Quantum Sections of Schematic Algebras

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Quantum Sections of Schematic Algebras.

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Abstract

The *Proj* of a non-commutative graded algebra may be defined as the quotientcategory of finitely generated graded modules modulo the subcategory of finite length modules. The schematic algebras we introduce are those for which this *Proj* satisfies an "affine" covering property bringing it closer to being a "geometric" object. The term schematic refers to the fact that the use of rings of quantum sections allows to obtain a kind of scheme structure. It is shown that enveloping algebras of finite dimensional Lie algebras, quantum Weyl Algebras as well as several types of gauge algebras are schematic. Quantum sections of enveloping algebras are calculated in an explicit way.

1 Introduction

Projective n-space $\mathbb{P}^n = Proj(\mathbb{C}[X_0, \ldots, X_n])$ may be covered by the standard affine open pieces $Spec(\mathbb{C}[x_o, \ldots, \check{x_i}, \ldots, x_n])$ where $x_j = \frac{X_j}{X_i}$. The ring $\Gamma(X_i, \mathcal{O}_{\mathbb{P}^n})$, generated by the x_j over $\mathbb{C}$, is the degree zero part of the localisation of $\mathbb{C}[X_0, \ldots, X_n]$ at the multiplicatively closed set generated by X_i. Modules, (coherent sheaves of modules) over $\mathbb{P}^n$ can be described by glueing modules on the basic open sets over the intersections.

This technique does not extend to non-commutative graded algebras because certain properties of localised rings can only be obtained if one can localise at an Ore-set. In fact, one needs the existence of "enough" localisations in the sense

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that one would like to obtain the affine covering property for $Proj$ necessary to describe modules by a suitable glueing process. So we call a positively graded algebra schematic if it contains "enough" Ore-sets in a sense to be made precise in the next section.

The essential idea behind the construction of schematic algebras, and in fact the reason why the quantum-aspect enters the picture, is that a filtered ring may be viewed as a deformation of the associated graded ring via the Rees-ring. Recently the Rees-rings ("homogenisations" or "central extensions" according to some authors) of Auslander regular rings have been studied in low dimensions, e.g. [7], [4] and [5]. We show that the schematic property lifts from the associated graded ring to the Rees-ring.

The second basic ingredient of the geometry of $Proj$ we try to extend to the non-commutative situation, using M. Artins definition of $Proj$ given in [2], is the sheaf theory. For this purpose one may use the quantum-sections connected to microlocalisations first introduced in [12]; the rings of quantum-sections have a completeness property that is useful in studying their module theory but it makes the determination of the defining relations for the algebras more cumbersome. For many concrete examples appearing in quantised-ring theory it is possible to obtain the quantum-sections in a completely algebraic way from certain Ore-sets. In fact, this technique may be extended further to almost complete generality by considering the minimal localisations at pseudo-Ore-sets in the sense of [14].

The second part of this paper is devoted to explicit calculations and consequences of the schematic approach for enveloping algebras of finite dimensional Lie algebras. We may use the Ore-sets described by W. Borho and R. Rentschler [3] in order to describe the rings of quantum-sections. As a consequence of the schematic approach, we may rederive the defining equations of the point-variety of the associated quantum-space, cf [4] and [6].

2 Schematic Algebras

A filtration FA on a ring A is given by an ascending chain $\ldots \subseteq F_nA \subseteq F_{n+1}A \subseteq \ldots$ such that $1 \in F_0A, \cup_n F_nA = A, F_nAF_mA \subseteq F_{n+m}A$ for $n, m \in \mathbb{Z}$. Unless otherwise stated we restrict attention to **separated** filtrations, i.e. we assume $\cap_n F_nA = 0$. We let $G(A) = \oplus_{n\in\mathbb{Z}} F_nA/F_{n+1}A$ be the associated $\mathbb{Z}$-graded ring and $\tilde{A} = \oplus_{n\in\mathbb{Z}} F_nA \simeq \sum_n F_nAX^n \subset A[X, X^{-1}]$ is the Rees-ring of the filtration FA. For full detail on filtered rings, in particular for Zariskian filtrations, we refer to the book [9]. We may identify $1 \in F_1A$ as the central regular element $X \in \tilde{A}_1$ and then we have the deformation principle :

