Nilpotent Representations

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Abstract

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Abstract

The Hesselink stratification is studied of the nullcone of m-tuples of $n \times n$ matrices. An algorithm is given to determine for given m and n the non-empty strata. Further, connections with moduli spaces of quiver representations are given.

1 Introduction

In this paper we will consider the problem of classifying n-dimensional nilpotent representations upto isomorphism. In matrix-terms we want to describe m-tuples of $n \times n$ matrices

$$X = (x_1, \dots, x_m) \in M_n(\mathbb{C})^{\oplus m}$$

which generate a nilpotent subalgebra of $M_n(\mathbb{C})$ under the action of GL_n by simultaneous conjugation, that is $g.X = (gx_1g^{-1}, \dots, gx_mg^{-1})$. This problem is known to be 'hopeless' as it implies the classification of nilpotent representations of arbitrary affine algebras.

The subvariety $N_{m,n} \subset X_{m,n} = M_n(\mathbb{C})^{\oplus n}$ of nilpotent m-tuples is the nullcone for the action by simultaneous conjugation. That is, it is determined to be the subset of $X = (x_1, \dots, x_m)$ satisfying

$$tr(x_{i_1} \dots x_{i_k}) = 0$$

for all $1 \leq i_j \leq m$ and $k \leq n^2$. Therefore, one can try to stratify the highly singular variety $N_{m,n}$ by smooth irreducible locally closed subvarieties using the refinements of the Hilbert-Mumford criterium [8] due to G. Kempf [3] and W. Hesselink [1]. We will recall their general results in the case of interest to us and show that they allow a reduction in complexity of the problem provided we can describe certain locally closed subsets U_s accurately.

The description of these subvarieties U_s and in particular the determination of the non-empty ones for given values of n and m is the main objective of this paper. Applying general results of F.C. Kirwan [5] and L. Ness [9] we reduce the determination of U_c to that of describing the semi-stable points for a specific quiver-representation problem and fixed character. An algorithmic method to solve this problem follows from recent work of A. King [4] based on the solution of some questions of V. Kač due to A. Schofield [10].

Let us outline our results and illustrate them in the only example in the literature where a complete description of the orbits in $N_{m,n}$ is known. This is the case of

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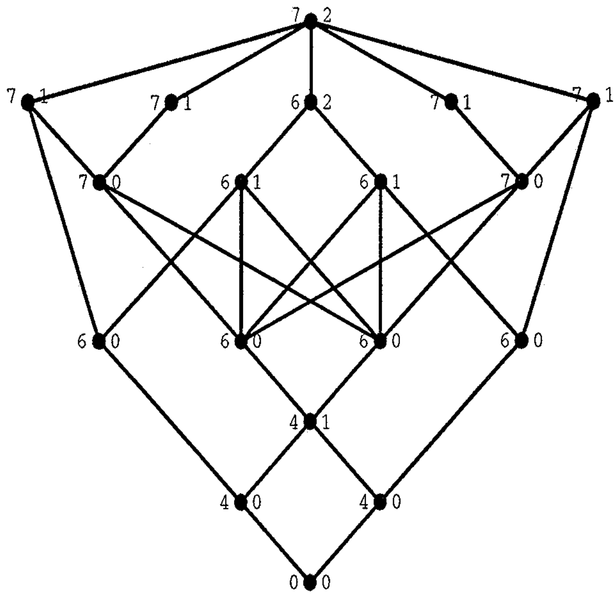
couples of 3×3 matrices studied by H. Kraft [7, p.202] (It is easy to verify that the orbits in $N_{m,2}$ are parameterized by a $\mathbb{P}^{m-1}$ together with the zero-orbit). Kraft obtained the following toric description of the orbit space



Here, each node represents a torus subvariety with the right hand number as its dimension. The left hand number is the dimension of the orbit determined by a point in the torus.
In this paper we describe the Hesselink stratification of the nullcones $N_{m,n}$ which collects together the points having the same set of optimal one-parameter subgroups. If we fix a maximal torus one can list the one-parameter subgroups which may appear as optimals. Each corresponds to a co-weight $s = (s_1, \ldots, s_n) \in \mathbb{Q}^n$ which is the disjoint union of strings

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

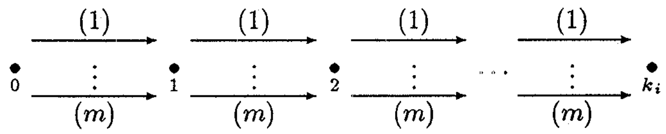
with $k_i \in \mathbb{N}$, the intermediate numbers $p_i + j$ appears in s with multiplicity $a_{ij} \geq 1$ and $\sum_{j \in \text{string}} s_j = 0$ for every string in s, see [1, Prop.6.8.a]. For given n one can compile a list $\mathcal{S}$ of dominant co-weights with these properties. For example,

$\mathcal{S}$ for 3×3 matrices

type	s_1	s_2	s_3	$\|s\|$
1	1	0	-1	2
2	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$
3	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
4	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$
5	0	0	0	0

For given m it is difficult to determine which $s \in \mathcal{S}$ correspond to an optimal one-parameter subgroup for a point in $N_{m,n}$. General theory tells us that for $m = 1$ the occurring coweights are the strongly balanced ones [1, Prop.6.8] (or equivalently, to the partitions of n) and that for m sufficiently large all s eventually appear [1, Prop. 5.2.b].
To solve this problem we associate to each $s \in \mathcal{S}$ a quiver- situation. For each string

in s we construct a quiver on k_i vertices



a dimension vector $\alpha_i = (a_{i0}, \ldots, a_{ik_i})$ and a character for the corresponding base-change group $GL(\alpha_i) = \prod GL_{a_{ij}}(\mathbb{C})$ determined by $\theta_i = (n_{i0}, \ldots, n_{ik_i})$ where $n_{ij} = d.(p_i + j)$ with d the l.c.m. of the numerators of the p_k's. The quiver-data is then the disjoint union over all strings in s and is called resp. Q_s, α_s and θ_s. The character θ_s can also be viewed as an additive function on the Grothendieck group of the modules over the path algebra of Q_s. Then, following [4] one can define θ_s-semi stable representations V of Q_s to be those such that $\theta_s(V) = 0$ and for all sub-representations $W \subset V$ we have $\theta_s(W) \geq 0$.
Using these notations the determination of the non-empty strata in the Hesselink stratification of $N_{n,m}$ is given in

