

Homological Properties of Braided Matrices

Lieven Le Bruyn*

Departement Wiskunde en Informatica

UIA,B-2610 Wilrijk (Belgium)

lebruyn@wins.uia.ac.be

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1 Introduction

S. Majid [9] associates to every solution R of the quantum Yang-Baxter equation a quadratic algebra $B(R)$ called the ring of braided matrices. If R is the classical solution corresponding to the semi-simple Lie algebra \mathfrak{g} , then the quantum enveloping algebra $U_q(\mathfrak{g})$ is an epimorphic image of $B(R)$, see [10]. In the simplest case when $\mathfrak{g} = \mathfrak{sl}_2$, $B(R)$ is a degeneration of the 4-dimensional Sklyanin algebras [11] studied extensively recently, a.o. [17], [18].

For every simple Lie algebra \mathfrak{g} there exist multi-parameter solutions of the Yang-Baxter equation. In this paper we will show that for $\mathfrak{g} = \mathfrak{sl}_n$ the

*research associate of the NFWO (Belgium)

associated braided matrices (which depend on $1 + \frac{(n-1)(n-2)}{2}$ parameters) have all the nice ringtheoretical properties : they are Auslander regular domains, Noetherian maximal orders and satisfy the Cohen-Macaulay property.

In view of these results and the classical \mathfrak{sl}_2 case mentioned above it would be interesting if these algebras turn out to be degenerations of even more interesting quadratic Auslander regular algebras.

The strategy of the proof has three main parts. First, we show that $B(R)$ has the same Hilbert series as the commutative polynomial ring on n^2 variables. This is essentially an application of Majid's transmutation theory [12]. First we show that we can always take R to be a regular solution of the Yang-Baxter equation, meaning that a quotient of the FRT-algebra $A(R)$ is a dual quasi-triangular Hopf algebra. Then transmutation asserts that $B(R)$ and $A(R)$ have the same Hilbert series and the result for $A(R)$ was proved by M. Artin, W. Schelter and J. Tate in [2].

The second step is to find a normalizing system of $2n - 1$ generators such that the quotient is braided matrices corresponding to \mathfrak{sl}_{n-1} . The main point is to show regularity of these elements at each stage which follows by a simple argument from our knowledge of the Hilbert series.

The final step is inducing all properties up through this normalizing system. This is a combination of filtered techniques with twisting á la [3].

Presumably the same strategy may be used to prove similar results for the other multi-parameter solution sets. In a separate paper we will investigate the geometry of these braided matrices and relate them to the classical case of the homogenized enveloping algebra of \mathfrak{sl}_n as studied in [4] and [5].

2 Dual quasi-triangularity of $A(R)$

The multi-parameter solution R to the Yang-Baxter equation of type A_{n-1} has the following form

$$R_{kl}^{ij} = \delta_k^i \delta_l^j M_{ij} + \delta_l^i \delta_k^j L_{ij}$$

where

$$M_{ij} = \delta^{ij} + \Theta^{ji} \frac{1}{q_{ij}} + \Theta^{ij} \frac{q_{ji}}{r^2}$$

$$L_{ij} = \Theta^{ij} (1 - r^{-2})$$

see a.o. [16]. In these formulas, Θ^{ij} equals 1 if $i > j$ and zero otherwise, q_{ij} with $i < j$ are the multi-parameters and r is the deforming parameter. One recovers the usual deformation by setting $q_{ij} = r = q$ for all $1 \leq i, j \leq n$.

The corresponding function type algebras $A(R)$ (quantum GL_n) are generated by n^2 non-commuting variables t_j^i (corresponding to the position (i, j) in an $n \times n$ matrix) and satisfying the following defining quadratic relations

$$eqn(i, j, k, l) : R_{mn}^{ik} t_j^m t_l^n = R_{jl}^{mn} t_n^k t_m^i$$

(sum over m, n). Using the above form for R this equation becomes

$$M_{ik} t_j^i t_l^k - M_{jl} t_l^k t_j^i = (L_{lj} - L_{ik}) t_j^k t_l^i$$

These algebras were studied independently by M. Artin, W. Schelter, J. Tate [2], A. Sudbery [20] and N. Reshitikhin [15]. Among the ringtheoretical properties proved in [2] we mention the fact that $A(R)$ is an iterated Ore extension and hence is an Auslander regular domain of global dimension n^2 having Hilbert series $\frac{1}{(1-t)^{n^2}}$. In order to translate the results of [2] to the ring $A(R)$ one has to take into account that $p_{ij} = \frac{r^2}{q_{ji}}$ and that $\lambda = \frac{1}{r^2}$.

