Central Singularities of Quantum Spaces

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Abstract

Let A be a positively graded, connected affine $\mathbb{C}$-algebra generated in degree one which is a finite module over a central subring R. Let $\mathcal{A}$ be the structure sheaf over $Proj(R)$ and **Spec** $\mathcal{Z}$ the central $Proj$. If $injdim(A) \leq \infty$, then **Spec** $\mathcal{Z}$ has singularities unless the ramification locus of $\mathcal{A}$ is pure of codimension one. If $gldim(A) \leq \infty$, then the codimension ≥ 2 parts of the ramification locus of $\mathcal{A}$ and the singular locus of **Spec** $\mathcal{Z}$ coincide.

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Central Singularities of Quantum Spaces

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Abstract

Let A be a positively graded, connected affine $\mathbb{C}$-algebra generated in degree one which is a finite module over a central subring R. Let $\mathcal{A}$ be the structure sheaf over $Proj(R)$ and **Spec** $\mathcal{Z}$ the central $Proj$. If $injdim(A) \leq \infty$, then **Spec** $\mathcal{Z}$ has singularities unless the ramification locus of $\mathcal{A}$ is pure of codimension one. If $gldim(A) \leq \infty$, then the codimension ≥ 2 parts of the ramification locus of $\mathcal{A}$ and the singular locus of **Spec** $\mathcal{Z}$ coincide.

1 Introduction

Throughout this paper, A will be a positively graded affine $\mathbb{C}$-algebra

$$A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$$

which is connected (that is, $A_0 = \mathbb{C}$), and generated as $\mathbb{C}$-algebra in degree one. Moreover, we will be primarely be interested in the case when A is a finite module over a central graded affine subalgebra R. Examples of current interest include quadratic quantum algebras at roots of unity and Sklyanin- or Odeskii-Feigin algebras associated to a torsion point on an elliptic curve, [13] and [20].

The structure sheaf of the graded R-module A is a sheaf of coherent algebras $\mathcal{A}$ over $Y = Proj(R)$. The center $\mathcal{Z}$ of $\mathcal{A}$ is a sheaf of coherent commutative $\mathcal{O}_Y$-algebras and we can associate to it a commutative scheme $X =$ **Spec** $\mathcal{Z}$ together with an affine morphism

$$f : X = \mathbf{Spec}\ \mathcal{Z} \rightarrow Y = Proj(R)$$

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such that $f_* \mathcal{O}_X \simeq \mathcal{Z}$ and f_* induces an equivalence between $\mathcal{O}_Y$-Modules and $\mathcal{Z}$-Modules, [5, Ex. II.5.17]. **Spec** $\mathcal{Z}$ is called the central *Proj* associated to A.

For 3-dimensional Sklyanin algebras M. Artin proved [2] that **Spec** $\mathcal{Z} \simeq \mathbb{P}^2$. However, for 4-dimensional Sklyanin algebras it was shown by S.P. Smith and J. Tate [17] that **Spec** $\mathcal{Z} \not\simeq \mathbb{P}^3$ as **Spec** $\mathcal{Z}$ has singularities.

In this paper we wil study the singularities of the central *Proj* in some generality. The main result is

Theorem 1 *1. If* $injdim(A) \leq \infty$, *then* **Spec** $\mathcal{Z}$ *either has singularities or the ramification locus of* $f^*\mathcal{A}$ *is of pure codimension one.*

2. If $gldim(A) \leq \infty$, *the codimension* ≥ 2 *part of the ramification locus of* $f^*\mathcal{A}$ *coincides with the codimension* ≥ 2 *part of the singular locus of* **Spec** $\mathcal{Z}$.

Sklyanin algebras have finite global dimension [20] and their ramification loci are pure of codimension one in dimension 3 and contain components of codimension two and three in dimension 4. So, the theorem is compatible with the results recalled above.

In proving the theorem we obtain a result which may be of independent interest. It follows from combining an old result of W. Vasconcelos [22] with recent work of J.T. Stafford and J. Zhang [19].

