Lie Stack Deformations

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Lie Stack Deformations

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May 30, 1996

Abstract

Enveloping algebras of Lie stacks, as introduced in [7], give irreducible Hopf algebra deformations of $U(\mathfrak{g})$ which are neither commutative nor cocommutative. In this paper we present and study a large class of examples of Lie stacks. In particular, we show that the PBW-bases of these Hopf algebras does not have to be finite in general. Further, we construct a non cocommutative Hopf structure on $U(\mathfrak{g})$ (usually with antipode of infinite order) whenever $\mathfrak{g}$ has a codimension one Lie ideal $\mathfrak{h}$ such that the quotient has the $\mathfrak{h}$-weight of an eigenvector of $\wedge^2 \mathfrak{h}$.

1 Introduction

Over the last decade several attempts have been made to construct and classify 'nice' algebra deformations of the commutative polynomial algebras $\mathbb{C}[x_1,\ldots,x_n]$. Even if we assume excellent homological properties (such as Auslander regularity and the Cohen-Macaulay property as in [1] and [9]) a classification seems to be out of reach at the moment, whenever $n \geq 4$.
Apart from homological properties, polynomial algebras have a lot of additional structure. For example, the corresponding affine variety $\mathbb{A}^n = Max(\mathbb{C}[x_1,\ldots,x_n])$ carries the structure of an Abelian group under componentwise addition. That is, $\mathbb{C}[x_1,\ldots,x_n] = \mathbb{C}[\mathbb{G}_a^n]$ the coordinate ring of the algebraic affine group scheme $\mathbb{G}_a^n$ (n copies of the additive group $\mathbb{G}_a$). As

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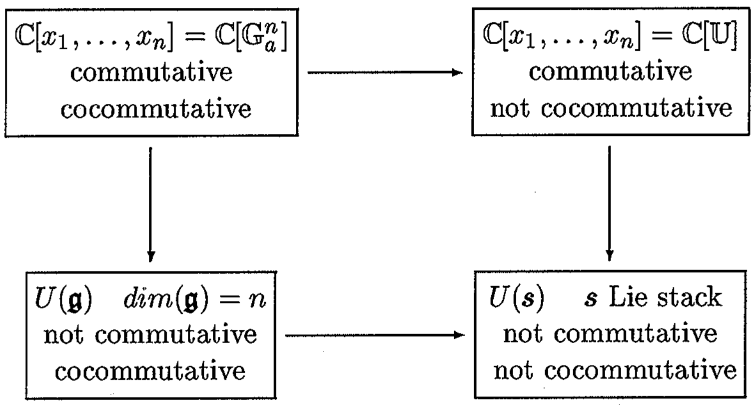
such, $\mathbb{C}[x_1, \ldots, x_n]$ has the structure of an Hopf algebra which is both commutative and cocommutative with comultiplication Δ and counit ϵ induced by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i \text{ and } \epsilon(x_i) = 0$$

that is, the algebra generators x_i are primitive elements. Further, this coalgebra structure is irreducible meaning that every sub-coalgebra of $\mathbb{C}[x_1, \ldots, x_n]$ contains the unique group-like element 1.

Hence, it is a natural question to construct and classify Hopf algebra deformations U of $\mathbb{C}[x_1, \ldots, x_n]$ which remain irreducible as coalgebra, but which may be neither commutative nor cocommutative. It will turn out that such deformations U then automatically have excellent homological properties if we can bound their growth, see [7].

If we restrict attention to deform either the multiplication or the comultiplications, then deformations are classically well known and are summarized in the diagram below



where $\mathbb{U}$ is an n-dimensional unipotent group and $\mathfrak{g}$ an n-dimensional Lie algebra.

If $\mathbb{U}$ is a non-commutative group, the corresponding Hopf algebra $\mathbb{C}[\mathbb{U}]$ is not cocommutative. Because $\mathbb{U}$ has a composition series all of which factors are isomorphic to $\mathbb{G}_a$, one can view $\mathbb{C}[\mathbb{U}]$ as a Hopf algebra deformation of $\mathbb{C}[\mathbb{G}_a^n]$.

Similarly, if $\mathfrak{g}$ is a non-Abelian Lie algebra its enveloping algebra $U(\mathfrak{g})$ is not commutative and carries a Hopf algebra structure by declaring all Lie algebra elements to be primitive elements. Since the associated graded Hopf algebra for the generator filtration is $\mathbb{C}[\mathbb{G}_a^n]$, $U(\mathfrak{g})$ again can be viewed as a Hopf algebra deformation of $\mathbb{C}[\mathbb{G}_a^n]$.

In this paper we focus attention on the remaining corner in the above diagram. That is, we want to construct and study Hopf algebra deformations of $\mathbb{C}[\mathbb{G}_a^n]$ which are neither commutative nor cocommutative but are still irreducible as Hopf algebras. Hence, these deformations are different from the extensively studied quantized enveloping algebras (as in for example [6]) which have more group-like elements.

A first idea would be to mimic the deformation procedure from $\mathbb{C}[\mathbb{G}_a^n]$ to $\mathbb{C}[\mathbb{U}]$ starting this time from the non-commutative enveloping algebra $U(\mathfrak{g})$. Hence, we would like to put a not cocommutative comultiplication on $U(\mathfrak{g})$. If this can be done, we have two Hopf algebras with the same category of representations (as they are isomorphic as algebras) but with different tensor product structures on these representations.

In our investigation we stumbled upon the following easy procedure to construct such Hopf algebra deformations of $U(\mathfrak{g})$.

Theorem : Let $\mathfrak{h}$ be a finite dimensional Lie algebra such that there is a non-zero $\mathfrak{h}$-eigenvector

$$y = \sum y_i \otimes y_i' - y_i' \otimes y_i \in \wedge^2 \mathfrak{h}$$

with character $\lambda \in \mathfrak{h}^*$.

Let $\mathfrak{g} = \mathfrak{h} + \mathbb{C}x$ be the Lie algebra extension determined by λ. That is, the remaining brackets are given by $[x, h] = \lambda(h)x$ for all $h \in \mathfrak{h}$.

For $t \in \mathbb{C}^*$ we define a Hopf algebra $U_t(\mathfrak{g})$ which coincides with $U(\mathfrak{g})$ as algebra, the usual coalgebra structure on the subalgebra $U(\mathfrak{h})$ and with modified maps on x as follows

$$\Delta(x) = x \otimes 1 + 1 \otimes x + ty \quad \epsilon(x) = 0 \text{ and } S(x) = -x + t \sum [y_i, y_i']$$

$U_t(\mathfrak{g})$ is a Hopf algebra which is not cocommutative and with antipode S of infinite order whenever $\sum [y_i, y_i'] \neq 0$ in $\mathfrak{h}$. Moreover, if $t \longrightarrow 0$ the Hopf algebra $U_t(\mathfrak{g})$ degenerates to the usual Hopf structure on $U(\mathfrak{g})$.

These deformations are particular examples of a more general construction introduced in [7]. The basic idea is to deform Lie algebras (and subsequently their enveloping algebras) in the larger categories of Lie stacks. The main purpose of this paper is to present and study a large class of examples of Lie stacks.

In particular we give counter-examples to a conjecture from [7] asking whether the Poincaré-Birkhoff-Witt bases of these algebras are necessarily finite. For example,

Theorem : Let $\mathfrak{h} = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3 + \mathbb{C}y_1$ with non-zero bracket $[x_1, y_1] = y_1$ and consider the subalgebra $U(\mathfrak{s})$ of the affine Lie algebra

$U(\mathfrak{h}[t])$ spanned by $\mathfrak{h}$, $x_1 \otimes t$ and the $y_{n+1} = y_1 \otimes t^n$ for all $n \in \mathbb{N}$. $U(\mathfrak{s})$ is an Hopf algebra with comultiplication induced by

$$\Delta(y_{n+1} = y_{n+1} \otimes 1 + 1 \otimes y_{n+1} + n y_n \otimes x_2 + \ldots + \binom{n}{i} y_{n+1-i} \otimes x_2^i + \ldots + y_1 \otimes y_1 \otimes x_2^n$$

and is the enveloping algebra of a Lie stack.

