Conformal sl_2 Enveloping Algebras

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Conformal $\mathbf{sl}_2$ Enveloping Algebras

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Abstract

A 3-parameter family of deformations of $U(\mathbf{sl}_2)$ is introduced. The finite dimensional simple representations are studied using non-commutative projective geometry.

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1 Introduction

In this paper we introduce and study a 3-parameter family of deformations of $U(\mathbf{sl}_2)$ which we call conformal $\mathbf{sl}_2$ enveloping algebras. These algebras $U_{abc}(\mathbf{sl}_2)$ have defining relations

$$\begin{aligned} XY - aYX &= Y \\ ZX - aXZ &= Z \\ YZ - cZY &= bX^2 + X \end{aligned} \tag{1}$$

with $ac \neq 0$. We argue that all sensible $U(\mathbf{sl}_2)$ deformations coming from physics will belong to this class up to isomorphism (and possibly twists). The reason is that they are characterized by the requirement that they should encode 'duality' in some conformal field theory (whence the name) and that they should have the same excellent homological properties as $U(\mathbf{sl}_2)$ (which we obtain from the requirement that the associated graded algebra is Auslander regular).

From a purely ring theoretical point of view these algebras are interesting because their homogenizations $H_{abc}(\mathbf{sl}_2)$ are quadratic Auslander regular algebras. Hence, they can be studied using the recently developed tools from non-commutative projective geometry, see for example [3], [4],[5],[27]. On the other hand, these algebras can also be studied along more traditional lines as they are iterated Ore extensions, they have Borel-like sub algebras and if $a \neq 1$ they have normal elements in degree one and two giving a Cartan-like description.

Therefore, these algebras provide test objects to appreciate the relative strengths and weaknesses of the two approaches. The test-problem which we want to solve is directly inspired from the connection with physics : classify all finite dimensional simple representations of $H_{abc}(\mathbf{sl}_2)$ (or, virtually equivalently, of $U_{abc}(\mathbf{sl}_2)$).

As in the case of quantum enveloping algebras there are two types of algebras in our family. For generic values of the parameters, the $H_{abc}(\mathbf{sl}_2)$ behave like enveloping algebras, their simple representations should be found by using highest weight modules and we expect a discrete family of solutions. For special cases, namely when a and c are roots of unity, $H_{abc}(\mathbf{sl}_2)$ is a finite module over its center. In this case, we expect for a specific dimension (the p.i.-degree of the algebra) a continuous family of simple representations which are usually studied via the module varieties and invariant theory.

As we will see below, these standard methods can indeed be applied effectively in this case. For the generic case, it is not difficult to use the Borel-like sub algebras to prove that there are at most two simple representations in each dimension and to characterize them. In the special cases, one can use the iterated Ore description to compute the p.i.-degree and to show that the center is rational. However, giving generators and relations of the center as well as describing all simple representations is substantially harder. The reason is that if one iterates the usual arguments to study an Ore extension, one is often forced to localize and therefore one obtains 'only' the generic answer rather than a full description. In handling these problems non-commutative geometry has proved to be a rather effective tool over the last couple of years.

The basic idea of non-commutative projective geometry is that in low dimensions certain quadratic algebras A are determined by (commutative) geometric data : a subscheme of $\mathbb{P}^n = \mathbb{P}(A_1^*)$ which is called the point-variety of the algebra A together with an automorphism on it. In the good cases one recovers the quadratic relations of A as the set of two-tensors vanishing on the graph of this automorphism. This means that all information about A (such as the classification of the finite dimensional simple representations) must be dictated by this purely commutative algebraic geometric data. Remains the problem of extracting the desired ring theoretical information from the geometry. Another advantedge of this geometrical approach is that it gives a uniform treatment to both the generic and the special cases. The dychotomy in the representation theory is caused by the order of the automorphism (infinite resp. finite).

As our algebras $U_{abc}(\mathbf{sl}_2)$ are deformations of $U(\mathbf{sl}_2)$, the approach is based on homogenized $\mathbf{sl}_2$, which was studied in [15]. As all simple $\mathbf{sl}_2$ representations are quotients of Verma modules we want a geometric description of them (or rather of their homogenizations). They turn out to be precisely the line modules of homogenized $\mathbf{sl}_2$, that is, the graded modules of $H(\mathbf{sl}_2)$ with same Hilbert series as a line in $\mathbb{P}^3$, namely $(1-t)^{-2}$. Moreover, these line modules can be visualized as all the lines lying in a pencil of quadrics in $\mathbb{P}^3$ determined by the point-variety and the automorphism. Hence, we have a class of graded modules (the line modules) which we can characterize using the geometric data and which have the property that all finite dimensional simple representations are quotients of such modules.

The idea is then for sufficiently nice 4-dimensional quadratic algebras to view their line modules as deformed Verma (or highest weight) modules, to classify them using the underlying geometry and to find all their finite dimensional simple quotients. This point of view was advocated (and used with great success) by S.P. Smith and J. Staniskis [27] and [26] in their study of the 4-dimensional Sklyanin algebras. Hence, we begin by studying the point-variety of $H_{abc}(\mathbf{sl}_2)$ and give a geometric description of its line modules by using joint work with S.P. Smith and M. Van den Bergh [12].

Whereas the line modules are sufficient to find all simple representations in the generic cases, they will not give us all simples in the cases when $H_{abc}(\mathbf{sl}_2)$ is finite over its center. The reason is that we expect a 3-parameter family of fat points (homogenizations of finite dimensional simples) in these cases, meaning that the line modules should span $\mathbb{P}^3$. However, we show that the lines usually only cut out a dimension two subvariety.

The good news is that this draws our attention to sub-families of algebras where the line modules indeed do span $\mathbb{P}^3$. We find that there are precisely two 1-parameter classes. One corresponding to the algebras obtained by E. Witten [31] (by specifying the field theory to be 3-dimensional Chern-Simons gauge theory), the other is a twist of $U_q(\mathbf{sl_2})$ in Drinfeld-Jimbo notation. This gives a Hopf-free approach to the best $\mathbf{sl_2}$ (quasi) quantum groups : they are those conformal $\mathbf{sl_2}$ enveloping algebras having enough Verma modules to obtain all finite dimensional simples.

The bad news is that we have to extend the Smith-Staniskis approach to handle the remaining (p.i.) cases. Observe that we cannot simply wipe them under the carpet. For example, in Wittens approach, the parameter $q = exp(\frac{2\pi i}{k})$ where $1/k$ is the coupling constant of the theory. However, in order to get a consistent quantum field theory one has to impose by [29] a quantization condition namely that $k \in \mathbb{Z}$. Hence, the p.i.-cases are not the odd cases out, they are the only relevant ones for physics !

As mentioned above the p.i.-cases occur when both a and c are roots of unity so each of them belong to certain two-dimensional sub-family $C_{ij} = \{H_{abc}(\mathbf{sl_2}) \mid a^i c^j = 1\}$. For each of these classes we can show that all finite dimensional simple representations are either quotients of line modules or of higher degree plane curve modules with degree determined by i and j. Therefore, replacing line modules by plane curve modules one retains the essence of the Smith-Staniskis approach. The complete description of these curve modules from the underlying geometry as well as their simple quotients will be the topic of the second part of this paper. To give the reader an idea of the method we include the easiest case of the Witten algebras (where line modules still suffice) and give a concrete realization of all finite dimensional simple representations. I hope that the results of this paper may motivate people to investigate higher degree modules for Auslander regular algebras.

As there seems to be no bound on the degree of the curves required, it is tempting to conjecture that there should exist deformations of the 4-dimensional Sklyanin algebras with similar properties. The examples found by J.T. Stafford [28] correspond to the case of plane elliptic curves. Further,it has not escaped my notice that one can use the algebras $H_{abc}(\mathbf{sl_2})$ as building blocks to construct large parameter families of conformal enveloping algebras for any semi-simple Lie algebra. We leave this as suggestions for further work.

2 From physics to algebras

It has been argued by several people (for example, M. Gerstenhaber in [8] and E. Witten in [31]) that a deformation of an enveloping algebra can be viewed as a method to break the total Lie group symmetry down to a maximal torus. Remains the problem of finding suitable deformations of $U(\mathbf{g})$ preferably coming from physical situations.

In this section we will briefly recall E. Witten's approach based on duality in conformal field theory, see [31], to obtain deformations of $U(\mathbf{sl_2})$. This gives a 7-parameter family of algebras. Clearly, not all of these algebras will have desirable properties like being a domain, having finite global dimension or a

Poincaré-Birkhoff-Witt basis. In the classical $U(\mathbf{sl_2})$ case these properties follow from the fact that the associated graded algebra is a commutative polynomial ring.

Whereas all these deformations will be filtered we can no longer assume that the associated graded algebra will be commutative. However, we can investigate the deformations having a quadratic Auslander-regular algebra for associated graded algebra. As in the classical case all good properties can then be lifted from the associated graded to the filtered deformation. The deformations of $U(\mathbf{sl_2})$ which have a regular associated graded algebra will be called **conformal** $\mathbf{sl_2}$ **enveloping algebras**. The main result of this section is :

Theorem 1 *Up to (filtration preserving) isomorphism there is a* 3*-parameter family of conformal* $\mathbf{sl_2}$ *enveloping algebras* $U_{abc}(\mathbf{sl_2})$ *with defining relations*

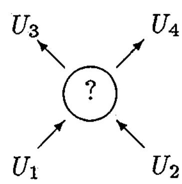
$$\begin{array}{rcl} XY - aYX & = & Y \\ ZX - aXZ & = & Z \\ YZ - cZY & = & bX^2 + X \end{array} \qquad (2)$$

If $ac \neq 0$, $U_{abc}(\mathbf{sl_2})$ *is Auslander-regular of global dimension* 3 *and satisfies the Cohen-Macaulay property.*

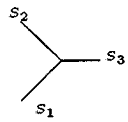
2.1 The physical vector space $\mathcal{H}_{\Sigma;U_i}$

What follows is meant as an introduction for ring theorists to some of the material of [29],[30] and [31]. We skate over some of the finer physical points (for example, taking limits for large spin) and the reader should consult the above papers for full enlightenment. For the heuristic origin of duality in string theory the reader may wish to read the first chapter of [9].

We want to investigate the easiest scattering process in some conformal field theory with underlying symmetry $\mathbf{sl_2}$, namely



two particles with spin U_1 and U_2 (that is, simple $\mathbf{sl_2}$ representations of dimension $2U_i + 1$, we will always identify a simple representation with its highest weight in $\frac{1}{2}I\!N$.) interact in some way and produce outgoing particles with spin U_3 and U_4. We can view this black-box as the surface of the Riemann sphere Σ with 4 punctures and representations $U_i, 1 \leq i \leq 4$. The actual process can then be represented by a Feynman graph inside this sphere with open edges at the four special points and all internal vertices represent admissible 3-valent couplings. That is, around each vertex the Feynman graph looks like

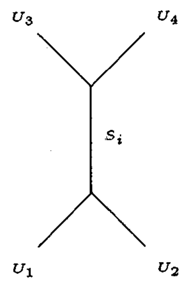


with the S_i simple representations of $\mathbf{sl_2}$ and (for example) S_3 a direct summand of $S_1 \otimes S_2$. The Clebsch-Jordan decomposition, which asserts that for $U \geq V$ the tensor product decomposes as

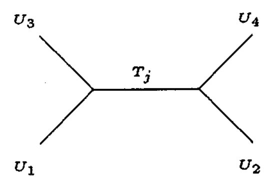
$$U \otimes V = (U - V) \oplus (U - V + 1) \oplus \dots \oplus (U + V)$$

allows to make this requirement explicit.

