# The Nullcone of Quiver Representations

Lieven Le Bruyn
(Senior Researcher at the NFWO)
Department of Mathematics & Computer Science
University of Antwerp, UIA, Belgium

June 1995 Report no. 95-14

Division of Pure Mathematics
Department of Mathematics & Computer Science

universitaire
instelling
antwerpen

Universiteitsplein 1, B-2610 Wilrijk-Antwerpen , BELGIUM

# The Nullcone of Quiver Representations

Lieven Le Bruyn
(Senior Researcher at the NFWO)
Department of Mathematics & Computer Science
University of Antwerp, UIA, Belgium

June 1995 Report no. 95-14

## Abstract

The Hesselink stratification of the nullcone is investigated in the case of representations of quivers. In particular, an algorithm is presented to describe the actually occuring strata. The orbit problem for a specific stratum is closely related to the moduli space of semi-stable representations of a given level-quiver representation problem.

# The Nullcone of Quiver Representations

Lieven Le Bruyn*
Departement Wiskunde UIA
B-2610 Wilrijk (Belgium)
email : lebruyn@uia.ua.ac.be

June 8, 1995

**Abstract**

The Hesselink stratification of the nullcone is investigated in the case of representations of quivers. In particular, an algorithm is presented to describe the actually occuring strata. The orbit problem for a specific stratum is closely related to the moduli space of semi-stable representations of a given level-quiver representation problem.

## 1 Introduction

Consider a quiver without oriented cycles $Q$, a dimension vector $\alpha$ and the representation space $R(Q,\alpha)$ (all relevant definitions are recalled in the next section). The key problem in the representation theory of finite dimensional hereditary algebras is to study the orbits in $R(Q,\alpha)$ under the reductive base-change group $GL(\alpha)$. For quivers of finite or tame type a satisfactory description is known, for the wild case this is considered to be a 'hopeless' problem.
A natural approach to this problem is to stratify $R(Q,\alpha)$ in more manageable locally closed subvarieties and to study the orbit-spaces for these strata. This idea was pursued by H. Kraft and Ch. Riedtmann in [8] where they took as the strata the 'sheets', that is, one collects together the representations with same orbit-dimension (or equivalently, such that their endomorphism rings have equal dimension). As all orbits in such a sheet are closed one can then hope to construct a good quotient variety as the orbit space.
Whereas this sheet-approach has a natural interpretation in representation theoretic terms, it does not seem to have sufficiently good properties in general. The strata are rarely irreducible or smooth and not much is known about the quotient varieties. Recently, some progress has been made to the construction of good moduli spaces for representations in sufficiently general position. A. King [5] introduced the notion of semi-stable representations with respect to a character of $GL(\alpha)$ and constructed a moduli space for them. Using the solution of A. Schofield [14] to some conjectures of V. Kač one can describe the open (but possibly empty) subsets of semi-stable representation for specific characters explicit. Ongoing research studies the geometrical properties of the corresponding moduli spaces. For example, if the dimension vector is indivisible and the character is sufficiently general, this moduli space is known to be a smooth projective variety with known cohomology ring [5]. It is to be expected that more information about these moduli spaces will be unraveled in the near future.

*Senior researcher at the NFWO

In this paper we will investigate a different stratification of $R(Q,\alpha)$. As we assumed that the quiver $Q$ has no oriented cycles, there are no polynomial invariants for the $GL(\alpha)$-action, see [9]. Hence, the representation space $R(Q,\alpha)$ is the nullcone for this action. Invariant theory gives us an excellent stratification of nullcones due to W. Hesselink [2]. It is based on the notion of optimal one-parameter subgroups due to G. Kempf [4] and G. Rousseau [15]. For example, the strata are known to be open subsets of vectorbundles on flag varieties and are in particular smooth and irreducible. Moreover, a reduction of complexity is known in order to study the orbits in a specific stratum.

However, the Hesselink stratification has some disadvantages from a practical point of view. For a specific point in the nullcone it is usually very hard to determine the set of optimal one-parameter subgroups corresponding to this point. Because of this it is also very difficult to determine which strata actually occur, that is, whereas one can give a combinatorial list of the strata that can occur it is hard to determine which of them are empty.

In this paper we will describe the Hesselink stratification of $R(Q,\alpha)$ with the emphasis on trying to overcome the above practical difficulties. In particular we will present an algorithm to determine for given $Q$ and dimension vector $\alpha$ a list of the occurring strata (or equivalently, of the relevant one-parameter subgroups). This algorithm is based on King's notion of semi- stability mentioned above and A. Schofield's algorithmic description of the dimension vectors of generic subrepresentations in terms of the Euler inner product [14].

A perhaps surprising consequence of our investigation is that at the hearth of every stratum there is a moduli-space problem for semi-stable representations with specific character (corresponding to the optimal one-parameter subgroup) of a quiver with very simple form, the so-called level quivers. In particular, there is a surjective map from the orbit-space of the stratum to one of King's moduli spaces for a specific level-quiver. The investigation of the fibers of this natural map seems to be a very interesting (but difficult) project for further research.

Another conundrum is the representation-theoretic interpretation of the optimal one-parameter subgroup belonging to a specific representation. It induces a filtration by subrepresentations but examples show that this filtration is neither the top- nor socle-filtration. What our results show is that the top part of this filtration is a semi-stable representation for a specific level-quiver. Whether an iteration of this gives a canonical filtration on representations and what its representation theoretic relevance is, remains to be seen. Also more examples should be worked out in some detail. A particularly interesting case is that of the $k$-arrow rank two quiver because of its connections with vectorbundles on projective curves.

The paper is organized as follows. In section 2 we recall the relevant definitions from representation- and invariant theory. Some effort has been made to describe the notion of optimal one-parameter subgroup as concrete as possible in the specific situation of interest. In section 3 we recall the definitions and main properties of the Hesselink stratification, again concentrating on the setting of quivers. Moreover, we give an algorithm to construct the list $L(Q,\alpha)$ of possibly occurring strata. In section 4 we then come to the hearth of this paper which consists in giving an algorithmic description of the non-empty strata and the relation with the underlying level-quiver situation. As all our results remain valid for the nullcone $N(Q,\alpha)$ of $R(Q,\alpha)$ in case the quiver $Q$ does have oriented cycles, results will be stated in this generality, the reader interested in finite dimensional algebras may want to replace $N(Q,\alpha)$ by $R(Q,\alpha)$.

**Acknowledgment :** This paper is a corrected version of talks given at the workshop on wild quiver representations in Krippen, march 1995. I like to thank the organizers (D. Happel,C.M. Ringel,L. Unger) for the invitation and the participants, in particular A. King and A. Schofield for helpful discussions.