Theorem 1 *The stratum S_s corresponding to $s \in \mathcal{S}$ is non-empty iff there are θ_s-semi stable representations of dimension-vector α_s for the quiver Q_s. Moreover, an algorithmic description of this property exists using only the Ringel form of the quiver Q_s.*

The algorithm depends heavily on the work of A. Schofield [10]. For 3×3 matrices we obtain the following quiver-data and dimensions of the occurring strata

Strata for 3×3 matrices

type	α_s			θ_s			$m = 1$	$m = 2$
1	1	1	1	-1	0	1	6	9
2	1	2		-2	1		-	6
3	2	1		-1	2		-	6
4	1	1		-1	1		4	5
	1			0				
5	3			0			0	0

For example, type 3 does not occur for $m = 1$ as every representation of

$$\bullet \longrightarrow \bullet$$

of dimension-vector $(2, 1)$ has a sub-representation of dimension-vector $(1, 0)$ (the kernel) for which $\theta_s = -1 < 0$. For $m = 2$, representations in general position of

$$\bullet \rightrightarrows \bullet$$

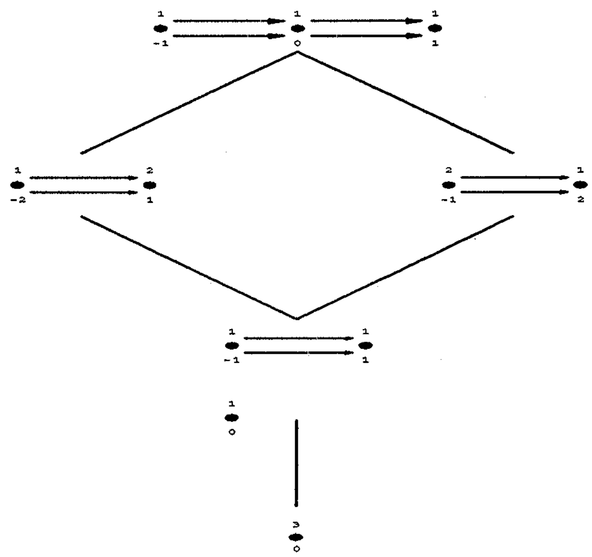
of dimension-vector $(2, 1)$ no longer have such sub-representations and are indeed θ_s-semi stable. The same holds for all $m \geq 2$.

Having determined which strata S_s do actually occur for given m and n we can also describe them explicitly and initiate the study of the orbits in a given stratum. The stratum S_s consists of the GL_n-translates of U_s where U_s is an open subset of the sub vectorspace $C_s^m \subset M_n(\mathbb{C})^m$ consisting of the matrix tuples with zeroes everywhere except perhaps at the entries (i, j) when $s_i - s_j \geq 1$. Moreover, $U_s = \pi^{-1}(V_s)$ where $\pi : C_s^m \longrightarrow B_s^m$ is the natural projection onto the sub vectorspace of all matrix tuples with zeroes except perhaps at entries (i, j) such that $s_i - s_j = 1$ and V_s the open subset of B_s^m of θ_s-semi stable tuples (in fact, the action of the Levi-group associated to the one-parameter subgroup on B_s^m coincides with the base-change action on $B_s^m = R(Q_s, \alpha_s)$). This allows us to describe U_s and hence the stratum S_s explicitly.

By general Hesselink theory, the GL_n-orbits in the stratum S_s are in natural one-to-one correspondence with the P_s-orbits in U_s where P_s is the parabolic associated to the one-parameter subgroup. The map π introduced above induces a projection

$$Orb(P_s, U_s) \xrightarrow{\pi} M(Q_s, \alpha_s; \theta_s)$$

from the P_s-orbit space of U_s to the moduli space of θ_s semi stable representations of the quiver Q_s with dimension vector α_s. The description of these moduli spaces can be obtained from the representation theory of quivers (see also [4]) and can be seen as the first approximation to the orbit classification. We leave a detailed discussion of the fibres of π as a suggestion for further research.
In the case of couples of 3×3 matrices we have the quiver-data associated to the strata and their degenerations as indicated below



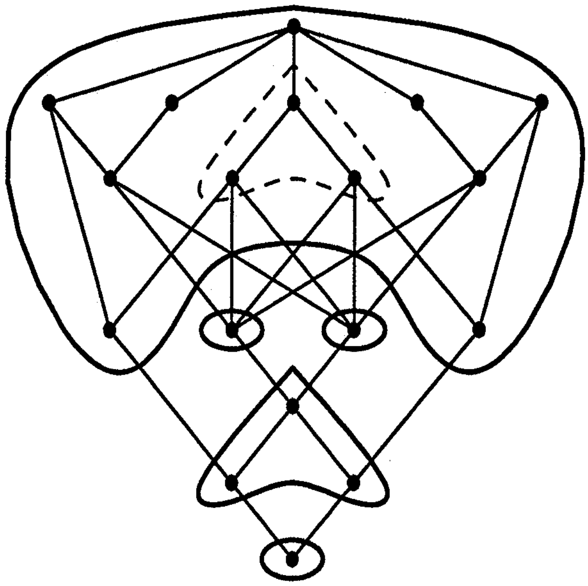
where the upper indices give the components of α_s and the lower ones those of θ_s. The moduli spaces $M(Q_s, \alpha_s; \theta_s)$ for these quiver situations are : for type 5 there is just the zero-orbit so we get $\mathbb{P}^0$, for type 4 we have to classify the indecomposable Kronecker modules of dimension $(1,1)$ and we get a $\mathbb{P}^1$ as moduli space. For types 2 resp. 3 we have to classify the indecomposable Kronecker modules of dimension $(1,2)$ resp. $(2,1)$. These are real Schur roots, so there is just one such orbit and we obtain a $\mathbb{P}^0$ in both cases. For the generic stratum we have to classify the indecomposables of dimension $(1,1,1)$ which have $\mathbb{P}^1 \times \mathbb{P}^1$ as its moduli space.
As $B_s = C_s$ for s of type $2, 3, 4$ and 5 we have classified the orbits in the corresponding Hesselink strata. For the generic stratum we have the projection

$$Orb(P_s, B_s) \twoheadrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

which one verifies to be a birational map but with $\mathbb{P}^1$ fibers along the open torus part of the diagonal with representants the matrix couples

$$(\begin{bmatrix} 0 & 1 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & c \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix})$$

with $a \in \mathbb{C}^*$ and $(b : c) \in \mathbb{P}^1$.
The connection between our description and the one obtained by Kraft is given in the picture below



Here, each circled region gives a toric description of the orbit space of the corresponding stratum, the dashed region gives the extra orbits (that is, those not coming from the quiver setting).

