zero on the linear terms we may assume that $m \in \mathbb{C}[S_i]_{\geq 2}$. But then,

$$lin(c_l) = c_l - m \in P(\mathbb{G}_a^n) = \mathbb{C}x_1 + \ldots \mathbb{C}x_n$$

and hence $lin(c_k)$ lies in the algebra generated by $\mathbb{C}[S_i]$ and the $c_l \in C_{i+1}$. Conversely, the c_l lie in the algebra spanned by S_{i+1}, finishing the proof of the result. □

In fact, we have proved the following result

Corollary 1 *Let C be a sub-coalgebra of $\mathbb{C}[\mathbb{G}_a^n]$. The sub-Hopf algebra generated by C coincides with the sub-Hopf algebra generated by $lin(C) = \{lin(c) \mid c \in C\}$.*

The proof of the theorem follows easily from this using the next result. Recall that Aff_n is the group of affine automorphisms of $\mathbb{C}^n$ that is such that each $\phi(x_i)$ has degree ≤ 1.

Proposition 2 *Let ϕ be an endo-morphism of $\mathbb{C}[x_1, \ldots, x_n]$ and $\alpha \in Aff_n$, then $C(\phi) \simeq C(\alpha \circ \phi) \simeq C(\phi \circ \alpha)$.*

Proof : As 1 in contained in any sub-coalgebra it suffices to prove the result for α a linear automorphism in GL_n. Now, use the fact that GL_n is the group of Hopf-algebra automorphisms of $\mathbb{C}[\mathbb{G}_a^n]$. □

Proof of theorem 1 : We can replace ϕ by $\tau \circ \phi$ where τ is the translation $x_i - \phi(0)_i$ to arrange that $\phi(0) = 0$. As the linear term $\phi_{(1)}$ of ϕ is invertible (it corresponds to the matrix $J(\phi)(0)$) we can further replace ϕ by $\phi_{(1)}^{-1} \circ \phi$ to arrange that $\phi(x_i) = x_i +$ higher degree terms. The result now follows from the above results.

Example 1 : Observe that we used only the fact that $J(\phi)(0)$ is invertible to prove the result. Hence, it is easy to construct non-invertible endo-morphisms ϕ such that $C(\phi)$ generates $\mathbb{C}[x_1, \ldots, x_n]$. For example, take $\phi(x_i) = x_i + x_i^2$.

3 A tameness result

A triangular or Jonquiére automorphism ϕ of $\mathbb{C}[x_1, \ldots, x_n]$ is one of the form

$$\phi(x_i) = x_i + r_i \text{ with } r_i \in \mathbb{C}[x_1, \ldots, x_{i-1}]$$

for all $i \geq 2$ and $\phi(x_1) = x_1$. Tame automorphisms are all automorphisms generated by the triangular and the affine automorphisms. Nagata has conjectured that for $n \geq 3$ there must exist wild (that is, non-tame) automorphisms and for $n = 3$ he even gave an explicit candidate for a wild automorphism.
In this section we show that the co-invariant $co(\phi)$ gives a numerical measure for the potential wildness of the automorphism. Let $deg(\phi) = k$ and denote with $m(\phi)$ the number of i such that $\phi(x_i)$ has degree k. We claim that the co-invariant must be bounded between the following two numbers

$$n + 1 \leq co(\phi) \leq \binom{n+k-1}{n} + m(\phi)$$

For, every $\phi(x_i)$ is contained in the sub-coalgebra $\mathbb{C}[x_1, \ldots, x_n]_{\leq k-1} + \sum_{deg\ \phi(x_k)=k} \mathbb{C}\ \phi(x_k)$. We will show the following tameness result

Theorem 2 *Let ϕ be an automorphism of $\mathbb{C}[x_1, \ldots, x_n]$ with $co(\phi) = n+1$, then ϕ is a tame automorphism.*