## 2 Filtrations and one-parameter subgroups

Throughout, we fix an algebraically closed field of characteristic zero and call it $\mathbb{C}$.
A **quiver** $Q$ is a 4-tuple $(Q_v, Q_a, t, h)$ where $Q_v$ is a finite set $\{1, ..., k\}$ of vertices, $Q_a$ a finite set of arrows $\phi$ between these vertices and $t, h : Q_a \to Q_v$ are two maps assigning to an arrow $\phi$ its tail $t_\phi$ and its head $h_\phi$ respectively. Note that we do not exclude loops or multiple arrows.
A **representation** $V$ of a quiver $Q$ consists of a family $\{V(i) : i \in Q_v\}$ of finite dimensional $\mathbb{C}$-vector spaces and a family $\{V(\phi) : V(t_\phi) \to V(h_\phi); \phi \in Q_a\}$ of linear maps between these vectorspaces, one for each arrow in the quiver. The dimension-vector $dim(V)$ of the representation $V$ is the $k$-tuple of integers $(dim(V(i)))_i \in \mathbb{N}^k$.
We have the natural notion of morphisms and isomorphisms between representations consisting of $k$-tuples of linear maps with obvious commutativity conditions.
For a fixed dimension-vector $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$ one defines the **representation space** $R(Q, \alpha)$ of the quiver $Q$ to be the set of all representations $V$ of $Q$ with $V_i = \mathbb{C}^{\alpha_i}$ for all $i \in Q_v$. Because $V \in R(Q, \alpha)$ is completely determined by the linear maps $V(\phi)$, we have a natural vector space structure

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_a} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ is the vector space of all $h_\phi \times t_\phi$ matrices over $\mathbb{C}$.
We consider the vector space $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and function field $\mathbb{C}(Q, \alpha)$. There is a canonical action of the linear reductive group

$$GL(\alpha) = \prod_{i=1}^{k} GL_{\alpha_i}(\mathbb{C})$$

on the variety $R(Q, \alpha)$ by base change in the $V_i$. That is, if $V \in R(Q, \alpha)$ and $g = (g(1), ..., g(k)) \in GL(\alpha)$, then

$$(g.V)(\phi) = g(h_\phi)V(\phi)g(t_\phi)^{-1}$$

The $GL(\alpha)$-orbits in $R(Q, \alpha)$ are precisely the isomorphism classes of representations.
By geometric invariant theory (see [12] or [7]) we have a quotient morphism

$$\pi : R(Q, \alpha) \to V(Q, \alpha)$$

where $V(Q, \alpha)$ is the affine variety corresponding to the ring of polynomial invariants $\mathbb{C}[Q, \alpha]^{GL(\alpha)}$. It is well know that the points of the quotient variety parameterize the closed orbits. In the special case under consideration, the closed $GL(\alpha)$ orbits correspond to the isomorphism classes of semi-simple representations in $R(Q, \alpha)$.
In [9, Theorem 1] it was shown that the invariant ring is generated by traces of oriented cycles in the quiver $Q$ of length at most $\sum \alpha_i^2$.
Moreover, a Luna - stratification of $V(Q, \alpha)$ in locally closed irreducible smooth subvarieties corresponding to the different representation-types of semi-simple representations was given in [9, §2]. The étale local structure of the quotient-variety was investigated in [9, Theorem 5]. In view of these results we have a fairly accurate description of $V(Q, \alpha)$. For a few remarks on the main remaining open problem, rationality of $\mathbb{C}(Q, \alpha)^{GL(\alpha)}$, we refer to [10] and [1].
In this paper we start the investigation of the complementary problem, that is, of the "kernel" of the quotient map. Define $N(Q, \alpha)$ the **nullcone** of $R(Q, \alpha)$ to be the set of $V \in R(Q, \alpha)$ such that for all polynomial invariants $f \in \mathbb{C}[Q, \alpha]^{GL(\alpha)}$ we have $f(V) = 0$. Clearly, $N(Q, \alpha)$ is a closed $GL(\alpha)$-stable subvariety of $R(Q, \alpha)$.
Observe that in the important special case when the quiver $Q$ has no oriented cycles

we have $N(Q,\alpha) = R(Q,\alpha)$. In this case, the stratification of the nullcone to be given in the next sections gives a novel stratification of the representation space. Recall that other stratifications of $R(Q,\alpha)$ were given, a.o. [8], according to the dimensions of the orbits. Observe that this stratification may have singular strata. By the results of [9] given above, $V \in N(Q,\alpha)$ iff for every oriented cycle $(\phi_1, \phi_2, \dots, \phi_l)$ in the quiver $Q$ we have

$$Tr(V(\phi_l)V(\phi_{l-1})\dots V(\phi_2)V(\phi_1)) = 0$$

Clearly, one would prefer a more manageable description of these nilpotent representations.

Such a description is provided by the Hilbert's criterium (see [7, III.2]). In our case it asserts that $V \in N(Q,\alpha)$ iff there is a one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(\alpha)$ such that

$$\lim_{t\to 0} \lambda(t).V = 0$$

where 0 is the zero-representation (all maps zero-matrices). In this setting, a torus action corresponds to a filtration on the representation $V$ (see for example [7, II.2.7] or [5, p. 520-521]). For each vertex $i \in Q_v$ we can decompose

$$\mathbb{C}^{\alpha_i} = \oplus_{z\in\mathbb{Z}} \mathbb{C}^{\alpha_i}_{(z)}$$

in weight-spaces where $\lambda(t)$ acts on $\mathbb{C}^{\alpha_i}_{(z)}$ by multiplication by $t^z$. This decomposition defines a filtration

$$\mathbb{C}^{\alpha_i}_{(\geq z)} = \oplus_{z'\geq z} \mathbb{C}^{\alpha_i}_{(z')}$$

One verifies that the above limit exists if and only if for all arrows $\phi \in Q_a$ the maps are filtration preserving, that is,

$$V(\phi) : \mathbb{C}^{\alpha_{t\phi}}_{(\geq z)} \to \mathbb{C}^{\alpha_{h\phi}}_{(\geq z)}$$

for all $z \in \mathbb{Z}$. Hence the subspaces $\mathbb{C}^{\alpha_i}_{(\geq z)}$ determine subrepresentations $V_z$ of $V$ and hence a $\mathbb{Z}$-filtration

$$\dots \supseteq V_{z-1} \supseteq V_z \supseteq V_{z+1} \supseteq \dots$$

which is of course bounded, that is $V_z = 0$ for $z >> 0$ and $V_z = V$ for $z << 0$. Conversely, any such $\mathbb{Z}$-filtration on $V$ is associated to some (but not necessarily unique) one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(\alpha)$ for which the limit $\lim_{t\to 0} \lambda(t).V$ exists. The limit-representation is then the associated graded representation

$$gr(V) = \oplus_{z\in\mathbb{Z}} V_z/V_{z+1}$$

of this filtration. Hence

**Lemma 1** *$V \in N(Q,\alpha)$ iff there exists a filtration by subrepresentations on $V$ such that the associated graded representation is the zero-representation.*

However, the filtration (or the one-parameter subgroup) having this property is by no means unique. For example, we can "blow up" the filtration (or replace $\lambda$ by a multiple) or change it by a filtration-preserving isomorphism (or conjugate $\lambda$ by an element of its associated parabolic subgroup $P(\lambda)$).