2 Examples of Lie stacks

The basic idea to construct Hopf algebra deformations of $U(\mathfrak{g})$ is to deform the finite dimensional Lie algebra $\mathfrak{g}$ in the category of Lie stacks as introduced in [7]. A Lie stack is an algebra morphism

$$\mathfrak{s} : A \longrightarrow A \otimes B$$

where A and B are finite dimensional $\mathbb{C}$-algebras and where B is assumed to be augmented local, that is, B has a unique nilpotent maximal ideal $\mathfrak{m}$ with $B/\mathfrak{m} \simeq \mathbb{C}$. Therefore, the dual coalgebra B^* is pointed irreducible, that is, B^* has a unique group like element (which we will denote with 1) contained in every sub-coalgebra.
One can associate to a Lie algebra $\mathfrak{g}$ a Lie stack $\mathfrak{s}_{\mathfrak{g}}$ as follows. Assume

$$\mathfrak{g} = \mathbb{C}g_1 + \ldots + \mathbb{C}g_n \hookrightarrow Der\ A$$

that is, $\mathfrak{g}$ is a Lie subalgebra of the $\mathbb{C}$-derivations $Der\ A$ of a finite dimensional algebra A. Then, we define B_0 to be the commutative 2-nilpotent algebra of dimension $n+1$

$$B_0 = \mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^2$$

and a linear map

$$\mathfrak{s}_{\mathfrak{g}} : A \longrightarrow A \otimes B_0 \quad a \mapsto a \otimes 1 + \sum_{i=1}^{n} g_i(a) \otimes x_i$$

It is easy to verify that $\mathfrak{s}_{\mathfrak{g}}$ is an algebra map using that the g_i are derivations and all $x_i.x_j$ vanish. The Lie stack $\mathfrak{s}_{\mathfrak{g}}$ contains all information necessary to construct the enveloping algebra $U(\mathfrak{g})$ as we will recall in the next section.
If we fix $m = dim\ A$ and $n+1 = dim\ B$ we can define the variety of Lie stacks

$$LieS(m,n) \hookrightarrow Alg_m \times AugmLocal_{n+1} \times M_{m,m(n+1)}(\mathbb{C})$$

which is affine. Here, Alg_m denotes the variety of all associative algebras of dimension m and $AugmLocal_{n+1}$ the Zariski closed subvariety of Alg_{n+1} consisting of the augmented local algebras. For more details on these varieties we refer the reader to [3], [5], [10], [11] and [7].

We now want to construct 'nearby' points $\boldsymbol{s}$ of the point $\boldsymbol{s}_{\mathfrak{g}} \in LieS(m,n)$, that is, we want to deform $\boldsymbol{s}_{\mathfrak{g}}$ in the category of Lie stacks. We can find this deformed Lie stack $\boldsymbol{s}$ by either deforming A in Alg_m or B_0 in $AugmLocal_{n+1}$ or the algebra map.

For a specific $\mathfrak{g}$ this study involves an infinitesimal obstruction given by calculating the normal space to the orbit of $\boldsymbol{s}_{\mathfrak{g}}$ under the base-change action by $GL_m \times GL_{n+1}$, or equivalently, to compute the 'Lie stack cohomology' group of $\boldsymbol{s}_{\mathfrak{g}}$. Next, one then has to verify that a non-trivial element in this group really determines a Lie stack $\boldsymbol{s}$. For specific $\mathfrak{g}$ (for example $\mathfrak{g}$ simple) it is conceivable that the obstruction will vanish and hence that $\boldsymbol{s}_{\mathfrak{g}}$ will be rigid in $LieS(m,n)$.

In this paper we want to bypass these technical problems by putting restrictions on the algebra A (and hence on the Lie subalgebras $\mathfrak{g} \hookrightarrow Der\ A$) such that we can deform $\boldsymbol{s}_{\mathfrak{g}}$ by deforming B_0 to an arbitrary $B \in AugmLocal_{n+1}$.

Restrictions on A **:** Let Q be a connected quiver without oriented cycles, that is, Q is an oriented finite graph with vertices $\{v_1, \ldots, v_k\}$ and arrows $\{w_1, \ldots, w_l\}$ whose underlying graph is connected. Consider the algebra A of dimension $m = k + l$

$$A = \mathbb{C}v_1 + \ldots + \mathbb{C}v_k + \mathbb{C}w_1 + \ldots + \mathbb{C}w_l$$

with multiplicative basis and defining relations

$$v_i.v_j = \delta_{ij} v_i \quad w_i.w_j = 0$$

and if $v_{s(w)}$ (resp. $v_{t(w)}$) is the starting vertex (resp. terminating vertex) of the arrow w we have

$$v_i.w = \delta_{s(w)i} w \quad w.v_i = \delta_{t(w)i} w$$

Rephrased in standard finite dimensional algebra terminology we have

$$A = \mathbb{C}\ Q/(Q_+)^2$$

where $\mathbb{C}\ Q$ is the path algebra of the quiver Q and Q_+ is the twosided ideal of $\mathbb{C}\ Q$ generated by the arrows in Q. As a consequence, A is basic (that is, all simple A-modules are one dimensional) and 2-nilpotent (that

is, $(rad\ A)^2 = 0)$. Conversely, every finite dimensional basic 2-nilpotent algebra has such a description.
Using results of Happel [5] we have a fairly tight control on the derivations of A :

Lemma 1 *With A as above, $Der\ A$ is a metabelian Lie algebra of dimension $2l$.*

Proof : For every finite dimensional algebra A we have the exact sequence

$$0 \longrightarrow Inn(A) \longrightarrow Der(A) \longrightarrow H^1(A) \longrightarrow 0$$

where $Inn(A)$ is the sub Lie algebra of inner derivations and $H^1(A)$ is the first Hochschild cohomology (which determines whether A has deformations in Alg_m). The inner derivations are generated by $x_i = [v_i, -]$ and $y_j = [w_j, -]$ subject to the relation that $x_1 + \ldots + x_k = 0$ (because the v_i are the primitive idempotents of A). The x_i commute with each other as do the y_j and further we have

$$[x_i, y_j] = (\delta_{i,s(w_j)} - \delta_{i,t(w_j)})y_j$$

That is, $Inn(A)$ is a metabelian Lie algebra f dimension $k + l - 1$.
As Q has no oriented cycles, we know from [4, Th. 2.2] that the dimension of $H^1(A)$ is equal to the Euler characteristic of Q which is using the above notation $1 - k + l$. In particular, all derivations of A are inner if and only if the underlying graph of Q has no cycles. If there are cycles, outer derivations arise as follows : let w_j be an arrow in a cycle of the underlying graph, then $\delta : A \longrightarrow A$ defined by

$$\delta(v_i) = 0 \quad \text{and} \quad \delta(w_i) = \delta_{ij} w_j$$

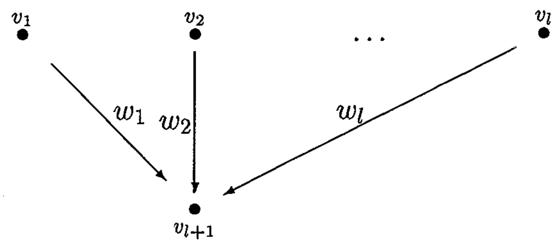
is a derivation and using connectivity of Q one can show that δ determines a non-zero element in $H^1(A)$, [4, Prop 2.3]. Observe that these outer derivations commute with the x_i and y_j.
Combining this information we have that the dimension of $Der\ A$ has dimension $2l$ and has an Abelian Lie ideal of dimension l

$$\mathbb{C}y_1 + \ldots + \mathbb{C}y_l \triangleleft Der\ A$$

with Abelian quotient. □

In particular, for all $\delta \in Der\ A$ and all $a \in A$ we have that $\delta(a)$ is either zero or lies in the linear span of the arrows $\mathbb{C}w_1 + \ldots + \mathbb{C}w_l$.

Example 1 : Consider the path algebra A_l of the l-subspace quiver



then A_l can be identified with the subalgebra of $M_{l+1}(\mathbb{C})$

$$A_l = \begin{bmatrix} \mathbb{C} & 0 & \ldots & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \ldots & 0 & \mathbb{C} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \ldots & 0 & \mathbb{C} \end{bmatrix}$$

via the map $v_i \mapsto E_{i,i}$ and $w_i \mapsto E_{i,l+1}$. As the underlying graph has no cycles all derivations are inner and