Proof : As $co(\phi) = n + 1$ we know that

$$C(\phi) = \mathbb{C}1 + \mathbb{C}\phi(x_1) + \ldots + \mathbb{C}\phi(x_n)$$

and as ϕ is an automorphism we know from the foregoing section that

$$\sum \mathbb{C}\ lin(\phi(x_i)) = \sum \mathbb{C}\ x_i$$

Consider the coradical filtration on C. As any $\alpha \in GL_n$ preserves the coradical filtration we can choose a linear automorphism α such that if we denote $\psi = \alpha \circ \phi$ we have that $\psi(x_1), \ldots, \psi(x_n)$ is ordered with respect to the coradical filtration on $C(\psi)$ and that

$$lin(\psi(x_i)) = x_i$$

for all $1 \leq i \leq n$.
As $\phi(x_1)$ is a primitive element of C (and hence lies in $P(\mathbb{C}[x_1, \ldots, x_n]) = \sum \mathbb{C}x_i$) we have that

$$\phi(x_1) = lin(\phi(x_1)) = x_1$$

Assume by induction that we have already shown that

$$\phi(x_i) = x_i + r_i \text{ with } r_i \in \mathbb{C}[x_1, \ldots, x_{i-1}]$$

for all $i < j$ and that $\phi(x_j) \in C(\psi)_k$ (the k-th part of the coradical filtration). Then, we know that the subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $C(\psi)_{k-1}$ is contained in $\mathbb{C}[x_1, \ldots, x_{j-1}]$.
By definition of the coradical filtration we know that

$$\Delta' \; \psi(x_j) \in C_{k-1} \otimes C_{k-1}$$

and hence we have as in the foregoing section the existence of a polynomial without linear term $r_j \in \mathbb{C}[x_1, \ldots, x_{j-1}]$ (in fact, in the subalgebra generated by $C(\psi)_{k-1}$) such that $\Delta' \; \psi(x_j) = \Delta' \; r_j$. But then, $\psi(x_j) - r_j$ is a primitive element of $\mathbb{C}[x_1, \ldots, x_n]$ and hence must be equal to its linear term whence

$$\psi(x_j) = x_j + r_j$$

Therefore, ψ is triangular, and hence ϕ must be tame. □

Example 2 : Clearly, not every tame automorphism has $co(\phi) = n + 1$. For example, take $\phi(x_i) = x_i$ for $1 \leq i \leq n - 1$ and

$$\phi(x_n) = x_n + l^k \text{ where } l \in \mathbb{C}[x_1, \ldots, x_{n-1}]_1$$

with $k \geq 2$, then one verifies that $C(\phi)$ is spanned by 1, the x_i with $i < n$, $l^2, \ldots, l^{k-1}$ and $\phi(x_n)$. Hence, $co(\phi) = n + k - 1$ and ϕ is clearly triangular. Still, for these triangular automorphisms the actual number $co(\phi)$ is small compared to the theoretical upper bound $\binom{n+k-1}{n} + 1$.

On the other hand, let us see what happens in the case of the Nagata automorphism

Example 3 : The automorphism η of $\mathbb{C}[x,y,z]$ that Nagata proposed as a candidate for a wild automorphism is given by

$$\eta(x) = x - 2(xz+y^2)y - (xz+y^2)^2 z$$
$$\eta(y) = y + (xz+y^2)z$$
$$\eta(z) = z$$

It is of degree 5 with $m(\eta) = 1$, hence the co-invariant must lie between the following two theoretical bounds

$$4 \leq co(\eta) \leq 36$$

One verifies that the sub-coalgebra $C(\eta)$ of $\mathbb{C}[x,y,z]$ must contain all linear and quadratic monomials. In degree three it contains the following 8 monomials

$$xy^2, xyz, x^2z, xz^2, y^3, y^2z, yz^2, z^3$$

In degree four $C(\phi)$ is spanned by the elements

$$y^2z^2 + 2xz^3, y^3z + xyz^2 \text{ and } 4xy^2z + 3x^2z^2$$

and finally, in degree five we have to add $\eta(x)$. Therefore,

$$co(\eta) = 22$$

which is fairly large with respect to the theoretical upper bound.

It would be interesting to compute for small degrees the maximal number $co(\phi)$ which do arise. In particular, is 22 the maximal number which can be obtained for a degree 5 automorphism of $\mathbb{C}[x,y,z]$. Such an investigation may lead to a large class of potential wild automorphisms.