Recall (for example [7, III.2.5]) that

$$P(\lambda) = \{g \in GL(\alpha) \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists } \}$$

is a parabolic subgroup of $GL(\alpha)$ with unipotent radical $U(\lambda)$ the subgroup of $P(\lambda)$ such that the limit is the unit element and Levi-subgroup $L(\lambda)$ which is the centralizer of $P(\lambda)$ and which is a product of $GL_m$'s.

Therefore, we would like to have a canonical choice of one-parameter subgroup $\lambda$ corresponding to a given $V \in R(Q,\alpha)$. To this end, let us begin by defining a **measure of instability** $m(V,\lambda)$. Consider the descending $\mathbb{Z}$-filtration $V_z$ by subrepresentations defined by $\lambda$ (always assuming the limit exists and is equal to 0). For each arrow $\phi \in Q_a$ we define its degree by

$$deg(\phi;V,\lambda) = min\{k \in \mathbb{Z} \mid V(\phi) : V_z \to V_{z+k} \text{ for all } z \in \mathbb{Z}\}$$

Because the associated graded representation is 0 all these degrees have to be $\geq 1$. Now,

$$m(V,\lambda) = min\{deg(\phi;V,\lambda) \mid \phi \in Q_a\}$$

So, the measure of instability $m(V,\lambda)$ of a one-parameter subgroup $\lambda$ with respect to a representation $V$ gives the minimal degree $k$ such that all linear maps $V(\phi)$ are filtration $+k$ maps for the descending filtration on $V$ determined by $\lambda$.
Let $X_*(G)$ be the set of all one-parameter subgroups of a reductive group $G$. If we fix a maximal torus $T \subset G$, it is well known that $X_*(G) = \cup_{g \in G} X_*(gTg^{-1})$. In our case, let $n = \sum_{i=1}^k \alpha_i$ the total dimension and consider a fixed embedding $GL(\alpha) \hookrightarrow GL_n(\mathbb{C})$

$$\begin{bmatrix} GL_{\alpha_1}(\mathbb{C}) & 0 & \dots & 0 \\ 0 & GL_{\alpha_2}(\mathbb{C}) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & GL_{\alpha_k}(\mathbb{C}) \end{bmatrix} \hookrightarrow GL_n(\mathbb{C})$$

In case our quiver has no oriented cycles it is advantageous to fix the ordering of the vertices such that it makes $Q$ into a directed quiver. With $I_i$ we will denote the interval of $[1,..,n]$ corresponding to the factor $\alpha_i$.
We fix the maximal torus $T_n$ of $GL(\alpha)$ (or of $GL_n(\mathbb{C})$),

$$\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{bmatrix}$$

with all $t_i \in \mathbb{C}^*$. It is well known that $X_*(T) \simeq \mathbb{Z}^n$ where the correspondence is given via the map

$$a = (a_1, \dots, a_n) \to \lambda_a(t) = \begin{bmatrix} t^{a_1} & & & \\ & t^{a_2} & & \\ & & \ddots & \\ & & & t^{a_n} \end{bmatrix}$$

Given a representation $V \in R(Q,\alpha)$, $m(V,\lambda_a)$ can be computed in the following way. Decompose $R(Q,\alpha)$ into weight-spaces for the action of the maximal torus $T_n$, then

$$R(Q,\alpha) = \bigoplus_{\pi_{ij} = \pi_j - \pi_i} R(Q,\alpha)_{\pi_{ij}}$$

where $\pi_{ij}$ occurs with a non-zero weightspace iff there is an arrow in $Q$ from $k \to l$ and $i \in I_k$, $j \in I_l$. Moreover, then the dimension of the weightspace equals the number of such arrows in $Q$. Accordingly, we can decompose any $V \in R(Q,\alpha)$ into weight-vectors

$$V = \sum_{\pi_{ij}} V_{ij}$$

Then, if $a = (a_1, ..., a_n)$ we have

$$m(V, \lambda_a) = min\{a_j - a_i \mid V_{ij} \neq 0\}$$

We have an inproduct and associated quadratic form on $X_*(T_n)$, namely, if $a, b \in \mathbb{Z}^n$ then we define

$$(\lambda_a, \lambda_b) = \sum_{i=1}^{n} a_i b_i \text{ and } q(\lambda_a) = \sum_{i=1}^{n} a_i^2$$

One verifies that this form is invariant under the action of the Weyl-group $S_n$ of $GL_n(\mathbb{C})$ and hence of the Weyl-group $S_{\alpha_1} \times ... \times S_{\alpha_k}$ of $GL(\alpha)$. Because of this we can extend the quadratic form to

$$X_*(GL(\alpha)) = \bigcup_{g \in GL(\alpha)} X_*(gT_ng^{-1})$$

by defining $q(g\lambda_a g^{-1}) = q(\lambda_a)$. We have now all the required data to define

**Definition 1** *A one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(\alpha)$ is* **optimal** *for a $V \in N(Q, \alpha)$ iff for all $\mu \in X_*(GL(\alpha))$ we have*

$$\frac{m(V, \lambda)}{\sqrt{q(\lambda)}} \geq \frac{m(V, \mu)}{\sqrt{q(\mu)}}$$

*For $V \in N(Q, \alpha)$ define $\Lambda_V$ to be the set of optimal $\lambda \in X_*(GL(\alpha))$ for $V$ which are primitive, that is, such that $\lambda$ is not of the form $b\mu$ with $b \geq 2$ a natural number and $\mu \in X_*(GL(\alpha))$.*

A key result due to Kempf [4] and Rousseau [15] (see also for example [16, §2]) states that $\Lambda_V$ is nonempty, there exists a parabolic subgroup $P(V) \subset GL(\alpha)$ such that for all $\lambda \in \Lambda_V$ we have $P(\lambda) = P(V)$ and if we take one $\lambda \in \Lambda_V$, then any other $\mu \in \Lambda_V$ is of the form $\mu = g\lambda g^{-1}$ with $g \in P(V)$.
For our purposes it is best to have a slightly different definition of optimal one-parameter subgroups or filtrations, due to Hesselink [2, §2]. Consider the set of **coweights**

$$X_{\mathbb{Q}}(GL(\alpha)) = \mathbb{Q} \otimes_{\mathbb{Z}} X_*(GL(\alpha))$$

then the quadratic form $q$ is extended to $X_{\mathbb{Q}}(GL(\alpha))$ and also the notion of measure of instability by

$$m(V, \mu) = \frac{m(V, b\mu)}{b}$$

if $b\mu \in X_*(GL(\alpha))$.