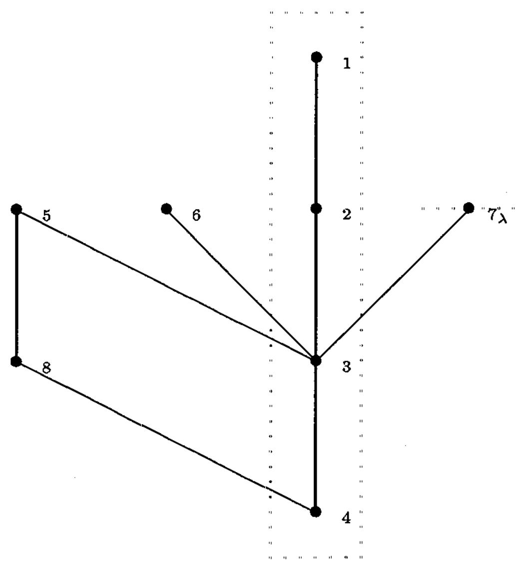
$$Der(A_l) = \mathbb{C}x_1 + \ldots + \mathbb{C}x_l + \mathbb{C}y_1 + \ldots + \mathbb{C}y_l$$

the only non vanishing brackets of which are

$$[x_i, y_j] = \delta_{ij} y_i$$

Data corresponding to B : As a consequence of the heavy restrictions on A we can take for $B = \mathbb{C}1 + \mathbb{C}b_1 + \ldots + \mathbb{C}b_n$ an arbitrary augmented local algebra of dimension $n+1$ with maximal ideal $\mathfrak{m} = \mathbb{C}b_1 + \ldots + \mathbb{C}b_n$. If the basis is ordered with respect to the radical filtration, that is, if $b_j \in \mathfrak{m}^z$ for some z then all $b_j \in \mathfrak{m}^z$ when $j \leq j'$ then we call it a standard basis. With e we denote the 'embedding dimension' of B, that is, $e = dim\ \mathfrak{m}/(\mathfrak{m})^2$.
For $n \leq 4$ these algebras and their degenerations were completely classified in [3], [10] and [5]. For example, the degeneration picture of all 4-dimensional

augmented local algebras can be depicted by



where the dotted region indicates the commutative algebras. Representations of the algebras and standard bases are given in the table below

type	*representation*	b_1	b_2	b_3
1	$\mathbb{C}[x]/(x^4)$	x	x^2	x^3
2	$\mathbb{C}[x,y]/(x^2,y^2)$	x	y	xy
3	$\mathbb{C}[x,y]/(x^3,xy,y^2)$	x	y	x^2
4	$\mathbb{C}[x,y,z]/(x,y,z)^2$	x	y	z
5	$\mathbb{C}\langle x,y\rangle/(y^2,x^2+yx,xy+yx)$	x	y	xy
6	$\mathbb{C}\langle x,y\rangle/(x^2,y^2,yx)$	x	y	xy
7_λ	$\mathbb{C}\langle x,y\rangle/(x^2,y^2,yx-\lambda xy)$	x	y	xy
8	$\mathbb{C}\langle x,y\rangle/(x^2,y^2,xy+yx)$	x	y	xy

For arbitrary n one can describe the augmented local algebras B of dimension $n+1$ inductively using the theory of Hochschild cocycles, see for example [11] or [7]. One obtains a similar degeneration picture : there is a central commutative tree rooted at the commutative 2-nilpotent algebra B_0 but which can have more components if $n \geq 7$ [11]. All non-commutative B eventually degenerate into this commutative tree. In particular, $AugmLocal_{n+1}$ is connected and every B degenerates into B_0 (the algebra occurring in the definition of $\boldsymbol{s}_{\mathfrak{g}}$).

With notations as above we have :

Lemma 2 *For A and B as above and $d \in Grass(n, 2l)$ representing an embedding $d : \mathfrak{m} \hookrightarrow Der\ A$, the linear map*

$$s_d : A \longrightarrow A \otimes B \quad a \mapsto a \otimes 1 + \sum_{i=1}^{n} b_i(a) \otimes b_i$$

is a Lie stack.

Proof : In order to verify that $s_d(aa') = s_d(a)s_d(a')$ one has to check that

$$(b_i(a) \otimes b_i).(b_j(a) \otimes b_j) = 0$$

for all i, j. From our description of $Der\ A$ it follows that $b_k(a) \in \mathbb{C}w_1 + \ldots + \mathbb{C}w_l = rad\ A$ and the claim follows because A is 2-nilpotent. □

As these Lie stacks depend only on a choice of basis of the maximal ideal $\mathfrak{m}$ of B and not on the specific algebra structure of B we have the following

Proposition 1 *Let $s_{d'} : A \longrightarrow A \otimes B'$ be a Lie stack of the above type. If B' deforms to B in $AugmLocal_{n+1}$ there exists a Lie stack $s_d : A \longrightarrow A \otimes B$ which degenerates to $s_{d'}$ in $LieS(m, n)$.*
In particular, if $\mathfrak{g} \hookrightarrow Der\ A$ we can find for any $B \in AugmLocal_{n+1}$ a Lie stack s_d which deforms $s_{\mathfrak{g}}$.

Proof : Let t be the deformation parameter and $\mathcal{B}$ the $\mathbb{C}[t]$-algebra such that $\mathcal{B}/(t-1) \simeq B$ and $\mathcal{B}/(t) \simeq B'$ defined by the deformation. Choose $\beta_1(t), \ldots, \beta_n(t) \in \mathcal{B}$ such that $\beta_1(0), \ldots, \beta_n(0)$ is a basis for the maximal ideal $\mathfrak{m}'$ of B'.
Using the embedding $d' : \mathfrak{m}' \hookrightarrow Der\ A$ we can now define a Lie stack

$$s_d : A \longrightarrow A \otimes B \quad a \mapsto a \otimes 1 + \sum_{i=1}^{n} \beta_i(0)(a) \otimes \beta_i(1)$$

which degenerates into $s_{d'}$. □

3 The Hopf algebras $U(s)$ and Lie coalgebras

In this section we will recall the definition of the enveloping algebra $U(s)$ of a Lie stack and supplement on some of the results of [7]. In particular we will indicate the connection with Lie coalgebras as introduced in [13].

Given a Lie stack $\boldsymbol{s} : A \longrightarrow A \otimes B$ we have the pointed irreducible coalgebra B^* and a linear map

$$m_{\boldsymbol{s}} : B^* \longrightarrow End(A) \quad \text{mapping} \quad \lambda \mapsto s_\lambda$$

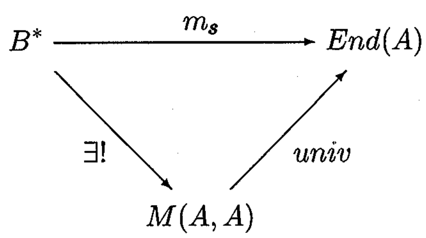
where s_λ is the composition

$$A \xrightarrow{s} A \otimes B \xrightarrow{id \otimes \lambda} A \otimes \mathbb{C} \simeq A$$

Because $\boldsymbol{s}$ is an algebra morphism, the map $m_{\boldsymbol{s}}$ is a measuring of the coalgebra B^* on A meaning that

$$s_\lambda(aa') = \sum_{(\lambda)} s_{\lambda_{(1)}}(a) s_{\lambda_{(2)}}(a') \text{ and } s_\lambda(1) = \epsilon(\lambda)1$$

From general results of M. Sweedler [15, Ch. VII] there exists a universal measuring $univ : M(A, A) \longrightarrow End(A)$ such that there is a uniquely determined coalgebra map $B^* \longrightarrow M(A, A)$ making the diagram



commutative. Crucial for our purposes is the fact that one can define a bialgebra structure on $M(A, A)$ induced by the composition of measurings.

Definition 1 *The enveloping algebra $U(\boldsymbol{s})$ of the Lie stack $\boldsymbol{s}$ is the subalgebra of $M(A, A)$ generated by the image of B^*.*

Clearly, $U(\boldsymbol{s})$ is a bialgebra measuring A via *univ*. Moreover, as $U(\boldsymbol{s})$ is generated by the pointed irreducible coalgebra B^*, $U(\boldsymbol{s})$ is irreducible as coalgebra and hence an Hopf algebra by [15, Thm. 9.2.2].
Because some of the structure results we need about $U(\boldsymbol{s})$ hold in greater generality we state them as such.

If U is an arbitrary irreducible Hopf algebra one can define an exhaustive Hopf filtration on it, namely the coradical filtration. This filtration is defined by

$$U_0 = \mathbb{C}1 \text{ and } U_n = U_0 + \{x \in Ker(\epsilon) \mid \Delta'(x) \in U_{n-1} \otimes U_{n-1}\}$$

where we define for every $x \in Ker(\epsilon)$

$$\Delta'(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$$

In particular, $U_1 = \mathbb{C}1 + P(U)$ where $P(U)$ is the space of primitive elements of U. With $gr\ U$ we will always denote the associated graded Hopf algebra with respect to the coradical filtration.
From [15, Ch XI] one deduces that $gr\ U$ is a commutative Hopf algebra. For a short crisp inductive proof of this fact we refer to [12]. The following result was proved in [7]. As it is crucial for the sequel we recall its short proof.