1. $\tilde{A}/X\tilde{A} \simeq G(A)$ as graded rings.
2. $\tilde{A}/(1-X)\tilde{A} \simeq A$ and FA is obtained from the gradation of $\tilde{A}$ via $F_nA = \tilde{A}_n \bmod (1-X)\tilde{A}$

3. $\tilde{A}_X \simeq A[X, X^{-1}]$

4. The category of filtered A-modules and filtered morphisms is equivalent to the full subcategory of $\tilde{A}$-gr consisting of the X-torsionfree graded $\tilde{A}$-modules.

Let k be an arbitrary field and let $R = k \oplus R_1 \oplus \ldots$ be generated by R_1 over k. The theory of filtered rings has benefitted a lot from this deformation principle, cf [9], e.g. for Zariskian filtrations it has been shown that several homological properties like Auslander regularity etc ... lift from the associated graded ring to the Rees-ring. Much of the secrets of the filtered ring A are hidden in the properties of the canonical graded epimorphism $\tilde{A} \longrightarrow \tilde{A}/X\tilde{A} = G(A)$; this explains why we now turn to the consideration of the following situation.

Proposition 1 *Assume that R has a central homogeneous element X of degree one. Denote the canonical epimorphism $R \to R/XR = \overline{R}$ by $\overline{\ }$ and suppose there exist homogeneous elements $t_j \in R$ (j in some index-set J) and a natural number n such that*

$$\overline{R}_{\geq n} \stackrel{def}{=} \bigoplus_{p \geq n} \overline{R}_p \subseteq \sum_{j \in J} \overline{Rt_j}$$

then for all $m \in \mathbb{N}$ there exists a natural number k such that :

$$R_{\geq k} \subseteq RX^m + \sum_{j \in J} Rt_j$$

Proof : Put $k = n + m - 1$ and consider a homogeneous element μ of degree $l \geq k$ in R. The claim is trivial when $m = 0$, so suppose $m \geq 1$. Then $l \geq n$, so by hypothesis $\overline{\mu} \in \sum_{j \in J} \overline{Rt_j}$ and thus $\exists (a_j)_{j \in J} \in R_{l-1}$ (almost all zero) , $\exists s_1 \in R_{l-1}$ such that $\mu = \sum_{j \in J} a_j t_j + s_1 X$. If $m = 1$, we are finished. Otherwise $l - 1 \geq n$, so we may apply the foregoing argument to s_1 and we obtain that s_1 can be written as $\sum_{j \in J} b_j t_j + s_2 X$ for some $b_j, s_2 \in R_{l-2}$ (almost all b_j zero). Thus $\mu = \sum_{j \in J} (a_j + b_j X) t_j + s_2 X^2$. After at most m steps, we find that μ is in $RX^m + \sum_{j \in J} Rt_j$. □

From now on, we suppose that R is a Noetherian domain, generated by a finite number of degree 1 elements and we denote $\sum_{j \geq 1} R_j$ by R_+. Define $\mathcal{L}(\kappa_+) = \{I \lhd_l R \mid \exists n \in \mathbb{N} \text{ such that } (R_+)^n \subseteq I\}$, then it is easy to check that this is a Gabriel-filter (cfr. [11] and references there for Gabriel-filters). If S is an Ore-set in R, we denote the corresponding Gabriel-filter of all left ideals of R having a non-trivial intersection with S by $\mathcal{L}(\kappa_S)$.

Definition 1 *Suppose there is a set I of homogeneous degree one elements of R such that for each $f \in I$ there exists a minimal homogeneous Ore-set S_f containing f. If $\mathcal{L}(\kappa_+) = \cap_{f \in I} \mathcal{L}(\kappa_{S_f})$, then we say that R is* **schematic**. *A schematic algebra for which the set I can be taken to be finite, is said to have the* **finite affine covering property** *(finite A.C. for short).*

Now it is easy to prove the following :