**Definition 2** *Let $V \in N(Q, \alpha)$ and define*

$$q^*(V) = inf\{q(\mu) \mid q \in X_{\mathbb{Q}}(GL(\alpha)) \text{ such that } m(V, \mu) \geq 1\}$$

*Then the optimal class of coweights corresponding to $V$ is*

$$\Gamma_V = \{\lambda \in X_{\mathbb{Q}}(GL(\alpha)) \mid m(V, \lambda) \geq 1 \text{ and } q(\lambda) = q^*(V)\}$$

Again, reformulating the Kempf-Rousseau result we have that $\Gamma_V$ is nonempty if $V \in N(Q, \alpha)$ and if $T$ is a torus of $GL(\alpha)$ then $\Gamma_V \cap X_{\mathbb{Q}}(T)$ consists of at most one element, see [2, lemma 2.3].

# 3 The Hesselink stratification of $N(Q,\alpha)$

In [2] W. Hesselink gave a stratification of the nullcone of a linear reductive group action on an affine variety using the Kempf-Rousseau theory of optimal one-parameter subgroups recalled above. In this section we will recall the main ideas in the special case of interest to us. For more details we refer the reader to [16] or to [6] and [13] for a more analytic approach linking the Hesselink stratification to the moment map.
We can use the optimal set of coweights $\Gamma_V$ of nilpotent representations $V \in N(Q,\alpha)$ to define

**Definition 3** *On $N(Q,\alpha)$ we define two equivalence relations*

1. *$V \approx V'$ iff $\Gamma_V = \Gamma_{V'}$, the equivalence class $[V] = \{V' \in N(Q,\alpha) \mid V \approx V'\}$ is called the* **blade** *of $V$.*
2. *$V \sim V'$ iff $\Gamma_V = \Gamma_{g\,V'}$ for some $g \in GL(\alpha)$. The equivalence class $GL(\alpha)[V] = \{V' \in N(Q,\alpha) \mid V \sim V'\}$ is called the* **stratum** *of $V$.*

Fix a representation $V \in N(Q,\alpha)$ and $\lambda \in \Gamma_V$. One defines

$$S(V) = \{V' \in R(Q,\alpha) \mid m(V',\lambda) \geq 1\}$$

It is clear from the Hilbert-criterium that $S(V)$ is a linear subspace of $N(Q,\alpha)$. We specialize the general results of [2, Prop. 4.2] to our case :

1. The blade $[V]$ of $V$ is the Zariski-open subset of $S(V)$ determined by
$$[V] = \{V' \in S(V) \mid q^*(V) = q^*(V')\}$$
2. The stratum $GL(\alpha)[V]$ of $V$ is the Zariski-open subset of the irreducible subset $GL(\alpha).S(V)$ of $N(Q,\alpha)$ determined by
$$GL(\alpha).[V] = \{V' \in GL(\alpha).S(V) \mid q^*(V) = q^*(V')\}$$

Further, the parabolic subgroup $P(\lambda)$ acts on $S(V)$ and hence on $GL(\alpha) \times S(V)$ by

$$p.(g,W) = (gp^{-1}, p.W)$$

Similarly, we have a $P(\lambda)$ action on the product $GL(\alpha) \times R(Q,\alpha)$ and the natural map

$$GL(\alpha) \times R(Q,\alpha) \rightarrow G/P \times R(Q,\alpha)$$

sending $(g,W)$ to $(gP(\lambda), g.W)$ is easily seen to be a geometric quotient for this action (that is, points of $G/P \times R(Q,\alpha)$ classify the $P(\lambda)$-orbits. We denote this quotient by $GL(\alpha) \times^{P(\lambda)} R(Q,\alpha)$ and see from the above that it is a trivial vector-bundle over the flag-variety $GL(\alpha)/P(\lambda)$ with fiber $R(Q,\alpha)$.
With $GL(\alpha) \times^{P(\lambda)} S(V)$ we denote the image of the $GL(\alpha) \times S(V)$ in this quotient. One then verifies that $GL(\alpha) \times^{P(\lambda)} S(V)$ is a vector-bundle over the flag-variety $G/P$ with typical fiber $S(V)$, in particular, it is a smooth variety of dimension $dim(GL(\alpha)) - dim(P(\lambda)) + dim(S(V))$.
We have the natural morphisms

$$\begin{array}{ccc} GL(\alpha) \times^{P(\lambda)} S(V) & \xrightarrow{\phi} & GL(\alpha).S(V) \\ \uparrow & & \uparrow \\ GL(\alpha) \times^{P(\lambda)} [V] & \xrightarrow{\phi'} & GL(\alpha).[V] \end{array}$$

and by [2, Th. 4.7] $\phi$ is birational and a resolution of singularities. Moreover, $\phi'$ is an isomorphism of $GL(\alpha)$-varieties.
Concluding, we have the following information about a stratum $GL(\alpha).[V]$ :

**Theorem 1 (Hesselink)** *With notations as above, we have :*

1. *The stratum $GL(\alpha).[V]$ is a smooth irreducible subvariety of $N(Q, \alpha)$*
2. *The closure of the stratum is $\overline{GL(\alpha).[V]} = GL(\alpha).S(V)$.*
3. *The desingularization of this closure is a vectorbundle of rank $dim(S(V))$ over the flag-variety $GL(\alpha)/P(\lambda)$.*
4. *$GL(\alpha)$-orbits in $GL(\alpha).[V]$ are in a natural one-to-one correspondence with $P(\lambda)$-orbits in the open subset $[V] \subset S(V)$.*

Hence, we have reduced the orbit-problem on the stratum $GL(\alpha).[V]$ to a smaller problem, namely that of the orbits of the smaller group $P(\lambda)$ on a smaller dimensional subspace $S(V)$ of $R(Q, \alpha)$. However, the main problem is that it is usually very difficult to determine the set of optimal one-parameter subgroup for a given representation. Therefore, the main objective of the present paper is to give another description of the strata.
By the results of Hesselink recalled above we know that the finite set of Hesselink-strata of $N(Q, \alpha)$ is in bijective correspondence with the set of $GL(\alpha)$-conjugacy classes of blades. Because $[V]$ is a dense open subset of $S(V)$, the set of blades is in injective correspondence with so called **saturated** subspaces of $N(Q, \alpha)$ (among which sets are the $S(V)$).
An arbitrary subset $X \subset N(Q, \alpha)$ is called uniformly unstable iff there exists a coweight $\lambda \in X_{\mathbb{Q}}(GL(\alpha))$ such that $m(x, \lambda) \geq 1$ for all $x \in X$. With $\Gamma(X)$ we denote the set of such coweights having the additional property that their norm is minimal among these. Then one defines the saturation of $X$ to be

$$S(X) = \{V \in R(Q, \alpha) \mid m(V, \lambda) \geq 1 \text{ for all } \lambda \in \Gamma(X)\}$$

and we call a subset saturated iff $X = S(X)$. By [2, lemma 2.8] the $S(V)$ defined above are saturated sets.
In order to determine the possible saturated subspaces of $N(Q, \alpha)$ we fix a maximal torus $T_n$ of $GL(\alpha)$, decompose $R(Q, \alpha)$ according to weightspaces wrt. $T_n$ and denote by $\Pi$ the set of weights with non-zero weightspace. As we mentioned before $\Pi = \{\pi_{ij} = \pi_j - \pi_i\}$ with $i \in I_k$ and $j \in I_l$ and there is an arrow in $Q$ from $k$ to $l$. If $R$ is a subset of $\Pi$, we call it unstable iff there exists a $\lambda \in X_{\mathbb{Q}}(T_n)$ such that $(\pi, \lambda) \geq 1$ for all $\pi \in R$, but then we know that there is a unique coweight $\delta = \delta(R)$ having the above property and the additional fact that its norm is minimal. With the saturation of $R$ we mean the set