Theorem 1 *If U is an irreducible Hopf algebra (over $\mathbb{C}$), then U is a domain and has a Poincaré-Birkhoff-Witt basis.*

Proof : If R is a finitely generated graded sub Hopf algebra of the commutative positively graded Hopf algebra $gr\ U$ it is smooth and hence must be isomorphic (as algebra) to the polynomial algebra on $R_+/(R_+)^2$. From this it follows that $gr\ U$ is a domain (whence so is U) and that $gr\ U$ is a polynomial algebra on $gr\ U_+/(gr\ U_+)^2$. □

The above result controls the algebra structure of $gr\ U$. Similarly, we would like to control the costructure. To do this we need to recall the definitions of Lie coalgebras and their co-enveloping bialgebras as introduced and studied in [13] and [14]. These notions are dual to those of Lie algebras and their enveloping algebras.
If V is a vectorspace, denote with τ the twist on $V \otimes V$ and with ζ the cyclic permutation on $V \otimes V \otimes V$. A Lie coalgebra $\mathfrak{f}$ is a vectorspace together with a cobracket $b : \mathfrak{f} \longrightarrow \mathfrak{f} \otimes \mathfrak{f}$ satisfying

$$Im(b) \subset Im(1-\tau) \text{ and } (1+\zeta+\zeta^2) \circ (1 \otimes b) \circ b = 0$$

where the second condition is dual to the Jacobi identity. For example, if C is a coalgebra one can make it into a Lie coalgebra via the cobracket $b = (1-\tau) \circ \Delta$.
Dualizing the universal property of enveloping algebras one gets a definition for the universal coenveloping bialgebra $U^c(\mathfrak{f})$ of a Lie coalgebra $\mathfrak{f}$ (in fact, we take the irreducible component of 1). It has a canonical structure of a commutative irreducible bialgebra. We refer to [13] and [14] for more details.

Theorem 2 *If U is an irreducible Hopf algebra (over $\mathbb{C}$), then the PBW-basis $\mathbf{u} = gr\ U_+/(gr\ U_+)^2$ is a Lie coalgebra such that $gr\ U \simeq U^c(\mathfrak{p})$ as Hopf algebras.*

Proof : Let B be an irreducible commutative bialgebra with augmentation ideal $I = Ker(\epsilon)$ and define $\mathfrak{u} = I/I^2$. The main observation necessary to showing that $\mathfrak{u}$ inherits a Lie coalgebra structure as a quotient of the canonical Lie coalgebra structure on B is that

$$\delta(x) \in Ker(1 \otimes \epsilon) \cup Ker(\epsilon \otimes 1) = I \otimes I$$

and that I^2 is a Lie coideal of I, that is,

$$\delta(x.y) \in I^2 \otimes I + I \otimes I^2$$

and hence via the identification $B = \mathbb{C}1 \oplus I$, $\mathfrak{u}$ inherits the structure of Lie coalgebra under the map

$$B \twoheadrightarrow I \twoheadrightarrow I/I^2 = \mathfrak{u}$$

Moreover, by [14] we know that $B \simeq U^c(\mathfrak{u})$ as Hopf algebras.
The result now follows from the fact that $gr\ U$ is an irreducible commutative bialgebra. □

In particular, $\mathfrak{u}^*$ will be a Lie algebra naturally associated to the irreducible Hopf algebra U which will be graded and when finite it will be nilpotent. Nilpotency follows from the fact (see [7]) that $gr\ U$ is then the coordinate ring of an affine unipotent group scheme.
If we denote $U_+ = Ker(\epsilon_U)$ then also $U_+/(U_+)^2$ has a canonical Lie coalgebra structure. However, it may be too small to be of any interest. For example, for $U = U(\mathfrak{sl}_2)$ it is zero !

After this general excursion let us return to the irreducible Hopf algebras $U(\mathfrak{s}_d)$ of interest to us.

Proposition 2 *With notations as before we have :*

1. *$U(\mathfrak{s}_d)$ is cocommutative iff B is commutative*
2. *$U(\mathfrak{s}_d)$ is commutative then $Im(d)$ is Abelian*

In particular, if we start from $\mathfrak{s}_{\mathfrak{g}}$ with $\mathfrak{g}$ a non-Abelian Lie subalgebra of $Der\ A$ and deform B_0 to a non-commutative B, we obtain an irreducible Hopf algebra $U(\mathfrak{s}_d)$ which is neither commutative nor cocommutative.

Proof : (1) : As $d : \mathfrak{m} \hookrightarrow Der\ A$ we know that the coalgebra map $B^* \longrightarrow M(A, A)$ is injective (because it is injective on the primitive

elements of B^*) so if B is non commutative, B^* and hence $U(\boldsymbol{s}_d)$ is non cocommutative. Conversely, if B is commutative, B^* is cocommutative and hence so is any bialgebra generated by it.
(2) : Degenerating B to B_0 we can degenerate $\boldsymbol{s}$ to a Lie stack $\boldsymbol{s}_0 : A \longrightarrow A \otimes B_0$. If $U(\boldsymbol{s})$ is commutative so must be $U(\boldsymbol{s}_0)$ but this last algebra is the enveloping algebra of the Lie subalgebra generated by $Im(d)$ by the structure results of cocommutative irreducible bialgebras, see also [7]. □

In the remaining part of this paper we try to understand the algebra structure of $U(\boldsymbol{s}_d)$. In particular we want to construct a PBW-basis and determine its size. Recall from [7] that the coradical filtration on $U(\boldsymbol{s}_d)$ is finite dimensional. Hence, if the PBW-basis is infinite the degrees of the generators of $gr\ U(\boldsymbol{s}_d)$ must be unbounded. Below we will outline an effective, though laborious way to determine the PBW-basis inductively.

Using the coalgebra structure of $C = B^*$ there is a unique pointed irreducible Hopf structure on $T(\mathfrak{m}^*)$ and as $U(\boldsymbol{s}_d)$ is generated by C we have an (Hopf algebra) epimorphism $T(\mathfrak{m}^*) \twoheadrightarrow U(\boldsymbol{s}_d)$. Assume we have already found an ideal I of relations which are valid in $U(\boldsymbol{s}_d)$ and we want to verify whether $T(\mathfrak{m}^*)/I \simeq U(\boldsymbol{s}_d)$, then it suffices to verify injectivity on the primitive elements of $T(\mathfrak{m}^*)/I$, see [15, 11.0.1]. Hence, we have to hunt for primitive elements and their relations.
At each step α in the procedure we have two lists of elements of $U(\boldsymbol{s}_d)$ resp. $T(\mathfrak{m}^*)/I_\alpha$. The first, $\mathcal{B}_\alpha = \{p_1, \ldots, p_{k_\alpha}\}$ is part of the PBW-basis and the second, $\mathcal{N}_\alpha = \{n_1, \ldots, n_{l_\alpha}\}$ are potential new basis elements. Both lists are ordered with respect to the coradical filtration on $U(\boldsymbol{s}_d)$ resp. $T(\mathfrak{m}^*)/I_\alpha$.
The first step consists of taking

$$\mathcal{B}_1 = \{p_1, \ldots, p_{k_1}\} \text{ and } \mathcal{N}_1 = \{c_{e+1}, \ldots, c_n\}$$

where e is the embedding dimension of B, where $\mathbb{C}p_1 + \ldots + \mathbb{C}p_{k_1} = \mathfrak{g}$ is the Lie subalgebra of $Der\ A$ generated by the image of d_1 which is the natural map

$$d_1 : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow Der\ A$$

and where I_1 is the ideal generated by all relations holding in $\mathfrak{g}$.
What actions do we perform at step $\alpha + 1$? First we see whether for any of the n_i we have that $\Delta'(n_i) = \Delta(w_i)$ for some w_i an ordered polynomial in the basis $\mathcal{B}_\alpha$. If this is the case, $n_i - w_i$ will be a primitive element of $T(\mathfrak{m}^*)/I_\alpha$ and therefore in $U(\boldsymbol{s}_d)$ we must have

$$n_i = w_i + g_i \text{ for some } g_i \in Der\ A$$

If g_i is spanned by the $\mathcal{B}_\alpha$ we can remove n_i from $\mathcal{N}_\alpha$ and continue.
If g_i does not belong to the span of $\mathcal{B}_1$ we do the following : take as $\mathcal{B}_{\alpha+1}$ a basis of the Lie algebra spanned by the degree one elements from $\mathcal{B}_\alpha$ and g_i, take for $\mathcal{N}_{\alpha+1}$ the remaining elements from $\mathcal{B}_\alpha$ together with the remaining elements from $\mathcal{N}_\alpha$ and for $I_{\alpha+1}$ the ideal generated by I_α and the relations valid in this new Lie algebra. Then go to step $\alpha+2$.
Assume that for none of the n_i we have that $\Delta'(n_i)$ is a polynomial in $\mathcal{B}_\alpha$, we compute