2 Hilbert's criterium and $N_{m,n}$

The Hilbert criterium, see a.o. [6, III.2], asserts that $X = (x_1, \ldots, x_m) \in N_{m,n}$ if and only if there is a one-parameter subgroup $\lambda : \mathbb{C}^* \hookrightarrow GL_n$ such that

$$\lim_{t \longrightarrow 0} \lambda(t).(x_1, \ldots, x_m) = (0, \ldots, 0)$$

Upto conjugation in GL_n (which amounts to possibly replacing X by another point in its orbit) we may assume that λ has the form

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & 0 \\ & \ddots & \\ 0 & & t^{r_n} \end{bmatrix}$$

with the $r_i \in \mathbb{Z}$ and $r_1 \geq r_2 \geq \ldots \geq r_n$. Clearly,

$$(\lambda(t).x_k)_{i,j} = t^{r_i - r_j}(x_k)_{i,j}$$

so $\underset{t \longrightarrow 0}{lim} \lambda(t).X = \underline{0}$ iff the only non-zero entries in the components x_k are at the entries (i, j) for which $r_i - r_j \geq 1$. In particular, the x_k are in the set N of all strictly upper triangular (nilpotent) $n \times n$ matrices. The action map μ induces a surjection

$$GL_n \times N^m \xrightarrow{\mu} N_{m,n}$$

Consider the action of the Borel subgroup B of GL_n (consisting of all upper triangular matrices) on $GL_n \times M_n(\mathbb{C})^m$ given by

$$b.(g, X) = (gb^{-1}, b.X)$$

then B-orbits in $GL_n \times N^m$ are mapped under μ to the same point in $N_{m,n}$.

Consider the morphism

$$GL_n \times M_n(\mathbb{C})^m \longrightarrow GL_n/B \times M_n(\mathbb{C})^m$$

determined by sending (g, X) to $(gB, g.X)$. One verifies that the fibers of this morphism are precisely the B-orbits, so there exist a quotient variety for the B-action which is a trivial vectorbundle over the flag variety GL_n/B.
With $GL_n \times^B N^m$ we will denote the image of the subvariety $GL_n \times N^m$ under this quotient map. We have the diagram

$$\begin{array}{ccc} GL_n \times N^m & \hookrightarrow & GL_n \times M_n(\mathbb{C})^m \\ \downarrow & & \downarrow \\ GL_n \times^B N^m & \hookrightarrow & GL_n/B \times M_n(\mathbb{C})^m \end{array}$$

and $GL_n \times^B N^m$ is a subbundle (but not necessarily trivial as the action of GL_n does not map N^m to itself) of rank $m\dim N$ over the flag variety GL_n/B. With these notations we have

Proposition 1 *(see a.o. [6, p.179] and [7, 2.8])*
Let U be the open subvariety of N^m consisting of the m-tuples of total rank $n-1$, then the action map μ induces the diagram

$$\begin{array}{ccc} GL_n \times^B U & \xrightarrow{\simeq} & GL_n.U \\ \downarrow & & \downarrow \\ GL_n \times^B N^m & \xrightarrow{\mu} & N_{m,n} \end{array}$$

where the upper map is an isomorphism of GL_n-varieties (the action on the fibre bundles is given by left multiplication in the first component). Hence, there is a natural one-to-one correspondence between GL_n-orbits in $GL_n.U$ and B-orbits in U.
In particular, μ is a desingularization of the nullcone and $N_{m,n}$ is irreducible of dimension $(m+1)\binom{n}{2}$.

Hence, this result gives a reduction in complexity from

$$(GL_n, M_n(\mathbb{C})^m) \text{ to } (B, N^m)$$

at least on the stratum $GL_n.U$. The aim of the Hesselink stratification is to have a similar type result for a stratification of the complement. That is, we want to cover $N_{m,n} - GL_n.U$ by strata $GL_n.U_s$ such that the orbits are in one-to-one correspondence with P_s orbits in $U_s \subset C_s^m$ where P_s is a parabolic subgroup of GL_n and C_s a subvectorspace of N.
The key idea behind such a stratification is the result due to G. Kempf [3] that each $X \in N_{m,n}$ has an essentially unique 'optimal' one parameter subgroup λ such that $\lim_{t \longrightarrow 0} \lambda(t).X = \underline{0}$. If λ lies in the maximal torus T_n with associated powers $(r_1, \ldots, r_n)$ then the components of X can have only non-zero entries on entries (i,j) with $r_i - r_j \geq 1$. Conversely if E_X is the set of entries (i,j) such that a component

of X has there a non-zero value, we can compute the n-tuple $s_X = (s_1, \ldots, s_n) \in \mathbb{R}^n$ satisfying

$$s_i - s_j \geq 1 \text{ for all } (i,j) \in E_X$$

minimal with respect to the norm

$$\| s_X \| = s_1^2 + \ldots + s_n^2$$

This $s \in \mathbb{Q}^n$ does not necessarily determine a one-parameter subgroup but there is a unique $\mu_X \in \mathbb{N}s \cup \mathbb{Z}^n$ with $\gcd(\mu_i) = 1$ which is then called the best one-parameter subgroup for X in T_n. However, we can repeat this procedure for $X' = g.X$ with $g \in GL_n$ and it may happen that $\| s_{X'} \| < \| s_X \|$. In fact, we can always find an $X' = g.X$ in the orbit where a minimal $\| s_{X'} \|$ is reached. An optimal one-parameter subgroup associated to X say $\lambda(X)$ is then defined to be

$$\lambda(X) = g^{-1} \mu_{X'} g$$

It is unique in the following sense : let P be the parabolic subgroup of GL_n associated to $\lambda(X)$ (see a.o. [6, III.2.5]) then every other optimal one-parameter subgroup associated to X (coming from another point $X" = h.X$ in the orbit where a minimal value is reached) is conjugated to $\lambda(X)$ under P. For more details we refer to [3],[1] or [11].