To each admissible physical process as above inside the sphere we associate a vector in the physical vector space $\mathcal{H}_{\Sigma;U_i}$ corresponding with the input-output data. The way to do this is to take the Feynman path integral over all $\mathbf{sl_2}$-connections modulo gauge transformations. Even if we dont (want to) know what this sentence means, we can work with this vectorspace because "duality" in conformal field theory gives us the dimension of this space as well as sets of base vectors. The dimension of $\mathcal{H}_{\Sigma;U_i}$ is equal to the number of common simple factors of $R_1 \otimes R_2$ and $R_3 \otimes R_4$. Moreover, a specific basis for $\mathcal{H}_{\Sigma;U_i}$ is given by the vectors v_i corresponding to the admissible Feynman graphs :

$v_i =$ 

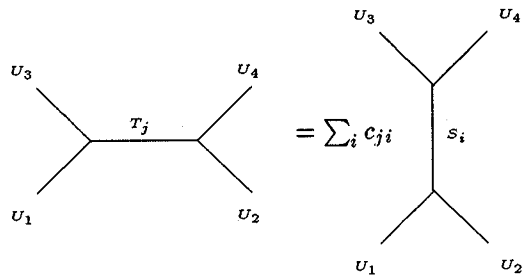
But we could have coupled the particles differently, say :

$w_j =$ 

As a matter of fact, all possible w's again form a basis for $\mathcal{H}_{\Sigma;U_i}$, the matrix describing the base change $\{v_i\} \rightarrow \{w_j\}$ is called the braiding matrix of the theory and its entries can be found by computing expectation values of certain processes, see for example [22]. Anyway, we obtain a linear relation in $\mathcal{H}_{\Sigma;U_i}$

$$w_j = \sum c_{ij} v_i$$

for $c_{ij} \in \mathbb{C}$. Such relations should be viewed similar to skein relations in knot theory. That is,



means that if we have Feynman graphs Γ_j resp. Λ_i that coincide outside the sphere Σ and are w_j resp. v_i inside Σ, then we have the linear relation

$$< \Gamma_j > = \sum_i c_{ji} < \Lambda_i >$$

between their expectation values.

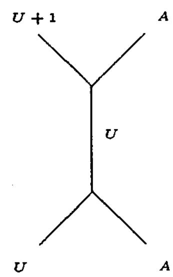
2.2 Deformations of $U(\mathrm{sl}_2)$

Let us apply the foregoing to find the deformations of $U(\mathbf{sl_2})$ of [31, §3]. As the enveloping algebra should encode couplings with the gauge bosons which live in the adjoint representation A (spin 1) it is natural to consider the vectorspaces $\mathcal{H}_{\Sigma;U_i}$ with two of the representations U_i equal to A.

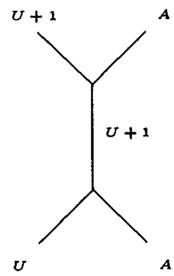
Consider the vector space $\mathcal{H} = \mathcal{H}_{U,A,A,U+1}$ with four charges in representations U, A, A and $U+1$. Because

$$\begin{aligned} U \otimes A &\simeq (U-1) \oplus U \oplus (U+1) \\ (U+1) \otimes A &\simeq U \oplus (U+1) \oplus (U+2) \end{aligned} \tag{3}$$

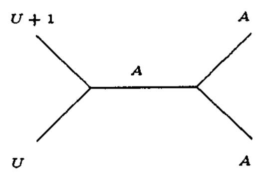
the dimension of $\mathcal{H}$ is two and suitable basis elements correspond to the graphs



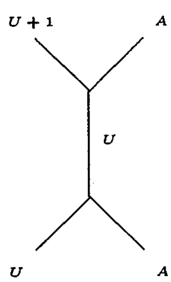
and

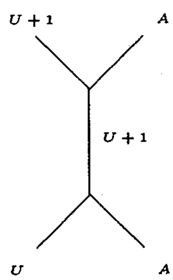


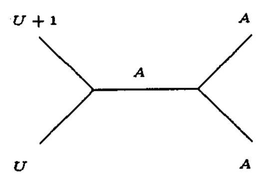
If we couple the particles differently :



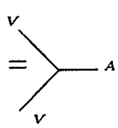
the corresponding element in $\mathcal{H}$ must be a linear combination of the base vectors. That is, there exist $u, v \in \mathbb{C}$ not both zero such that

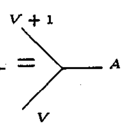
u 

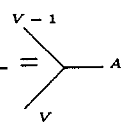
$+v$ 

$=$ 

If we use the raising and lowering operators in $\mathbf{sl}_2$:

$T_0 =$ 

$T_+ =$ 

$T_- =$ 

the above relation given a commutation relation

$$uT_+T_0 + vT_0T_+ = T_+$$

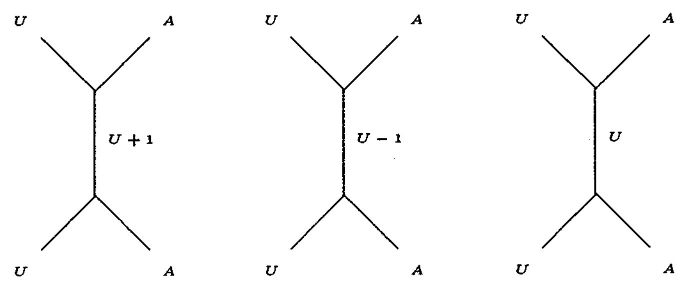
and as at least one of u or v is nonzero, we can rewrite this as

$$T_0T_+ = \alpha T_+T_0 + \beta T_+$$

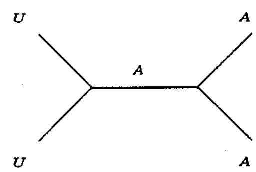
In a similar way, using the vector space $\mathcal{H}_{\Sigma;U,A,A,U-1}$ (which again is two-dimensional) we obtain the commutation relation

$$T_-T_0 = \varepsilon T_0T_- + \zeta T_-$$

The third commutation relation follows from the vector space $\mathcal{H} = \mathcal{H}_{\Sigma;U,A,A,U}$ which is three dimensional with basis vectors corresponding to the graphs



If we couple the particles differently, then the vector corresponding to the graph



can be expressed as a linear combination of the base vectors above, leading to a commutation relation of the form

$$T_+T_- - \gamma T_-T_+ = \delta T_0^2 + \epsilon T_0$$

Returning to more ring theoretical notation where $X = T_0, Y = T_+$ and $Z = T_-$, these three commutation relations define a 7-parameter family of deformations of $U(\mathbf{sl_2})$:

$$\begin{aligned} XY - \alpha YX &= \beta Y \\ YZ - \gamma ZY &= \delta X^2 + \epsilon X \\ ZX - \varepsilon XZ &= \zeta Z \end{aligned} \qquad (4)$$

Clearly, in a specific physical theory (such as three dimensional Chern-Simons gauge theory considered in [31]) all the coefficients will be functions of the coupling constant and they can be calculated by evaluating certain expectation values.

If one allows couplings with spin $\frac{1}{2}$ particles one obtains in an analogous way deformations of the super enveloping algebra $U(\mathbf{osp}(1,2))$, see [6].

2.3 Conformal $\mathbf{sl}_2$ enveloping algebras

Recall that a finitely generated filtered ring R of finite global dimension n is said to

- be **Auslander regular** if for every finitely generated left R-module M, every integer $j \geq 0$ and every (right) R-submodule N of $Ext^j_R(M,R)$ we have that $j(N) \geq j$. Here, $j(N)$ is the grade number of N which is the least integer i such that $Ext^i_R(M,R) \neq 0$.
- satisfy the **Cohen-Macaulay property** if for every finitely generated left R-module M we have the equality : $GKdim(M) + j(M) = n$ where $GKdim$ denotes the Gelfand-Kirillov dimension.

For enveloping algebras these properties follow from the fact that the associated graded ring (with respect to the standard filtration) is a commutative polynomial ring and hence satisfies all these homological requirements. As our deformations are filtered algebras it is natural to define :

Definition 1 *A deformation $A(\alpha,..,\zeta)$ of $U(\mathbf{sl}_2)$ with defining relations (4) is said to be a* **conformal $\mathbf{sl}_2$ enveloping algebra** *if the associated graded algebra $gr(A(\alpha,..,\zeta))$ is a three dimensional Auslander regular quadratic algebra.*

Characterizing all conformal $\mathbf{sl}_2$ enveloping algebras implies solving two subproblems :

1. Is the quadratic algebra $B(\alpha,\gamma,\delta,\varepsilon)$ with defining relations

$$\begin{aligned} XY - \alpha YX &= 0 \\ YZ - \gamma ZY &= \delta X^2 \\ ZX - \varepsilon XZ &= 0 \end{aligned} \qquad (5)$$

   Auslander regular of dimension 3 ?

2. Is the natural morphism

$$B(\alpha,\gamma,\delta,\varepsilon) \rightarrow gr(A(\alpha,..,\zeta))$$

   an isomorphism ? That is, is the quadratic algebra B really the associated graded algebra of A ?

Both problems have been studied in great generality. M. Artin and W. Schelter classified 3-dimensional quadratic Auslander regular algebras in [2]. Later, M. Artin, J. Tate and M. Van den Bergh gave a more geometrical classification of these algebras in [4] and [5]. They also proved that the 'regular' algebras studied in [2] are actually Auslander regular and satisfy the Cohen-Macaulay property.

The second problem of recognizing when a filtered algebra has an Auslander regular quadratic algebra as its associated graded algebra was solved in [12] where such algebras were called central extensions.

For each of the two subproblems we will briefly recall the results and perform the required computations in the case of interest here.

In the Artin-Schelter approach $B(\alpha, .., \varepsilon)$ will be Auslander-regular if we can put the defining relations in standard form. This means that we can choose a basis for the 3-dimensional vectorspace of quadratic relations say f_1, f_2, f_3 such that there is a 3×3 matrix M with linear entries in $B(\alpha, .., \varepsilon)$ and an invertible matrix $Q \in GL_3(\mathbb{C})$ such that for $F = (f_1, f_2, f_3)^t$ and $x = (X, Y, Z)^t$ we have the following matrix equations over the free algebra $\mathbb{C} < X, Y, Z >$:

$$F = M.x \quad and \quad x^t.M = (Q.F)^t$$

For $B(\alpha, \gamma, \delta, \varepsilon)$ we can take

$$\begin{array}{rcl} f_1 & = & YZ - \gamma ZY - \delta X^2 \\ f_2 & = & ZX - \varepsilon XZ \\ f_3 & = & \frac{\gamma}{\alpha}(XY - \alpha YX) \end{array} \tag{6}$$

in which case the matrix becomes

$$M = \begin{pmatrix} -\delta X & -\gamma Z & Y \\ Z & 0 & -\varepsilon X \\ -\gamma Y & \frac{\gamma}{\alpha} X & 0 \end{pmatrix}$$

and hence $x^t.M$ is

$$\begin{pmatrix} YZ - \gamma ZY - \delta X^2 \\ -\gamma XZ + \frac{\gamma}{\alpha} ZX \\ XY - \epsilon YX \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\gamma}{\alpha} & 0 \\ 0 & 0 & \frac{\alpha}{\gamma} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

if and only if $\alpha = \varepsilon$. Therefore, we have

Lemma 1 *The quadratic algebra $B(\alpha, \gamma, \delta, \varepsilon)$ is Auslander regular of dimension 3 iff $\alpha.\gamma \neq 0$ and $\alpha = \varepsilon$.*

We now turn to our second sub problem. This problem was solved in [12] for arbitrary three generated filtered algebras A with B a 3-dimensional Auslander-regular algebra with defining quadratic relations f_i in standard form. Assume the defining equations of A are of the form

$$g_i = f_i + l_i + \alpha_i$$

where l_i (resp. α_i) are the linear (resp. constant) parts, then a necessary and sufficient condition for the $gr(A) = B$ and hence for A to be Auslander-regular of dimension 3 is by [12, Thm 3.1.3] that we can find $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3$ that are a solution to the following system of linear equations in $B_1^{\otimes 2}$ (resp. B_1 and $\mathbb{C}$)

$$\begin{aligned} \sum_i \gamma_i f_i &= \sum_i (x_i l_i - l_i x_i^*) \\ \sum_i \gamma_i l_i &= \sum_i \alpha_i (x_i - x_i^*) \\ \sum_i \gamma_i \alpha_i &= 0 \end{aligned} \tag{7}$$

where $x^* = (X^*, Y^*, Z^*) = Q^t.(X, Y, Z)$.

In the case of $\mathbf{sl_2}$-deformations with $\alpha = \varepsilon$ and $\alpha\gamma \neq 0$ the defining equations are brought in standard form as above and we have

$$\alpha_i = 0, \quad l_1 = -\epsilon X, \quad l_2 = -\zeta Z, \quad l_3 = -\frac{\beta\gamma}{\alpha} Y$$

$$(X^*, Y^*, Z^*) = (X, \frac{\gamma}{\alpha} Y, \frac{\alpha}{\gamma} Z)$$

As $\beta\epsilon\zeta \neq 0$ (if one of them is zero, it would give a nontrivial linear relation between the base vectors of one of the physical vector spaces $\mathcal{H}$) we see that the equations can only be satisfied if all $\gamma_i = 0$ and then the first equation gives the extra requirement that $\beta = \zeta$. We therefore obtain

Lemma 2 *If $\alpha\beta\gamma\epsilon\varepsilon\zeta \neq 0$, then the deformation $A(\alpha, .., \zeta)$ is a conformal $\mathbf{sl_2}$ enveloping algebra iff $\alpha = \varepsilon$ and $\beta = \zeta$. That is , the defining relations are*

$$\begin{aligned} YZ - \gamma ZY &= \delta X^2 + \epsilon X \\ ZX - \alpha XZ &= \beta Z \\ XY - \alpha YX &= \beta Y \end{aligned} \tag{8}$$

The theorem now follows as we can force $\beta = \epsilon = 1$ using the homotheties $X \to \lambda X$ and $Y \to \mu Y, Z \to \mu Z$. Observe that, with notations as in the theorem, $U_{101}(\mathbf{sl_2}) = U(\mathbf{sl_2})$.