$$c_{ij} = \Delta'([p_i, n_j]) \in T(\mathfrak{m}^*)/I_\alpha^{\otimes 2}$$

for $1 \leq j \leq l_\alpha$ and for $1 \leq i \leq k_\alpha$. For fixed j, assume that $c_{ij} = \Delta'(w_{ij})$ for some polynomial in the basis $\mathcal{B}_\alpha$, then again as before we must have for the image in $U(\boldsymbol{s}_d)$ that

$$[p_i, n_j] = w_{ij} + g_{ij} \text{ for some } g_{ij} \in Der\ A$$

If the g_{ij} does not belong to the span of $\mathcal{B}_\alpha$ we proceed as before. Otherwise, we have a new relation in $U(\boldsymbol{s}_d)$ which we may add to I_α. If we get such a relation for every i we may remove n_i from the list of potential basis-elements.
If however, c_{ij} is not the Δ' for a polynomial in $\mathcal{B}_\alpha$ we have to add a new PBW-basis element p with

$$\Delta(p) = p \otimes 1 + 1 \otimes p + c_{ij}$$

and we have the relation $[p_i, n_j] = p$ in $U(\boldsymbol{s})$.
Because the coradical filtration on $U(\boldsymbol{s}_d)$ is finite we get after a finite number of steps the correct PBW-basis up to a required degree, together with all their commutation relations up to this degree.
However, it is not clear that the above procedure must terminate eventually. In principle it is possible that we have to keep on adding new basis-elements of arbitrarily large degree. In fact, we will give such examples below.

4 PBW-basis for $U(\boldsymbol{s}_d)$

In this section we will apply the foregoing general inductive method to construct explicit PBW-bases for the Hopf algebras $U(\boldsymbol{s}_d)$. As every augmented local algebra B degenerates to a commutative one, let us first consider this case.

4.1 B commutative

If $\boldsymbol{s}_d : A \longrightarrow A \otimes B$ is a Lie stack with B commutative, we know that $U(\boldsymbol{s}_d) \simeq U(\mathfrak{g})$ for a Lie subalgebra $\mathfrak{g}$ of $Der(A)$. In this subsection we will give an inductive procedure to construct explicitly the embedding $C \hookrightarrow U(\mathfrak{g})$ of the dual coalgebra $C = B^*$. Observe that this will give us rather special sets of generators (namely, spanning a sub-coalgebra) of enveloping algebras.
C is pointed irreducible of dimension $n+1$, with unique group like element $1 = 1^*$ and basis $c_i = b_i^*$. If we start with a standard basis of B, this basis is ordered with respect to the coradical filtration C_z on C, that is, if $c_j \in C_z$ then also $c_{j'} \in C_z$ for all $j' \leq j$. In particular,the primitive elements $Prim(C)$ of C is the span $\mathbb{C}c_1 + \ldots + \mathbb{C}c_e$ for e the embedding dimension of B.
If B is 2-nilpotent, every c_i is primitive and hence acts as the derivation $d(b_i)$ on A. The embedding $C \hookrightarrow U(\boldsymbol{s}_d)$ then maps c_i to $d(b_i)$ and $\mathfrak{g}$ is the Lie subalgebra generated by the image of d.
In the general case we may assume inductively that we have already constructed for the quotient algebra $\phi : B \twoheadrightarrow \overline{B} = B/(b_n)$ and the corresponding Lie stack

$$\overline{\boldsymbol{s}_d} : A \xrightarrow{\boldsymbol{s}_d} A \otimes B \xrightarrow{1\otimes\phi} A \otimes \overline{B}$$

the embedding

$$\overline{B}^* = \overline{C} \hookrightarrow U(\overline{\boldsymbol{s}_d}) = U(\mathfrak{h}) \hookrightarrow U(Der\ A)$$

Hence, $\Delta'(c_n) \in \overline{C} \otimes \overline{C} \hookrightarrow U(\mathfrak{h}) \otimes U(\mathfrak{h})$. As $C \hookrightarrow U(Der\ A)$ we can use the classical version of PBW for enveloping algebras to deduce that there must exist an element $w \in U(\mathfrak{h})$ such that $\Delta'(c_n) = \Delta'(w)$. Hence, $c_n - w$ is a primitive element of $U(\boldsymbol{s}_d)$ and since $U(\boldsymbol{s}_d)$ measures A there is a derivation $g \in Der\ A$ such that $c_n = w + g$. One can compute g as we know the action of c_n on A via the Lie stack $\boldsymbol{s}_d$ and the action of w on A via the measuring of $U(\mathfrak{h})$ induced by $\overline{\boldsymbol{s}_d}$.

Proposition 3 *With notations as above, let $\mathfrak{g}$ be the Lie subalgebra of Der A generated by $\mathfrak{h}$ and g, then there are coalgebra embeddings*

$$\begin{array}{ccc} \overline{C} & \hookrightarrow & U(\mathfrak{h}) \\ \downarrow & & \downarrow \\ C = \overline{C} + \mathbb{C}c_n & \hookrightarrow & U(\mathfrak{g}) \end{array}$$

which sends $c_n \mapsto w + g$.

The element $\Delta'(c_n) \in \overline{C} \otimes \overline{C}$ satisfies a condition which implies that $C = \overline{C} + \mathbb{C}c_n$ is a coalgebra. The required condition is obtained by dualizing the treatment of (symmetric) Hochschild cocycles in [11]. For future reference we state the general result here.

Proposition 4 *Let* $\overline{C}$ *be a finite dimensional pointed irreducible coalgebra with unique group like element* 1 *and let* $y = \sum c \otimes c' \in \overline{C} \otimes \overline{C}$. *Define* $C = \overline{C} + \mathbb{C}x$ *and* $\Delta(x) = x \otimes 1 + 1 \otimes x + y$. *Then,*

1. *C is a coalgebra if and only if y is a Hochschild co-cocycle, that is,*

$$\sum c_{(1)} \otimes c_{(2)} \otimes c' = \sum c \otimes c'_{(1)} \otimes c'_{(2)} \in \overline{C}^{\otimes 3}$$

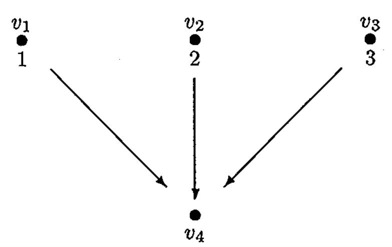
2. *C is cocommutative if and only if* $\overline{C}$ *is cocommutative and y is a symmetric Hochschild co-cocycle, that is,*

$$\sum c \otimes c' = \sum c' \otimes c \in \overline{C} \otimes \overline{C}$$

In this case, the Lie coalgebra $\mathfrak{u} = gr\ U(\mathfrak{s})_+/(gr\ U(\mathfrak{s})_+)^2 = \mathfrak{g}$ has zero cobrackets. Hence, $\mathfrak{u}^*$ is the Abelian Lie algebra.

Observe that the images of the c_i give a generating set for $U(\mathfrak{g})$. Even in the special case when $\mathfrak{m} \hookrightarrow Der\ A$ gives a commutative sub Lie algebra this gives us interesting tame automorphisms of the polynomial algebra. Let us give some examples

Example 1 (continued) : Let us compute these automorphisms for the commutative 4-dimensional local algebras. Their representations and standard basis were given before.
Consider the embedding $d : \mathfrak{m} \hookrightarrow Der\ A_3$ (where A_3 is the path algebra of the 3-subspace quiver) depicted by



By this we mean that $d(b_i) = x_i$ the inner derivation determined by v_i. Then the embeddings $C \hookrightarrow \mathbb{C}[x_1, x_2, x_3]$ are given by the following tame automorphisms

type	c_1	c_2	c_3
1	x_1	$x_2 + \frac{1}{2}(x_1^2 - x_1)$	$x_3 + x_1x_2 + \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2 + \frac{1}{3}x_1$
2	x_1	x_2	$x_3 + x_1x_2$
3	x_1	x_2	$x_3 + \frac{1}{2}(x_1^2 - x_1)$
4	x_1	x_2	x_3

In a forthcoming paper [8] we will study the applications of Lie stacks on automorphism problems of polynomial and enveloping algebras.

4.2 B not commutative

If B is commutative, we have seen that $U(\mathfrak{s}_d) \simeq U(\mathfrak{g})$ as Hopf algebras for some Lie subalgebra $\mathfrak{g} \hookrightarrow$ *Der A*. In particular, the PBW-basis of $U(\mathfrak{s}_d)$ is finite. We will now investigate what happens if B is no longer commutative. First, let us state a few general facts. If $\mathfrak{p} = Prim(U(\mathfrak{s}_d))$, then $\mathfrak{p} \hookrightarrow$ *Der A* and $U(\mathfrak{p}) \hookrightarrow U(\mathfrak{s}_d)$.