4 Constructing special automorphisms

In this section we will use Lie stacks, as introduced and studied in [4] and [5], to construct lots of automorphisms ϕ of $\mathbb{C}[x_1,\ldots,x_n]$ with $co(\phi) = n+1$. We will also give a connection with the classification problem of local commutative $\mathbb{C}$-algebras of dimension $n+1$.

Lemma 1 *If ϕ is an automorphism of $\mathbb{C}[x_1, \ldots, x_n]$ with $co(\phi) = k+1$, then $deg(\phi) \leq k$.*

Proof : Consider the coradical filtration on $C(\phi)$

$$\mathbb{C}1 = C_0 \subset C_1 \subset \ldots \subset C_l = C(\phi)$$

Since $dim\ C_i/C_{i-1} \geq 1$ we have that $l \leq k$. Finally, observe that the coradical filtration on $C(\phi) \hookrightarrow \mathbb{C}[\mathbb{G}_a^n]$ is induced by that on $\mathbb{C}[\mathbb{G}_a^n]$ which is the canonical filtration given by $deg\ x_i = 1$. $\square$

Fix an integer $d \geq 1$ and define

$$E_{(d)} = \{\phi \in End\ \mathbb{C}[x_1, \ldots, x_n] \mid deg\ \phi \leq d\}$$

Clearly, $E_{(d)}$ is n times the space of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of degree $\leq d$. Hence, $E_{(d)}$ is an affine space of dimension $n \binom{n+d}{n}$. In this space we will be interested in the following subset

$$G_{(d)} = \{\phi \in E_{(d)} \mid \phi \text{ is an automorphism } \}$$

By [2, Cor. 1.6] we know that $G_{(d)}$ is a closed subvariety of $E_{(d)}$. Moreover, the varieties $E_{(d)}$ and $G_{(d)}$ have a canonical action of GL_n via composition with linear automorphisms.

Lemma 2 *$G_{(d)}(k) = \{\phi \in G_{(d)} \mid co(\phi) \geq k\}$ is an open GL_n-stable subvariety of $G_{(d)}$.*

Proof : From the construction of $C(\phi)$ it follows that the corresponding set $E_{(d)}(k)$ is an open subvariety of $E_{(d)}$. As $G_{(d)}(k) = E_{(d)}(k) \cap G_{(d)}$ the statement follows. For GL_n-invariance it suffices to observe that GL_n is the group of Hopf-algebra automorphisms of $\mathbb{C}[\mathbb{G}_a^n]$. $\square$

With the above notations we are interested in the GL_n-variety $G_{(n)}(n+1)$. First we need to recall some facts on finite dimensional commutative local $\mathbb{C}$-algebras.
Consider the variety $N_{(n)}$ parameterizing the multiplication rules

$$\zeta : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

of commutative, associative $\mathbb{C}$-algebra structures on $\mathbb{C}^n$ such that $a_1 a_2 \ldots a_{n+1} = 0$ for all $a_i \in \mathbb{C}^n$. As ζ is bilinear it can be identified

with a point of $Hom(\mathbb{C}^n \otimes \mathbb{C}^n, \mathbb{C}^n) \simeq \mathbb{C}^{n^3}$. Hence $N_{(n)}$ is a closed subvariety of the n^3-dimensional affine space.
If $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$, a point ζ of $N_{(n)}$ determines an $n+1$-dimensional $\mathbb{C}$-algebra

$$L_\zeta = \mathbb{C}1 \oplus \mathbb{C}e_1 \oplus \ldots \oplus \mathbb{C}e_n$$