Moreover, $A(R)$ becomes a bialgebra if $\Delta(t_j^i) = t_k^i \otimes t_j^k$ (sum over k) and $\epsilon(t_j^i) = \delta_j^i$. There is a grouplike determinant element

$$D = \sum_{\pi \in S_n} \sigma(q, \pi) \prod_{i=1}^n t_{\pi(i)}^i$$

which is normalizing. Here, the coefficient is determined for each partition $\pi \in S_n$ to be

$$\sigma(q, \pi) = \prod_{i < j, \pi(i) > \pi(j)} (M_{\pi(i), \pi(j)})^{-1}$$

We want to determine when R is regular, that is, the quotient $A = A(R)/(D-1)$ is a dual quasi-triangular Hopf algebra. A necessary condition clearly is that D has to be a central element of $A(R)$. This restriction is easily deduced from [2, Th.3] (see also [16]). In terms of M_{ij} these conditions can be described as

$$\prod_{i=1}^n M_{ij} = r^{1-n}$$

for every $j = 1, \dots, n$. The number of remaining free parameters is $1 + \frac{(n-1)(n-2)}{2}$.

If the above conditions are satisfied, $A = A(R)/(D - 1)$ is a well defined Hopf algebra (e.g. use [2] for the antipode). It is obvious from the explicit form of the dual quasi-triangular structure of $A(R)$ as an array of R -matrices [14, §5.2] that the dual quasi-triangular structure descends to the quotient if D acts as the identity on the fundamental representation ρ^+ , on the conjugate fundamental representation ρ^- of $A(R)$ and on their respective anti-representations ρ_a^+ and ρ_a^- . These representations $A(R) \rightarrow M_n(\mathcal{C})$ are defined by

$$\rho^+(t_j^i)_l^k = \lambda R_{jl}^{ik} \text{ and } \rho^-(t_j^i)_l^k = \lambda^{-1}(R^{-1})_{lj}^{ki}$$

and the anti-representations

$$\rho_a^+(t_j^i)_l^k = \lambda R_{lj}^{ki} \text{ and } \rho_a^-(t_j^i)_l^k = \lambda^{-1}(R^{-1})_{jl}^{ik}$$

for some $\lambda \in \mathcal{C}^*$ which corresponds to a suitable normaliation of the R -matrix. Our job will be to verify that we can choose λ such that D acts as the identity in all four representations. Perhaps the following result is known among specialists.

Proposition 1 *Let R be a multi-parameter solution of type A_{n-1} to the quantum Yang-Baxter equation such that D is central in $A(R)$. Then, the quotient $A = A(R)/(D - 1)$ is a dual quasi-triangular Hopf algebra.*

Proof : We have to verify that D acts as 1 on the fundamental (anti)representation and on the conjugate fundamental (anti)representation. Now,

$$\rho^+(D) = \lambda^n \left(\sum_{\pi \in S_n} c(q, \pi) R_{\pi(1), m_1}^{1, k} R_{\pi(2), m_2}^{2, m_1} \dots R_{\pi(n), l}^{n, m_{n-1}} \right)_l^k$$

We claim that only the terms with $\pi = id$ contribute. For, look at the minimal i s.t. $\pi(i) > i$ then according to the form of the R -matrix the term

$$R_{\pi(i), m_i}^{i, m_{i-1}} = \delta_{m_i}^i \delta_{\pi(i)}^{m_{i-1}} L_{i\pi(i)} = 0$$

Therefore, $\rho^+(D) =$

$$\lambda^n (R_{1, m_1}^{1, k} R_{2, m_2}^{2, m_1} \dots R_{n, l}^{n, m_{n-1}})_l^k = \lambda^n (M_{1k} M_{2k} \dots M_{nk})_k^k = \lambda^n (r^{1-n})_k^k$$

where the last equality follows from the centrality conditions on D . Hence, if we choose $\lambda = \sqrt[n]{r^{n-1}}$ then $\rho^+(D) = 1$. The same argument gives $\rho_a^+(D) = 1$.