Proposition 1 *1. If* $injdim(A) \leq \infty$, *then* $f^*\mathcal{A}$ *is a sheaf of Cohen-Macaulay modules over* **Spec** $\mathcal{Z}$.

2. If $gldim(A) \leq \infty$, *then* $f^*\mathcal{A}$ *is a sheaf of Cohen-Macaulay maximal orders over* **Spec** $\mathcal{Z}$ *which is itself a normal Cohen-Macaulay scheme.*

The study of the ramification locus of $f^*\mathcal{A}$ turns out to be equivalent to the study of fat point modules of A of certain multiplicity. It is known that fat point modules are closely tied to finite dimensional simple A-modules [15] and that the latter ones can be studied via invariant theory, see for example [1], [14] or [6]. If our graded algebra A is a quotient of a Noetherian graded algebra of finite global dimension having Hilbert series $(1-t)^{-k}$ for some k, we give the following description of fat point modules

Proposition 2 *With assumption as above, let* $Mod_m(A)$ *be the variety of m-dimensional representations of A. There is a natural* $PGL_m \times \mathbb{C}^*$*- action on it, the* PGL_m*-component encoding isomorphism as A-modules and the* $\mathbb{C}^*$*-component coming from the gradation on A. Then,*

1. *If an orbit has a finite isotropy group of order l, it determines a fat point module of A of multiplicity* $\frac{m}{l}$.
2. *If A is a finte module over a central subalgebra, every fat point module of A arises in this way.*

In the last section we illustrate these results on a class of Auslander regular Clifford algebras associated to an n-dimensional linear system of quadrics in $\mathbb{P}^{n-1}$. The theory of quadratic forms describes their finite dimensional simple modules and hence via proposition 2 their fat point modules and so via the main theorem the ramification of the central $Proj$'s.

Proposition 3 *If the n-dimensional linear system of quadrics in $\mathbb{P}^{n-1}$ has no base point, then the associated Clifford algebra is a quadratic Auslander-regular algebra of gldim $= n$ satisfying the Cohen-Macaulay property. Moreover, if n is even the central Proj has singularities.*

This result ties up with work of M. Van den Bergh [21].

2 Singularities and ramification

Throughout this section, $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$ will be a positively graded affine $\mathbb{C}$-algebra generated in degree 1 which is a finite module over a graded central subalgebra R.

If A has finite global or injective dimension, it follows from recent work of J.T. Stafford and J. Zhang [19] that A has excellent homological properties. To be precise, we say that A

- satisfies the **Auslander-condition** if for every finitely generated left A-module M, every integer $j \geq 0$ and every (right) A-submodule N of $Ext^j_A(M,A)$ we have that $j(N) \geq j$. Here, $j(N)$ is the grade number of N which is the least integer i such that $Ext^i_A(M,A) \neq 0$.
- satisfies the **Cohen-Macaulay property** if for every finitely generated left A-module M we have the equality : $GKdim(M)+j(M) = GKdim(A)$ where $GKdim$ denotes the Gelfand-Kirillov dimension.

The Stafford-Zhang result asserts that if $gldim(A) = n+1$ (resp. $injdim(A) = n+1$), A satisfies the Auslander-condition and the Cohen-Macaulay property. In short, we say that A is Auslander-regular (resp. Auslander-Gorenstein).

Let $Y = Proj(R)$ the projective variety associated to R. It is covered by affine open subsets $X(c) = Spec(S_c)$ for every homogenous element $c \in R$ where $S_c = (R_c)_0$ the degree zero part of the localization of R at the homogenous multiplicative set $\{1, c, c^2, \ldots\}$. The structure sheaf $\mathcal{A}$ of the graded R-module A has sections $\Gamma(X(c), \mathcal{A}) = (A_c)_0$ which we will denote by Λ_c. Λ_c is an order with center Z_c (which contains $(Z(A)_c)_0$ but may be larger) and the Z_c's determine a coherent sheaf $\mathcal{Z}$ of $\mathcal{O}_Y$-algebras. By [5, Ex. II.5.17] we can associate to $\mathcal{Z}$ a commutative scheme $X = \mathbf{Spec}\ \mathcal{Z}$ together with an affine morphism

$$f : X = \mathbf{Spec}\ \mathcal{Z} \rightarrow Y = Proj(R)$$

such that $f_*\mathcal{O}_X \simeq \mathcal{Z}$. The commutative scheme $X = \mathbf{Spec}\ \mathcal{Z}$ is called the central *Proj* of A, see for example [2] and [17]. Using the above notation we have :