3 From algebras to geometry

Rather than studying modules over the filtered algebra $U_{abc}(\mathbf{sl_2})$ it will turn out to be better to study graded modules over its homogenization $H_{abc}(\mathbf{sl_2})$ with respect to an additional central element T. That is, $H_{abc}(\mathbf{sl_2})$ is the quadratic graded algebra generated by $V = \mathbb{C}X + \mathbb{C}Y + \mathbb{C}Z + \mathbb{C}T$ and with defining relations

$$(g_1) : 0 = YZ - cZY - bX^2 - XT$$

$$
\begin{aligned}
(g_2): 0 &= ZX - aXZ - ZT \\
(g_3): 0 &= \frac{c}{a}(XY - aXY - YT)
\end{aligned}
\tag{9}
$$

and T central. From the result of the foregoing section and [12] we have

Proposition 1 *The homogenized conformal enveloping algebra $H = H_{abc}(\mathbf{sl_2})$ is a quadratic Auslander regular algebra of dimension 4 satisfying the Cohen-Macaulay property provided $ac \neq 0$. In particular, H is a domain, a maximal order and has Hilbert series $\mathcal{H}(H,t) = \sum_{i=0}^{\infty} dim(H_n) = (1-t)^{-4}$.*

In going from $U_{abc}(\mathbf{sl_2})$ to $H_{abc}(\mathbf{sl_2})$ we do not loose any information as the homogenization functor

$$h : U_{abc}(\mathbf{sl_2}) - filt \to H_{abc}(\mathbf{sl_2}) - gr$$

gives an equivalence of categories between the category $U_{abc}(\mathbf{sl_2}) - filt$ of all finitely generated filtered left $U_q(\mathbf{sl_2})$ with proper morphisms and the full subcategory $T - tf$ of all T-torsion free objects in $H_{abc}(\mathbf{sl_2})$-gr, the category of all finitely generated graded left $H_{abc}(\mathbf{sl_2})$-modules with gradation preserving morphisms, see for example [18] or [14] for more details.

In this section we will assign to $H_{abc}(\mathbf{sl_2})$ certain geometrical objects in $\mathbb{P}^3 = \mathbb{P}(V^*) = \mathbb{P}(H_1^*)$. The **point-variety** $(\mathcal{P}_H, \sigma)$ consists of a closed subscheme $\mathcal{P}$ of $\mathbb{P}^3$ together with an automorphism σ on it. Its importance is eminent from the fact that one recovers the quadratic algebra $H_{abc}(\mathbf{sl_2})$ from this purely commutative algebraic geometrical data. So, every result on $U_{abc}(\mathbf{sl_2})$ or $H_{abc}(\mathbf{sl_2})$ can be expressed in terms of the scheme $\mathcal{P}$ and the automorphism σ. We will give only a brief outline of the required calculations, for more details we refer to [12, §4].

Using [12, §5] we then describe all the line modules of $H_{abc}(\mathbf{sl_2})$. In analogy with the case of homogenized $\mathbf{sl_2}$, which was studied in [15], line modules can be viewed as deformed Verma modules or, equivalently, correspond to polarizations. To every line module one associates a uniquely determined line in $\mathbb{P}^3$ and hence one can speak of the subvariety of $\mathbb{P}^3$ cut out by the collection of line modules. As every $f \in \mathbf{sl_2}^*$ has polarizations one would expect that for "good" deformations of $U(\mathbf{sl_2})$ the line modules span $\mathbb{P}^3$. We will see, however, that for generic values of (a,b,c) this line module variety has only dimension 2 and one obtains the following result

Theorem 2 *Among the (homogenized) conformal $\mathbf{sl_2}$ enveloping algebras $H_{abc}(\mathbf{sl_2})$ there are two one-parameter families having the property that their line modules span $\mathbb{P}^3$:*

1. *$c = 1$ and $b = a - 1$ in which case $H_{abc}(\mathbf{sl_2})$ can be identified with the homogenized quantum $\mathbf{sl_2}$ enveloping algebra of E. Witten [31, (5.2)].*

2. *$ac = 1$ and $b = \frac{1}{2}(a-1)$ in which case $H_{abc}(\mathbf{sl_2})$ can be identified with the algebra studied by Jing and Zhang [10]. Moreover, this algebra can be shown to be a twist of homogenized $U_q(\mathbf{sl_2})$ in Drinfeld-Jimbo notation.*

Hence, this can be viewed as a Hopf-algebra free characterization of the most important $U(\mathbf{sl_2})$ deformations for physics : they are those conformal $\mathbf{sl_2}$ enveloping algebras having enough polarizations or deformed Verma modules. The calculations in this section will be used in the study of the representation theory of conformal enveloping algebras.

3.1 The point variety

M. Artin, J. Tate and M. Van den Bergh have shown that three dimensional Auslander-regular algebras are determined by geometric data : a cubic divisor $C \hookrightarrow \mathbb{P}^2$ in the projective plane together with an automorphism $\sigma : C \to C$.

Let $A = \mathbb{C} < X, Y, Z > /(f_1, f_2, f_3)$ be Auslander-regular in standard form as described before, then the associated cubic divisor is

$$C = V(det(M)) \hookrightarrow \mathbb{P}^2 = \mathbb{P}(A_1^*)$$

and the automorphism σ associates to a point $p \in C$ the unique point $q \in \mathbb{P}^2 = \mathbb{P}(A_1^*)$ determined by the matrix equation $M(p).q^t = 0$.

Conversely, to data $\sigma : C \to C \hookrightarrow \mathbb{P}^2 = \mathbb{P}(V^*)$ we can associate all $f \in V \otimes V$ vanishing on the graph of the automorphism in $\mathbb{P}^2 \times \mathbb{P}^2$. The Auslander-regular algebra A is then the quotient of $T(V)$ by all such f. So, the defining equations are determined by the geometrical data.

A more ring theoretical description of the divisor C and automorphism σ is as follows. We define a **point-module** $P(p)$ of A to be a graded left A-module generated in degree zero with constant Hilbert series $\frac{1}{1-t}$. It is clear that $P(p) \simeq A/(Al_1 + Al_2)$ with l_1, l_2 linear independent elements of A_1 hence $P(p)$ determines a point $p = V(l_1, l_2) \in \mathbb{P}(A_1^*)$. The collection of these points is C. If $P(p)$ is a point-module for A then $P(p)_{\geq 1}(1)$ is again a point-module, namely $P(\sigma^{-1}(p))$.

By the definition of a conformal $\mathbf{sl_2}$ enveloping algebra we can apply the foregoing to the 3-dimensional Auslander regular algebra $gr(U_{abc}(\mathbf{sl_2}))$. For later use we record here some of the relevant data for $H_{abc}(\mathbf{sl_2})$:

$$\begin{aligned} f_1 &= YZ - cZY - bX^2 \\ f_2 &= ZX - aXZ \\ f_3 &= \frac{c}{a}(XY - aYX) \\ l_1 &= -X \\ l_2 &= -Z \\ l_3 &= -\frac{c}{a}Y \\ \alpha_i &= 0 \end{aligned} \tag{10}$$

and the matrices

$$M = \begin{pmatrix} -bX & -cZ & Y \\ Z & 0 & -aX \\ -cY & \frac{c}{a}X & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{c}{a} & 0 \\ 0 & 0 & \frac{a}{c} \end{pmatrix}$$

and finally

$$(X^*, Y^*, Z^*) = (X, \frac{c}{a}Y, \frac{a}{c}Z)$$

With this data one computes :

Lemma 3 *The point-variety of $gr(U_{abc}(\mathbf{sl_2}))$ has defining relation $x(abx^2 + (a^2c-1)yz) = 0$ that is, generically a conic and a line. The automorphism σ is given*

1. *On the line : $\sigma(0:y:z) = (0:y:cz)$*
2. *On the conic : $\sigma(x:y:z) = (x:ay:\frac{z}{a})$*

If $a^2c = 1$ we are in a triple line situation with automorphism as above.
If $b = 0$ we have a triangle $V(xyz)$ with automorphism on the x-line as above and with automorphism

1. *On the line $y = 0$: $\sigma(x:0:z) = (ax:0:z)$*
2. *On the line $z = 0$: $\sigma(x:y:0) = (x:ay:0)$*

Finally, if $b = 0$ and $a^2c = 1$ the cubic is indeterminate which corresponds to the case that $gr(U_{abc}(\mathbf{sl_2}))$ is a twist of a polynomial ring.

We will now show that there is an analogous geometric object associated to $H = H_{abc}(\mathbf{sl_2})$. Again, we can consider the set of all point-modules of H that is, all graded quotient modules

$$P(p) = H/(Hl_1 + Hl_2 + Hl_3)$$

with Hilbert series $\mathcal{H}(P(p),t) = \frac{1}{1-t}$ and where the l_i are linearly independent elements of H_1. To $P(p)$ we can then associate the point $p \in \mathbb{P}^3 = \mathbb{P}(V^*)$. The set of all $P(p)$ is easy to determine. As point-modules are critical modules we have that either $T.P(p) = 0$ or T acts as a non-zero divisor on $P(p)$. In the first case, $P(p)$ is a point-module over the quotient algebra $H/(T) = gr(U_{abc}(\mathbf{sl_2}))$ which we described above, the $\mathbb{P}^2$ is now the hyperplane $V(T)$ at infinity. In the second case we see from the equivalence of categories that they correspond to homogenizations of one-dimensional representations of $U_{abc}(\mathbf{sl_2})$ which give points $(x:y:z:1)$ such that

$$\begin{aligned} ((a-1)x+1)y &= 0 \\ ((a-1)x+1)z &= 0 \end{aligned}$$

$$(1-c)yz = x(bx+1) \quad (11)$$

We see that these points either lie on the conic $yz = \frac{b-a+1}{(1-a)^2(1-c)}$ in the plane $V((a-1)X+T)$ (again there will be degenerate cases) or they are the two points $(0:0:0:1)$ and $(-\frac{1}{b}:0:0:1)$.

Again, we have an automorphism on this set of point modules. For points lying at infinity it is the automorphism determined on point-modules for $gr(U_{abc}(\mathbf{sl_2}))$ and on all other points it is the identity.

Clearly, one prefers and needs the scheme structure on this set of point-modules. This was computed in [12, §4] for general central extensions of three dimensional Auslander-regular algebras in standard form. With the notations introduced before, i.e. with $g = (g_1, g_2, g_3)^t, l = (l_1, l_2, l_3)^t, \alpha = (\alpha_1, \alpha_2, \alpha_3)^t$ and with M_1, M_2, M_3 the columns of M, we define

$$\begin{aligned} h_1 &= X det(M) + T det[lM_2M_3] + T^2 det[\alpha M_2 M_3] \\ h_2 &= Y det(M) + T det[M_1 l M_3] + T^2 det[M_1 \alpha M_3] \\ h_3 &= Z det(M) + T det]M_1 M_2 l] + T^2 det[M_1 M_2 \alpha] \end{aligned} \quad (12)$$

Then according to [12, thm. 4.2.2] the defining equations for the point-variety $\mathcal{P}$ have affine presentations

- On $X(t) : V(g1, g2, g3)$
- On $X(x) : V(tg_1, tg_2, tg_3, h_1)$
- On $X(y) : V(tg_1, tg_2, tg_3, h_2)$
- On $X(z) : V(tg_1, tg_2, tg_3, h_3)$

at least if the cubic divisor of the 3-dimensional Auslander-regular is determinate. If not,there is a vector μ of independent linear forms such that $\mu^t M = 0$ and then the defining equations of $\mathcal{P}$ are

$$V(tq, tg_1, tg_2, tg_3)$$

where $q = \mu^t(l + \alpha t)$.