As for the conjugate fundamental representation, observe first that

$$(R^{-1})_{jl}^{ik} = \delta_j^i \delta_l^k M_{ik}^{-1} + \delta_l^i \delta_j^k \Theta^{ij} (1 - r^2)$$

where $M_{ij}^{-1} = \delta^{ij} + \Theta^{ij} q_{ij} + \Theta^{ij} \frac{r^2}{q_{ji}}$ so $M_{ij}^{-1} = r^2 M_{ji}$. A similar argument as above shows that $\rho^-(D)$ is equal to

$$\lambda^{-n} ((R^{-1})_{m_1,1}^{k,1} \dots (R^{-1})_{l,n}^{m_{n-1},n})_l^k = (M_{k_1}^{-1} M_{k_2}^{-1} \dots M_{k_n}^{-1})_k^k = (1)_k^k$$

and similarly for $\rho_a^-(D)$, finishing the proof. \square

3 Defining equations of $B(R)$

We will now turn to the braided matrices $B(R)$ formed from these type A_{n-1} R -matrices. Their properties will resemble those of enveloping algebras of semi-simple Lie algebras. In this section we aim to prove the following

Theorem 1 *Let R be a multi-parameter solution of type A_{n-1} of the quantum Yang-Baxter equation. Then, there exists a regular solution R' such that $B(R) = B(R')$.*

We recall from [13] that $B(R)$ is generated by the n^2 variables u_j^i (again represented by the entries of an $n \times n$ matrix) satisfying the defining quadratic relations

$$eqn(i, j, k, l) : R_{ab}^{ki} u_c^b R_{jd}^{ca} u_l^d = u_a^k R_{bc}^{ai} u_d^c R_{jl}^{db}$$

(sum over a, b, c, d). As any R -tensor term has two terms

$$\begin{array}{c} i & k \\ | & | \\ j & l \end{array} \delta_j^i \delta_l^k M_{ik} \quad \begin{array}{c} i & k \\ \diagdown & / \\ j & l \end{array} \delta_l^i \delta_j^k L_{ik}$$

and as any proper cross term involves a Θ^{ik} term leading to an inequality condition, it is easy to eliminate and limit the summation indices in each of the 4 different cross symbols.

Lemma 1 *The equation $eqn(i, j, k, l)$ can be rewritten as*

$$M_{ki}M_{jk}u_j^i u_l^k - M_{li}M_{jl}u_l^k u_j^i + (L_{ki} - L_{lj})M_{ji}u_j^k u_l^i = \\ \delta_j^i \sum_{a>i} (L_{ai}L_{li} - L_{ki}L_{ai})u_a^k u_l^a + \delta_l^i \sum_{a>i} M_{ji}L_{ai}u_a^k u_j^a - \delta_j^k \sum_{a>j} M_{ki}L_{ak}u_a^i u_l^a$$

Proof : Let us compute the left hand term $R_{ab}^{ki}R_{jd}^{ca}u_c^b u_l^d$ by cross-symbols.

We obtain

$\begin{array}{cc} k & i \\ & \\ a & b \end{array}$	$\begin{array}{cc} c & a \\ & \\ j & d \end{array}$	$M_{ki}M_{jk}u_j^i u_l^k$
$\begin{array}{cc} k & i \\ & \\ a & b \end{array}$	$\begin{array}{cc} c & a \\ \diagdown & \diagup \\ j & d \end{array}$	$\delta_j^i \sum_{c>j} M_{ki}L_{cj}u_c^i u_l^c$
$\begin{array}{cc} k & i \\ \diagdown & \diagup \\ a & b \end{array}$	$\begin{array}{cc} c & a \\ & \\ j & d \end{array}$	$L_{ki}M_{ji}u_j^k u_l^i$
$\begin{array}{cc} k & i \\ \diagdown & \diagup \\ a & b \end{array}$	$\begin{array}{cc} c & a \\ \diagdown & \diagup \\ j & d \end{array}$	$\delta_j^i \sum_{c>i} L_{ki}L_{ci}u_c^k u_l^c$

Similarly, one calculates the right hand side leading to the claimed expression.