Lemma 1 *If A is Auslander-Gorenstein (resp. Auslander-regular) satisfying the Cohen-Macaulay property, then so does $\Lambda_c = \Gamma(X(c), \mathcal{A})$.*

Proof : A_c is a strongly graded ring, that is, $(A_c)_{-1}(A_c)_1 = (A_c)_0 = \Lambda_c$. For, let $deg(c) = k$ and write $c = \sum_i d_i l_i$ with $d_i \in A_{k-1}$ and $l_i \in A_1$ which is possible as A is generated in degree 1. Then, $1 = c^{-1}.c = \sum_i (c^{-1}d_i)l_i \in (A_c)_{-1}(A_c)_1$ and as the $(A_c)_{-1}(A_c)_1$ is an ideal of $(A_c)_0 = \Lambda_c$ this proves the claim.

Using [12, Th.I.3.4] we know that taking degree zero parts induces an equivalence of categories

$$(-)_0 : A_c - gr \rightarrow \Lambda_c - mod$$

from graded left A_c-modules to left Λ_c-modules (the inverse functor is $A_c \otimes_{\Lambda_c} -$).

Finally, all required properties first go up under central localization at c to A_c and as they can be expressed in category theoretical terms the above equivalence induces them on Λ_c. □

The following result rests on work of W. Vasconcelos [22]. Let Λ be an algebra which is a finite module over a local commutative ring S with maximal ideal m. Clearly, there are only a finite number of maximal ideals $\{P_1, ..., P_k\}$ of Λ lying over m. We call Λ **moderated Gorenstein** if and only if

- $injdim(\Lambda) = Kdim(\Lambda) = n < \infty$.
- For all $1 \leq i \leq k$ we have that $\max\{p : Ext^p_\Lambda(\Lambda/P_i, \Lambda) \neq 0\} = n$.

Vasconcelos' result states that if Λ is a moderated Gorenstein algebra over a local domain S, Λ is a Cohen-Macaulay module over its center. For more details we refer to [22] or [10, IV.1].

Proposition 4 *Let A be a positively graded affine algebra generated in degree 1 which is a finite module over a central subring. Then,*

1. *If $injdim(A) = n + 1$, then $f^*\mathcal{A}$ is a sheaf of Cohen-Macaulay modules over its center* **Spec** $\mathcal{Z}$.
2. *If $gldim(A) = n + 1$, then $f^*\mathcal{A}$ is a sheaf of maximal Cohen-Macaulay orders over* **Spec** $\mathcal{Z}$ *which is a normal Cohen-Macaulay scheme.*

Proof : Of course, it suffices to verify these claims locally in an open affine subset $X(c)$ of Y. We know already that Λ_c is an Auslander- Gorenstein ring (resp. Auslander-regular if $gldim < \infty$) satisfying the Cohen-Macaulay property. The same facts hold for $\Lambda_x = \Lambda_c \otimes_{S_c} (S_c)_x$ for every point $x \in X(c)$. From the Cohen-Macaulay property we deduce that

$$j(\Lambda_x/P_i) = n$$

for each of the maximal ideals P_i of Λ_x over m_x. As $Ext^k_{\Lambda_x}(\Lambda_x/P_i), \Lambda_x) = 0$ for all $k \geq n+1$, we see that Λ_c is moderated Gorenstein and therefore locally a Cohen-Macaulay module over its center.