3 Hesselink stratification for $N_{m,n}$

Two points $X, Y \in N_{m,n}$ lie in the same stratum provided they have representants in their orbits $X' = g.X$ and $Y' = h.Y$ such that X' and Y' have the same set of optimal one-parameter subgroups. For a fixed maximal torus in GL_n it is easy to compute the best optimal one-parameter subgroup for a given point in the nullcone. However, finding an optimal one-parameter subgroup is usually a very difficult job roughly equivalent to giving a canonical form for the m-tuple.
This forces us to describe the strata in a different way. We restrict to one-parameter subgroups in T_n and investigate the problem of describing the set of points for which this subgroup is optimal. To begin with, observe that the minimality condition puts severe restrictions on the n-tuple $s = (s_1, \ldots, s_n) \in \mathbb{Q}^n$. To be precise, s has to be the disjoint union of strings

$$\{p_i, p_i + 1, \ldots, p_i + k_i\}$$

with $k_i \in \mathbb{N}$ such that all these numbers appear in s possibly with multiplicities a_{ij} (the multiplicity of $p_i + j$) and such that

$$\sum_{j \in \text{string}} s_j = 0$$

for every string in s, see [1, Prop. 6.8.a]. Given n it is easy to write down the list $\mathcal{S}$ of all dominant (i.e. $s_1 \geq \ldots \geq s_n$) such n-tuples. The set $\mathcal{S}$ will be the combinatorial object underlying the classifications of the occurring strata. For $n = 2$ (resp. $3, 4, 5$) the cardinality of $\mathcal{S}$ is 2 (resp. $5, 11, 28$). For example

$\mathcal{S}$ for 4×4 matrices

type	s_1	s_2	s_3	s_4	$\|s\|$
1	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	5
2	$\frac{5}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{11}{4}$
3	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{5}{4}$	$\frac{11}{4}$
4	1	0	0	-1	2
5	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1
6	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$
7	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$
8	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
9	$\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
10	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
11	0	0	0	0	0

From now on we fix one $s = (s_1, \ldots, s_n) \in \mathcal{S}$, and associate to it a tableaux $T(s) = (a_{ij})$ where i runs over the distinct strings $(p_i, p_i + 1, \ldots, p_i + k_i)$ of s and a_{ij} is the multiplicity with which $p_i + j$ occurs in s. To s we associate certain data

- The **corner** C_s which is the sub-vectorspace of N consisting of all matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j \geq 1$
- The **parabolic subgroup** P_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j \geq 0$
- The **Levi-subgroup** L_s which is the subgroup of GL_n consisting of matrices with zero entries except perhaps at entry (i,j) when $s_i - s_j = 0$. Observe that $L_s = \prod GL_{a_{ij}}$.

Example 1 *Consider the following* 5*-tuple*

$$s = (\frac{2}{3}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2})$$

which has tableaux

$$T(s) = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

The associated corner, parabolic and Levi are resp.

$$C_s = \begin{bmatrix} \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad P_s = \begin{bmatrix} * & * & * & * & * \\ \cdot & * & * & * & * \\ \cdot & \cdot & * & * & * \\ \cdot & \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot & * \end{bmatrix} \quad L_s = \begin{bmatrix} * & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & * & \cdot \\ \cdot & \cdot & * & * & \cdot \\ \cdot & \cdot & \cdot & \cdot & * \end{bmatrix}$$

The corner C_s will replace the role of N and the parabolic P_s that of the Borel subgroup B in the formulation of proposition 1. The substitute for the open set U will be

$$U_s = \{X \in C_s^m \mid \mu_s \text{ is optimal for } X\}$$

which is an open set of C_s^m. Observe that U_s may very well be empty.
For example, if $m = 1$ we know that the strata should correspond to the finite set of orbits in $N_{1,n}$ which correspond via the Jordan normal form blocks to partitions

of n. We will see below that the strata are labeled by the $s \in \mathcal{S}$ for which $U_s \neq \emptyset$, hence if $m = 1$ 'most' of the U_s will be empty.
In fact, describing for given m and n the set of s for which $U_s \neq \emptyset$ is one of the main aims of this paper. General theory [1, Prop. 5.2.b] only tells us that for large m all s will eventually appear.
On a more intuitive level, U_s is the set of m-tuples of matrices with non-zero entries in the corner C_s which cannot be simultaneously conjugated to an m-tuple corresponding to a 'smaller' corner s'. Here, 'smaller' means that $\| s' \| < \| s \|$.
From now on we will assume that $s \in \mathcal{S}$ is such that the open set $U_s \subset C_s^m$ is nonempty. Then, the corresponding stratum will be the GL_n-translate $S_s = GL_n.U_s$ and we will investigate its properties, all of which follow from results of W. Hesselink [1].
Similar to the discussion in the preceding section we equip the product $GL_n \times M_n(\mathbb{C})^m$ with a P_s-action via the rule $p.(g,X) = (gp^{-1}, p.X)$ and show that there is a quotient variety which is a trivial vectorbundle over the flag variety GL_n/P_s. This bundle has a not necessarily trivial subbundle $GL_n \times^{P_s} C_s^m$ of rank $m dim C_s$. We then have the following generalization of proposition 1

Proposition 2 *(Hesselink, [1])*
With notations as before we have the diagram

$$\begin{array}{ccc} GL_n \times^{P_s} U_s & \xrightarrow{\simeq} & S_s \\ \cap\downarrow & & \cap\downarrow \\ GL_n \times^{P_s} C_s^m & \xrightarrow{\mu} & \overline{S_s} \end{array}$$

where μ is the action map, $\overline{S_s}$ is the Zariski closure of the stratum S_s in $N_{n,m}$ and the upper map is an isomorphism of GL_n-varieties.
Hence, the stratum S_s is irreducible, smooth of dimension

$$dim(S_s) = dim(GL_n/P_s) + rk(GL_n \times^{P_s} C_s^m) = n^2 - dim(P_s) + m dim(C_s)$$

Moreover, the vectorbundle $GL_n \times^{P_s} C_s^m$ is a desingularization of the closure $\overline{S_s}$ of the stratum S_s. In other words, this vector bundle 'feels' the gluing of S_s to the other strata.
Further, we have a natural one-to-one correspondence the GL_n-orbits in $GL_n \times^{P_s} C_s^m$ and the P_s-orbits in C_s^m which is given by the

$$GL_n.\overline{(g,X)} = GL_n.\overline{(1,X)} = GL_n \times^{P_s} P_s.X$$

In particular, the study of GL_n-orbits in the stratum S_s reduces to the study of P_s-orbits in U_s.