□

We can divide both sides of $eqn(i, j, k, l)$ by M_{ji} and define for all $u, v, w \in \{1, 2, \dots, n\}$: $M_{uvw} = M_{uv} \cdot M_{vw} \cdot (M_{uw})^{-1}$. Then, the foregoing expression becomes

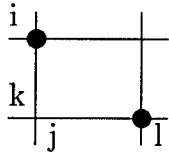
$$M_{jki}u_j^i u_l^k - M_{jli}u_l^k u_j^i + (L_{ki} - L_{lj})u_j^k u_l^i = \\ (1 - r^{-2})(\delta_l^i \sum_{a>i} u_a^k u_j^a - \delta_j^k \sum_{a>j} u_a^i u_l^a + \delta_j^i \sum_{a>i} (L_{lj} - L_{ki})u_a^k u_l^a)$$

Thus, the defining relations of $B(R)$ depend only upon r and the $\frac{(n-1)(n-2)}{2}$ parameters M_{nij} as $M_{uvw} = M_{nuv}M_{nvw}(M_{nuw})^{-1}$. One verifies that one can always find M'_{ij} for $1 \leq i, j \leq n-1$ such that $M_{nij} = M'_{nij}$ where the M'_{nk} are

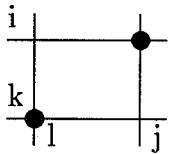
determined by satisfying the centrality restrictions of the foregoing section. If R' is the associated multi-parameter type A_{n-1} solution of the quantum Yang-Baxter equation, then R' is regular by proposition 1 and $B(R) = B(R')$ by the above, finishing the proof of the theorem.

From the form of the defining equations we see that they not only depend on the relative position of the entries in the matrix but also on their position with respect to the main diagonal of the matrix.

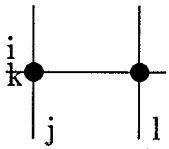
If none of (i, j) , (k, j) or (i, l) lie on the main diagonal, the situation is very similar to the $A(R)$ case. In this case the defining relations are



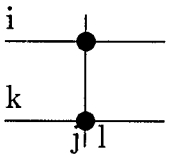
$$M_{jki} u_j^i u_l^k = M_{jli} u_l^k u_j^i$$



$$M_{jki} u_j^i u_l^k = M_{jli} u_l^k u_j^i + (r^{-2} - 1) u_j^k u_l^i$$

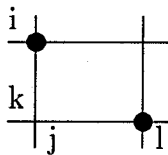


$$u_j^i u_l^k = r^2 M_{jli} u_l^k u_j^i$$



$$r^2 M_{jki} u_j^i u_l^k = u_l^k u_j^i$$

However, if one of the vertices of the square lie on the main diagonal we get extra terms. For example, if $j = k$ we get the following commutation relations



$$u_j^i u_l^k - M_{jli} u_l^k u_j^i = (r^{-2} - 1) \sum_{a>j} u_a^i u_l^a$$

And similar expressions for the other cases. From the defining equations it also follows that the diagonal matrix variables form a commutative subalgebra (cfr. the usual torus in GL_n). In fact, these braided matrices have similar properties as the homogenized enveloping algebra of \mathfrak{sl}_n although we may recover other rings in the semi-classical limit.

4 The Hilbert series of $B(R)$

In this section we will prove the following result

Theorem 2 *Let R be a multi-parameter solution of the Yang-Baxter equation of type A_{n-1} . Then, $B(R)$ has Hilbert series $(1-t)^{-n^2}$.*

As usual, this is essentially an application of Bergman's diamond lemma [1]. For the corresponding FRT-algebras $A(R)$ the Hilbert series was determined in [2]. From the specific form of the defining equations of $A(R)$ it follows that it is sufficient to restrict to the case of 3×3 matrices to solve all the overlap-ambiguities. As this limits the calculations to just a few cases these were then carried out by hand. For $B(R)$ however we have seen that the defining equations do not force trinomials to be expressible in the entries of a 3×3 submatrix of the matrix (u_j^i) . We will circumvent this problem by Majid's theory of transmutations [11].

By the result of the foregoing section we may restrict attention to the case when R is a regular multi-parameter solution of type A_{n-1} . This means that there is a dual quasi-triangular Hopf algebra A defining a braided monoidal category (the A -comodules) such that $B(R)$ is a bialgebra in this category [13, §5]. In fact, the original motivation for introducing $B(R)$ was that A does not act covariantly on $A(R)$ (by the adjoint coaction induced by the projection $A(R) \rightarrow A$). However, if we keep the coalgebra structure of $A(R)$ but define a different algebra multiplication then we obtain $B(R)$ on which A acts covariantly. This, in turn, clearly distorts the bialgebra structure which

we can save however if we change the tensor products. For more details on transmutation theory we refer the reader to [12] and [13].