Proposition 2 *Let R be as above and assume that we are given a family of Ore-sets $(S_f)_{f\in I}$ generated by a degree one element f, then*

$$\mathcal{L}(\kappa_+) = \cap_{f\in I}\mathcal{L}(\kappa_{S_f})$$

$$\Updownarrow$$

$$\forall (s_f)_{f\in I} \in \prod_{f\in I} S_f, \exists n \in I\!N \text{ such that } (R_+)^n \subseteq \sum_{f\in I} Rs_f$$

Proof : Use the foregoing proposition and remark that for a graded ring $A = k \oplus A_1 \oplus \ldots$ which is generated by A_1 the following holds : $(A_+)^n = A_{\geq n}$. □

Example 1 *By means of the previous proposition, it is easy to see that if R is commutative, then R has the finite A.C.-property : for each generator f we choose S_f equal to $\{f^n \mid n \in I\!N\}$. This is just a restatement of the fact that $Proj(R)$ is covered by the basic affine open sets $\{P \in Proj(R) \mid f \notin P\}$ and justifies our terminology.*

Theorem 1 *Let R be a positively filtered ring with $k = F_0R$ being a field. If the associated graded ring $G(R)$ is schematic, then $\tilde{R}$, the Rees-ring, is schematic and if $G(R)$ has finite A.C. then $\tilde{R}$ has finite A.C. Conversely, if $G(R)$ is a domain and $\tilde{R}$ is schematic or has finite A.C. then the same holds for $G(R)$.*

Proof : To each given Ore-set $\bar{S}_f$ in $G(R)$, we can associate an Ore-set $\tilde{S}_f$ in $\tilde{R}$, namely (using the identification of $\tilde{R}$ as a subring of $R[X, X^{-1}]$) $\tilde{S}_f = \{sX^{\deg\sigma(s)} \mid \sigma(s) \in \bar{S}_f\}$ where σ denotes the principal map. Because $\tilde{S}_f$ maps onto $\bar{S}_f$ under the canonical epimorphism $\pi : \tilde{R} \longrightarrow \tilde{R}/X\tilde{R} \simeq G(R)$, the theorem follows easily from the two previous propositions. For the converse, given Ore-sets $\tilde{S}_f$ for $\tilde{R}$, taking their image under the canonical map π provides us with Ore-sets in $G(R)$. If we start now with $(t_f) \in \prod \pi(S_f)$, then we choose a representative $s_f \in \tilde{R}$ of t_f and thus we find a natural number n such that $\tilde{R}_{\geq n} \subseteq \sum \tilde{R}s_f$. By letting π act on this, we find $G(R)_{\geq n} \subseteq \sum G(R)t_f$. □

Example 2 *The Rees-ring of an almost commutative ring has finite A.C. (A ring R is called almost commutative if there exists a filtration on R such that the associated graded ring is commutative)*

Theorem 2 *Let R be as in the previous theorem but suppose also that the filtration is Zariskian. If $G(R)$ is a schematic domain and a maximal order, then there exist Ore-sets S_f such that*

$$R = \cap R_{S_f}$$

Proof : Set A=$\tilde{R}$. The foregoing theorem tells us that A is schematic, so $\mathcal{L}(\kappa_+) = \cap\mathcal{L}(\kappa_{\bar{S}_f})$, thus $Q_{\kappa_+}(A) = \cap A_{\bar{S}_f}$. We claim that $Q_{\kappa_+}(A) = A$ i.e. that A is $\mathcal{L}(\kappa_+)$-closed. So we have to prove that $\forall I \in \mathcal{L}(\kappa_+)$:

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Hom}_A(I, A) \\ x & \mapsto & (a \mapsto ax) \end{array}$$

is an isomorphism. The map is injective since $\tilde{R}$ is a domain. Since A is a maximal order in its quotientring Q (see [13]), we may identify $\mathrm{Hom}_A(I, A)$ with $B = \{q \in Q \mid Iq \subseteq A\}$, so $A \subseteq B \subseteq Q$. If we choose a non-zero element a of A, then $aB \subseteq Q$. Thus A and B are equivalent orders and $A = B$ follows. We now have obtained that $\tilde{R} = \cap\tilde{R}_{\bar{S}_f}$. If we set $S_f = \{s \in R \mid \sigma(s) \in \bar{S}_f\}$ then $R = \cap R_{S_f}$ follows easily from this. □

Definition 2 *We call a filtered ring* **schematic** *if the associated Rees-ring is schematic.*

It follows from theorem 1 that if the associated graded ring of a filtered ring R is a domain, then R is schematic if and only if $G(R)$ is schematic.