Let us calculate this in the case of the algebras $H = H_{abc}(\mathbf{sl_2})$.

$$\begin{aligned} h_1 &= \frac{c}{a}x((1-a^2c)xyz - (1+ac)tyz - ax^2(bx+t)) \\ h_2 &= \frac{c}{a}y((1-a^2c)xyz - (1+a)tyz - ax^2(bx+(a-b)t)) \\ h_3 &= \frac{c}{a}z((1-a^2c)xyz - c(1+a)tyz - x^2(abx+(b-1)t)) \end{aligned} \quad (13)$$

In the indeterminate case $b = 0$ and $c = \frac{1}{a^2}$

$$\mu = (x : \frac{y}{a} : az)$$

$$q \;=\; a^2x^2 + (1+a)yz \tag{14}$$

From this information and the general result we deduce

Lemma 4 *In the generic case, the pointvariety $\mathcal{P}$ of $H_{abc}(\mathbf{sl_2})$ consists of two plane conics and a line (the intersection of these two planes) together with 2 points. The embedded points are the common intersection of the three one dimensional components which are also the fixed points of the automorphism at infinity.*
If $c = 1$ and $b = a-1$ the pointvariety consists of a conic at infinity and the plane $V(t+(a-1)x)$ and one extra point. The embedded points are the 2 intersection points of the conic and the plane.
If $b = 0$ and $c = \frac{1}{a^2}$ the divisor at infinity is indeterminate and the pointvariety is the plane at infinity together with a plane conic and two extra points. Again, the embedded points are the intersection points of the plane and the conic.
If moreover $a = 1$ the pointvariety is the plane at infinity together with an embedded component $V(t, x^2+2yz)$ and one extra point $(0:0:0:1)$.

The last case $(a,b,c) = (1,0,1)$ corresponds to the homogenized enveloping algebra [15]. The embedded conic at infinity can be identified with the flag variety, that is with the $\mathbb{P}^1$ of Borel subalgebras (using the identification $\mathbb{P}(\mathbf{sl_2}) \simeq \mathbb{P}(\mathbf{sl_2}^*)$ via the Killing form).

Recall from [12, Prop. 4.2.5] that one recovers the quadratic algebra $H_{abc}(\mathbf{sl_2})$ from the point variety $\mathcal{P}$ and the automorphism σ. So, this commutative algebraic geometric data should determine all relevant properties of $H_{abc}(\mathbf{sl_2})$.

We will now give a ring theoretical interpretation of the plane $V(T+(a-1)X)$ containing the extra conic, and of the embedded points in the scheme structure of $\mathcal{P}$.

The investigation of the point-variety draws our attention to the special degree one element

$$N = T + (a-1)X$$

A direct computation shows that this element is normalizing in $H = H_{abc}(\mathbf{sl_2})$

$$\begin{aligned} NX &= XN \\ NY &= aYN \\ aNZ &= ZN \\ NT &= TN \end{aligned} \tag{15}$$

If $a \neq 1$ this gives us an additional normalizing element in degree one which illustrates the breaking of symmetry of $\mathbf{sl_2}$ down to a maximal torus. In this case we can replace X by N and the defining equations of $H_{abc}(\mathbf{sl_2})$ become :

$$NY \;=\; aYN$$

$$\begin{aligned}
aNZ &= ZN \\
NT &= TN \\
TY &= YT \\
TZ &= ZT \\
YZ - cZY &= \frac{1}{(a-1)^2}(bN^2 + (a-1-2b)NT + (b+1-a)T^2) \quad (16)
\end{aligned}$$

An immediate consequence of this is :

Corollary 1 *$H_{abc}(\mathbf{sl_2})/(N)$ is a three dimensional Auslander-regular algebra with defining quadratic relations*

$$\begin{aligned}
TZ &= ZT \\
TY &= YT \\
YZ - cZY &= \frac{b+1-a}{(a-1)^2}T^2 \quad (17)
\end{aligned}$$

Proof : We have to find the degree three divisor and its automorphism. In this case this is

$$det \begin{pmatrix} 0 & T & -Z \\ T & 0 & -Y \\ -cZ & Y & -\frac{b+1-a}{(a-1)^2}T \end{pmatrix} = T(\frac{b+1-a}{(a-1)^2}T^2 + (c-1)YZ)$$

The automorphism is

- on the line : $\sigma(y : z : 0) = (y : cz : 0)$
- on the conic : $\sigma(y : z : t) = (y : z : t)$

which is of course compatible with the fact that this automorphism coincides with the one of $H_{abc}(\mathbf{sl_2})/(T)$ on the line component and is the identity on points not lying on the plane $V(T)$. □

The point-variety of $H_{abc}(\mathbf{sl_2})/(N)$ is for :

- a, b, c generic : conic+line
- $b = a - 1$: triangle $V(YZT)$
- $c = 1$: triple line $V(T)^3$
- $c = 1$ and $b = a - 1$: plane

As in [13] we can use the additional normalizing element to relate the module category of $U_{abc}(\mathbf{sl_2})$ to that of another filtered algebra obtained via twisting. The interested reader is referred to [13] for more details.

Recall that in the case of homogenized $\mathbf{sl_2}$ the embedded points correspond to Borel subalgebras of $\mathbf{sl_2}$, see [15]. Because our deformations correspond to symmetry breaking, most Borel subalgebras do not survive in $H_{abc}(\mathbf{sl_2})$. However, there are two obvious three dimensional Auslander regular subalgebras of $H_{abc}(\mathbf{sl_2})$ which correspond to the two embedded points :

$$B = \mathbb{C} < Y, T, N > \qquad B' = \mathbb{C} < Z, T, N >$$

As they are symmetric we restrict to B.

Lemma 5 *B is a three dimensional Auslander-regular subalgebra of $H_{abc}(\mathbf{sl_2})$.*

Proof : The point-variety of B is a triangle

$$det \begin{pmatrix} N & 0 & -aY \\ 0 & N & -T \\ T & -Y & 0 \end{pmatrix} = (a-1)YTN$$

The automorphism has the form

- On $V(Y)$: $\tau(0:t:n) = (0:t:n)$
- On $V(T)$: $\tau(y:0:n) = (ay:0:n)$
- On $V(N)$: $\tau(y:t:0) = (y:t:0)$

□

3.2 The line modules

In this subsection we will classify all the line modules of $H_{abc}(\mathbf{sl_2})$. The importance of these modules is clear from homogenized $\mathbf{sl_2}$ where they are either lines at infinity or homogenizations of Verma modules, see [15]. Hence, line modules for conformal $\mathbf{sl_2}$ enveloping algebras can be viewed as deformations of Verma modules.

In the case of homogenized $\mathbf{sl_2}$ there is also a nice geometric description of the line modules. Consider the pencil of quadrics

$$Q = \alpha T^2 + \beta(X^2 - 4YZ)$$

spanned by the double plane at infinity and the cone on the Casimir element. The line modules of $H(\mathbf{sl_2})$ are by [15] precisely the lines lying on a quadric in this pencil. In particular, if $f \in \mathbf{sl_2}^*$ represents a point in $\mathbb{P}^3 - V(T)$, then there is a unique quadric Q in the pencil passing through f. If $f \notin V(X^2-4YZ)$ this quadric is smooth and hence there are precisely two line modules of $H(\mathbf{sl_2})$ passing through f. These line modules correspond to the homogenizations of the two polarizations of f. These line modules intersect the base locus of the pencil of quadrics $V(T, X^2 - 4YZ)$ at infinity in a point which represents the Borel subalgebra subordinate to f. By using the Killing form we can identify

$\mathbb{P}(\mathbf{sl_2}) = \mathbb{P}(\mathbf{sl_2}^*) = V(T)$ and so the base locus can be interpreted as the flag variety of $\mathbf{sl_2}$. Moreover, this base locus is also the embedded component of the point variety of $H(\mathbf{sl_2})$.

Observe that the Lie algebras $\mathbf{sl_n}$ are characterized among the semi-simple Lie algebras by the property that every linear functional has a polarization. So, in geometric terms, the fact that the line modules of $H(\mathbf{sl_2})$ span $\mathbb{P}^3$ is characteristic for $\mathbf{sl_2}$ and should be preserved in "good" deformations. For this reason we will characterize all line modules of the homogenized conformal $\mathbf{sl_2}$ enveloping algebras $H_{abc}(\mathbf{sl_2})$.

The calculations are exercises to the general results of [12] and we refer to that paper for full details. We begin with an easy but important observation.

Proposition 2 *Line modules of $H = H_{abc}(\mathbf{sl_2})$ correspond to secant lines. That is, every line contains at least two points of the point variety (counted with multiplicities).*

Proof : As line modules are critical (see [17]) normalizing elements either act on them as zero or as non zero divisors.

Let l be a line module. Assume first that either $T.l = 0$ or $N.l = 0$. Then l is a line module of $H/(T)$ or $H/(N)$ both of which are Auslander-regular algebras of dimension 3 and so l contains at least 3 points (counted with multiplicities) by [5].

If $T.l \neq 0$ then $l/T.l$ has Hilbert series $\frac{1}{1-t}$ and hence is a point module p_T of W. Similarly, if $N.l \neq 0$, then $p_N = l/N.l$ is a point module. So, l contains at least two points. □

Line modules of $H_{abc}(\mathbf{sl_2})$ either correspond to lines lying in the plane at infinity $V(T)$ or they intersect this plane in a point of the point variety of $H_{abc}(\mathbf{sl_2})/(T) = gr(U_{abc}(\mathbf{sl_2})) = A$. By [12, Thm 5.1.6] we know that a line in $\mathbb{P}^3$ passing through $p \in \mathcal{P}_A \subset V(T)$ corresponds to a line module of $H_{abc}(\mathbf{sl_2})$ if and only if it is contained in either $V(T)$ or a quadric $Q_p \subset \mathbb{P}^3$ uniquely determined by p.

Given the data recalled in the foregoing section (that is, the equations f_i, g_i in standard form and the matrices M and Q) [12, Prop.5.1.7] asserts that this quadric is defined by

$$Q_p = V(\sigma(p_T)^t.Q.g)$$

where $g^t = (g_1, g_2, g_3)$. We will investigate these quadrics in the case when $\mathcal{P}_A$ is a conic and line situation (that is, whenever $ab \neq 0$ and $a^2c \neq 1$), the remaining cases can be treated similarly.

In this case we have three different types of points $p_T \in \mathcal{P}_A$:

1. p_T is an intersection point of line and conic
2. p_T lies on the line but not on the conic

3. p_T lies on the conic but not on the line

and we will handle these cases separately :

the intersection points

For the intersection points (fixed points under σ) we have

- If $p_T = (0:1:0:0)$, then $Q_p = Z.((a-1)X+T)$
- If $p_T = (0:0:1:0)$, then $Q_p = Y.((a-1)X+T)$

Hence, all line modules passing through such a point can be viewed as either lines lying in $V(T)$ passing through p_T or in the plane determined by the tangent line in p_T to the conic and going through the 'origin' $o = (0:0:0:1)$ or in the plane $V(N)$ determined by the normalizing element N. Observe that this description coincides with the description of point modules of the corresponding Borel subalgebra. In this case the line modules are really induced point modules from B to $H_{abc}(\mathbf{sl}_2)$.

points on the line

If $p_T = (0:v:w:0)$ with $vw \neq 0$ then we compute that

$$Q_p = ((a-1)X+T)(awY+vZ)$$

If $a \neq -1$, then p_T lies only on the plane component corresponding to the normalizing element N, that is all line-modules passing through p_T either lie in the plane $V(T)$ or in the plane $V(N)$.

If, $a = -1$ then for every point p_T there is an extra $\mathbb{P}^1$ of line modules passing through p_T namely those lying in the plane $V(vZ-wY)$.

points on the conic

If $p_T = (1 : v : \frac{ab}{(1-a^2c)v} : 0)$, then the corresponding quadric Q_p is determined by the symmetric matrix

$$\begin{pmatrix} b(1-a^2c)v & \frac{1}{2}(a-1)b & \frac{1}{2}(a-1)c(1-a^2c)v^2 & \frac{1}{2}(1-a^2c)v \\ \bullet & 0 & \frac{1}{2}(c-1)(1-a^2c)v & \frac{1}{2}b \\ \bullet & \bullet & 0 & \frac{1}{2}c(1-a^2c)v^2 \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}$$

The determinant of this matrix is

$$\frac{1}{16}(c-1)(a^2c-1)^3(1-c-a^2c-4bc+4abc-4b^2c+a^2c^2)v^4$$

If this determinant is non-zero (which depends only on the parameters (a,b,c) and not on the particular point !), the quadric is smooth and so the line modules passing through p_T are either the lines at $V(T)$ or one of the two lines lying on Q_p going through p_T. Let us have a closer look at the degenerate cases (recall that we discart the cases when $a^2c = 1$ or $ab = 0$)

- If $c = 1$, then for generic (a, b) the quadric is singular but p_T is a smooth point on it. In this case there is just one extra line-module through p_T. The quadric will be of rank 2 (that is, Q_p will be a plane-pair) if and only if $(a-1)(a+1)(a-b-1) = 0$. In these cases all lines lying in the plane(s) containing p_T will correspond to line modules, that is for each point there is an extra $\mathbb{P}^1$ family of line modules.