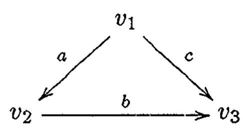
$$R^s = \{\pi \in \Pi \mid (\pi, \delta) \geq 1\}$$

By [2, Prop. 5.5] we have a bijective correspondence between the $GL(\alpha)$-conjugacy classes of saturated subsets of $N(Q, \alpha)$ and the conjugacy classes of the saturated subsets of $\Pi$ under the action of the Weyl-group $S_{\alpha_1} \times \ldots \times S_{\alpha_k}$. This correspondence associates to a saturated $R$ the subspace of all eigenvectors of $R(Q, \alpha)$ with weight $\pi \in R$ and to a saturated subset $X \subset N(Q, \alpha)$ the set of non-zero weights.
Clearly, a saturated subset $R \subset \Pi$ is determined by the coweight $\delta = \delta(R)$ and we will now describe the possible occurring coweights following [2, 6.8]. Let $\lambda = \sum a_i \pi_i$ be a coweight in $X_{\mathbb{Q}}(T_n)$, then we can partition $\{1, \ldots, n\}$ into a disjoint union of **segments** $I$ determine by the properties that there exist $p \leq q$ rational numbers such that

- $\{a_i \mid i \in I\} = \{x \in p + \mathbb{Z} : p \leq x \leq q\}$
- $I = \{i \in \{1, \ldots, n\} \mid a_i \in p + \mathbb{Z}, p - 1 \leq a_i \leq q + 1\}$

Call a coweight $\lambda$ balanced iff $\sum_{i \in I} a_i = 0$ for every of its segments $I$.
Repeating the argument of [2, Prop. 6.8.a] one shows that if $\delta$ determines a saturated subset $R(\delta) \subset \Pi$, then $\delta$ has to be a balanced coweight. However, unlike in the adjoint case of $GL_n$ considered by Hesselink, we can no longer assume that this coweight is dominant, that is $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$). We have to distribute the coweights over the vertices and use the Weyl-group action to assume that the local vertex-coweights are all dominant. This gives us a combinatorially determined list of possibly occurring coweights.
However, contrary to [2, Prop. 6.8.b] it is no longer true that every balanced coweight determines a saturated subset $R \subset \Pi$. For each coweight $\delta$ in the list obtained, we determine the associated set $R(\delta)$ and the norm of $\delta$. For fixed $R$ we only retain the $\delta$ such that $R \subset R(\delta)$ and $q(\delta)$ minimal. This gives us a list $L(Q, \alpha)$ of coweights determining the saturated subsets and hence the finite list of all saturated subspaces of $N(Q, \alpha)$.
The list $L(Q, \alpha)$ will be the combinatorial object underlying the Hesselink-stratification of the nullcone $N(Q, \alpha)$.

**Example 1** *For example, for $n = 4$ the balanced coweights and their norms are given in [11]. Now, consider the 'triangle' quiver*




*with $a$ arrows from 1 to 2, $b$ arrows from 2 to 3 and $c$ arrows from 1 to 3 and consider the dimension-vector $\alpha = (1, 2, 1)$. Then, for example,*

$$\delta = (0 \mid 1, -1 \mid 0)$$

*is a dominant (with respect to the smaller Weyl-group) balanced coweight with $q(\delta) = 2$ and $R(\lambda) = \{\pi_{12}, \pi_{34}\}$. However, this set is* **not** *saturated as we have another balanced coweight, $\mu = (-1/2 \mid 1/2, -1/2 \mid 1/2)$ with $q(\mu) = 1$ and $R(\mu) = \{\pi_{12}, \pi_{34}, \pi_{14}\}$.*
*By the above remarks we can compile the list of saturated subsets $R \subset \Pi = \{\pi_{12}, \pi_{13}, \pi_{24}, \pi_{34}, \pi_{14}\}$*

$L(Q, \alpha)$

| *name* | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $R(\delta)$ | $\parallel \delta \parallel$ |
|---|---|---|---|---|---|---|
| *1* | $-1$ | 0 | 0 | 1 | $\Pi$ | 2 |
| *2* | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\{\pi_{12}, \pi_{14}, \pi_{34}\}$ | 1 |
| *3* | $-\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\{\pi_{12}, \pi_{13}, \pi_{14}\}$ | $\frac{3}{4}$ |
| *4* | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{3}{4}$ | $\{\pi_{14}, \pi_{24}, \pi_{34}\}$ | $\frac{3}{4}$ |
| *5* | $-\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | $\{\pi_{12}, \pi_{13}\}$ | $\frac{2}{3}$ |
| *6* | $-\frac{2}{3}$ | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\{\pi_{12}, \pi_{14}\}$ | $\frac{2}{3}$ |
| *7* | $-\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | $\frac{2}{3}$ | $\{\pi_{14}, \pi_{34}\}$ | $\frac{2}{3}$ |
| *8* | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{2}{3}$ | $\{\pi_{24}, \pi_{34}\}$ | $\frac{2}{3}$ |
| *9* | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\{\pi_{12}\}$ | $\frac{1}{2}$ |
| *10* | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\{\pi_{14}\}$ | $\frac{1}{2}$ |
| *11* | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\{\pi_{34}\}$ | $\frac{1}{2}$ |
| *12* | 0 | 0 | 0 | 0 | $\emptyset$ | 0 |

Having the list $L(Q,\alpha)$ of possibly occurring strata in the Hesselink stratification of $N(Q,\alpha)$ we now have to determine which strata actually do occur. We will solve this problem in the next sections by assigning to each dominant coweight $\lambda \in L(Q,\alpha)$ another quiver problem.

## 4 Algorithmic description of the strata

Given a balanced coweight $\lambda \in L(Q,\alpha)$, it is easy to determine the associated saturated subset $S(\lambda)$ of $N(Q,\alpha)$ namely

$$\bigoplus_{\pi\in\Pi|(\pi,\lambda)\geq 1} R(Q,\alpha)_\pi$$

Substantially harder is the problem to determine when this saturated set determines a Hesselink stratum, that is, to determine whether $S(\lambda)$ is of the form $S(V)$ and if so to determine the Zariski-open blade $[V] \subset S(V)$.