For our purposes it is important to note that $A(R) = B(R)$ as graded vectorspaces and hence have the same Hilbert series. As the Hilbert series of $A(R)$ is $(1-t)^{-n^2}$ by [2] this finishes the proof of the theorem.

One can also give a quantum-group free proof of this result as follows. For bi-invertible solutions to the Yang-Baxter equation which are not necessarily regular S. Majid has given an inductive procedure to relate monomials in $A(R)$ and $B(R)$ in each degree [13]. The first three maps are

$$T : B(R) \rightarrow A(R)$$

$$\begin{aligned} T_1(u_j^i) &= t_j^i \\ T_2(u_j^i u_l^k) &= R_{ad}^{ic} \bar{R}_{jc}^{bk} t_b^a t_l^d \\ T_3(u_j^i u_l^k u_n^m) &= R_{aq}^{ip} R_{dy}^{aw} \bar{R}_{cw}^{bv} \bar{R}_{jp}^{ck} R_{sz}^{qy} \bar{R}_{lv}^{um} t_b^{dts} t_u^{zn} \end{aligned} \quad (1)$$

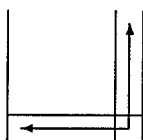
where $\bar{R} = ((R^{t_2})^{-1})^{t_2}$ where t_2 denotes transposition in the second factor.

It is then an amusing exercise in cross-symbol computations to show that these maps are well-defined and that they are triangular when restricted to ordered trinomials. From this fact when then deduces that all overlap ambiguities in $B(R)$ can be solved and hence that ordered (in the lexicographic ordering) monomials form a basis for $B(R)$.

5 A normalizing reduction system for $B(R)$

An ordered set $a_1, \dots, a_k \in A$ is said to be a normalizing reduction system iff the image of a_i is normalizing and regular in the quotientring $\bar{A}_{i-1} = A/(Aa_1 + \dots + Aa_{i-1})$ for each $i = 1, \dots, n$. In this section we will prove

Theorem 3 *Let R be a multi-parameter solution of the Yang-Baxter equation of type A_{n-1} . Then, $\{u_{i,n}, u_{n,i} : 1 \leq i \leq n\}$ ordered under inverse lexicographic order is a normalizing reduction system for $B(R)$ with total quotient braided matrices associated to a solution of type A_{n-2} .*



$$u_{n,n}, u_{n,n-1}, \dots, u_{n,1}, u_{n-1,n}, \dots, u_{1,n}$$

We will first show that this set is a normalizing system. As $eqn(n, n, k, l)$ is

$$M_{nkn}u_n^n u_l^k = M_{nl n}u_l^k u_n^n$$

we see that $u_{n,n}$ is a normalizing element. Next, $eqn(n, j, k, l)$ is

$$M_{jkn}u_j^n u_l^k - M_{jln}u_l^k u_j^n = L_{lj}u_j^k u_l^n + (1 - r^{-2})\delta_j^k \sum_{a>j} u_a^n u_l^a$$

As the right hand side lies in the ideal $(u_{n,n}, u_{n,n-1}, \dots, u_{n,j+1})$ whence the image of u_j^n is normalizing in the quotient. Finally, $eqn(i, n, k, l)$ is

$$M_{nki}u_n^i u_l^k - M_{nli}u_l^k u_n^i = -L_{ki}u_n^k u_l^i + (1 - r^{-2})\delta_n^i \sum_{a>i} u_a^k u_n^a$$

and again the right hand side belongs to the ideals $(u_{n,n}, u_{n-1,n}, \dots, u_{i+1,n})$ proving the claim.