  In the conic and line situation this happens iff $b = a - 1$. In this case

  $$Q_p = (a-1)(T + (a-1)X)(Y - v(1+a)X - v^2(1+a)Z)$$

  and there are extra line-modules namely those lines passing through p_T and lying in the plane $Y = v(1+a)X + v^2(1+a)Z$. This plane can be interpreted geometrically as the plane passing through $p_T, \sigma(p_T)$ and the origin o.

- The remaining case when

  $$1 = c + a^2c + 4bc - 4abc + 4b^2c - a^2c^2$$

  For generic (a, c) the quadric Q_p will be singular but smooth at p_T giving rise to one extra line-module. The ring theoretical interpretation for the existence of this special surface of parameters is unclear to me (however, see the next section).

Summarizing we have :

Proposition 3 *For generic (a, b, c) the line modules of $H_{abc}(\mathbf{sl_2})$ cut out a two dimensional subvariety of $\mathbb{P}^3$ with 4 plane components $V(TNYZ)$ and a surface described by the lines passing through the conic at infinity.*

The remaining cases when $b = 0$ or $a^2c = 1$ are left to the interested reader.

3.3 Some like it Hopf

The fact that every $f \in \mathbf{sl_2}^*$ has a polarization can be phrased in geometric terms by stating that the line modules of $H(\mathbf{sl_2})$ span $\mathbb{P}^3$. In the foregoing subsection we have seen that for generic values of (a, b, c) the line modules only span a dimension two subvariety.

In this subsection we will determine those conformal $\mathbf{sl_2}$ enveloping algebra having enough line modules. It will turn out that we recover the more important quantum enveloping algebras such as Witten's $W_q(\mathbf{sl_2})$ and (twisted) homogenized Drinfeld-Jimbo $U_q(\mathbf{sl_2})$. So, these considerations can be viewed as given an Hopf algebra free characterization of these (quasi)- quantum groups. Moreover, we will give a ring theoretical interpretation of these special cases in terms of central and normalizing degree two elements.

In view of the above calculations, a special two dimensional class of conformal enveloping algebras occurs when $a+1=0$. In this case we have that every point $(0:v:w:0)$ on the line has an additional $\mathbb{P}^1$ of line modules passing through it, the lines lying in the plane $V(vZ-wY)$. Hence, the line modules of the algebras S_{bc} with relations

$$\begin{aligned} XY+YX &= Y \\ ZX+XZ &= Z \\ YZ-cZY &= bX^2+X \end{aligned} \tag{18}$$

span $\mathbb{P}^3$. As $U(\mathbf{sl_2})$ does not belong to this class we leave a detailed analysis of these conformal algebras as a suggestion for further work.

If we view the conic at infinity as a relic of the flagvariety in the classical $U(\mathbf{sl_2})$ case it is more important to determine the conformal algebras with the property that every point p on the conic has an additional $\mathbb{P}^1$ of line modules through is. This happens in two cases :

1. Q_p is of rank two, that is, a plane pair
2. Q_p is singular with top p

The first possibility occurs when $c=1$ **and** $b=a-1$. In this case all the lines passing through p which lie in the plane through $p,\sigma(p)$ and the origin o correspond to line modules of $H_{abc}(\mathbf{sl_2})$. The defining relations become in this case

$$\begin{aligned} XY-aYX &= Y \\ ZX-aXZ &= Z \\ YZ-ZY &= (a-1)X^2+X \end{aligned} \tag{19}$$

and one verifies that this algebra is up to filtration preserving isomorphism equal to the quantum $\mathbf{sl_2}$ enveloping algebra found by E. Witten [31] by evaluating expectation values of tetrahedra in a specific conformal field theory namely three dimensional Chern-Simons gauge theory with group SU_2. This algebra $W_q(\mathbf{sl_2})$ has defining equations :

$$\begin{aligned} \frac{1}{\sqrt{q}}XY-\sqrt{q}YX &= (q+q^{-1})Y \\ \sqrt{q}XZ-\frac{1}{\sqrt{q}}ZX &= -(q+q^{-1})Z \\ YZ-ZY &= \frac{\sqrt{q}(q-1)}{q^2+1}X^2+X \end{aligned} \tag{20}$$

Remark 1 *Observe that in Witten's approach* $q = e^{\frac{2\pi i}{k}}$ *where* $1/k$ *is the coupling constant of the theory. In [29] it is shown that the condition to have a consistent quantum field theory is that* $k \in \mathbb{Z}$ *so the case when* q *is a root of unity is implied by physics and hence the description of the representation theory of* $W_q(\mathbf{sl_2})$ *in these cases is important.*

For the second class of special conformal enveloping algebras we have to investigate when all $p = (1 : v : \frac{ab}{(1-a^2c)v} : 0)$ on the conic are singular points of Q_p. We have

$$\begin{aligned}\frac{\partial Q}{\partial X}(p_T) &= (1+a)b(1-ac)v \\ \frac{\partial Q}{\partial Y}(p_T) &= b(ac-1) \\ \frac{\partial Q}{\partial Z}(p_T) &= (1-ac)(a^2c-1)v^2 \\ \frac{\partial Q}{\partial T}(p_T) &= (1+b-a^2c+abc)v \end{aligned} \tag{21}$$

So, p_T will always be a smooth point of Q_p except when

$$ac = 1 \quad and \quad b = \frac{a-1}{2}$$

In this case, every point p has an extra $\mathbb{P}^1$ of line modules passing through it. The relations of the corresponding algebra are (in the generators N, T, Y and Z) :

$$\begin{aligned} NY &= aYN \\ aNZ &= ZN \\ YZ - \frac{1}{a}ZY &= \frac{1}{2(a-1)}(N^2 - T^2) \end{aligned} \tag{22}$$

and T central. The morphism induced by

$$\tau(N) = N \quad \tau(Y) = \frac{1}{\sqrt{a}}Y \quad \tau(Z) = \sqrt{a}Z \quad \tau(T) = T$$

induces a gradation preserving automorphism τ and we can twist the above algebra with respect to it as in [5] or [13]. We obtain the quadratic algebra with relations (if we denote $a = q$) :

$$\begin{aligned} NY &= \sqrt{q}YN \\ \sqrt{q}NZ &= ZN \\ NT &= TN \\ \sqrt{q}TY &= YT \\ TZ &= \sqrt{q}ZT \end{aligned}$$

$$YZ - ZY = \frac{1}{2(\sqrt{q} - \sqrt{q}^{-1})}(N^2 - T^2) \tag{23}$$

Observe that NT is a central degree two element in this twisted algebra and we can dehomogenize by quotienting out $NT - 2i$. If we denote $Y = x, Z = y$ and $N = \sqrt{2}k$, this quotient algebra is then the Drinfeld-Jimbo quantum $\mathbf{sl_2}$:

$$\begin{aligned} xk &= \frac{1}{\sqrt{q}}kx \\ yk &= \sqrt{q}ky \\ xy - yx &= \frac{k^2 + k^{-2}}{\sqrt{q} - \sqrt{q}^{-1}} \end{aligned} \tag{24}$$

So, we obtain also the (twisted) $U_q(\mathbf{sl_2})$ from this conformal approach.

Both $W_q(\mathbf{sl_2})$ and homogenized $U_q(\mathbf{sl_2})$ have two central degree two elements, one being the homogenization parameter T^2 resp. NT, the other the quantum Casimir element. This raises the question whether the conformal $\mathbf{sl_2}$ enveloping algebras $H_{abc}(\mathbf{sl_2})$ all have a deformed Casimir element. It is easy to see that one cannot expect a central Casimir element in general as some of the $H_{abc}(\mathbf{sl_2})/(T)$ do not have central degree two elements. However, this quotient always has a degree two normalizing element. This can be shown either by direct calculation or by using the general fact that any three dimensional Auslander-regular algebra has a normalizing degree 3 element corresponding to the degree three divisor of the point-variety. As the point-variety decomposes into two components and the line component corresponds to a degree one normalizing element this shows that also the conic should correspond to a normalizing element of degree two. Perhaps surprisingly, this elements lifts to a normal degree two element of $H_{abc}(\mathbf{sl_2})$.

Lemma 6 *Let*

$$C = (a-1)^2(c-1)(ac-1)(a^2c-1)YZ + b(c-1)(ac-1)N^2 +$$
$$(a-1-2b)(c-1)(a^2c-1)TN + (1-a+b)(ac-1)(a^2c-1)T^2$$

Then, C is normalizing with commutation relations

$$CN = NC \quad cCY = YC \quad CZ = cZC \quad CT = TC$$

which is the deformed Casimir operator in $H_{abc}(\mathbf{sl_2})$.

For the two special cases ($c = 1$ or $ac = 1$) substitution only gives the obvious normalizing elements T^2 resp. NT. However, we can divide by $c - 1$ resp. $ac - 1$ and remove the pole that arises by requiring the corresponding numerator to vanish which then gives us the condition on b. So, in these special cases we obtain the normalizing elements :

- If $c = 1$ and $b = a - 1$, then $C = (a-1)^2(a+1)YZ + N^2 - (a+1)TN$
- If $ac = 1$ and $b = \frac{1}{2}(a-1)$, then $C = (a-1)^2 YZ + \frac{1}{2}N^2 + \frac{a}{2}T^2$

A more conceptual result on the importance of this special normalizing element is

Proposition 4 *There is a $\mathbb{Z}$-gradation by eigenspaces of $H_{abc}(\mathbf{sl_2})$ over the 'Cartan' subalgebra $R = \mathbb{C}[T, N, C]$ of the form*

$$\ldots \oplus RY^2 \oplus RY \oplus R \oplus RZ \oplus RZ^2 \oplus \ldots$$

where R is the commutative polynomial ring and YZ is by the above description of C an element of R.

Therefore, our deformations of $H(\mathbf{sl_2})$ are such that not only the maximal torus survives but also the Cartan decomposition. I thank M. Van den Bergh for this observation.

4 From geometry to representations

In this section we begin to collect the fruits of our labors. The final aim is to characterize all finite dimensional simple representations of the conformal $\mathbf{sl_2}$ enveloping algebras $U_{abc}(\mathbf{sl_2})$.

In the case when either a or c is not a root of unity this is a fairly routine matter in view of the existence of Borel-like sub algebras of $U_{abc}(\mathbf{sl_2})$ or its homogenization $H_{abc}(\mathbf{sl_2})$, or because of the Cartan decomposition given above.

Here, we give yet another more geometrical approach which was introduced in [13]. The problem of classifying all finite simple representations of $U_{abc}(\mathbf{sl_2})$ is equivalent to that of finding all fat point modules of $H_{abc}(\mathbf{sl_2})$. Using the three dimensional Auslander regular subalgebras B we can show that such fat point modules are quotients of line modules passing through the intersection points (at least if a is not a root of unity). It is then fairly easy to characterize those line modules which do have fat point quotients. In fact, due to the additional normalizing element N in degree one (which arises from the breaking of $\mathbf{sl_2}$ symmetry) the required calculations are in fact easier than in the classical $U(\mathbf{sl_2})$ case. We show :

Theorem 3 *If a is not a root of unity, then for all $n > 1$ there are at most two simple n-dimensional representations of $U_{abc}(\mathbf{sl_2})$.*

But we have seen before that the cases when a and c are roots of unity are the more interesting ones for physics. In this case, $U_{abc}(\mathbf{sl_2})$ (or $H_{abc}(\mathbf{sl_2})$) is a finite module over its center and using the description of $H_{abc}(\mathbf{sl_2})$ as an iterated Ore extension we can compute the p.i.-degree and show that the center is rational.

Unlike the case of Sklyanin algebras it is no longer true that all fat point modules are quotients of line modules for $H_{abc}(\mathbf{sl}_2)$. This is clear from the fact that the cases where $H_{abc}(\mathbf{sl}_2)$ is finite over its center form a Zariski dense subset and for generic abc the line variety is only of dimension two. Still, we will show :

Theorem 4 *If $a^i c^j = 1$, then every fat point module of $H_{abc}(\mathbf{sl}_2)$ is a quotient of a plane curve module of degree determined by i and j.*

Hence, in order to classify all finite dimensional simple modules in the cases when $U_{abc}(\mathbf{sl}_2)$ is a finite module over its center one has to classify higher degree curve modules and subsequently its fat point quotients. This program will be carried out in full detail in the second part of the paper. In order to give the reader an idea of the method we will briefly scetch the main steps in the (for physics) most interesting case of the Witten algebras $W_q(\mathbf{sl}_2)$ where all fat point modules are still quotients of line modules.