In order to achieve these goals we have to recall results of F. Kirwan [6, 12.18-12.26] and P. Slodowy [16, Prop. 1].

Given a balanced coweight $\lambda$ for $R(Q,\alpha)$ we will denote with

- $Y_\lambda = \oplus_\pi R(Q,\alpha)_\pi$ with $(\pi,\lambda) \geq 1$
- $Z_\lambda = \oplus_\pi R(Q,\alpha)_\pi$ with $(\pi,\lambda) = 1$

then we have a natural projection map

$$Y_\lambda \to Z_\lambda$$

with vectorspaces as fibers. Further, we have subgroups of $GL(\alpha)$ with respect to $\lambda$, namely, $P(\lambda)$ the associated parabolic subgroup and $L(\lambda)$ its Levi-subgroup. The action of $GL(\alpha)$ on $R(Q,\alpha)$ induces an action of

- $P(\lambda)$ acts on $Y_\lambda$
- $L(\lambda)$ acts on $X_\lambda$

Now, there is a Zariski-open (but possibly empty) subset $X_\lambda^{ss}$ of points $x \in X_\lambda$ such that $\lambda \in \Gamma(x)$, that is, for which the one-parameter subgroup determined by $\lambda$ is optimal. Specializing [6, 12.24 & 12.26] to our setting we obtain.

**Lemma 2** *Let $\lambda$ be a balanced coweight in $L(Q,\alpha)$ and $S(\lambda)$ its associated saturated subspace of $N(Q,\alpha)$, then $S(\lambda)$ determines a Hesselink stratum, that is, $S(\lambda) = S(V)$ for some $V \in N(Q,\alpha)$ iff $X_\lambda^{ss} \neq \emptyset$. Moreover, we have,*

1. $[V] = \{y \in Y_\lambda \mid pr(y) \in X_\lambda^{ss}\}$
2. *$X_\lambda^{ss}$ is an $L(\lambda)$-stable subset of $X_\lambda$*
3. *the fibers of $pr : [V] \to X_\lambda^{ss}$ are vectorspaces*
4. *$[V]$ is a $P(\lambda)$-stable subset of $S(\lambda) = S(V)$*

Hence, remains the problem to determine when $X_\lambda^{ss}$ is non-empty and to describe it as precisely as possible. Following [6, 12.21 and 12.24] and[16, Prop 1] this problem reduces to describing the semi-stable points of the action of a certain subgroup $G \subset L(\lambda)$ on the vectorspace $Z_\lambda$. We recall from [11, Rem. 1] that one must change the definition of this subgroup in the definition of [16, p. 124] slightly.

Let $\mu$ be the uniquely determined element in $\mathbb{Z}^n$ determined by $\mathbb{N}.\lambda \cap \mathbb{Z}^n = \mathbb{N}\mu$, then $\mu = (\mu_1, \ldots, \mu_n)$ determines a character on the Levi group $L(\lambda)$

$$\chi_\lambda : L(\lambda) \longrightarrow \mathbb{C}^*$$

determined by sending an element $(g_k)_k \in L(\lambda)$ to

$$\prod det(g_k)^{n_k}$$

where $n_k$ is the constant number among the $\mu_i$ belonging to the $k$-th component of $L(\lambda)$. Recall that $L(\lambda) = \prod GL_{a_k}$ where the $a_k$ are the multiplicities of components of $\mu$ (or $\lambda$). With $G(\lambda)$ we denote the kernel of this character. Observe that it has the property of [16, p. 123] necessary for the proof of [16, Prop. 1] namely that $T_n \cap G(\lambda)$ has as charactergroup those $\chi \in \mathbb{Z}^n$ such that $(\chi, \mu) = 0$. But then [16, Prop. 1] asserts the following

**Lemma 3** *$X_\lambda^{ss}$ is the set of semi-stable points of $X_\lambda$ with respect to the action of the linear group $G(\lambda)$. Hence, $X_\lambda^{ss} \neq \emptyset$ iff there exists a non-constant $G(\lambda)$-invariant polynomial function on the affine space $X_\beta$.*

An alternative way to formulate this in terms of $L(\lambda)$ is as follows. $X_\lambda^{ss}$ is the open subset of $X_\lambda$ of points $x$ such that there is a semi-invariant function $f : X_\lambda \longrightarrow \mathbb{C}$ such that $f(x) \neq 0$ corresponding to the character $\chi_\lambda$. That is, such that for all $g \in L(\lambda)$ we have

$$g.f = \chi_\lambda(g)^k f$$

for some integer $k \in \mathbb{N}$.
We will now work all this out in our case and reduce all the above problems to the study of quiver-representations again, this time only of level-quivers.

**Definition 4** *A quiver $Q$ is said to be a* **level-quiver** *if we can partition its vertices in subsets $S_1, S_2, \ldots, S_l$ such that the only arrows in the quiver are from a vertex in $S_i$ to one in $S_{i+1}$.*

The next result is of crucial importance to this paper :

**Theorem 2** *Let $\lambda$ be a balanced coweight of $L(Q, \alpha)$. Then, the action of $L(\lambda)$ on $X_\lambda$ is the representation problem of a disjoint union of level quivers.*

**Proof :** By definition $X_\lambda$ consists of eigenspaces with weight $= 1$ so the structure of $X_\lambda$ is the product over the segments of $\lambda$. So assume that $(i_1, \ldots, i_z)$ are the numbers of $\pi_i$ such that $a_i$ belongs to the segment $I$ of $\lambda$. Moreover we can order them such that all $i_j$ with equal $a_j$ and lying in the same vertex $I_k$ are consecutive.

An investigation learns us that the action of $L(\lambda)$ on this part of $X_\lambda$ is that of a level quiver with $S_1$ the set of vertices from $Q$ having a $\pi_i$ with minimal $a = a_i$ in the segment, $S_1$ those with $a + 1$ etc. . The associated dimension vector is the number of equal $a_j$ lying in the same vertex of $Q$ and the number of arrows from a vertex belonging to a vertex $i$ from $Q$ to a vertex belonging to $j$ from $Q$ is the number of arrows from $i$ to $j$ in $Q$. □

Hence, if $\lambda \in L(Q, \alpha)$ the associated level quiver $Q_\lambda$ is a finite subquiver of the infinite quiver $\Gamma(Q)$ with vertices

$$\Gamma_v = Q_v \times \mathbb{Z}$$

and arrows : for each $\phi : t_\phi \longrightarrow h_\phi$ there are $\mathbb{Z}$ arrows in $\Gamma$

$$\phi_n : (t_\phi, n) \longrightarrow (h_\phi, n+1)$$

To find the dimension vector $\alpha_\lambda$ for this subquiver $Q_\lambda$ decompose $\lambda$ in its disjoint segments