with multiplication rules

$$e_i.1 = 1.e_i = e_i \text{ and } e_i.e_j = \zeta(e_i, e_j)$$

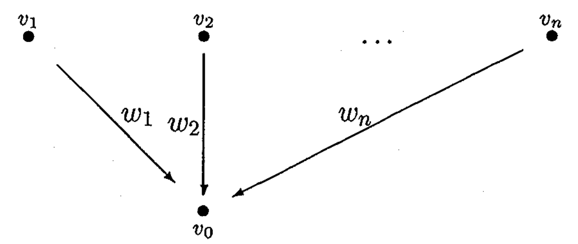
L_ζ is a commutative local algebra with radical $\sum \mathbb{C}e_i$.
By transport of structure, $g \in GL_n$ acts on $N_{(n)}$ from the right in such a way that $g : \zeta^g \simeq \zeta$ becomes a $\mathbb{C}$-algebra automorphism : $\zeta^g(x, y) = g^{-1}.\zeta(g.x, g.y)$.
The variety $N_{(n)}$ can be studied using Hochschild-cocycles as in [6]. In particular, it was shown in [6] that $N_{(n)}$ is an irreducible rational variety of dimension $n^2 - n$ when $n \leq 6$. In fact, every local algebra of dimension $n+1$ is in these cases a degeneration of $\mathbb{C}[x]/(x^{n+1})$. However, for $n \geq 7$ this is no longer the case and $N_{(n)}$ has other irreducible components.
If $C \hookrightarrow \mathbb{C}[\mathbb{G}_a^n]$ is a sub-coalgebra of dimension $n+1$, then C is a pointed irreducible cocommutative coalgebra, whence the dual algebra C^* is a local commutative algebra of dimension $n+1$. For this reason we are interested in the GL_n-variety $N^*_{(n)}$ dual to $N_{(n)}$ whose points are the pointed irreducible cocommutative $\mathbb{C}$-coalgebras of dimension $n+1$.
With notations as above we have

Theorem 3 *There is a GL_n-equivariant embedding*

$$i : N^*_{(n)} \hookrightarrow G_{(n)}(n+1)$$

such that $C(\phi) \simeq C$ if $i(C) = \phi$.

Proof : Consider the path algebra A_n of the n-subspace quiver



then A_n can be identified with the subalgebra of $M_{n+1}(\mathbb{C})$

$$A_n = \begin{bmatrix} \mathbb{C} & 0 & \dots & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \dots & 0 & \mathbb{C} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \dots & 0 & \mathbb{C} \end{bmatrix}$$

via the map $v_i \mapsto E_{i,i}$ and $w_i \mapsto E_{i,l+1}$. As the underlying graph has no cycles all derivations are inner and

$$Der(A_n) = \mathbb{C}x_1 + \dots + \mathbb{C}x_n + \mathbb{C}y_1 + \dots + \mathbb{C}y_n$$

where $x_i = [v_i, -]$ and $y_i = [w_i, -]$. The only non vanishing brackets are

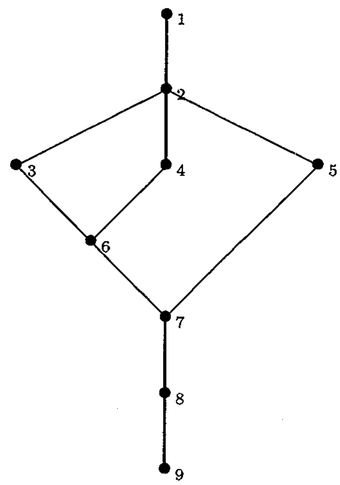
$$[x_i, y_j] = \delta_{ij} y_i$$

Let $L_\zeta = \mathbb{C}1 + \mathbb{C}e_1 + \dots + \mathbb{C}e_n$ be a local commutative algebra of dimension $n+1$ determined by $\zeta \in N_{(n)}$, then we have a Lie stack, that is an algebra morphism

$$\boldsymbol{s}_\zeta : A_n \longrightarrow A_n \otimes L_\zeta \quad a \mapsto a \otimes 1 + \sum_{i=1}^{n} x_i(a) \otimes e_i$$