Lemma 3 *If U_i denotes the i-th part in the coradical filtration on $U(\mathfrak{s}_d)$, then, U_i is a finite dimensional $\mathfrak{p}$-module and so is gr $U(\mathfrak{s}_d)_i = U_i/U_{i-1}$.*

Proof : The $\mathfrak{p}$-action is given by commutation in $U(\mathfrak{s}_d)$. Because the associated graded algebra is commutative we have for $p \in \mathfrak{p}$ and $u \in U_i$ that

$$[h, u] = hu - uh \in U_i$$

from which the claims follow. □

Lemma 4 *The generators of degree two of gr $U(\mathfrak{s}_d)$ can be chosen such that they have preimages $z_i \in U_2$ such that the $\Delta'(z_i)$ span a $\mathfrak{p}$-submodule of $\wedge^2(\mathfrak{p})$.*

Proof : Let $z \in U_2$ be such that its image in *gr* $U(\mathfrak{s}_d)$ is a generator. Clearly,

$$\Delta'(z) \in \mathfrak{p} \otimes \mathfrak{p} = S^2(\mathfrak{p}) \oplus \wedge^2(\mathfrak{p})$$

say $\Delta'(z) = s + a$. As $s \in S^2(\mathfrak{p})$ there is a quadratic element $w \in U(\mathfrak{p})$ such that $\Delta'(w) = s$. If we replace z by the generator $z' = z - w$ then $\Delta'(z') \in \wedge^2(\mathfrak{p})$.

For every $p \in \mathfrak{p}$ we can compute that $\Delta'([p, z']) \in \wedge^2(\mathfrak{p})$ and the claim follows from the foregoing lemma. $\square$

These facts enable us to give a first example of a Lie stack $\boldsymbol{s}_d$ such that $U(\boldsymbol{s}_d)$ has an infinite PBW-basis.

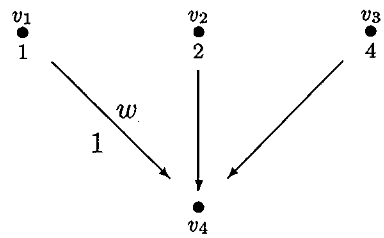
Example 1 (continued) : Consider the augmented local noncommutative five dimensional algebra

$$B = \mathbb{C}\langle x, y, z\rangle / (x^2, yx, y^2, z^2, xz, zx, yz, zy)$$

which has as a standard basis $b_1 = x, b_2 = y, b_3 = z$ and $b_4 = xy$. Its dual algebra C is such that c_1, c_2 and c_3 are primitive and

$$\Delta(c_4) = c_4 \otimes 1 + 1 \otimes c_4 + c_1 \otimes c_2$$

Consider the Lie stack $\boldsymbol{s}_d : A_3 \longrightarrow A_3 \otimes B$ depicted by



That is, we have the embedding $\mathfrak{m} \hookrightarrow Der\ A_3$ sending $b_1 \mapsto x_1, b_2 \mapsto x_2, b_3 \mapsto y_1$ and $b_4 \mapsto x_3$.
In fact we could have taken any 2-nilpotent algebra A and construct the embedding in such a way that v_1 is the starting point of w_1 and that v_2 and v_3 do not lie on w_1. A similar remark can be made for all the examples given below where we took for concreteness sake the special case when $A = A_l$ the path algebra of the l-subspace quiver.
Let $\mathfrak{h} = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}x_3 + \mathbb{C}y_1$ be the four dimensional sub Lie algebra of $Der\ A$ with brackets

$$[x_i, x_j] = 0 \quad [x_1, y_1] = y_1 \quad [x_2, y_1] = 0 \quad [x_3, y_1] = 0$$

then $U(\mathfrak{h}) \hookrightarrow U(\boldsymbol{s}_d)$. If we denote the image of c_4 in $U(\boldsymbol{s}_d)$ by $c + x_3$, we have

$$\Delta(c) = c \otimes 1 + 1 \otimes c + x_1 \otimes x_2$$

As $\Delta'(c) \notin S^2(\mathfrak{h})$ we must add c to the PBW-basis of $U(\mathfrak{s}_d)$. Next, we want to find the commutation relations of c with $h \in \mathfrak{h}$. In order to do this we compute $\Delta[c, h] = [\Delta(h), \Delta(h)]$ using the formula valid in any bialgebra

$$[a_1 \otimes a_2, a_3 \otimes a_4] = [a_1, a_3] \otimes a_2 a_4 + a_3 a_1 \otimes [a_2, a_4]$$

We obtain, using the commutation relations in $\mathfrak{h}$ that

$$[c, x_i] \text{ are primitive elements in } U(\mathfrak{s}_d)$$

so they must act as derivations on A. As c acts as the zero derivation we deduce that

$$[c, x_i] = 0 \text{ in } U(\mathfrak{s}_d)$$

However, if we calculate the commutator of c and y_1 we find

$$\Delta[c, y_1] = [c, y_1] \otimes 1 + 1 \otimes [c, y_1] + y_1 \otimes x_2$$

whose Δ' does not belong to $S^2(\mathfrak{h})$. Therefore, we have to add yet another element y_2 to our PBW-basis with

$$\Delta(y_2) = y_2 \otimes 1 + 1 \otimes y_2 + y_1 \otimes x_2 \text{ and } [c, y_1] = y_2$$

as we know the actions of c and y_1 on A we can compute by composition that of y_2 which is in this case the zero map.
Observe that we could have deduced the existence of another PBW-basis element of degree two also from the fact that they must span a $\mathfrak{h}$-module and indeed $\mathbb{C}\Delta'(c) + \mathbb{C}\Delta'(y_2)$ is an $\mathfrak{h}$-submodule of $\mathfrak{h} \otimes \mathfrak{h}$.
Again, we have to compute commutation relations of y_2 with the other base-elements. For the elements of $\mathfrak{h}$ we obtain as before

$$[x_1, y_2] = y_2 \quad [x_2, y_2] = 0 \quad [x_3, y_2] = 0 \quad [y_1, y_2] = 0$$

but if we compute the commutator of c and y_2 we find

$$\Delta'([c, y_2]) = 2y_2 \otimes x_2 + y_1 \otimes x_2^2$$

and one verifies that this cannot be the Δ' of a polynomial in the ordered PBW-basis $\{x_1, x_2, x_3, y_1, c, y_2\}$. Hence, we have to add a degree element y_3 to our PBW-basis with

$$\Delta(y_3) = y_3 \otimes 1 + 1 \otimes y_3 + 2y_2 \otimes x_2 + y_1 \otimes x_2^2 \text{ and } [c, y_2] = y_3$$

Using the foregoing computations a a basis for induction we obtain

Lemma 5 *There exist elements y_n in $U(\mathfrak{s}_d)$ such that*

$$\Delta(y_{n+1}) = y_{n+1}\otimes 1+1\otimes y_{n+1}+ny_n\otimes x_2+\ldots+\binom{n}{i} y_{n+1-i}\otimes x_2^i+\ldots+y_1\otimes x_2^n$$

and which act on A as the zero map.
In $U(\mathfrak{s}_d)$ these elements satisfy the following commutation relations

$$[x_1, y_n] = y_n \quad [x_2, y_n] = 0 \quad [x_3, y_n] = 0 \quad [y_i, y_n] = 0 \quad [c, y_n] = y_{n+1}$$

Proof : Assume we proved the commutation relations already for y_n, then using centrality of x_2^i which simplifies the required computations we find that

$$\Delta' [x_1, y_{n+1}] = \Delta' y_{n+1} \quad \Delta' [x_i, y_{n+1}] = 0 \quad \Delta' [y_i, y_{n+1}] = 0$$

from which the required commutation relations follow using that y_{n+1} acts as the zero map on A. Moreover,

$$\Delta' [c, y_{n+1}] = [c\otimes 1, \Delta' y_{n+1}] + [1\otimes c, \Delta' y_{n+1}] + [x_1\otimes x_2, \Delta y_{n+1}]$$

and by the above mentioned general formula one computes that

$$[c\otimes 1, \Delta' y_{n+1}] = ny_{n+1}\otimes x_2 + \binom{n}{2} y_n\otimes x_2^2 + \ldots + y_2\otimes x_2^n$$

$$[1\otimes c, \Delta' y_{n+1}] = 0$$

$$[x_1\otimes x_2, \Delta y_{n+1}] = y_{n+1}\otimes x_2 + ny_n\otimes x_2^2 + \ldots + y_1\otimes x_2^{n+1}$$

from which the definition of y_{n+2}, the commutation relation $[c, y_{n+1}] = y_{n+2}$ and the fact that y_{n+2} acts as zero on A follows. □

We have the following elements in $U(\mathfrak{s}_d)$ in the coradical filtration

deg	0	1	2	3	...	n	...
elem	1	x_1	c				
		x_2					
		x_3					
		y_1	y_2	y_3	...	y_n	...