Lemma 1 *Given a positively graded ring R which is generated by R_1 and which is schematic by means of Ore-sets S_f, given σ a graded automorphism of R and δ a σ-derivation of degree 1, then $\forall (s_f) \in \prod S_f, \forall m \in \mathbb{N}, \exists p \in \mathbb{N}$ such that*

$$(R[x, \sigma, \delta]_+)^p \subseteq M \stackrel{def}{=} \sum R[x, \sigma, \delta]s_f + R[x, \sigma, \delta]x^m$$

where $R[x, \sigma, \delta]$ denotes the Ore-extension considered with gradation $(R[x, \sigma, \delta])_n = \oplus_{i=0}^n R_i x^{n-i}$.

Proof : Because R is schematic, we know there is a $n \in \mathbb{N}$ such that $(R_+)^n \subseteq \sum Rs_f$. Put $p = n + m$ and choose $l \geq p$. We proceed by induction on t, the number of x occurring in a monomial of $R[x, \sigma, \delta]$ of length l. The case $t = 0$ follows from $l \geq m$. Suppose all monomials of length l with occurrences of x smaller than t belong to M. If $t \geq m$, then, modulo elements of M, we may permute all x to the last place and we end with a term in $R[x, \sigma, \delta]x^m$. If $t < m$, then again modulo M we can rewrite our monomial in the form $x^t a$ with $a \in R_{l-t}$ and this one is in M because $l - t \geq n$ □

Example 3 *Quantum Weyl algebras as defined by J. Alev and F. Dumas in [1] are schematic. We briefly recall the definition. Given a $n \times n$ matrix $\Lambda = (\lambda_{ij})$ ($n \geq 2$) with $\lambda_{ij} \in k^*$ and a row vector $\overline{q} = (q_1, \ldots, q_n)$, all $q_i \neq 0$, one defines the Quantum Weyl algebra $A_n = A_n^{\overline{q},\Lambda}$ as the algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ and subjected to relations ($i < j$):*

$$x_i x_j = \mu_{ij} x_j x_i$$

$$\begin{aligned} x_i y_j &= \lambda_{ji} y_j x_i \\ y_j y_i &= \lambda_{ji} y_i y_j \\ x_j y_i &= \mu_{ij} y_i x_j \\ x_j y_j &= 1 + q_j y_j x_j + \sum_{i<j} (q_i - 1) y_i x_i \end{aligned}$$

where $\mu_{ij} = \lambda_{ij} q_i$.

Proof : We can view A_n as an iterated Ore-extension where the variables are joined in the order $x_1, y_1, x_2, y_2, \ldots$ With respect to the standard filtration, the successive associated graded rings have one of the following forms : $G(A_{k-1})[x_k, \sigma]$ or $G(A_{k-1})[x_k, \sigma][y_k, \tau, \delta] = G(A_k)$. It is easy to see that $\{x_k^n \mid n \in I\!N\}$ is an Ore-set in $G(A_{k-1})[x_k, \sigma]$ so the lemma entails that $G(A_{k-1})[x_k, \sigma]$ is schematic. This implies by turns that $G(A_{k-1})[x_k, \sigma]$ matches the conditions of the lemma. So we are left to prove that $\{y_k^n \mid n \in I\!N\}$ is an Ore-set in $G(A_{k-1})[x_k, \sigma][y_k, \tau, \delta]$. Since this is a commuting set, it suffices to prove the exchange condition for all pairs (x_i, y_k) , (y_i, y_k) $(1 \leq i \leq k-1)$ and for (x_k, y_k). This follows immediately from the relations, except for the last one, so let us do that one explicitely. In $G(A_k)$, we have

$$x_k y_k = q_k y_k x_k + \sum_{i<k} (q_i - 1) y_i x_i$$

or

$$y_k x_k = q_k^{-1} x_k y_k - q_k^{-1} \sum_{i<k} (q_i - 1) y_i x_i$$

Multiplying on the left by y_k and using the relation $y_k y_i x_i = \lambda_{ki} y_i y_k x_i = y_i x_i y_k$, we find