We thus have a reduction of complexity similar to that of proposition 1. The Hesselink stratification of the nullcone $N_{m,n}$ is then given by

$$N_{m,n} = \bigcup_{s \in \mathcal{S}'} GL_n.U_s$$

where $\mathcal{S}'$ is the set of $s \in \mathcal{S}'$ for which $U_s \neq \emptyset$. Further,

$$\overline{S_s} \subset \bigcup_{\|s'\| \leq \|s\|} S_{s'}$$

Hence, we have an accurate description of $N_{m,n}$ and a reduction of the orbit-problem provided we can determine the s for which $U_s \neq \emptyset$ and give a precise description of this open set. This will be the topic of the following sections.

4 Reduction to the quiver Q_s

In this section we reduce the description of U_s to a certain problem on representations of a quiver Q_s. In the next section we will then use recent results on quiver-representations to solve this problem.
As U_s is the set of m-tuples out of C_s which cannot be conjugated to an m-tuple of 'smaller' corner-type, it is intuitively clear that the border-region of C_s will be important.

- The **border** B_s is the sub-vectorspace of C_s consisting of all matrices with zero entries except perhaps on entry (i, j) where $s_i - s_j = 1$

Observe that the reductive Levi-group L_s acts on B_s and we are aiming to reduce the parabolic action of P_s on C_s^m to that of L_s on B_s^m. Before we do this, let us give a representation theoretic interpretation of the latter action which is crucial to this paper.
A quiver Q is a 4-tuple (Q_v, Q_a, t, h) where Q_v is a finite set $\{0, \dots, k\}$ of vertices, Q_a a finite set of arrows ϕ between these vertices and $t, h : Q_a \to Q_v$ are two maps assigning to an arrow ϕ its tail t_ϕ and its head h_ϕ respectively. A representation V of a quiver Q consists of a family $\{V(i) : i \in Q_v\}$ of finite dimensional $\mathbb{C}$-vector spaces and a family $\{V(\phi) : V(t_\phi) \to V(h_\phi); \phi \in Q_a\}$ of linear maps between these vectorspaces, one for each arrow in the quiver. The dimension-vector $\dim(V)$ of the representation V is the k-tuple of integers $(\dim(V(i)))_i \in \mathbb{N}^{k+1}$. We have the natural notion of morphisms and isomorphisms between representations consisting of $k+1$-tuples of linear maps with obvious commutativity conditions. For a fixed dimension-vector $\alpha = (\alpha_0, \dots, \alpha_k) \in \mathbb{N}^{k+1}$ one defines the representation space $R(Q, \alpha)$ of the quiver Q to be the set of all representations V of Q with $V_i = \mathbb{C}^{\alpha_i}$ for all $i \in Q_v$. Because $V \in R(Q, \alpha)$ is completely determined by the linear maps $V(\phi)$, we have a natural vector space structure

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_a} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ is the vector space of all $h_\phi \times t_\phi$ matrices over $\mathbb{C}$. There is a canonical action of the linear reductive group

$$GL(\alpha) = \prod_{i=0}^{k} GL_{\alpha_i}(\mathbb{C})$$

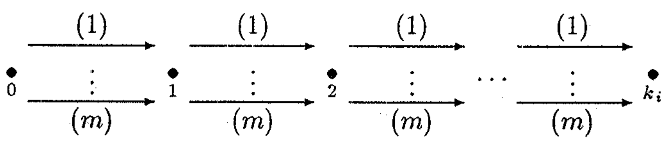
on the variety $R(Q, \alpha)$ by base change in the V_i. That is, if $V \in R(Q, \alpha)$ and $g = (g(0), \dots, g(k)) \in GL(\alpha)$, then

$$(g.V)(\phi) = g(h_\phi)V(\phi)g(t_\phi)^{-1}$$

It is clear that the $GL(\alpha)$-orbits in $R(Q, \alpha)$ are precisely the isomorphism classes of representations.
To $s \in \mathcal{S}$ we will now associate a quiver Q_s. Recall that s is the disjoint union of strings

$$\{\underbrace{p_i, \dots, p_i}_{a_{i0}}, \underbrace{p_i + 1, \dots, p_i + 1}_{a_{i1}}, \dots, \underbrace{p_i + k_i, \dots, p_i + k_i}_{a_{ik_i}}\}$$

satisfying $\sum a_{ij}(p_i + j) = 0$. For each string we define the quiver Q_i on $k_i + 1$ vertices of type A_{k_i} but with m arrows between the consecutive vertices, that is,



and consider the dimension vector

$$\alpha_i = (a_{i0}, a_{i1}, \ldots, a_{ik_i})$$

which is the i-th row in the tableaux $T(s)$.
The quiver Q_s is the disjoint union of the string-quivers Q_i and we define the dimension vector α_s to be the vector determined by the α_i.
Using these conventions, the following result is readily verified

Proposition 3 *The action of the Levi-group $L_s = \prod GL_{a_{ij}}$ on the border B_s coincides with that of the base-change group $GL(\alpha_s)$ on the quiver-representations $R(Q_s, \alpha_s)$.*

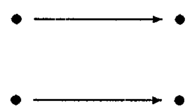
Example 2 *(continuation of example 1)*
Computing the action of the Levi-group L_s by conjugation on the border B_s

$$L_s = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & \beta_{11} & \beta_{12} & 0 \\ 0 & 0 & \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & 0 & 0 & \delta \end{bmatrix} \qquad B_s = \begin{bmatrix} 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

gives the transformation-maps

$$\begin{array}{ccc} \begin{bmatrix} x & y \end{bmatrix} & \mapsto & \alpha . \begin{bmatrix} x & y \end{bmatrix} . \beta^{-1} \\ \begin{bmatrix} z \end{bmatrix} & \mapsto & \gamma . \begin{bmatrix} z \end{bmatrix} . \delta^{-1} \end{array}$$

which is the natural base-change action on the representation space of the quiver Q_s



and dimension vector $\alpha_s = \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}$. Clearly, for the action of L_s on B_s^m the only change that has to be made is that there are m arrows between the indicated vertices.