Proposition 5 *The set $\{x_1, x_2, x_3, c, y_1, y_2, \ldots\}$ is an infinite PBW-basis for $U(\mathfrak{s}_d)$. With respect to it the defining relations of $U(\mathfrak{s}_d)$ are*

$$[x_1, x_2] = 0 \quad [x_1, c] = 0 \quad [x_2, c] = 0$$

$$[x_1, y_n] = y_n \quad [x_2, y_n] = 0 \quad [c, y_n] = y_{n+1}$$

$$[y_i, y_j] = 0$$

Proof : From the above computations it follows that the $y_i \in U(\boldsymbol{s}_d)$ and satisfy the required commutation relations. As the image of C is contained in its span, they generate $U(\boldsymbol{s}_d)$ as algebra. Assume the basis would not be a PBW-basis, then after factoring out the Hopf-ideal (x_2) (which makes all the elements primitive in the quotient) there should be a linear relation among the base-elements in the quotient and hence a relation in $U(\boldsymbol{s}_d)$

$$y_{n+1} = \alpha x_1 + \beta x_3 + \gamma c + \sum_{i \leq n} \beta_i y_i + x_2.f$$

with n minimal and f a polynomial in the ordered basis. From our knowledge of $\Delta(y_i)$ it follows that $f \neq 0$. But then, there should be a term $x_2 \otimes f$ in $\Delta(y_{n+1})$, a contradiction. □

From the commutation relations we deduce that there is an embedding

$$U(\boldsymbol{s}_d) \hookrightarrow U(\mathfrak{h}[t])$$

where $\mathfrak{h}[t]$ is the infinite dimensional Lie algebra spanned by $h \otimes t^n$ for all $h \in \mathfrak{h}$ and $n \in \mathbb{N}$. Brackets in $\mathfrak{h}[t]$ are given by $[h \otimes t^k, h' \otimes t^l] = [h, h'] \otimes t^{k+l}$. The embedding is given via

$$c \mapsto x_1 \otimes t \text{ and } y_{n+1} \mapsto y_1 \otimes t^n$$

The cobrackets of the Lie coalgebra $\mathfrak{u}$ spanned by the PBW-basis are given by

$$b(x_i) = 0 \quad b(c) = x_1 \otimes x_2 - x_2 \otimes x_1$$
$$b(y_{n+1}) = n(y_n \otimes x_2 - x_2 \otimes y_n)$$

Hence, $\mathfrak{u}^*$ is the Lie algebra with non-zero brackets

$$[x_1^*, x_2^*] = c^* \text{ and } [y_n^*, x_2^*] = n y_{n+1}$$

These Lie brackets resemble those of the Virasoro algebra (see for example [2].

If we consider the subalgebra of $U(\boldsymbol{s}_d)$ generated by the elements $x_1, x_2, x_3, y_1, \ldots, y_n$ we see that they span a finite dimensional Lie algebra and we obtain a non cocommutative Hopf algebra structure on the enveloping algebra they generate.

Hence, even in the cases when $U(\boldsymbol{s})$ can be shown to have an infinite PBW-basis some sub-Hopf algebras may be regular and they correspond to enveloping algebras of modified Lie stacks. In fact we have

Proposition 6 *Let H be an affine Hopf-subalgebra of $U(\boldsymbol{s})$ for some Lie stack $\boldsymbol{s} : A \longrightarrow A \otimes B$. Then, there exists a Lie stack $\boldsymbol{s}' : A \longrightarrow A \otimes B'$ such that $H \simeq U(\boldsymbol{s}')$ as Hopf algebras.*

Proof : As algebra H is generated by $h_1, \ldots, h_z$. Let C be a finite dimensional sub coalgebra of H containing the h_i, then as a sub coalgebra of $U(\boldsymbol{s})$ there is a measuring of C on A and C is pointed irreducible. As we have seen before, a measuring on A is equivalent to an algebra morphism

$$\boldsymbol{s}' : A \longrightarrow A \otimes C^*$$

hence we can take $B' = C^*$. From the construction of enveloping algebras of Lie stacks, the conclusion then follows. □

Remark however that if we started with a Lie stack $\boldsymbol{s}_d$ with $d : \mathfrak{m} \hookrightarrow Der\ A$ it will not always be the case that the new Lie stack $\boldsymbol{s}'$ is of a similar type. As an example, consider the sub Hopf algebra mentioned above. Then we can take as a subcoalgebra

$$C = \mathbb{C}x_1 + \mathbb{C}x_3 + \mathbb{C}x_2 + \mathbb{C}x_2^2 + \ldots + \mathbb{C}x_2^{n-1} + \mathbb{C}y_1 + \ldots + \mathbb{C}y_n$$

and the corresponding algebra is

$$B' = \mathbb{C}\langle x_1, x_2, x_3, y_1 \rangle / (x_1^2, x_3^2, x_2^n, y_2, x_2 y_1)$$

with $dim\ \mathfrak{m}' = 2n+1$ which becomes eventually larger that $dim\ Der(A)$.

The computations performed in the above example can be used to construct other enveloping algebras of Lie stacks with infinite PBW-basis.

Proposition 7 *Let B be a noncommutative augmented local algebra of dimension 4, then there exists a Lie stack $\boldsymbol{s}_d : A_m \longrightarrow A_m \otimes B$ such that $U(\boldsymbol{s}_d)$ has an infinite PBW-basis.*

Proof : Representations and standard bases for the non-commutative 4-dimensional augmented local algebras are given by

type	*representation*	b_1	b_2	b_3
5	$\mathbb{C}\langle x, y \rangle / (y^2, x^2 + yx, xy + yx)$	x	y	xy
6	$\mathbb{C}\langle x, y \rangle / (x^2, y^2, yx)$	x	y	xy
7_λ	$\mathbb{C}\langle x, y \rangle / (x^2, y^2, yx - \lambda xy)$	x	y	xy
8	$\mathbb{C}\langle x, y \rangle / (x^2, y^2, xy + yx)$	x	y	xy

where the infinite family 7_λ is called the family of Scorza algebras [10]. For the dual coalgebras $C = B^*$ we have that c_1 and c_2 are primitive and the comultiplication on c_3 is easily verified to be determined by

type	$\Delta' \, c_3$
5	$c_1 \otimes c_1 + c_1 \otimes c_2 - c_2 \otimes c_1$
6	$c_1 \otimes c_2$
7_λ	$c_1 \otimes c_2 + \lambda c_2 \otimes c_1$
8	$c_1 \otimes c_2 - c_2 \otimes c_1$

Consider the map $d : \mathfrak{m} \longrightarrow Der\, A_m$ which sends $b_1 \mapsto x_1$ and $b_2 \mapsto y_1 + x_2$, then for each of the above algebras we have that the image of d_1 generates a three dimensional Lie algebra $\mathfrak{h} = \mathbb{C}x_1 + \mathbb{C}x_2 + \mathbb{C}y_1$. Moreover, using the explicit form of $\Delta(c_3)$ we can in each case assign an element of $\mathfrak{h}$ to c_3 and modify the generator by a quadratic element of $U(\mathfrak{h})$ as before to obtain c_3' which acts as zero on A_m and such that

$$\Delta(c_3') = c_3' \otimes 1 + 1 \otimes c_3' + x_1 \otimes (y_1 + x_2)$$

But then there must be a PBW-base element d_3 with $\Delta' \, d_3 = x_1 \otimes x_2$ and which acts as zero on A (for, compute $\Delta' \, [x_1, c_3']$).
If we replace the role of c by d_3 in the foregoing example we can make the same calculations and arguments and obtain that $U(s_d)$ has an infinite PBW-basis. □

Theorem 1 *If B is an augmented local algebra having a noncommutative local quotient of dimension 4, then there exist Lie stacks $s_b : A \longrightarrow A \otimes B$ such that $U(s_d)$ has an infinite PBW-basis.*

Proof : Clearly, if $\pi : B \twoheadrightarrow B'$ is an epimorphism and $s' : A_m \longrightarrow A_m \otimes B'$ is a Lie stack, then we can extend s' to a Lie stack $s : A_m \longrightarrow A_m \otimes B$ such that $s' = (1 \otimes \pi) \circ s$. For, we can assign arbitrary derivations to basiselements in $Ker(\pi)$. Further, we can choose m large and such that $\mathfrak{m} \hookrightarrow Der\, A$. But then we have $U(s') \hookrightarrow U(s)$ and the foregoing result finishes the claim. □

4.3 Break-off conditions

Hence, most Lie stacks with B non-commutative will have an infinite PBW-basis. However, we have seen before that certain Hopf subalgebras may have a finite PBW-basis and are again enveloping algebras of Lie stacks. It is therefore a very interesting problem to characterize the Lie stacks whose

enveloping algebras have generators of bounded degree. This will be a difficult problem in general.
To finish this paper let us give a characterization in the easiest case, that is, we assume that all the generators of $U(s)$ are in degree one except for one generator in degree two.