$$y_k^2 x_k = q_k^{-1} (y_k x_k - \sum_{i<k} (q_i - 1) y_i x_i) y_k$$

Now the lemma says that $G(A_k)$ is schematic, thus A_k is schematic. □

3 Enveloping Algebras are Schematic

Starting with a finite dimensional Lie algebra $\mathfrak{g} = \mathbb{C}\, x_1 \oplus \cdots \oplus \mathbb{C}\, x_n$ with

$$[x_i, x_j] = \sum_{k=1}^{n} \beta_{ij,k} x_k$$

one defines $H(\mathfrak{g})$, the homogenization of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, to be the positively graded algebra, generated by elements $X_1, \cdots, X_n$ together with a central element X_0 and subject to relations

$$[X_i, X_j] = \sum_{k=1}^{n} \beta_{ij,k} X_k X_0 \quad \forall i, j \in \{1, \ldots, n\}.$$

One may view $H(\mathfrak{g})$ as the Rees-ring of $U(\mathfrak{g})$ with respect to the canonical filtration, thus $H(\mathfrak{g})/(X_0-1)H(\mathfrak{g}) \simeq U(\mathfrak{g})$ and $H(\mathfrak{g})/X_0H(\mathfrak{g}) \simeq G(U(\mathfrak{g})) \simeq S(\mathfrak{g})$. $H(\mathfrak{g})$ is a quadratic Auslander-regular algebra of global dimension $n+1$ and it satisfies the Cohen-Macaulay property,[9].

The quantum space $I\!P_q(\mathfrak{g})$ of $\mathfrak{g}$ is by definition $Proj(H(\mathfrak{g}))$ in Artins sense [2]. Fix a degree one element of $H(\mathfrak{g})$, say $f = X + \gamma X_0$, where X corresponds to $x \in \mathfrak{g}$ and $\gamma \in \mathbb{C}$. (We will always use the convention that corresponding upper- and lower-case letters will denote corresponding elements of $H(\mathfrak{g})$ and $\mathfrak{g}$). Let $\mathbb{Z}E$ be the additive subgroup of $\mathbb{C}$ generated by the set E of all eigenvalues of the adjoint representation of $x : [x,-] : \mathfrak{g} \to \mathfrak{g}$. Then the minimal Ore-set in $H(\mathfrak{g})$ containing powers of X is the multiplicatively closed set generated by the set

$$S_f = \{X + (\gamma - e)X_0 : e \in \mathbb{Z}E\}$$

Hence, we can form the localization $H(\mathfrak{g})_{S_f}$ which is again an Auslander-regular algebra. As S_f consists of homogeneous elements, $H(\mathfrak{g})_{S_f}$ is a graded algebra and we define :

Definition 3 *The* **quantum sections** $\Gamma(f) = \Gamma(f, \mathcal{O}_q(\mathfrak{g}))$ *of the quantum space* $I\!P_q(\mathfrak{g})$ *associated to the 'open' set corresponding to* $f \in \mathbb{C}X_0 \oplus \mathfrak{g}$ *is the homogeneous part of degree zero of the graded algebra* $H(\mathfrak{g})_{S_f}$.

As S_f contains elements of degree one, the algebra $H(\mathfrak{g})_{S_f}$ is strongly graded meaning that $(H(\mathfrak{g})_{S_f})_i(H(\mathfrak{g})_{S_f})_j = (H(\mathfrak{g})_{S_f})_{i+j}$. Then, by the equivalence of categories ([10])

$$\Gamma(f, \mathcal{O}_q(\mathfrak{g})) - mod \simeq H(\mathfrak{g})_{S_f} - gr$$

we deduce that $\Gamma(f)$ is an Auslander-regular algebra. However, we will see that it does not have to be an affine algebra.