Moreover, it is easy to see from the defining equations that the quotient $B(R)/(u_{i,n}, u_{n,i} : 1 \leq i \leq n)$ is indeed braided matrices associated to a solution of type A_{n-2} . Therefore, it remains to show that the system is a reduction system i.e. that each of the images is a regular element in the corresponding quotients. This follows from the knowledge of the Hilbert series of $B(R)$ and of its total quotient (which is braided matrices of type A_{n-2}) and the following lemma

Lemma 2 *Let A be a graded algebra with Hilbert series $(1 - t)^m$ and let $\{a_1, \dots, a_k\}$ be a normalizing system of degree one elements such that the quotient $A/(a_1, \dots, a_k)$ has Hilbert series $(1 - t)^{m-k}$, then $\{a_1, \dots, a_k\}$ is a normalizing reduction system.*

Proof : For each i let $Ann(a_i)$ denote $\{q \in \bar{A}_{i-1} : q \cdot \bar{a}_i = 0\}$. For each i we have an exact sequence of graded vectorspaces

$$0 \rightarrow Ann(a_i) \rightarrow \bar{A}_{i-1} \rightarrow \bar{A}_{i-1} \rightarrow \bar{A}_i \rightarrow 0$$

where the middle map is right multiplication with the normalizing degree one element \bar{a}_i . This gives the following relations between the Hilbert series

$$\mathcal{H}(\bar{A}_i, t) = (1 - t)\mathcal{H}(\bar{A}_{i-1}, t) + t\mathcal{H}(Ann(a_i), t)$$

for every i . Combining we obtain,

$$\mathcal{H}(\overline{A}_k, t) - (1-t)^k \mathcal{H}(A, t) = \\ (1-t)^{k-1} t \mathcal{H}(\text{Ann}(a_1), t) + (1-t)^{k-2} t \mathcal{H}(\text{Ann}(a_2), t) + \dots + t \mathcal{H}(\text{Ann}(a_k), t)$$

Using the assumption the left hand side is zero. Hence, because Hilbert series have positive integer coefficients it follows that

$$\mathcal{H}(\text{Ann}(a_i), t) = 0$$

for every i finishing the proof. \square

6 The main result

In this section we will prove

Theorem 4 *Let R be a multi-parameter solution of the Yang-Baxter equation of type A_{n-1} , then $B(R)$ is an Auslander regular domain, a Noetherian maximal order and satisfies the Cohen-Macaulay property.*

In view of results of Th. Levasseur [7] and J.T. Stafford [19] we only have to show that $B(R)$ is Noetherian, Auslander regular and satisfies the Cohen-Macaulay property.

We recall that an algebra A is said to be Auslander regular of dimension n iff $\text{gldim}(A) = n$ and for every submodule N of $\text{Ext}_A^i(M, A)$ (M any f.g. left A -module and $0 \leq i \leq n$) we have that $j(N) = \min\{j : \text{Ext}_A^j(N, A) \neq 0\} \geq i$. For a Noetherian algebra A of finite Gelfand-Kirillov dimension m we say that A satisfies the Cohen-Macaulay property iff $\text{GKdim}(M) + j(M) = m$ for every f.g. right A -module M .

As $\mathbb{C}[x]$ clearly satisfies all properties we will prove the theorem by induction on n using the regular normalizing system of the foregoing section. Therefore, the result follows from the following

Proposition 2 *Let A be a positively graded algebra with a regular normalizing element n of degree 1. If $B = A/nA$ is Noetherian, Auslander regular satisfying the Cohen-Macaulay property, then so is A .*

Proof : As n is a regular normalizing element it induces a graded automorphism τ on A . Let A_τ be the twisted algebra with respect to this automorphism as in [3]. As there is a natural equivalence between $A - gr$ and $A_\tau - gr$ (the categories of graded f.g. onesided modules), A satisfies the required properties iff A_τ does. Now, n_τ is a regular central element of degree 1 in A_τ . Then, A_τ is the homogenization (or Rees ring) of the filtered algebra $A_\tau/(n_\tau - 1) = R$. The associated graded ring of the filtered ring R is $A_\tau/(n_\tau)$ which is a twist of B so is Noetherian, Auslander-regular satisfying the Cohen-Macaulay property by assumption. Now, standard filtered techniques as in [8] entail that A_τ (and hence A) is Noetherian and Auslander-regular. For the lift of the Cohen-Macaulay property see [7]. \square

This combination of twisting and filtered techniques can also be used to determine inductively all linear subspaces in Artin's quantum space $Proj(B(R)) = B(R) - gr/(B(R)_+ - torsion)$ much as in [6]. In a forthcoming paper we will see that one can use Majid's Lie-like bracket on $B(R)$ [10] to get a more transparent description of linear subspaces as in the case of homogenizations of enveloping algebras of Lie algebras as in [4] and [5].

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