Homological Properties of Braided Matrices

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Central singularities of certain quantum spaces

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Abstract

Let A be a positively graded Auslander-regular algebra satisfying the Cohen-Macaulay property which is a finite module over its center. We show that $Proj(A)$ is a sheaf of tame orders which are Cohen-Macaulay modules over their centers which in turn are integrally closed Cohen-Macaulay algebras. Moreover, the singular locus of the center coincides with the non-Azumaya locus provided the last one has codimension at least 2. Weaker results are shown in case A is Auslander-Gorenstein.

Throughout, $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \ldots$ is a positively graded affine $\mathbb{C}$-algebra generated in degree 1 which is a finite module over its center C which is itself a positively graded affine $\mathbb{C}$-algebra (though not necessarely generated in degree 1). Examples of current interest are Sklyanin algebras associated to a torsion point on an elleptic curve [6] and quadratic quantum algebras at roots of unity.

All these examples have nice homological properties, they are Auslander-regular satisfying the Cohen-Macaulay property. A Noetherian ring R is

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said to satisfy the Auslander condition if for each $q \geq 0$ and every R-submodule N of $Ext^q_R(M,R)$ (M a f.g. R-module) one has that $j_R(N) = inf\{p : Ext^p_R(N,R) \neq 0\}$ is $\geq q$. R is said to be Auslander-Gorenstein (resp. Auslander-regular) if it satisfies the Auslander condition and has finite injective (resp. global) dimension. If R has finite Gelfand-Kirillov dimension $GKdim(R) = n$ then R is said to satisfy the Cohen-Macaulay property if for every f.g. nonzero R-module M one has that $GKdim(M) + j_R(M) = n$. We refer to [3] for more details.

For the positively graded algebra A we denote by $\mathcal{O}_A$ the sheaf of algebras over $Proj(C)$ associated to A, i.e. on the affine open piece $X(c)$ corresponding to a homogenous element $c \in C$ we put the algebra $(A_c)_0$ the part of degree zero of the central localization of A at powers of c. Let $\mathcal{O}_Z$ be the central sheaf of $\mathcal{O}_A$ and we are interested in local properties of this 'center of $Proj(A)$'.

Hence, we may restrict to an affine open set $X(c)$ of $Proj(C)$ and we consider $Z_c = \Gamma(X(c), \mathcal{O}_Z)$ and $\Lambda_c = \Gamma(X(c), \mathcal{O}_A)$. By definition, Z_c is the center of Λ_c and contains $C_c = \Gamma(X(c), \mathcal{O}_C)$. We have

Lemma 1 *If A is Auslander-Gorenstein (resp. Auslander-regular) satisfying the Cohen-Macaulay property then so is Λ_c.*

Proof : As A is generated in degree 1 we have that A_c is a strongly graded ring (i.e. $(A_c)_{-1}.(A_c)_1 = (A_c)_0 = \Lambda_c$). For, if $deg(c) = k$ and $c = \sum_i d_i l_i$ with $d_i \in A_{k-1}$ and $l_i \in A_1$ then $1 = c^{-1}.c = \sum_i (c^{-1}.d_i).l_i \in (A_c)_{-1}.(A_c)_1$ and as the right hand side is an ideal of $(A_c)_0$ we are done. Now, by e.g. [4] there is a natural equivalence of categories between $A_c - gr$ and $\Lambda_c - mod$ (the categories of graded resp. usual modules). As all our properties first go up under central localization at one element to A_c and are merely category theoretical they give the corresponding results on Λ_c. □

Proposition 1 *If A is Auslander-Gorenstein satisfying the Cohen-Macaulay property, then Λ_c is a Cohen-Macaulay module over its center Z_c. If A is Auslander-regular satisfying the Cohen-Macaulay property, then Z_c is an integrally closed Cohen-Macaulay domain over which Λ_c is a tame order.*

Proof : As all prime ideals P_i of Λ_c lying over the same central prime p have the same $GKdim(\Lambda_c/P_i)$ and hence the same grade number $j_{\Lambda_c}(\Lambda_c/P_i)$,

Λ_c is locally a moderated Gorenstein algebra as in [2, IV.1.1]. But then by [2, Th. IV.1.2] (which is a variation on a result of Vasconcelos [7]) we have that Λ_c is locally a Cohen-Macaulay module over its center. The properties of Z_c in the Auslander-regular case follow in a similar manner from [2, Prop. IV.1.5 and IV.1.6]. The last property follows from [2, Prop.IV.1.7]. $\square$

After these preliminaries we want to relate the non-singular locus of Z_c with the non-Azumaya locus of Λ_c whenever this last locus is of codimension greater than 1 in $Spec(Z_c)$. Observe that this (strong) condition is satisfied in many interesting examples provided $GKdim(A)$ is large enough (e.g. in the Sklyanin case it is often satisfied if $GKdim(A) \geq 4$, e.g. [5]). Let us call the singular locus $Proj(Z)_{sing}$ and the non-Azumaya locus $Proj(A)_{ramif}$,then we have the following

Theorem 1 *Let A be a positively graded algebra satisfying the Auslander-condition and the Cohen-Macaulay property. Assume (with notations as above) that for all c the non-Azumaya locus of Λ_c in $p^{-1}(X(c))$ has codimension > 1 where $p : Proj(Z) \to Proj(C)$.*

1. *If A is Auslander-Gorenstein, then $Proj(Z)_{sing} \supset Proj(A)_{ramif}$.*
2. *If A is Auslander-regular, then $Proj(Z)_{sing} = Proj(A)_{ramif}$.*

Proof : (1) : If $x \in p^{-1}(X(c))$ a non-singular point of $Proj(Z)$, then $(\Lambda_c)_x$ is a Cohen-Macaulay module over the regular center $(Z_c)_x$ and hence is a projective $(Z_c)_x$- module. On the other hand, by the assumption $(\Lambda_c)_x$ is a reflexive Azumaya algebra (meaning that it is a reflexive module and all localizations at height one primes of $(Z_c)_x$ are Azumaya, e.g. [1]). Hence as a reflexive Azumaya algebra which is projective over its center $(\Lambda_c)_x$ is Azumaya by e.g. [1] and so x does not belong to $Proj(A)_{ramif}$.

(2) : If $x \in p^{-1}(X(c))$ not lying in $Proj(A)_{ramif}$, then $(\Lambda_c)_x$ is an Azumaya algebra of finite global dimension with center $(Z_c)_x$ which then has to be a regular local ring,so x does not belong to $Proj(Z)_{sing}$. $\square$

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