The reduction of the description of U_s to this quiver representation now follows from applying general results due to F.C. Kirwan [5] and L. Ness [9] to our setting. Let V_s be the open subset of B_s^m consisting of those points for which μ_s is an optimal one-parameter subgroup. From [5, §12] we obtain the diagram

$$\begin{array}{ccc} U_s & \hookrightarrow & C_s^m \\ & & \downarrow \pi \\ V_s & \hookrightarrow & B_s^m \end{array}$$

where the horizontal maps are the natural inclusions and π the projection whose fibers are vectorspaces. The result states that $U_s = \pi^{-1}(V_s)$, that is, we can describe V_s we know U_s. Moreover, one can describe V_s as the set of semi-stable points in B_s^m under the action of a subgroup of L_s, see [5, Rem.12.21], [9, Pf. of Th.9.2] or [11, Prop.1] (but note remark 1 below).
Consider the character

$$\chi_s : L_s = \prod GL_{a_{ij}} \to \mathbb{C}^*$$

which maps an element (g_{ij}) to

$$\prod \det(g_{ij})^{n_{ij}}$$

where $n_{ij} = d.(p_i + j)$ and d is the least common multiple of the numerators of the p_k's. Equivalently, the n_{ij} are the integers occurring in μ_s grouped together at the corresponding vertex of the quiver Q_s. Using these notations we have

Proposition 4 *The open set $U_s = \pi^{-1}(V_s)$ where $\pi : C_s^m \to B_s^m$ is the natural projection. Moreover, V_s is the open subset of points $X \in B_s^m$ such that there is a semi-invariant function $f : B_s^m \to \mathbb{C}$ for which $f(X) \neq 0$ corresponding to the character χ_s, that is, such that for all $g \in L_s$ we have*

$$g.f = \chi_s(g)^k f$$

for some integer $k \in \mathbb{N}$.

Proof : This is [5, 12.21 and 12.24] and [11, Prop. 1] adapted to our situation and modified according to the remark below. □

Remark 1 *The group Z_n occurring in the formulation of [11, Prop. 1] is in general not the one defined on [11, p. 123] as it does not satisfy the requirement on the characters given at the bottom of [11, p. 123] which is needed in the proof. Let us give an example.*
Consider $s = (\frac{8}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{7}{5})$, then L_s (Z in terminology of [11]) is

$$L_s = \mathbb{C}^* \times \mathbb{C}^* \times GL_2 \times \mathbb{C}^*$$

and B_s (V_1 in the notation of [11, p.123]) is

$$\begin{bmatrix} . & * & . & . & . \\ . & . & * & * & . \\ . & . & . & . & * \\ . & . & . & . & * \\ . & . & . & . & . \end{bmatrix}$$

Then, Z_1 as defined in [11, p. 123] is the kernel of the character $(1, 1, 0, -2)$ *and has semi-stable points in B_s for example*

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

But, μ_s cannot be an optimal one-parameter subgroup for X as X has rank three and so should lie in the stratum determined by $s' = (\frac{3}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2})$.
However, for the kernel of the correct character given in proposition 4 and which is in this case $(-8, -3, 2, 7)$ there are no semi-stable points in B_s.
In fact, the distinction between the required character χ_s and the 'expected' character of [11, p.123] and which coincides in our case with Schofield's canonical character [10] for quiver situations, is responsible for most of the subtleties in describing the strata.

5 Algorithmic determination of the strata $\mathcal{S}'$

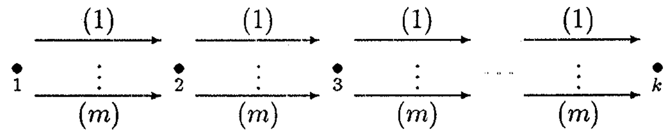
In this section we will give an algorithm to determine for given m and n the set $\mathcal{S}'$ of the s for which $U_s \neq \emptyset$.
In the foregoing section we have seen that $U_s \neq \emptyset$ if and only if $V_s \neq \emptyset$ and that V_s is the open subset of quiver-representations $R(Q_s, \alpha_s)$ for which a semi-invariant corresponding to the character χ_s does not vanish.
One of the advantages of reducing to this quiver situation is that we can view points of B_s^m as objects in the Abelian category of all representations of Q_s, that is, the category of modules over the path algebra $\mathbb{C}Q_s$. Therefore, we can associate to the character χ_s (which is determined by the integers (n_{ij}) defined above) an additive function on the Grothendieck group of the path algebra

$$\theta_s : K_0(\mathbb{C}Q_s) \to \mathbb{Z}$$

which is determined by sending the class of a representation of dimension-vector $\beta = (b_{ij})$ to $\sum n_{ij} b_{ij}$.
Using the analogy with vector bundles on projective varieties, A. King [4] defines a representation V of Q_s to be θ-semistable (for any additive function θ on the Grothendieck group) if $\theta(V) = 0$ and every sub-representation $V' \subset V$ satisfies $\theta(V') \geq 0$. Similarly, a representation V is called θ-stable if the only subrepresentations V' with $\theta(V) = 0$ are 0 and V. Using [4, Prop.3.1] we then have

Proposition 5 *V_s is the open subset of $R(Q_s, \alpha_s)$ which are θ_s-semistable.*

Hence, in order to verify whether $X \in B_s^m = R(Q_s, \alpha_s)$ lies in V_s it suffices to know the dimension vectors of all subrepresentations of X and verify that their values under θ_s are ≥ 0. If $V_s \neq \emptyset$ it is an open subvariety in $R(Q_s, \alpha_s)$ and so it suffices to know the dimension vectors of all sub-representations for a representation in general position.
Precisely this problem had to be addressed by A. Schofield [10] in his solution of some conjectures of V. Kač on the generic decomposition. Recall that V. Kač showed [2] that the dimension vectors of indecomposable quiver-representations form an infinite root system with associated generalized Cartan matrix the symmetrization of the Ringel form. In our case, where each of the component quivers is of the shape



the Ringel form is the bilinear map

$$r : \mathbb{Z}^k \times \mathbb{Z}^k \to \mathbb{Z}$$

with corresponding matrix

$$\begin{bmatrix} 1 & -m & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -m \\ 0 & & & 1 \end{bmatrix}$$

The Ringel form encodes a lot of information on representations. If V resp. W are representations of dimension-vector α resp. β then

$$r(\alpha, \beta) = \dim \operatorname{Hom}(V, W) - \dim \operatorname{Ext}(V, W)$$