4.1 Preliminary observations

In this subsection we will classify the finite dimensional simple representations of $U_{abc}(\mathbf{sl}_2)$ in the generic case and draw some consequences from the description of $H_{abc}(\mathbf{sl}_2)$ as an iterated Ore extension. First, we will prove

Proposition 5 *For generic a, b, c and $n > 1$ there will be precisely two simple n-dimensional representations of $U_{abc}(\mathbf{sl}_2)$. For $H_{abc}(\mathbf{sl}_2)$ there are two $\mathbb{C}^*$ families of simple n- dimensional representations.*

More important than the result is the method used to obtain it. To begin, let us note that the homogenization of a simple n-dimensional representation of a filtered ring R is a graded $h(R)$-module which is critical and has Hilbert series $\frac{n}{1-t}$. Such graded modules we will call **fat point-modules of multiplicity** n. In the case of a central extension of a three dimensional Auslander regular algebra we note two important facts

- For a three dimensional Auslander-regular algebra such that the corresponding automorphism on the point variety has infinite order there are no fat point modules of multiplicity > 1. In particular, there are no fat point-modules annihilated by T.
- The other fat point-modules (i.e. those for which T acts as a non-zero divisor) are all isomorphic to homogenizations of simple n-dimensional representations of the filtered algebra. In fact, all simple n-dimensional representations of the central extension are obtained as $F/(T-\lambda)F$ for a fat point-module F and $\lambda \in \mathbb{C}^*$

Therefore it suffices to find all fat point-modules over $H = H_{abc}(\mathbf{sl_2})$. Here, we will use the fact that we have a three dimensional Auslander regular subalgebra B (generated by X, Y, T) which allows us to do the following variation of the classical Borel-type argument, see also [13] for more details.

Let F be a fat point-module for H of multiplicity $n > 1$. Restricting F to B gives a graded B-module with Hilbert-series $\frac{n}{1-t}$. However, we know that B has no fat point-modules of multiplicity $\neq 1$ (at least in the generic case, or rather when a is not a root of unity). This means that there is a point module $P(p)$ over B such that $P(p)$ is a graded B-submodule of F. By tensoring with H we obtain a graded H-module morphism

$$H \otimes_B P(p) \rightarrow F$$

which is surjective in $Proj(H)$. Here, $Proj$ is the quotient category of all finitely generated graded left modules modulo the Serre subcategory of graded finite dimensional left modules. In particular, two graded left modules M and M' represent the same object in $Proj$ if and only if their tails are isomorphic as graded left modules. By the above argument we have that all fat point modules are (in the generic case, or rather when neither a nor c are roots of unity) quotients of line modules of a particular shape.

Further, we know all the point modules of B because the point-variety was calculated to be the triangle $V(YTN)$. Now, assume $P(p)$ were a point lying on either the T or N component. Then F would be a fat point-module of multiplicity > 1 over the quotient algebra $H/(T)$ (resp. $H/(N)$), but as these are three dimensional Auslander-regular algebras with automorphism of infinite order (in the generic case) this is impossible. So, the only remaining possibility is that $P(p)$ is a point-module on the Y-component, that is $P(p) = B/(BY + B(N - \lambda T)$ for $\lambda \in \mathbb{C}^*$. Tensoring with H then gives

Lemma 7 *In the generic case (in fact, if neither a nor c are roots of unity) every fat point module of $H = H_{abc}(\mathbf{sl_2})$ is a quotient of a line module of the form $L_\lambda = H/(HY + H(N - \lambda T))$ for $\lambda \in \mathbb{C}^*$.*

Observe that in the case of homogenized $\mathbf{sl_2}$, this line module is just the homogenization of a Verma module. We now have to describe for which values of λ these 'highest weight' modules have a fat point module quotient of multiplicity n. As one expects this can be solved by a variation on the classical $\mathbf{sl_2}$-computation.

We claim that this computation is in fact easier in this case as we can use the normalizing element $N = T + (a-1)X$ (at least in the case that $a \neq 1$). We have the following commutation relations

Lemma 8 *1.* $NY^k = a^k Y^k N$

2. $NZ^k = a^{-k} Z^k N$

3. $TY^k = Y^kT$

4. $TZ^k = Z^kT$

The commutator relations between Y and Z are slightly harder.

Lemma 9 *If we denote for every* $k \in \mathbb{Z}, u \in \mathbb{C}$ *by* $[k]_u = 1+u+u^2+...+u^{k-1}$, *then*

1. $YZ^k - c^kZ^kY$ *is equal to*

$$\alpha[k]_cZ^{k-1}T^2 + \beta a^{1-k}[k]_{ca}Z^{k-1}TN + \gamma a^{2-2k}[k]_{ca^2}Z^{k-1}N^2$$

2. $ZY^k - c^{-k}Y^kZ$ *is equal to*

$$-\alpha/c[k]_{c^{-1}}Y^{k-1}T^2 - \beta\frac{a}{c}[k]_{(ca)^{-1}}Y^{k-1}TN - \gamma\frac{a^2}{c}[k]_{(ca^2)^{-1}}Y^{k-1}N^2$$

where $\alpha = \frac{b+1-a}{(a-1)^2}, \beta = \frac{a-1-2b}{(a-1)^2}$ *and* $\gamma = \frac{b}{(a-1)^2}$.

Proof : Clearly, the relations hold for $k = 1$ so let us assume they hold for $k-1$, that is YZ^{k-1} is equal to

$$c^{k-1}Z^{k-1}Y+\alpha[k-1]_cZ^{k-2}T^2+\beta a^{2-k}[k-1]_{ca}Z^{k-2}TN+\gamma a^{4-2k}[k-1]_{ca^2}Z^{k-2}N^2$$

Multiplying both expressions on the right by Z yields

$$c^{k-1}Z^{k-1}(cZY + \alpha T^2 + \beta TN + \gamma N^2)+$$

$$\alpha[k-1]_cZ^{k-1}T^2 + \beta a^{1-k}[k-1]_{ca}Z^{k-1}TN + \gamma a^{2-2k}[k-1]_{ca^2}Z^{k-1}N^2$$

from which the claim follows. □

Proposition 6 L_λ *has a fat point module quotient of multiplicity* n *if and only if* λ *is a solution to the quadratic equation*

$$\gamma a^{2-2n}[n]_{ca^2}\lambda^2 + \beta a^{1-n}[n]_{ca}\lambda + \alpha[n]_c = 0$$

Proof : As in the classical case we have to find conditions to ensure that $YZ^nv = 0$ where v is the generator of $L_\lambda/(T-1)$ i.e. $Yv = 0$ and $Nv = \lambda v$. This condition then follows from the foregoing lemma. □

Remark 2 *Even in case* a *and* c *are roots of unity some of the above argument can be used in order to describe the fat point modules of intermediate multiplicity.*

Remark 3 *Computing the discriminants of the quadratic equations gives hypersurfaces in the* (a, b, c) *space for which there is just one simple* n*-dimensional representation and for most other dimensions there are two. It should be investigated how this affects the 'fusion-of-lines' argument of Witten in [31] to define tensor products of the finite dimensional simple representations.*

Remark 4 *In the study of the line modules we found a special surface defined by*

$$1 - c - a^2c - 4bc + 4abc - 4b^2c + a^2c^2 = 0$$

For a point lying on this surface the discriminant of the quadratic equation of n-dimensional simples is

$$(a-1)^2c^{n-1}P_n(a)^2$$

where P_n is the minimal polynomial of a primitive n-th root of unity.

However, the most interesting cases occur when a and c are roots of unity. We can draw some immediate consequences from the fact that $H_{abc}(\mathbf{sl_2})$ is an iterated Ore extension

$$\mathbb{C}[T,N][Z,\sigma_1][Y,\sigma_2,\delta_2]$$

where $\sigma_1 \in Aut_{\mathbb{C}}(\mathbb{C}[T,N])$ is determined by

$$\begin{aligned} \sigma_1(N) &= aN \\ \sigma_1(T) &= T \end{aligned} \tag{25}$$

$\sigma_2 \in Aut_{\mathbb{C}}(\mathbb{C}[T,N][Z,\sigma_1])$ and δ_2 is a σ_2- derivation (that is one satisfying $\delta_2(fg) = \sigma_2(f)\delta_2(g) + \delta_2(f)g$) determined by

$$\begin{aligned} \sigma_2(T) &= T \\ \sigma_2(N) &= a^{-1}N \\ \sigma_2(Z) &= cZ \\ \delta_2(T) &= 0 \\ \delta_2(N) &= 0 \\ \delta_2(Z) &= \frac{1}{(a-1)^2}(bN^2 + (a-2b-1)NT + (b+1-a)T^2) \end{aligned} \tag{26}$$

Observe that δ_2 is a degree $+1$ derivation.

Let $R = \mathbb{C}[T,N][Z,\sigma_1]$. Then, $Z(R) = \mathbb{C}[T,N^r,Z^r]$ if a is a primitive r-th root of one.

Proposition 7 *If $a^r = 1$ and c not a root of unity. Then, $Z(H_{abc}) = \mathbb{C}[T,N^r]$.*

Proof : σ_2 does not act trivially on the central element Z^r of R and we can consider the element $u = \delta_2(Z^r)(\sigma_2(Z^r) - Z^r)^{-1}$ which is

$$(\alpha[r]_c Z^{r-1}T^2 + \frac{\beta}{a^{r+1}}[r]_{ca}Z^{r-1}TN + \frac{\gamma}{a^{2r+2}}[r]_{ca^2}Z^{r-1}N^2)\cdot\frac{1}{c^r-1}\cdot Z^{-r} =$$

$$\frac{1}{c^r-1}\cdot(\alpha[r]_c Z^{-1}T + \frac{\beta}{a}[r]_{ca}Z^{-1}TN + \frac{\gamma}{a^2}[r]_{ca^2}Z^{-1}N^2)$$

and we can apply [1, lemme 1.4] to obtain that

$$\mathbb{C}(T,N)(Z,\sigma_1)[Y,\sigma_2,\delta_2] = \mathbb{C}(T,N)(Z,\sigma_1)[Y',\sigma_2]$$

where $Y' = Y + u$. As no power of σ_2 acts trivially on the center of R (and hence cannot be an inner automorphism on R) we have that

$$Z(\mathbb{C}(T,N)(Z,\sigma_1)[Y',\sigma_2]) = (\mathbb{C}(T,N^r,Z^r)^{\sigma_2}$$

and the latter field is $\mathbb{C}(T,N^r)$ from wich the result follows. □

Part of the previous argument can be used in case c is a root of unity but $c^r \neq 1$.

Proposition 8 *If a is a primitive r-th root of unity and c a primitive s-th root of unity with $c^r \neq 1$. Then, the p.i.-degree of $H_{abc}(\mathbf{sl}_2)$ is equal to $\frac{rs}{(r,s)}$ and has a rational center.*

Proof : As in the foregoing proof we have to study

$$\mathbb{C}(T,N)(Z,\sigma_1)[Y',\sigma_2]$$

But this time σ_2 has order $\frac{s}{(r,s)}$ on the center of R. So, there is a $v \in Q(R)$ such that conjugation with v equals the action of $\sigma_2^{\frac{s}{(r,s)}}$. But then the center of $Q(H)$ equals

$$(\mathbb{C}(T,N^r,Z^r)^{\sigma_2}(v(Y')^{\frac{s}{(r,s)}}) = \mathbb{C}(T,N^r,Z^{\frac{rs}{(r,s)}},v(Y^{\frac{s}{(r,s)}}))$$

which is rational and computing the dimension of $Q(W)$ over it gives the square of the p.i.-degree. □

The hardest (but for applications in physics the most interesting) case occurs when

- a is a primitive r-th root of unity
- c is a primitive s-th root of unity
- s divides r

Proposition 9 *If $a^r = 1$ and $a^k = c$ for some k. Then, the p.i.-degree of $H_{abc}(\mathbf{sl}_2)$ is r and it has a rational center.*

Proof : Consider the element $u = ZN^k$ and consider conjugation with u in $\mathbb{C}(T,N)(Z,\sigma_2)$. We have

$$\begin{array}{rcl} u^{-1}Tu & = & T \\ u^{-1}Nu & = & a^{-1}N \\ u^{-1}Zu & = & a^kZ \end{array} \quad (27)$$

so by assumption σ_2 is an inner automorphism. Replacing Y by $Y' = uY$ we then have

$$\mathbb{C}(T,N)(Z,\sigma_1)[Y,\sigma_2,\delta_2] = \mathbb{C}(T,N)(Z,\sigma_1)[Y',\delta_2']$$

where δ_2' is the derivation (!) $\delta_2'(f) = u\delta_2(f)$ on $\mathbb{C}(T,N)(Z,\sigma_1)$.