Proposition 8 *Let $s_d : A \longrightarrow A \otimes B$ be a Lie stack such that gr $U(s_d)$ is generated in degree one and one generator in degree two. Then, $U(s_d) \simeq U(\mathfrak{g})$ as algebras (but not as coalgebras) where $\mathfrak{g}$ is a finite dimensional Lie algebra.*

Proof : By lemma 4 we can take the extra generator x to be such that $y = \Delta'(x)$ is a $\mathfrak{p}$-eigenvector of $\wedge^2(\mathfrak{p})$ with character $\lambda \in \mathfrak{p}^*$. For every $p \in \mathfrak{p}$ we obtain

$$\Delta'([h,x]) = \lambda(h)\Delta'(x) \text{ and hence } [h,x] - \lambda(h)x \in Prim(U(s_d)) = \mathfrak{p}$$

Therefore, there is a derivation $g_h \in \mathfrak{p} \hookrightarrow Der\ A$ such that

$$[h,x] = \lambda(h)x + g_h$$

As all these brackets are obtained via commutators in $U(s_d)$ it is clear that $\mathfrak{g} = \mathfrak{p} + \mathbb{C}x$ is a Lie algebra and that $U(s_d) = U(\mathfrak{g})$ as algebras. □

This result shows in particular that the examples of [7, §5] are enveloping algebras in disguise.
The above proof gives the following procedure to construct not cocommutative Hopf algebra deformations of certain enveloping algebras.

Theorem 2 *Let $\mathfrak{h}$ be a finite dimensional Lie algebra such that*

$$0 \neq y = \sum (h_i \otimes h_i' - h_i' \otimes h_i) \in \wedge^2(\mathfrak{h})$$

is an $\mathfrak{h}$-eigenvector of weight $\lambda \in \mathfrak{h}^*$. Consider the extended Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathbb{C}x$ with extra brackets $[h,x] = \lambda(h)x$.
For every $t \neq 0$ there is a non cocommutative Hopf structure $U_t(\mathfrak{g})$ on the enveloping algebra $U(\mathfrak{g})$ determined by

$$\Delta(h) = h \otimes 1 + 1 \otimes h \text{ and } \Delta(x) = x \otimes 1 + 1 \otimes x + t \sum (h_i \otimes h_i' - h_i' \otimes h_i)$$

$$S(h) = -h \text{ and } S(x) = -x + t \sum [h_i, h_i']$$

If $\sum [h_i, h_i'] \neq 0$ in $\mathfrak{h}$ then the antipode S has infinite order. Clearly $U_t(\mathfrak{g}) \mapsto U(\mathfrak{g})$ as Hopf algebras if $t \longrightarrow 0$.

Proof : The only remaining point is the order of the antipode. If $0 \neq h = \sum [h_i, h'_i]$ then an inductive computation shows that

$$S^n(x) = (-1)^n x - (-1)^n nh$$

from which the claim follows. □

In this case, the Lie coalgebra $\mathfrak{u} = \mathfrak{g}$ has non-zero cobracket $b(x) = ty$. Hence, $\mathfrak{u}^*$ is the nilpotent Lie algebra with non-zero brackets

$$[h_i^*, h_i'^*] = tx$$

When $\mathfrak{h}$ is simple there are no $\mathfrak{h}$-eigenvectors in $\wedge^2(\mathfrak{h})$, however, if $\mathfrak{h}$ is solvable there always are. Let us make the easiest example explicit.

Example 2 : Let $\mathfrak{b} = \mathbb{C}a + \mathbb{C}b$ with $[a, b] = b$ the two- dimensional non-Abelian Lie algebra, then $\wedge^2(\mathfrak{b}) = \mathbb{C}(a \otimes b - b \otimes a) = \mathbb{C}y$ is one-dimensional hence must be an eigenspace. Clearly, $[a, y] = y$ and $[b, y] = 0$.
Then, $\mathfrak{g} = \mathbb{C}a + \mathbb{C}b + \mathbb{C}x$ is the Lie algebra with brackets $[a, b] = b$, $[a, x] = x$ and $[b, x] = 0$ and the Hopf structure $U_t(\mathfrak{g})$ on $U(\mathfrak{g})$ is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x + t(a \otimes b - b \otimes a) \text{ and } S(x) = -x + tb$$

so the antipode has infinite order.

The Lie algebra $\mathfrak{h}$ does not have to be solvable in order to satisfy the requirements of the theorem. For example, consider the twisted Lie algebra extension $\mathfrak{h}$ of $\mathfrak{sl}_2$ with respect to its four dimensional simple representation V_3.
One can iterate the procedure of the theorem to obtain non cocommutative structures on certain $U(\mathfrak{g})$ such that the degrees of the generators of *gr* $U(\mathfrak{g})$ with respect to the coradical filtration become arbitrarily large.

Proposition 9 *Assume we have a chain of Lie ideals*

$$\mathfrak{h} = \mathfrak{h}_0 \triangleleft \mathfrak{h}_1 \triangleleft \ldots \triangleleft \mathfrak{h}_k = \mathfrak{g}$$

where for every $1 \leq i < k$ *we have that* $\mathfrak{h}_{i+1}/\mathfrak{h}_i = \mathbb{C}x_i$ *where the* $\mathfrak{h}_i$*-weight* λ_i *of* x_i *is the weight of an* $\mathfrak{h}_i$*-eigenvector* y_i *in* $\wedge^2(\mathfrak{h}_i)$*. Then, we can define a non cocommutative Hopf algebra structure* $U_{\underline{t}}(\mathfrak{g})$ *on the enveloping algebra* $U(\mathfrak{g})$ *for* $\underline{0} \neq \underline{t} \in \mathbb{C}^k$ *and such that* $U_{\underline{t}}(\mathfrak{g}) \mapsto U(\mathfrak{g})$ *as Hopf algebras if* $\underline{t} \longrightarrow \underline{0}$.

Another application of the characterization is that the enveloping algebras of certain Lie stacks are enveloping algebras in disguise when the dimension of B is small.

Proposition 10 *Let* $\mathbf{s}_d : A \longrightarrow A \otimes B$ *be a Lie stack such that* $d_1 : \mathfrak{m}/\mathfrak{m}^2 \hookrightarrow Der\ A$ *is a Lie subalgebra. If* $dim(B) \leq 4$*, there is an algebra isomorphism* $U(\mathbf{s}_d) \simeq U(\mathfrak{g})$.

Proof : If B is commutative such a result holds in general. If B is non commutative, then $dim(B) = 4$ and has a standard basis with $b_1, b_2 \in \mathfrak{m}$ and $b_3 \in \mathfrak{m}^2$. By assumption $d(b_1)$ and $d(b_2)$ span a Lie algebra $\mathfrak{h}$ of $Der\ A$ which must be Abelian or the two-dimensional non-Abelian Lie algebra $\mathfrak{b}$. In either case, we have that

$$\Delta'(c_3) = s + a$$

where $s \in S^2(\mathfrak{h})$ and a an $\mathfrak{h}$-eigenvector in $\wedge^2(\mathfrak{h})$ (which is one-dimensional). But then by the arguments used before $U(\mathbf{s}_d) \simeq U(\mathfrak{g})$ as algebras for some Lie algebra $\mathfrak{g}$. □

Remark that we cannot bound the dimension of the Lie algebra $\mathfrak{g}$ as this depends on the dimension of the Lie subalgebra of $Der\ A$ generated by $\mathfrak{h}$ and $d(b_3)$.
We like to close with a suggestion for further research :

Question 1 *Assume* $\mathbf{s} : A \longrightarrow A \otimes B$ *is a Lie stack such that* $U(\mathbf{s})$ *has a finite PBW-basis. Does there exist a Lie algebra* $\mathfrak{g}$ *such that* $U(\mathbf{s}) \simeq U(\mathfrak{g})$ *as algebras ?*

In other words, is there a Lie subalgebra $\mathfrak{g}$ of $U(\mathbf{s})$ the images of which in $gr\ U(\mathbf{s}) = U^c(\mathfrak{u})$ span the Lie coalgebra $\mathfrak{u}$. The examples given in this paper perhaps motivate a further study of these 'Lie-bialgebras' and their enveloping Hopf algebras.

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