By results of T. Levasseur [11] and J.T. Stafford [18] we know that if A is Auslander-regular it is a domain and a maximal order. Whereas the maximal order property goes up to A_c it is in general not true that the degree zero part of a strongly graded maximal order is maximal, see [10, p.105] for an example. However, here we can use the fact that Λ_c is a tame order by [10, IV.1.7] and tame domains are maximal orders. In particular, the center is a normal domain and the Cohen-Macaulayness follows from [10, IV.1.6]. □

We want to apply the foregoing to the study of the singular locus of **Spec** $\mathcal{Z}$. In order to do so we need an extra condition on the ramification locus $\mathcal{A}$.

$\mathcal{A}$ is a sheaf of orders in a central simple algebra D which we assume to be of dimension e^2 over its center, that is, e is the p.i.-degree of the sheaf of algebras $\mathcal{A}$. Globalizing well known results we can view **Spec** $\mathcal{Z}$ as parametrizing isoclasses of e-dimensional semi-simple representations of $\mathcal{A}$. Hence, there is an open set $U \subset$ **Spec** $\mathcal{Z}$ consisting of points corresponding to simple e-dimensional representations of $\mathcal{A}$. U is called the Azumaya-locus of $\mathcal{A}$ because for every point $x \in U$ we have that $\mathcal{A}_x$ is an Azumaya algebra over its center. Its complement V is called the ramification locus of $\mathcal{A}$.

If Λ is an order over its center S which is a normal domain, we say that Λ is a reflexive Azumaya algebra provided Λ_p is Azumaya for every height one prime p of S. Or geometrically, the ramification locus of Λ in $Spec(S)$ has codimension ≥ 2. A reflexive Azumaya algebra is Azumaya if and only if it is a projective S-module, see for example [9]. Combining these facts with the foregoing results we have :

Proposition 5 *Let A be a positively graded affine algebra, generated in degree 1 which is a finite module over a central subring R. Let $c \in R$ be an homogenous element such that the ramification locus V_c of Λ_c is of codimension at least 2.*

1. *If $injdim(A) = n+1$, then $V_c \subset f^{-1}(X(c)) \cap (\mathbf{Spec}\ \mathcal{Z})_{sing}$.*
2. *If $gldim(A) = n+1$, then $V_c = f^{-1}(X(c)) \cap (\mathbf{Spec}\ \mathcal{Z})_{sing}$.*

Proof : Let $x \in f^{-1}(X(c))$ be a smooth point of **Spec** $\mathcal{Z}$, then $\mathcal{A}_x$ is a Cohen-Macaulay module over its center $\mathcal{Z}_x$ which is a regular domain. Hence, $\mathcal{A}_x$ is a projective $\mathcal{Z}_x$-module. By assumption, Λ_x is also a reflexive Azumaya algebra over the normal domain $\mathcal{Z}_x$ hence it must be Azumaya. Hence $x \notin V_c$ proving the first part.

Conversely, if $x \notin V_c$ then $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{Z}_x$. By assumption we know that $gldim(\mathcal{A}_x) = n$, but then also $gldim(\mathcal{Z}_x) = n$. Hence, x is a smooth point of **Spec** $\mathcal{Z}$. □

From this, the main theorem follows immediately.

3 Fat point-modules and invariants

In order to study the singular locus of **Spec** $\mathcal{Z}$ it is important to determine the ramification locus of the sheaf of algebras $\mathcal{A}$. That is, we have to determine the open subset U of **Spec** $\mathcal{Z}$ corresponding to simple e-dimensional representations of $\mathcal{A}$ where e is the p.i.-degree of $\mathcal{A}$. Here, the notion of fat point-module over the positively graded algebra A comes into the picture.

A **fat point-module** F of A is a 1- critical graded A-module. If A is a quotient of a graded Noetherian algebra of finite global dimension with Hilbert series $(1-t)^{-k}$ for some k, then the dimension of the homogenous parts F_i are constant for i sufficiently large. This constant number is then called the multiplicity of F.

For every homogenous central element $c \in R$ we can use the fact that A_c is strongly graded as in the foregoing section to see that there is a functorial bijection between the finite dimensional simple $(A_c)_0 = \Lambda_c$-modules and the fat points for A which have no c-torsion. Under this bijection a fat point module F corresponds to $(F[c^{-1}])_0$ and the multiplicity of the fat point equals the dimension of the corresponding simple module.