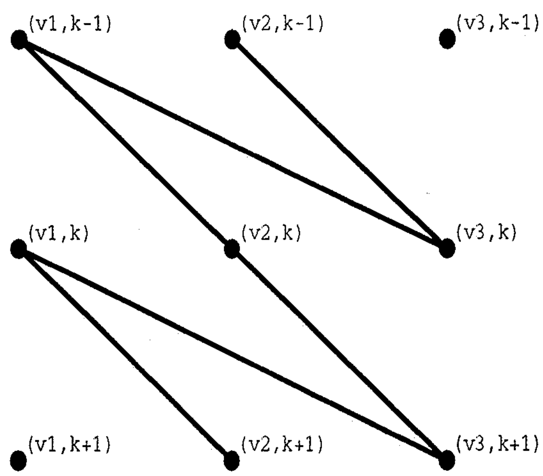
$$\{\underbrace{p_i, \ldots, p_i}_{a_0}, \underbrace{p_i+1, \ldots, p_i+1}_{a_1}, \ldots, \underbrace{p_i+k_i, \ldots, p_i+k_i}_{a_{k_i}}\}$$

and for each segment $i$ take a part of $\Gamma(Q)$ consisting of $k_i+1$ columns say starting at integer $s_i$ separated from the parts belonging to the other segments. The dimension vector for $(v, s_i+j)$ is the number of entries $k$ of $I_v$ such that $\lambda_k = p_i + j$.
Also verify that with these conventions we have that

$$L(\lambda) = GL(\alpha_\lambda)$$

and the character $\chi_\lambda$ is determined by associating to the vertex $(v, s_i+j)$ the number $n_{ij} = d.(p_i+j)$ where $d$ is the least common multiple of the numerators of the $p_k$'s determining the segments of $\lambda$.

**Example 2** *(continuation of example 1)*
*In this case the infinite quiver $\Gamma(Q)$ has following shape*




**Remark 1** *The erroneous definition of the character of [16, p. 123] would give a much better representation theoretic character to this level quiver situation. Recall that for a quiver $Q'$ the Schofield character is defined via*

$$\chi_\gamma(\beta) = \epsilon(\beta, \gamma) - \epsilon(\gamma, \beta)$$

*where $\epsilon$ is the Euler inner product*

$$\epsilon(\gamma, \beta) = \sum_{v \in Q'_v} \gamma_v \beta_v - \sum_{a \in Q'_a} \gamma_{ta} \beta_{ha}$$

*whose properties are closely linked to the description of Schur roots and canonical; decomposition of dimension-vectors [14]. The distinction between the Schofield-character and the required character $\chi_\lambda$ is responsible for the apparent lack of natural representation theoretic interpretation of the Hesselink stratification of $N(Q, \alpha)$ (at this moment) as well as for many of the computational complexities which otherwise could be side-stepped by using reflection functors.*

Above we have seen that the Hesselink stratum corresponding to the balanced coweight $\lambda \in L(Q,\lambda)$ is nonempty if and only if $X_\lambda^{ss} \neq \emptyset$ and this is the open subset of the level-quiver representations $R(Q_\lambda,\alpha_\lambda)$ for which a semi-invariant corresponding to the character $\chi_\lambda$ does not vanish.
One of the advantages of reducing to this quiver situation is that we can view points of $X_\lambda$ as objects in the Abelian category of all representations of $Q_\lambda$, that is, the category of modules over the path algebra $\mathbb{C}Q_\lambda$. Therefore, we can associate to the character $\chi_\lambda$ (which is determined by the integers $(n_{ij})$ defined above) an additive function on the Grothendieck group of the path algebra

$$\theta_\lambda : K_0(\mathbb{C}Q_\lambda) \longrightarrow \mathbb{Z}$$

which is determined by sending the class of a representation of dimension-vector $\beta = (b_{ij})$ to $\sum n_{ij} b_{ij}$.
Using the analogy with vector bundles on projective varieties, A. King [5] defines a representation $V$ of $Q_\lambda$ to be $\theta$-semistable (for any additive function $\theta$ on the Grothendieck group) if $\theta(V) = 0$ and every sub-representation $V' \subset V$ satisfies $\theta(V') \geq 0$. Similarly, a representation $V$ is called $\theta$-stable if the only subrepresentations $V'$ with $\theta(V) = 0$ are 0 and $V$. Using [5, Prop.3.1] we then have

**Proposition 1** *$X_\lambda^{ss}$ is the open subset of $R(Q_\lambda,\alpha_\lambda)$ which are $\theta_\lambda$-semistable.*

Hence, in order to verify whether $x \in X_\lambda = R(Q_\lambda,\alpha_\lambda)$ lies in $X_\lambda^{ss}$ it suffices to know the dimension vectors of all subrepresentations of $x$ and verify that their values under $\theta_\lambda$ are $\geq 0$. If $X_\lambda^{ss} \neq$ it is open in $R(Q_\lambda,\alpha_\lambda)$ and it suffices to know the dimension vectors of subrepresentations of a general representation.
Precisely this problem had t be addressed by A. Schofield [14] in his solution of some conjectures of V. Kač on the generic decomposition. Recall that V. Kač showed [3] that the dimension vectors of indecomposable quiver-representations form an infinite root system with associated generalized Cartan matrix the symmetrization of the Ringel form or the Euler inner product defined in remark 1. This form encodes a lot of information on representations. If $V$ resp. $W$ are representations of dimension-vector $\alpha$ resp. $\beta$ then

$$\epsilon(\alpha,\beta) = \dim \mathrm{Hom}(V,W) - \dim \mathrm{Ext}(V,W)$$

For fixed dimension vector $\beta$ and any quiver $Q$, there is an open subset of representations $V$ in $R(Q,\beta)$ such that the dimension vectors of its indecomposable components are constant, say $\beta_i$. Then,

$$\beta = \beta_1 + \ldots + \beta_l$$

is called the canonical decomposition of $\beta$ into Schur roots $\beta_i$ (Schur roots are roots $\gamma$ such that there is an open set of indecomposable representations in $R(Q,\gamma)$).
Kač asked for a combinatorial description of the set of Schur roots and of the canonical decomposition in terms of the Ringel form. Solutions to these problems were presented by A. Schofield [14] and depend heavily on being able to describe the dimension vectors of sub-representations of a general representation. Denote with

$$\beta \hookrightarrow \alpha$$

that a general representation of dimension-vector $\alpha$ has a sub-representation of dimension-vector $\beta$. Schofield gave an inductive way to find the dimension-vectors of these generic sub-representations using the Ringel form

$$\beta \hookrightarrow \alpha \quad \text{iff} \quad \underset{\beta' \hookrightarrow \beta}{Max} - \epsilon(\beta',\alpha-\beta) = 0$$