In [4] it was shown that one can associate to any Lie stack an enveloping algebra which is an irreducible Hopf algebra domain. In the special case under consideration (when L_ζ is commutative) the enveloping algebra $U(\boldsymbol{s}_\zeta)$ is the subalgebra of $U(Der\ A_n)$ generated by the images of L_ζ^* under a canonical universal coalgebra map.
In fact, as the action of L_ζ^* on A_n depends only on the x_i, $U(\boldsymbol{s}_\zeta)$ is really isomorphic to the commutative subalgebra $\mathbb{C}[x_1, \dots, x_n] \hookrightarrow U(Der\ A_n)$.
There is an inductive explicit procedure to determine the images of e_i^* in $\mathbb{C}[x_1, \dots, x_n]$ under the universal coalgebra map χ (which we will recall in the example below). As the elements $\chi(e_i^*)$ give a set of algebra generators of $\mathbb{C}[x_1, \dots, x_n]$ they determine an automorphism $\phi_\zeta(x_i) = \chi(e_i^*)$.
From the construction and the fact that L_ζ^* is a sub coalgebra of $\mathbb{C}[\mathbb{G}_a^n]$ of dimension $n+1$, it follows that $C(\phi_\zeta) = L_\zeta^*$ and $co(\phi_\zeta) = n+1$. □

Example 4 : Let us explain the above construction in the special case when $n = 4$. It is well known (see [6] or [3]) that there are 9 types of commutative local algebras f dimension 5 and they have the following degeneration picture



These algebras have the following presentations and standard bases (that is, such that the $f_i = e_i^*$ are ordered with respect to the coradical filtration on L_ζ^*)

type	representation	e_1	e_2	e_3	e_4
1	$\mathbb{C}[x]/(x^5)$	x	x^2	x^3	x^4
2	$\mathbb{C}[x,y]/(xy, y^2+x^3)$	x	y	x^2	x^3
3	$\mathbb{C}[x,y]/(x^3, y^3, xy)$	x	y	x^2	y^2
4	$\mathbb{C}[x,y]/(x^4, y^2, xy)$	x	y	x^2	x^3
5	$\mathbb{C}[x,y,z]/(xz, xy, yz-x^2, y^2, z^2)$	x	y	z	x^2
6	$\mathbb{C}[x,y]/(x^3, y^2, x^2y)$	x	y	x^2	xy
7	$\mathbb{C}[x,y,z]/(xz, yz, x^2, y^2, z^2)$	x	y	z	xy
8	$\mathbb{C}[x,y,z]/(xy, xz, yz, x^3, y^2, z^2)$	x	y	z	x^2
9	$\mathbb{C}[x,y,z,t]/(x,y,z,t)^2$	x	y	z	t

The multiplication rule of L_ζ determines the coalgebra structure of L_ζ^*. For example, with respect to the dual basis $f_i = e_i^*$ the comultiplication of L_2^* is determined by

$$\begin{aligned}
\Delta(f_1) &= f_1 \otimes 1 + 1 \otimes f_1 \\
\Delta(f_2) &= f_2 \otimes 1 + 1 \otimes f_2 \\
\Delta(f_3) &= f_3 \otimes 1 + 1 \otimes f_3 + f_1 \otimes f_1 \\
\Delta(f_4) &= f_4 \otimes 1 + 1 \otimes f_4 + f_1 \otimes f_3 + f_3 \otimes f_1 - f_2 \otimes f_2
\end{aligned}$$

In order to find the universal embedding $L_\zeta^* \hookrightarrow \mathbb{C}[x_1, x_2, x_3, x_4]$ one proceeds as in the proofs in section two. For a primitive element of L_ζ^* one maps the element to the corresponding derivation in $\mathbb{C}x_1 + \ldots + \mathbb{C}x_4$ of A_4. If $f_i \in (L_\zeta^*)_{j+1}$ for $j \geq 1$ and if we have already found the universal image of the f_k with $k < i$ one finds the image of f_i as follows. Consider the subalgebra R of $\mathbb{C}[x_1, \ldots, x_4]$ generated by the images of the f_k $(k < i)$. There exists an element $r \in R$ such that

$$\Delta'(f_i) = \Delta(r)$$

and hence $f_i - r$ is a primitive element of R which are all contained in $\mathbb{C}x_1 + \ldots + \mathbb{C}x_4$. To find the correct element d_i we have to compute the action of $f_i - r$ on A_4 (we find the action of r by composition of its terms). The universal image of f_i is then $d_i + r$.