Therefore, the study of the open set U is equivalent to the study of fat point modules of A of multiplicity e.

There is a close though subtle connection between fat point modules for A and finite dimensional simple representations of A. Much of what follows is at least implicit in [15, §1 and 2] but we give a slightly different account.

Let $Mod_m(A)$ be the affine variety parametrizing m-dimensional A-modules. If $X_0, \ldots, X_n$ is a basis for A_1, then $Mod_m(A)$ is the subvariety of $M_n(\mathbb{C})^{\oplus n+1}$ determined by the entries of the matrices we obtain by substituting generic m by m matrices for the X_i in the defining equations of A, see for example [6, II.2.7]. There is a natural PGL_m-action on $Mod_m(A)$, the orbits of which are the isomorphism classes of m-dimensional A-modules. Because A is a graded algebra, there is an additional $\mathbb{C}^*$ action on $Mod_m(\mathbb{C})$ induced by sending X_i to λX_i for $\lambda \in \mathbb{C}^*$.

Proposition 6 *Let A be a connected affine graded $\mathbb{C}$-algebra generated in degree one and assume that A is a quotient of a Noetherian graded algebra B with Hilbert series $(1-t)^{-k}$ and finite global dimension. If a $PGL_m \times \mathbb{C}^*$-orbit in $Mod_m(A)$ has finite isotropy group of order l, then this orbit determines a fat point-module of A of multiplicity $\frac{m}{l}$. Conversely, we recover the m-dimensional* *representations in the orbit as the finite dimensional simple quotients of the fat point-module.*

Proof : Observe first that the isotropy group can only be finite for simple m-dimensional A-modules in which case the isotropy group has the form $1 \times \mu_l$ where μ_l is the subgroup of $\mathbb{C}^*$ of l-th roots of unity. Let S be a simple m-dimensional A-module in the orbit, then $P = Ann(S)$ is a maximal ideal of A. Then, using [12] the largest graded ideal P_g contained in P is a maximal graded ideal of A. Thus, A/P_g is an order of $Kdim$ one and hence inverting a central homogenous element we obtain its ring of quotients $Q^g(A/P_g)$ which is a graded matrix ring

$$Q^g(A/P_g) \simeq M_m(\mathbb{C}[x,x^{-1}])(e_1.0, e_2.1, \ldots, e_l.(l-1))$$

(that m is the p.i.-degree of A/P_g is [12, I.2.10]) where x is a central element of degree l and where e_i is the number of base vectors in degree $0 \leq i \leq l$ in the graded vectorspace V over the graded field $\mathbb{C}[x,x^{-1}]$ such that $END(V)$ is the graded matrix ring. For more details on this we refer again to [12, I.4,I.5 and II.6].

The positive part of V, $F = V_{\geq 0}$ is a graded A-module with periodic Hilbert series with periods $(e_1, e_2, \ldots, e_l)$. As A s a quotient of a graded regular algebra we now that this Hilbert series must eventually become constant, say of multiplicity e. Therefore, all $e_i \neq 0$ and are all equal to e which is then m/l. The fact that F determines a fat point module of A of multiplicity e and that all simple A-quotients of it lie in the orbit of S follow (or see [15, §1,2]. □

In case A is a finite module over a central subalgebra, every fat point-module of A has finite dimensional simple quotients and hence the above argument can be applied and we obtain proposition 2. The method also enables us to relate the p.i.-degree of A (say N) and that of $\mathcal{A}$, e. If d is the greatest common divisor of the degrees of central homogenous elements in A, then $e = \frac{N}{d}$. Finally, observe that the special form of graded quotient rings of maximal graded ideals of A in case it is a quotient of a suitable ring B can sometimes be used as a criterium to show that no such B exists.