For fixed dimension vector α there is an open subset of representations V in $R(Q,\alpha)$ such that the dimension vectors of its indecomposable components are constant, say β_i. Then,

$$\alpha = \beta_1 + \ldots + \beta_l$$

is called the canonical decomposition of α into Schur roots β_i (Schur roots are roots β such that there is an open set of indecomposable representations in $R(Q,\beta)$). Kač asked for a combinatorial description of the set of Schur roots and of the canonical decomposition in terms of the Ringel form. Solutions to these problems were presented by A. Schofield [10] and depend heavily on being able to describe the dimension vectors of sub-representations of a general representation. Denote with

$$\beta \hookrightarrow \alpha$$

that a general representation of dimension-vector α has a sub-representation of dimension-vector β. Schofield gave an inductive way to find the dimension-vectors of these generic sub-representations using the Ringel form

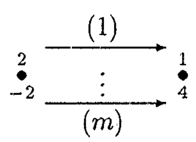
$$\beta \hookrightarrow \alpha \quad \text{iff} \quad \underset{\beta' \hookrightarrow \beta}{Max} - r(\beta', \alpha - \beta) = 0$$

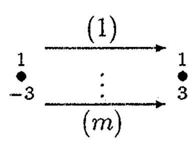
For example, the description of the Schur roots [10, Th.6.1] is then : α is Schur iff for all $\beta \hookrightarrow \alpha$ we have $r(\beta,\alpha) - r(\alpha,\beta) > 0$. A combinatorial description of the canonical decomposition was also given in [10].
These facts enable us to give the promised algorithmic description of the occurring strata :

Theorem 2 *For $s \in \mathcal{S}$, μ_s is an optimal one-parameter subgroup for a point $X \in N_{m,n}$ if and only if for the associated quiver Q_s all $\beta \hookrightarrow \alpha_s$ satisfy $\theta_s(\beta) \geq 0$.*
The open set U_s consists of those $X \in C^m_s$ for which the projection $\pi(X) \in B^m_s = R(Q_s, \alpha_s)$ is a θ_s-semistable representation.

In view of Schofield's inductive procedure to determine the dimension-vectors of generic subrepresentations, the first part allows us for given n and m to compile the list of actually occurring strata. The second part allows us to describe U_s for we can determine V_s by considering the 'bad' dimension vectors $\gamma < \alpha_s$ such that $\theta(\gamma) < 0$ and then V_s is the complement of those representations in $R(Q,\alpha_s)$ having a subrepresentation of dimension-vector γ (which is a closed condition and easy to express).

Example 3 *(continuation of example 1)*
The quiver Q_s is





where from now on we let the upper number denote the entry of the dimension-vector α_s and the lower number the component of θ_s.
When $m = 1$ every representation contains a subrepresentation of dimension-vector $\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}$ (the kernel of the upper map) for which θ_s gives -2. Hence, $U_s = \emptyset$.
When $m \geq 2$ a general representation no longer has such a subrepresentation and one verifies that $\theta_s(\beta) \geq 0$ for all $\beta \hookrightarrow \alpha_s$. Hence $U_s \neq \emptyset$.

Our results allow some immediate consequences. We want to recover the classical result that for $m = 1$ the strata correspond to the partitions of m. Following [1, 6.8] call s strongly balanced if the multiplicity function

$$m(x) = \{i \mid s_i = x\}$$

satisfies the following conditions

- if $m(x) \neq 0$ then $2x \in \mathbb{Z}$
- if $x \geq 0$ then $m(-x) = m(x) \geq m(x+1)$

Proposition 6 *For $m = 1$, $s \in \mathcal{S}'$ iff s is strongly balanced.*

Proof : When $m = 1$ all quivers are of type A_k and we can use the representation theory of this finite type case to get the result. The crucial observation is that all representations are direct sums of indecomposables which have dimension vectors

$$(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$$

for some uninterrupted string of 1's. □

The next result solves a very special case of a question of W. Hesselink [1, Rem.5.3].

Proposition 7 *For every n, all strata occur when $m = n - 1$ and this bound is optimal.*

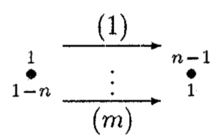
Proof : It is easy to see that when all dimensions are $< n$ and there are $m \geq n - 1$ arrows between the consecutive vertices the only dimension-vectors of subrepresentations of represetations in general position are of the form

$$(0, \ldots, 0, n_k, a_{ik+1}, a_{ik+2}, \ldots, a_{ik_i})$$

for $n_k \leq a_{ik}$. For such dimension-vectors the image under θ_s is clearly positive. Therefore, all strata occur if $m \geq n - 1$. Moreover, this bound is optimal for consider

$$s = (\frac{1}{n}, \ldots, \frac{1}{n}, -\frac{n-1}{n})$$

Then the associated quiver Q_s, dimension-vector α_s and θ_s are



If $m < n - 1$ any representation has a subrepresentation of dimension vector $(1, k)$ with $k \leq m$ and θ_s is negative on it. □

6 Examples

In this section we will initiate the study of the orbit-spaces for the strata in the Hesselink stratification of $N_{m,n}$. We have see before that there is a natural one-to-one correspondence between

- GL_n-orbits in S_s
- P_s-orbits in U_s

Moreover, under the natural projection map

$$U_s \xrightarrow{\pi} V_s \hookrightarrow R(Q_s, \alpha_s)$$

points lying in the same P_s-orbit in U_s are mapped to points lying in the same $L_s = GL(\alpha_s)$ orbit in V_s. Therefore, we have an induced projection map

$$Orb(P_s, U_s) \xrightarrow{\pi} M(Q_s, \alpha_s; \theta_s)$$

from the orbit-space of U_s under P_s to the 'moduli' space of θ_s-semi stable representations of Q_s of dimension-vector α_s, see [4] for some results on these moduli spaces. We will mean in this section by $M(Q_s, \alpha_s; \theta_s)$ the orbit-space of V_s under action of $GL(\alpha_s)$. The precise connection with King's moduli spaces has to be explored further as is a thorough investigation of the fibres of π. For low values of n one can describe both the moduli spaces and the fibres explicitly, see the introduction for $n \leq 3$ and $n = 4, 5$ below.