In the example L_2^*, f_1 and f_2 are primitive elements and as e_i acts as x_i on A_4, we know that under the universal map

$$f_1 \mapsto x_1 \quad \text{and} \quad f_2 \mapsto x_2$$

For f_3 we have that

$$\Delta'(f_3) = f_1 \otimes f_1 = \Delta'(\frac{1}{2}f_1^2)$$

and therefore $f_3 - \frac{1}{2}f_1^2$ must act as a derivation on A_4. Computing its action on a basis we see that $d_3 = x_3 - \frac{1}{2}x_1$ and hence under the universal map

$$f_3 \mapsto x_3 + \frac{1}{2}x_1^2 - \frac{1}{2}x_1$$

Finally, in a similar way we find that

$$\Delta'(f_4) = f_1 \otimes f_3 + f_3 \otimes f_1 - f_2 \otimes f_2 = \Delta'(f_1 f_3 - \frac{1}{2}f_2^2)$$

and $f_4 - f_1 f_3 + \frac{1}{2}f_2^2$ acts on A_4 as $d_4 = x_4 + \frac{1}{2}x_2$ whence we have that under the universal map

$$f_4 \mapsto x_4 + x_1 x_3 + \frac{1}{2}(x_1^3 - x_1^2) - \frac{1}{2}(x_2^2 - x_2)$$

and the assignment

$$\phi_2(x_i) = f_i$$

determines a tame automorphism of $\mathbb{C}[x_1, x_2, x_3, x_4]$ with $co(\phi_2) = 5$.

In a similar way one computes the images under the universal map in the other cases. For L_1^* we get the automorphism ϕ_1 determined by

$$
\begin{aligned}
f_1 &\mapsto x_1 \\
f_2 &\mapsto x_2 + \frac{1}{2}(x_1^2 - x_1) \\
f_3 &\mapsto x_3 + x_1x_2 + \frac{1}{2}(x_1^3 - x_1^2) \\
f_4 &\mapsto x_4 + x_1x_3 + \frac{1}{2}(x_2^2 - x_2) + \frac{1}{2}(3x_1^2x_2 - x_1x_2) + \frac{1}{8}(5x_1^4 - 6x_1^3 + x_1^2)
\end{aligned}
$$

And the images (or automorphisms) of the remaining cases are given in the table below

type	$\phi(x_1)$	$\phi(x_2)$	$\phi(x_3)$	$\phi(x_4)$
3	x_1	x_2	$x_3 + \frac{1}{2}(x_1^2 - x_1)$	$x_4 + \frac{1}{2}(x_2^2 - x_2)$
4	x_1	x_2	$x_3 + \frac{1}{2}(x_1^2 - x_1)$	$x_4 + x_1x_3 + \frac{1}{2}(x_1^3 - x_1^2)$
5	x_1	x_2	x_3	$x_4 + x_2x_3 + \frac{1}{2}(x_1^2 - x_1)$
6	x_1	x_2	$x_3 + \frac{1}{2}(x_1^2 - x_1)$	$x_4 + x_1x_2$
7	x_1	x_2	x_3	$x_4 + x_1x_2$
8	x_1	x_2	x_3	$x_4 + \frac{1}{2}(x_1^2 - x_1)$
9	x_1	x_2	x_3	x_4

As the embedding $N^*_{(4)} \hookrightarrow G_{(4)}(5)$ is GL_4-equivariant, we see from the degeneration picture of $N_{(4)}$ that all ϕ_i constructed above lie in the closure of the GL_4-orbit of the special automorphism ϕ_1 (which is a degree 4 automorphism of $\mathbb{C}[x_1, \ldots, x_4]$ with $co(\phi_1) = 5$).

Observe that any automorphism ϕ of $\mathbb{C}[x_1, \ldots, x_n]$ of degree n with $co(\phi) = n + 1$ must have corresponding $C(\phi) = L^*$ where $L = \mathbb{C}[x]/(x^{n+1})$. Therefore, if $n \geq 7$ we have automorphisms χ with $co(\chi) = n + 1$ which are not degenerations of degree $n + 1$ automorphisms in $G_{(n)}(n + 1)$.

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