In the important special case that A is (a quotient of) a quadratic algebra, the foregoing proposition can be rephrased in quiver-terms as

Corollary 1 *A fat point-module of a quadratic algebra A (as above) of multiplicity e corresponds to the $\mathbb{C}^*$-orbit of a simple representation of dimension vector $(e, e, \ldots, e)$ of a circular quiver of length l on the vertices $\{v_1, \ldots, v_l\}$ with arrows*

$$\begin{array}{ccccc} & \overset{X_0^{(i-1)}}{\longrightarrow} & & \overset{X_0^{(i)}}{\longrightarrow} & \\ \cdots \bullet_{v_{i-1}} & \vdots & \bullet_{v_i} & \vdots & \bullet_{v_{i+1}} \cdots \\ & \overset{X_n^{(i-1)}}{\longrightarrow} & & \overset{X_n^{(i)}}{\longrightarrow} & \end{array}$$

with relations of the form

$$\sum a_{kl} X_k^{(i-1)} X_l^{(i)} = 0$$

whenever $\sum a_{kl} X_k X_l$ *is a quadratic relation of* A.

4 An example

In this section we will introduce a large class of Auslander-regular algebras which are finite modules over their centers and study their fat point-module strata.

Let $R = \mathbb{C}[Y_1, \ldots, Y_n]$ be the commutative polynomial ring and let $A(M)$ be the Clifford algebra over R associated to a symmetric $n \times n$ matrix M with all its entries linear terms in the Y_i. That is, let

$$M = M_1 Y_1 + \ldots + M_n Y_n$$

with all $M_i \in M_n(\mathbb{C})$ symmetric matrices, then $A(M)$ is generated by X_i, Y_i $1 \leq i \leq n$ with defining relations

$$X_i X_j + X_j X_i = \sum_{k=1}^{n} (M_k)_{ij} Y_k$$

and the Y_i central. Observe that one defines a gradation on $A(M)$ by giving $deg(X_i) = 1$ and $deg(Y_i) = 2$.

The algebra $A(M)$ is associated to an n-dimensional linear system

$$\mathcal{Q} = \mathbb{C} Q_1 + \ldots + \mathbb{C} Q_n$$

of quadrics $Q_i \subset \mathbb{P}^{n-1}$ where Q_i is the zero set of $\sum_{kl} (M_i)_{kl} Y_k Y_l$. A base point of the linear system of quadrics $\mathcal{Q}$ is a common zero of the quadrics Q_i $1 \leq i \leq n$. Observe that if we fix a basis for $\mathcal{Q}$ we can associate to the system a symmetric $n \times n$ matrix M with linear entries in R and hence the Clifford algebra $A(M)$.

Proposition 7 *Let* $\mathcal{Q} = \mathbb{C} Q_1 + \ldots + \mathbb{C} Q_n$ *be an* n*-dimensional linear system of quadrics in* $\mathbb{P}^{n-1}$ *and* $A(M)$ *the corresponding Clifford algebra. Equivalent are,*

1. $A(M)$ *is a quadratic Auslander-regular algebra of* $gldim = n$ *satisfying the Cohen-Macaulay property.*

2. *The system of quadrics* $\mathcal{Q}$ *has no base points in* $\mathbb{P}^{n-1}$.

Proof : The fact that $A(M)$ is generated in degree one and has $gldim(A(M)) = n$ if and only if $\mathcal{Q}$ has no base points is merely translating [3, §1] to current terminology. As $A(M)$ is a finitely generated module over the central subring R the result follows from the Stafford-Zhang result [19]. □

Remark 1 *As a generic n-dimensional system of quadrics in $\mathbb{P}^{n-1}$ has no base points we recover a result of M. Van den Bergh [21] who proved Auslander regularity of $A(M)$ for generic M by a deformation argument. The above result has the advantedge in applications of clarifying what we mean by a 'generic' M.*

We will now study the fat point-module strata of the coherent sheaf of algebras $\mathcal{A}$ over $Y = Proj(R)$ which is a weighted projective $n-1$ space. To a point $(y_1, \ldots, y_n) \in Y$ corresponds a maximal graded ideal p of R such that $R/p \simeq \mathbb{C}[y]$ with y a degree two element. We can specialize the symmetric matrix M at p and obtain a symmetric matrix $M(p)$ over $\mathbb{C}[y]$. The relevant stratification of X will be given by the sets

$$Y_k = \{p \in Y \mid rkM(p) = k\}$$

The following result is based on standard results on quadratic forms over fields and principal ideal domains, see [7, Ch.5],[4, §7.4] and [8, §II.2 and §II.4] for more details.