Moreover, as c, ca and ca^2 are r-th roots of unity we have that $[r]_c, [r]_{ca}$ and $[r]_{ca^2}$ are all zero. But then, $\delta_2(Z^r) = 0$ and hence δ_2' acts trivially on the center of $\mathbb{C}(T,N)(Z,\sigma_1)$ which is $\mathbb{C}(T,N^r,Z^r)$ so it must be an inner derivation. There exists $v \in \mathbb{C}(T,N)(Z,\sigma_1)$ such that

$$\delta_2'(f) = fv - vf$$

Finally, changing Y' by $Y" = Y' + v$ we have that

$$\mathbb{C}(T,N)(Z,\sigma_1)[Y,\sigma_2,\delta_2] = \mathbb{C}(T,N)(Z,\sigma_1)[Y"]$$

and hence the center equals $\mathbb{C}(T,N^r,Z^r,Y")$ from which the statements of the proposition follow. □

4.2 Fat points as curve quotients

The basic strategy to use 'non commutative projective geometry' in the study of finite dimensional simple representations is as follows :

- finite dimensional simple representations are quotients of fat point modules
- find a classical geometric object G such that all fat point modules are quotients of graded modules with the same Hilbert series as G
- use the point variety and automorphism to describe all graded modules of type G
- similarly, classify in geometric terms the kernels of the surjection from type G module to fat point and the maps between them
- use these descriptions to get a geometric classifiaction of all fat point modules
- find all finite dimensional simple quotients of a fat point

In the case of the 4-dimensional Sklyanin algebra this approach pursued by S.P. Smith and J. Staniskis [27], [26] was very succesfull as one could take for type G-modules the line modules and these are easy to describe.

In the remaining case of interest here, that is $H_{abc}(\mathbf{sl_2})$ with a and c roots of unity, we have seen that we can no longer expect all fat point modules to be quotients of line modules. In this subsection we will show that for the two dimensional families of conformal enveloping algebras

$$\{H_{abc}(\mathbf{sl_2}) \mid a^ic^j = 1 \quad i,j \in \mathbb{Z}\}$$

the corresponding geometric object can be taken to be plane curves of degree determined by i and j.

We will first prove that a quadratic Auslander regular algebra of dimension 4 with additional properties has this property and subsequently that the $H_{abc}(\mathbf{sl_2})$ satisfy these properties when $a^i c^j = 1$.

Definition 2 *Let A be a quadratic Auslander regular algebra of dimension 4 with Hilbert series $(1-t)^{-4}$. A* **plane degree d curve module** *for A is a graded left cyclic A-module M with Hilbert series*

$$\mathcal{H}(M,t) = \frac{1+t+\ldots+t^{d-1}}{(1-t)^2}$$

The following result is an extension of [26, Thm 4.3] :

Proposition 10 *Let A be a quadratic Auslander regular algebra of dimension 4 with Hilbert series $(1-t)^{-4}$. Assume that the following properties hold*

1. *A has two central elements Ω_1 and Ω_2 of degree $d+1$*
2. *For every $\Omega = \alpha\Omega_1 + \beta\Omega_2$ with $\alpha, \beta \in \mathbb{C}$ there is a line module $L_\Omega = A/AU$ with U a 2-dimensional subspace of A_1 such that*
   - $\Omega.L_\Omega = 0$
   - *For every $u \in U : \Omega.(A/Au) \neq 0$*
   - *L_Ω has a point module $p \in \mathcal{P}_A$ as quotient*

Then, every fat point module of A of multiplicity > 1 is either a quotient of a line module or of a plane degree d curve module for A.

Proof : Let D be a fat point module of multiplicity $e > 1$. By [26, 4.2] the shifted module $D(-1)$ embeds in a uniquely determined fat point module F. Since F is critical $Ann(F)$ is a graded prime ideal of A which by a GK-dimension argument must intersect the commutative subring $\mathbb{C}[\Omega_1, \Omega_2]$ in a non zero graded prime ideal which then has to be generated by a element of the form $\Omega = \alpha\Omega_1 + \beta\Omega_2$. Thus, $\Omega.F = 0$.

By assumption there is a line module L_Ω such that $\Omega.L_\Omega = 0$. Now, $L = A/U$ where U is a 2-dimensional subspace of A_1 and so the homogeneous morphism

$$U \to Hom_{\mathbb{C}}(F_0, F_1) \simeq M_e(\mathbb{C}) \stackrel{det}{\to} \mathbb{C}$$

has a non-trivial zero. That is, there exist $u \in U$ and $m \in F_0$ such that $u.m = 0$. We have the following exact diagram with gradation preserving maps

$$\begin{array}{ccccccccc} 0 & \to & M & \to & A/(Au + A\Omega) & \to & L_\Omega & \to & 0 \\ & & & & \downarrow \phi & & & & \\ & & & & F & & & & \end{array}$$

where $\phi(1) = m$.

$P = A/Au$ is a plane module which is critical, so the central element Ω either acts on it as zero or as a non-zero divisor. By assumption $\Omega.P \neq 0$ but then the Hilbert series of $A/(Au + A\Omega)$ can be deduced from the exact sequence

$$0 \rightarrow P(-d-1) \xrightarrow{\Omega.} P \rightarrow A/(Au + A\Omega) \rightarrow 0$$

and hence is equal to

$$\frac{1 - t^{d+1}}{(1-t)^3} = \frac{1 + t + \dots + t^d}{(1-t)^2}$$

But then, the Hilbert series of M has to be

$$\mathcal{H}(M, t) = \frac{t + \dots + t^d}{(1-t)^2}$$

As M is cyclic (being generated by $U/\mathbb{C}u$) we see that $M' = M(1)$ is a plane degree d curve module for A.

If $\phi(M) \neq 0$, then there is a non-zero gradation preserving morphism $M \rightarrow F$ whose image is contained in $F_{\geq 1} = D(-1)$. Hence there is a non-zero map $M' \rightarrow D$, the cokernel of which has to be finite dimensional (D is critical). Thus, in $Proj(A)$ the fat point module D is a quotient of the plane degree d curve module M'.

If $\phi(M) = 0$, then ϕ factorizes over L_Ω giving a non-zero gradation preserving morphism $L_\Omega \rightarrow F$. By assumption L_Ω has a point module quotient say $p \in \mathcal{P}_A$. We have the diagram :

$$\begin{array}{ccccccccc} 0 & \rightarrow & L'(-1) & \rightarrow & L_\Omega & \rightarrow & p & \rightarrow & 0 \\ & & & & \downarrow & & & & \\ 0 & \rightarrow & D(-1) & \rightarrow & F & & & & \end{array}$$

where the kernel of the epimorphism $L_\Omega \rightarrow p$ is easily verified to be a twist of a line module L' of A. The image of $L'(-1)$ in F cannot be zero because F is craitical and of multiplicity > 1 so we cannot have a non-zero induced morphism $p \rightarrow F$. But then, by an argument as before, D is in $Proj(A)$ a quotient of the line module L'. □

Remark 5 *This result can be used to show that fat point modules of the deformations of Sklyanin algebras found by J. T. Stafford [28] (these algebras have 2 central elements of degree 4) are either quotients of line modules or of plane elliptic curve modules. In view of the result we are about to prove and which shows that for deformations of $U(\mathbf{sl}_2)$ there is no bound on the degree of the plane curves occuring in the proposition, one is tempted to conjecture that there are deformations of the 4-dimensional Sklyanin algebra with similar properties. We leave this as a suggestion for further work.*

We will now show that in case $a^i c^j = 1$, the algebras $H_{abc}(\mathbf{sl_2})$ satisfy the requirements of the proposition. Recall that we have the following basic normalizing elements :

- a central degree one element T
- a normalizing degree one element N with commutation relations $NY = aYN$, $NZ = a^{-1}ZN$ and $NT = TN$
- a normalizing degree two element C with commutation relations $CN = NC$, $CT = TC$, $CY = c^{-1}YC$ and $CZ = cZC$

From this data one immediately obtains

Lemma 10 *Let $i, j \in I\!N$ with $d = max(i, 2j)$. Then :*

1. *If $a^i = c^j$, then $H_{abc}(\mathbf{sl_2})$ has 2 central elements of degree $i + 2j$ namely $\Omega_1 = T^{i+2j}$ and $\Omega_2 = N^i C^j$*
2. *If $a^i c^j = 1$, then the twisted algebra $H_{abc}(\mathbf{sl_2})_\tau$ with respect to the graded automorphism τ defined by*

$$\tau(N) = N \quad \tau(T) = T \quad \tau(Y) = {}^{-}\sqrt[d]{a}Y \quad \tau(Z) = \sqrt[d]{a}Z$$

has two central elements of degree d namely $\Omega_1 = N^i T^{d-i}$ and $\Omega_2 = C^j T^{d-2j}$

Proof : Only the second part requires some explanation. In $H_{abc}(\mathbf{sl_2})$, Ω_1 and Ω_2 are normalizing elements of degree d and automorphism $\phi(N) = N$, $\phi(T) = T$, $\phi(Y) = a^{-d}Y$ and $\phi(Z) = a^d Z$. As both N and C are fixed under τ, the result follows. □

Lemma 11 *If $a^i = c^j$ and with notation as above every $\Omega = \alpha\Omega_1 + \beta\Omega_2$ has a line module of $H = H_{abc}(\mathbf{sl_2})$, $L_\lambda = H/(H.Z + H.(N - \lambda T))$ satisfying the requirements of the proposition.*

Proof : Observe first that the L_λ are indeed line modules for $H_{abc}(\mathbf{sl_2})$ (they are the induced modules from point modules of the Borel subalgebra B'). As the quotients $H/(T)$ and $H/(N)$ are Auslander regular of dimension three (and hence have line modules with the required properties) we may assume that $\alpha\beta \neq 0$.

From the description of C it follows that C acts on L_λ as $f(\lambda)T^2$ where

$$f(\lambda) = b(c-1)(ac-1)\lambda^2 + (a-1-2b)(c-1)(a^2c-1)\lambda + (1-a+b)(ac-1)(a^2-1)$$

and hence Ω_2 acts as $\lambda^i f(\lambda)^j \Omega_1$ whence L_λ is annihilated by $\Omega_2 - \lambda^i f(\lambda)^j \Omega_1$. As $f(\lambda)$ is not identically zero (unless in the special cases treated before where we can modify the argument) we have a non-constant map from the $\mathbb{P}^1$ of line modules L_λ to the $\mathbb{P}^1$ of elements Ω which is therefore surjective.

Therefore, every Ω annihilates a line module. The fact that $\Omega.(H/Hu) \neq 0$ for every $u \in \mathbb{C}Z + \mathbb{C}(N - \lambda T)$ can be shown by going to the domain $H/(T, N)$ and the fact that L_λ has a point quotient is clear from the fact that line modules are secants. □

As a twist has the same line modules, the above argument can be modified slightly to give the same result for the case when $a^i c^j = 1$ with $i, j \in \mathbb{N}$. So, we have a discrete family of two-parameter families of algebras for which the fat point modules are either quotients of line modules (which will alwas be the case for the generic values in the family) or quotients of plane curve modules (which is necessary for the Zariski-dense subset of algebras in the family which are finite modules over their center).

4.3 The Witten algebras $W_q(\mathbf{sl}_2)$

The complete classification of the finite dimensional simple representations of these two-parameter families of conformal $\mathbf{sl}_2$ enveloping algebras will be given in the second part of this paper. To give the reader some idea of the method involved we will finish by presenting a scetch of the easiest case when $c = 1$ and $b = a - 1$, that is the case of the Witten enveloping algebras $W_q(\mathbf{sl}_2)$. From the discussion on normalizing and central degree two elements given before we obtain :

Lemma 12 *There is a unique one-parameter family of conformal* $\mathbf{sl}_2$ *enveloping algebras with a non-trivial central degree two element* C. *This family is determined by* $c = 1$ *and* $b = a - 1$ *and hence is isomorphic to the Witten algebras* $W_q(\mathbf{sl}_2)$.