6.1 Nullcone for 4×4 matrices

In section 3 we gave the list $\mathcal{S}$ of 11 possibly occurring strata in the nullcone of m 4×4 matrices. The corresponding quiver-data is summarized in the table below

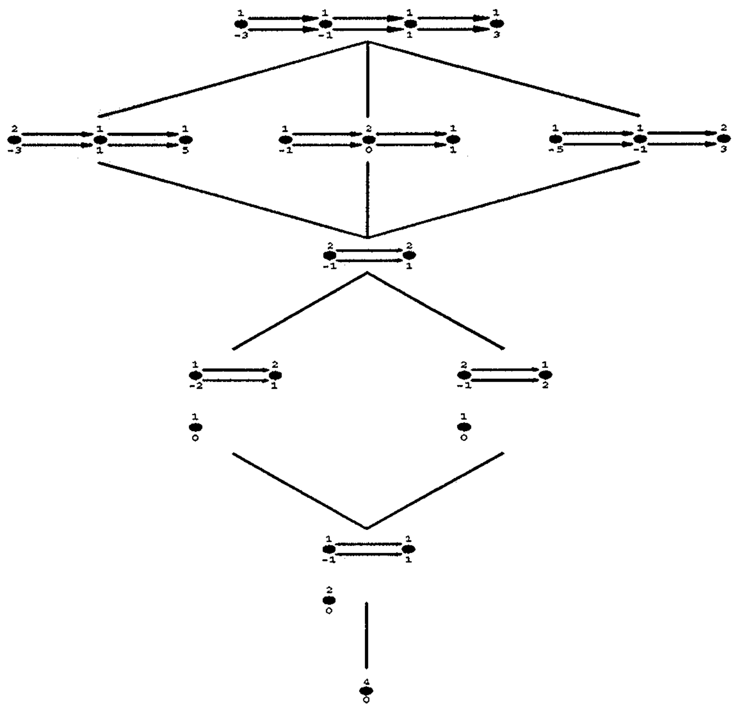
strata for 4×4 matrices

type	α_s				θ_s				$m=1$	$m=2$	$m=3$
1	1	1	1	1	-1	-1	-1	-1	12	18	24
2	2	1	1		-3	1	5		-	15	20
3	1	1	2		-5	-1	3		-	15	20
4	1	2	1		-1	0	1		10	15	20
5	2	2			-1	1			8	12	16
6	3	1			-1	3			-	-	12
7	1	3			-3	1			-	-	12
8	1 1	2			-2 0	1			-	9	11
9	2 1	1			-1 0	2			-	9	11
10	1 2	1			-1 0	1			6	7	8
11	4				0				0	0	0

where the last columns give the dimensions of the strata when they occur. For $m = 1$ only the strata occur corresponding to the 5 partitions of 4. For $m = 2$ four new strata occur, the only ones missing are the ones corresponding to the corners

$$\begin{bmatrix} \cdot & * & * & * \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

which appear for all $m \geq 3$.
For couples of 4×4 matrices we get the following degeneration picture of the nullcone where each stratum is represented by the underlying quiver-data



Representation theory of quivers, in particular the theory of Kronecker modules, allows us to determine the moduli spaces $M(Q_s, \alpha_s; \theta_s)$ explicitly. In the following table we distinguish between the components in these moduli spaces corresponding to indecomposable (resp. decomposable) representations.

Moduli spaces $M(Q_s, \alpha_s; \theta_s)$

type	indec	dec
1	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	-
2	$\mathbb{P}^1$	-
3	$\mathbb{P}^1$	-
4	$\mathbb{P}^3 \cup \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{P}^1 \times \mathbb{P}^1$
5	$\mathbb{P}^1$	$S^2(\mathbb{P}^1)$
8	-	$\mathbb{P}^0$
9	-	$\mathbb{P}^0$
10	-	$\mathbb{P}^1$
11	-	$\mathbb{P}^0$

For all types ≥ 5 we have $C_s = B_s$ so the above table also gives the orbit-spaces $Orb(P_s, U_s)$ for them. For type 4 (resp. 2,3) the map π is an isomorphism (resp. a rank one vectorbundle). The precise description of the fibers of π for type 1 is more complicated and similar to the 3×3 case given in the introduction.

6.2 Nullcone for 5×5 matrices

In the following table we collect the relevant data for the Hesselink strata of $N_{m,5}$, the last four columns give the dimensions of the strata when they occur.

strata for 5 × 5 matrices

type	α_s	θ_s	$m=1$	$m=2$	$m=3$	$m=4$
1	1 1 1 1 1	-2 -1 0 1 2	20	30	40	50
2	2 1 1 1	-6 -1 4 9	-	27	36	45
3	1 1 1 2	-9 -4 1 6	-	27	36	45
4	1 1 2 1	-8 -3 2 7	-	27	36	45
5	1 2 1 1	-7 -2 3 8	-	27	36	45
6	1 1 1 1 1	-3 -1 1 3 0	18	26	34	42
7	2 1 2	-1 0 1	-	24	32	40
8	1 1 3	-7 -2 3	-	-	28	35
9	3 1 1	-3 2 7	-	-	28	35
10	1 2 2	-6 -1 4	-	24	32	40
11	2 2 1	-4 1 6	-	24	32	40
12	2 1 1 1	-3 1 5 0	-	21	27	33
13	1 1 2 1	-5 -1 3 0	-	21	27	33
14	1 1 1 1 1	-2 0 2 -1 1	16	22	28	34
15	1 3 1	-1 0 1	14	21	28	35
16	2 3	-3 2	-	18	24	30
17	3 2	-2 3	-	18	24	30
18	2 1 1 1	-1 2 -1 1	-	17	21	25
19	1 2 1 1	-2 1 -1 1	-	17	21	25
20	2 2 1	-1 1 0	12	16	20	24
21	1 4	-4 1	-	-	-	20
22	4 1	-1 4	-	-	-	20
23	3 1 1	-1 3 0	-	-	16	19
24	1 3 1	-3 1 0	-	-	16	19
25	1 2 2	-2 1 0	-	12	14	16
26	2 1 2	-1 2 0	-	12	14	16
27	1 1 3	-1 1 0	8	9	10	11
28	5	0	0	0	0	0

For $m=1$ only 7 strata occur, corresponding to the partitions. For $m=2$ there are 22 strata. For $m=3$ we obtain 26 strata, the extra ones corresponding to the corners

$$\begin{bmatrix} . & * & * & * & * \\ . & . & * & * & * \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \text{ and } \begin{bmatrix} . & . & * & * & * \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

and their mirror images along the other diagonal.
For $m \geq 4$ all strata occur, the last two of which have corner

$$\begin{bmatrix} . & * & * & * & * \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}$$

and its mirror image along the other diagonal.

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