Proposition 8 *Over every $p \in Y$ there is a unique maximal graded prime ideal P of $A(M)$ such that $A(p) = A(M)/P \otimes_{\mathbb{C}[y]} \mathbb{C}[y, y^{-1}]$ has the following structure*

- *If $p \in X_{2l}$, then $A(p) \simeq M_{2^l}(\mathbb{C}[y, y^{-1}])(2^{l-1}.0, 2^{l-1}.1)$, that is, $A(p)$ is a graded central simple algebra of p.i.-degree 2^l with center $\mathbb{C}[y, y^{-1}]$.*
- *If $p \in X_{2l-1}$, then $A(p) \simeq M_{2^{l-1}}(\mathbb{C}[x, x^{-1}])(2^{l-1}.0)$, that is, $A(p)$ is a graded central simple algebra of p.i.-degree 2^{l-1} with center $\mathbb{C}[x, x^{-1}]$ where x is a degree 1 element.*

Proof: Over $\mathbb{C}[y, y^{-1}]$ every symmetric $n \times n$ matrix of rank k is equivalent to one of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is an invertible symmetric $k \times k$ matrix with determinant $\in \mathbb{C}^* y^k$. Hence, every prime ideal P of $A(M)$ lying over p corresponds to a prime factor of the Clifford algebra of the quadratic form associated to A over $\mathbb{C}[y, y^{-1}]$. If $k = 2l$ is even such Clifford algebras are orders in central simple algebras of p.i.-degree 2^l. If $k = 2l-1$ we use the fact that $det(A)$ is not a square in $\mathbb{C}[y, y^{-1}]$ and hence the Clifford algebra is an order in a central simple algebra of p.i.-degree 2^{l-1} with center $\mathbb{C}[y, y^{-1}, \sqrt{det(A)}]$. Clearly, over $\mathbb{C}$ all graded central simple algebras are graded matrix rings and as we have a central element of degree ≤ 2 they are easy to determine : □

Combining these observations with the results of the foregoing sections we have

Proposition 9 *Let Q be an n-dimensional linear system of quadrics in $\mathbb{P}^{n-1}$ without base point. If $A(M)$ is the associated Clifford algebra over R. Then,*

1. *A point $p \in Y = Proj(R)$ corresponds to a fat point-module of $A(M)$ of multiplicity 2^{l-1} if $p \in Y_k$ and $k = 2l$ or $k = 2l - 1$.*

2. *The ramification locus for $f^*\mathcal{A}$ is $f^{-1}(Y_{n-2})$ if n is even and is $f^{-1}(Y_{n-1})$ if n is odd.*

3. *If n is even, the singular locus of* **Spec** $\mathcal{Z}$ *is $f^{-1}(Y_{n-2})$.*

Remark 2 *If $n = 4$, then points on X_4 or X_3 determine fat point-modules of multiplicity 2. X_2 which is the zero set of all 3×3 minors of M determines the fat point-modules of multiplicity 1 (i.e. the point-variety of $A(M)$). For generic M one verifies that $codim(X_2) = 3$ so there can only be a finite number of points and* **Spec** $\mathcal{Z}$ *is a rational threefold with a finite number of isolated singularities. The fact that $A(M)$ has a finite number of point-modules was obtained by M. Van den Bergh [21] who further shows that for generic M there are precisely 20 points.*

Sklyanin algebras at torsion points seem to have some properties in common with Clifford algebras. Therefore, it is tempting to conjecture that the central *Proj* of Sklyanin algebras will be singular in even dimensions with a component of the singular locus equal to the zero set of the two central elements in degree $\frac{n}{2}$. We leave this as a suggestion fur further work.

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