This fact can be used to identify several seemingly unrelated quantum algebras discovered by physicists. It should be noted that some of these connections were observed by S. Majid [19] or [20] in the investigation of his braided matrices [21]. The connection between $W_q(\mathbf{sl}_2)$ and the algebras of Kulish and Sklyanin was brought to my attention by M. Vancliff (unpublished).

By a generic isomorphism we mean that there is an isomorphism for general values of the parameters. However, the formulas used in this isomorphism can have poles in a finite number of special values where the symmetry between the different objects is broken.

Proposition 11 *There is a generic isomorphism between the following classes of algebras :*

1. *Homogenizations of the Witten algebras* $W_q(\mathbf{sl}_2)$ *of [31] and [13].*

2. *Majid's braided matrices $BM_q(2)$ of [21] which are quadratic algebras generated by A, B, C and D satisfying the relations*

$$BA = q^2AB, \quad CA = q^{-2}AC, \quad DA = AD, \quad BC = CB+(1-q^{-2})A(D-A)$$

$$DB = BD + (1-q^{-2})AB, \quad CD = DC + (1-q^{-2})CA$$

3. *The algebras related to reflection equations of P.P. Kulish and E.K. Sklyanin [11] which are quadratic algebras generated by A, B, C and D satisfying the relations*

$$[A, B] = (q-q^{-1})AC, \quad AC = q^2CA, \quad [A, D] = (q-q^{-1})(qB+C)C$$

$$[B, C] = 0, \quad [B, D] = (q-q^{-1}CD, \quad CD = q^2DC$$

4. *The q-deformed Minkowski space algebra of W.B. Schmidke, J. Wess and B. Zumino [25] and others such as [19], [7], [23] a.o. which is a quadratic algebra generated by A, B, C and D satisfying the relations*

$$AB = BA - q^{-1}\lambda CD + q\lambda D^2, \quad BC = CB - q^{-1}\lambda BD, \quad BD = q^2DB$$

$$AC = CA + q\lambda AD, \quad AD = q^{-2}DA, \quad CD = DC$$

with $\lambda = q - q^{-1}$.

Hence, the surprising fact about the homogenized Witten algebras, or in our notation $H_a(\mathbf{sl_2})$ with defining equations

$$NY = aYN, \quad aNZ = ZN, \quad YZ - ZY = \frac{1}{a-1}(N^2 - NT)$$

and T central, is that it is both a deformed enveloping algebra and a deformed Minkowski space algebra. This point has been made by S. Majid in [20, §3.5]. Therefore, investigation of the finite dimensional simple representations of $W_q(\mathbf{sl_2})$ or $H_a(\mathbf{sl_2})$ may be important.

In physics, one usually imposes a real or Hermitian structure on the algebras and their representations. In this case this is usually done with demanding that a is real and

$$\overline{N} = N, \quad \overline{T} = T, \quad \overline{Y} = Z, \quad \overline{Z} = Y$$

see for example [24]. However, then it is clear from the discussions on finite dimensional representations of $H_{abc}(\mathbf{sl_2})$ in the generic case that for $a \neq \pm 1$ one has a discrete family of simples which can hardly be viewed as points in a deformed Minkowski space !

However, there is another possibility to impose an Hermitian structure on $H_a(\mathbf{sl_2})$. Assume $a \in \mathbb{C}$ to lie on the unit circle and define

$$\overline{N} = N, \quad \overline{T} = T, \quad \overline{Y} = \sqrt{a}Y, \quad \overline{Z} = \sqrt{a}Z$$

then the central element

$$C = \frac{1}{a}((a-1)(a^2-1)YZ + N^2 - (a+1)TN)$$

is a self conjugated operator playing the role of a deformed metric. In particular, if a is a primitive n-th root of unity, then $H_a(\mathbf{sl_2})$ is a finite module over its center Z. The variety associated to Z is 4- dimensional and parametrizes semi-simple n-dimensional representations of $H_a(\mathbf{sl_2})$ and hence can be viewed as a deformed Minkowski space (at least the 'real' points of it).

We will indicate how one can find all fat point modules of $H_a(\mathbf{sl_2})$ in case a is a primitive n-th root of unity. Observe that one obtains all n-dimensional simple representations of $H_a(\mathbf{sl_2})$ by specializing T to a constant $\lambda \in \mathbb{C}^*$ in these fat points. Further, it should be mentioned that because in this case all fat point modules are quotients of line modules we follow the Smith-Staniskis approach of [27] or [26].

To begin, the fat point modules of multiplicity one correspond to points on the point variety which consists of the plane $V(N)$ (where the automorphism is the identity) together with a conic $V(T,(a-1)(a^2-1)YZ+N^2)$ (where the automorphism is given by sending $p = (n:y:z:0)$ to $\sigma(p) = (n:ay:\frac{z}{a}:0)$ and one extra point $o = (1:0:0:1)$ (here, we use the coordinates N,Y,Z and T).

Next, let us characterize the fat point modules F_e of intermediate multiplicity $1 < e < n$. F_e as a B-module where B is the Borel subalgebra generated by Y,T and N must be an extension of point modules as B localized at YTN is an Azumaya algebra of p.i.-degree n. Hence, F_e must be a quotient of a line module of the form $L_\lambda = H_a(\mathbf{sl_2})/(H.Y + H.(N-\lambda T))$ and we did already compute which of them can have such a fat point quotient, namely the zeros of a quadratic relation. Here there is just one non-trivial zero for each e (the other zero is $\lambda = 0$ but for the same reason as above for B, the quotient $H/(N)$ can have no fat point modules of multiplicity e). Hence, there is a unique fat point module of multiplicity $1 < e < n$.

The remaining (and more interesting) case will be

Proposition 12 *If a is a primitive n-th root of unity, then every fat point module of $H = H_a(\mathbf{sl_2})$ of multiplicity n is of the form*

$$F = H/(Hl_1 + Hl_2 + H(N^n - \lambda T^n))$$

where $\lambda \neq 0$ and $V(l_1,l_2)$ determines a line module of H.

Proof : We will scetch the main arguments. Details can be filled in from [26]. We know already that F is a quotient of a line module $L = H/(Hl_1+Hl_2)$ and we will consider the case here where L is the secant line determined by a point p on the conic and a point n on the plane $V(N)$, the remaining cases reduce to some three dimensional regular algebra and are easier. Observe, that we already characterized the line modules of H going through p as all the lines lying in the plane determined by $p, \sigma(p)$ and o so we have complete knowledge of the possible l_i.

In $Proj(H)$, the kernel of the epimorphism $L \longrightarrow F$ can be shown to be a shifted line module

$$0 \longrightarrow L'(-n) \longrightarrow L \longrightarrow F \longrightarrow 0$$

where L' is the line passing through $\sigma^{-n}(p) = p$ and n. Hence $L' = L$ and we have to find degree n maps from L to itself.

This can either be done by projecting to p or n and iterating this process as in [3] or [26] or by noticing that we have already two degree n central elements T^n and N^n and hence a $\mathbb{P}^1$ of degree n maps from L to itself given by multiplication with $\Omega = \alpha T^n + \beta N^n$. As Ω acts as a non-zero divisor on L, the Hilbert series of $L/\Omega L$ is $\frac{n}{1-t}$.

If $\alpha\beta = 0$, the quotient $L/\Omega L$ cannot be a fat point module as it has point module quotients (either p or n). However, if $\alpha\beta \neq 0$ we claim that the quotient is indeed a fat point module of multiplicity n. Assume otherwise, then $L/\Omega L$ would have a fat point quotient F_e of multiplicity $e < n$. In $Proj(H)$ we would then have a sequence

$$0 \longrightarrow L'(-e) \longrightarrow L \longrightarrow F_e \longrightarrow 0$$

where L' should be a line module corresponding to the line passing through the points n and $\sigma^{-e}(p)$. However, for all but a dimension one family of n (where a different argument is needed) n does not lie in the plane determined by $\sigma^{-e}(p), \sigma^{-e+1}(p)$ and o and so L' is not a line module for H so the above sequence cannot exist.

Finally, by a dimension argument one concludes that all fat point module quotients of L of multiplicity n are of the prescribed form. $\square$

Using the terminology of [16] one deduces from the foregoing proof that the ramification locus of H is not pure of codimension one (for example, the conic at infinity is a codimension two component) and hence the central $Proj$ has singularities by [16, Thm. 1]. Hence, the 4-dimensional variety corresponding to the center Z of $H_a(\mathbf{sl}_2)$ has lots of singularities. We leave the interpretation of these singularities in deformed Minkowski space to your imagination.

References

[1] J. Alev and F. Dumas. Sur le corps des fractions de certaines algèbres quantiques. J.Alg. (to appear)

[2] M. Artin and W. Schelter, Graded algebras of global dimension 3, Advances in Math. 66 (1987) 171-216

[3] M. Artin. Geometry of quantum planes. Contemp. Math. 124 (1992) 1-15

[4] M. Artin, J. Tate and M. Van den Bergh, Some algebras related to automorphisms of elliptic curves, The Grothendieck Festschrift Vol. 1, 33-85, Birkhauser, Boston (1990)

[5] M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991) 335-388

[6] K. Bauwens and L. Le Bruyn. Conformal $\mathbf{osp}(1,2)$. (in preparation)

[7] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura. A quantum Lorentz group. Intern. J. Mod. Phys. A 6 (1991) 3081-3108

[8] M. Gerstenhaber.

[9] M. Green, J. Schwarz and E. Witten. "Superstring theory, Volume 1". Cambridge University Press (1987)

[10] N. Jing and J. Zhang. Quantum Weyl algebras and deformations of $U(\mathbf{g})$. Michigan preprint (1993)

[11] P.P. Kulish and E.K. Sklyanin. Algebraic structures related to reflection equations. J. Phys. A : Math. Gen. 25 (1992) 5963-5975

[12] L. Le Bruyn, S.P. Smith and M. Van den Bergh, Central extensions of three dimensional Artin-Schelter regular algebras, Math.Zeitschrift (to appear)

[13] L. Le Bruyn. Two remarks on Witten's quantum $\mathbf{sl}_2$ enveloping algebras. Comm.Alg. 22 (1994) 865-876

[14] L. Le Bruyn and F. Van Oystaeyen. Quantum sections and gauge algebras. Publ. Mathématiques 36 (1992) 693-714

[15] L. Le Bruyn and S.P. Smith. Homogenized $\mathbf{sl}_2$. Proc. AMS 118 (1993) 725-730

[16] L. Le Bruyn. Central singularities of quantum spaces. UIA-preprint 94-06.

[17] T. Levasseur and S.P. Smith. Modules over the 4-dimensional Sklyanin algebra. Bull.Soc.Math. de France 121 (1993) 35-90

[18] Li Huishi and F. Van Oystaeyen. Global dimension and Auslander regularity of Rees rings. Bull.Soc.Math.Belg. 43 (1991) 59-87

[19] S. Majid. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. Commun. Math. Phys. 156 (1993) 607-638

[20] S. Majid. Braided geometry : a new approach to q-deformations. DAMTP-preprint (1993)

[21] S. Majid. Examples of braided groups and braided matrices. J. Math. Phys. 32 (1991) 3246-3253

[22] S. Naculich, H.A. Riggs and H.J. Schnitzer. Simple-current symmetries, rank-level duality, and linear skein relations for Chern-Simons graphs. Nuclear Physics B 394 (1993) 445-506

[23] O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino. q-Deformed Poincaré algebra. Commun. Math. Phys. 150 (1992) 495-518

[24] M. Pillin, W.B. Schmidke and J. Wess. q-Deformed relativistic one-particle states. Nuclear Phys. B 403 (1993) 223-237

[25] W.B. Schmidke, J. Wess and B. Zumino. A q-deformed Lorentz algebra. Z. Phys. C - Particles and Fields 52 (1991) 471-476

[26] S.P. Smith. The 4-dimensional Sklyanin algebra at points of finite order. Univ. Washington preprint (1992)

[27] S.P. Smith and J. Staniskis. Irreducible representations of the 4-dimensional Sklyanin algebra. J.Alg. (1993)

[28] J. T. Stafford. Regularity of algebras related to the Sklyanin algebra. Univ. Michigan preprint (1991)

[29] E. Witten. Quantum field theory and the Jones polynomial. Commun. Math. Phys. 121 (1989) 351-399

[30] E. Witten. Gauge theories and integrable lattice models. Nuclear Physics B 322 (1989) 629-697

[31] E. Witten. Gauge theories, vertex models and quantum groups. Nuclear Physics B 330 (1990) 285-346