For example, the description of the Schur roots [14, Th.6.1] is then : $\alpha$ is Schur iff for all $\beta \hookrightarrow \alpha$ we have $r(\beta, \alpha) - r(\alpha, \beta) > 0$. A combinatorial description of the canonical decomposition was also given in [14].
These facts enable us to give the promised algorithmic description of the occurring strata :

**Theorem 3** *For a balanced coweight $\lambda \in L(Q, \alpha)$, $\mu_\lambda$ is an optimal one-parameter subgroup for a point $x \in N(Q, \alpha)$, or equivalently, the Hesselink stratum corresponding to $\lambda$ is non-empty if and only if for the associated level quiver $Q_\lambda$ all $\beta \hookrightarrow \alpha_\lambda$ satisfy $\theta_\lambda(\beta) \geq 0$.*
*The Hesselink stratum consists of those $x \in Y_\lambda$ for which the projection $pr(x) \in X_\lambda = R(Q_\lambda, \alpha_\lambda)$ is a $\theta_\lambda$-semistable representation.*

In view of Schofield's inductive procedure to determine the dimension-vectors of generic subrepresentations, the first part allows us for given quiver $Q$ and dimension vector $\alpha$ to compile the sublist of $L(Q, \alpha)$ corresponding to the actually occurring strata in the Hesselink stratification. The second part allows us to describe these strata for we can determine $X_\lambda^{ss}$ by considering the 'bad' dimension vectors $\gamma < \alpha_\lambda$ such that $\theta_\lambda(\gamma) < 0$ and then $X_\lambda^{ss}$ is the complement of those representations in $R(Q_\lambda, \alpha_\lambda)$ having a subrepresentation of dimension-vector $\gamma$ (which is a closed condition and easy to express).

**Example 3** *(continuation of example 1)*
*With the conventions introduced before, we will for each of the 12 types give the dimension vectors $\alpha_\lambda$ and the characters $\theta_\lambda$ as well as the restrictions imposed on the number of arrows $a, b$ and $c$ in the quiver $Q$ for the stratum to be non-empty (we will assume $abc \neq 0$). In this table a starred line means a break between two segment component quivers.*

*level-quiver data*

| *name* | $\alpha_\lambda$ | | | $\theta_\lambda$ | | | *cond* |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | -1 | 0 | 0 | - |
| | 0 | 2 | 0 | 0 | 0 | 0 | |
| | 0 | 0 | 1 | 0 | 0 | 1 | |
| 2 | 1 | 1 | 0 | -1 | -1 | 0 | - |
| | 0 | 1 | 1 | 0 | 1 | 1 | |
| 3 | 1 | 0 | 0 | -3 | 0 | 0 | $a \geq 2$ |
| | 0 | 2 | 1 | 0 | 1 | 1 | |
| 4 | 1 | 2 | 0 | -1 | -1 | 0 | $b \geq 2$ |
| | 0 | 0 | 1 | 0 | 0 | 3 | |
| 5 | 1 | 0 | 0 | -2 | 0 | 0 | $a \geq 2$ |
| | 0 | 2 | 0 | 0 | 1 | 0 | |
| | * | * | * | * | * | * | |
| | 0 | 0 | 1 | 0 | 0 | 0 | |
| 6 | 1 | 0 | 0 | -2 | 0 | 0 | - |
| | 0 | 1 | 1 | 0 | 1 | 1 | |
| | * | * | * | * | * | * | |
| | 0 | 1 | 0 | 0 | 0 | 0 | |
| 7 | 1 | 1 | 0 | -1 | -1 | 0 | - |
| | 0 | 0 | 1 | 0 | 0 | 2 | |
| | * | * | * | * | * | * | |
| | 0 | 1 | 0 | 0 | 0 | 0 | |
| 8 | 0 | 2 | 0 | 0 | -1 | 0 | $b \geq 2$ |
| | 0 | 0 | 1 | 0 | 0 | 2 | |
| | * | * | * | * | * | * | |
| | 1 | 0 | 0 | 0 | 0 | 0 | |
| 9 | 1 | 0 | 0 | -1 | 0 | 0 | - |
| | 0 | 1 | 0 | 0 | 1 | 0 | |
| | * | * | * | * | * | * | |
| | 0 | 1 | 1 | 0 | 0 | 0 | |
| 10 | 1 | 0 | 0 | -1 | 0 | 0 | - |
| | 0 | 0 | 1 | 0 | 0 | 1 | |
| | * | * | * | * | * | * | |
| | 0 | 2 | 0 | 0 | 0 | 0 | |
| 11 | 0 | 1 | 0 | 0 | -1 | 0 | - |
| | 0 | 0 | 1 | 0 | 0 | 1 | |
| | * | * | * | * | * | * | |
| | 1 | 1 | 0 | 0 | 0 | 0 | |

*level-quiver data*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *12* | *1* | *2* | *1* | *0* | *0* | *0* | - |

# References


[1] Ch. Bessenrodt, L. Le Bruyn, Stable rationality of certain $PGL_n$-quotients, Invent. Math. 104 (1991) 179-199

[2] W. Hesselink, Desingularization of varieties of nullforms, Invent. Math. 55 (1977) 141-163

[3] V. Kač, Infinite root systems, representations of graphs and invariant theory, Invent. Math. 56 (1980) 57-92

[4] G. Kempf, Instability in invariant theory, Ann. Math. 108 (1978) 299-316

[5] A.D. King, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford 45 (1994) 515-530

[6] F.C. Kirwan, "Cohomology of quotients in symplectic and algebraic geometry", Princeton Math. Notes 31 (1984)

[7] H. Kraft, "Geometrische Methoden in der Invariantentheorie", Aspects of Mathematics D1, Vieweg (1984)

[8] H. Kraft, Ch. Riedtmann, Geometry of representations of quivers, in "Representations of algebras" London Math. Soc. Lecture Notes 116 (1986) 109-145

[9] L. Le Bruyn, C. Procesi, Semisimple representations of quivers, Trans. AMS 317 (1990) 585-598

[10] L. Le Bruyn, A. Schofield, Rational invariants of quivers and the ring of matrix-invariants, in "Perspectives in Ring theory" NATO-ASI Ser. C-233 (1988) 21-30

[11] L. Le Bruyn, Nilpotent representations, UIA-preprint 95- , to appear

[12] D. Mumford, J. Fogarty, "Geometric Invariant Theory" (2nd edition) Springer (1981)

[13] L. Ness, A stratification of the null-cone via the moment map, Amer. J. Math. 106 (1984) 1281-1329

[14] A. Schofield, General representations of quivers, Proc. LMS 65 (1992) 46-64

[15] G. Rousseau, Immeubles sphériques et théorie des invariants, C.R. Acad. Sci. Paris 286 (1978) 247-250

[16] P. Slodowy, Die theorie der optimalen einparametergruppen für instabile vektoren, in "Algebraic transformation groups and invariant theory" DMV Seminar 13, Birkhäuser (1989)