It is clear that one can "glue" the quantum sections $\Gamma(f, \mathcal{O}_q(\mathfrak{g}))$ with $\Gamma(g, \mathcal{O}_q(\mathfrak{g}))$ $(g \in (H(\mathfrak{g}))_1)$ over their "intersection" which has as its sections the part of degree zero of the localization of $H(\mathfrak{g})$ at the multiplicative system generated by S_f and S_g (which is automatically a twosided Ore-set). Hence, to $I\!P_q(\mathfrak{g})$ we can associate a family of section-algebras $\Gamma(f, \mathcal{O}_q(\mathfrak{g}))$ together with glueing data as in [2] for the case of a quantum space finite over its center. Using the fact that the algebras $H(\mathfrak{g})_{S_f}$ are strongly graded, it follows as in the classical case that any $M \in I\!P_q(\mathfrak{g})$ is determined by glueing the $\Gamma(X_f, \mathcal{O}_q(\mathfrak{g}))$-modules $(M_{S_f})_0$. In particular we see that a point-module is determined by a one-dimensional representation of one of the section-algebras $\Gamma(f, \mathcal{O}_q(\mathfrak{g}))$. Summarizing :

Proposition 3 *The quantum sections* $\Gamma(f)$ *are Auslander-regular algebras and there is a unique sheaf on* $I\!P_q(\mathfrak{g})$ *coinciding with* $\Gamma(f)$ *on the basic open set corresponding to* f. *Moreover, any element* M *of* $I\!P_q(\mathfrak{g})$ *is determined by the* $\Gamma(f)$*-modules* $(M_{S_f})_0$.

However, there are also noticeable differences between the quantumsections $\Gamma(f)$ and $U(\mathfrak{g})$. For instance, with the filtration used by the canonical generators (as in the next subsection) they are virtually never almost commutative. Further, they usually have larger Gelfand-Kirilov dimension by [8, Theorem 8]. We have already noticed that $H(\mathfrak{g})$ has the finite A.C.-property. It follows from theorem 2 that $U(\mathfrak{g})$ can be written as an intersection of certain localisations at Ore-sets generated by degree one elements. We now prove a stronger version of this theorem, namely that $U(\mathfrak{g})$ can be written as an intersection of two localisations.

Proposition 4 *If $i \neq j$ then $U(\mathfrak{g}) = S_{x_i}^{-1}U(\mathfrak{g}) \cap S_{x_j}^{-1}U(\mathfrak{g})$ where S_{x_k} is the multiplicatively closed set generated by all $x_k - \alpha$ with α in the abelian group generated by all eigenvalues of x_k $(k = i, j)$, cfr [3].*

Proof : Suppose $z \in T \stackrel{def}{=} S_{x_i}U(\mathfrak{g}) \cap S_{x_j}U(\mathfrak{g})$. Then $z = s^{-1}a = t^{-1}b$ for some $a, b \in U(\mathfrak{g})$ and $s \in S_{x_i}, t \in S_{x_j}$, hence $\sigma(s) = x_i^n, \sigma(t) = x_j^m$ by definition of S_{x_i} and S_{x_j}. $G(U(\mathfrak{g}))$ is a domain, so σ is multiplicative and $\sigma(z) = x_i^{-n}\sigma(a) = x_j^{-m}\sigma(b)$, or $x_j^m\sigma(a) = x_i^n\sigma(b) \in (x_i^n)$. Since $j \neq i$, we must have that $\sigma(a) \in (x_i^n)$ and thus $\deg\sigma(a) \geq n$. Therefore $\deg\sigma(z) \geq 0$ holds for all z in T. Suppose now that z is in T but not in $U(\mathfrak{g})$. We can write $\sigma(a) = x_i^n\sigma(c_1)$ for some c_1 in $U(\mathfrak{g})$. Then $\sigma(z) = \sigma(c_1)$, $0 \neq z - c_1 \in T$ and $\deg\sigma(z - c_1) < \deg\sigma(z)$. After applying this argument a finite number of steps, we find $z - c_1 - \ldots - c_r \in T$ but $\deg\sigma(z - c_1 - \ldots - c_r) < 0$, a contradiction. □

3.1 Defining equations of $\Gamma(f)$

The idea to obtain the relations of the section algebra $\Gamma(f, \mathcal{O}_q(\mathfrak{g}))$ is as follows : just write down how $f = X + \gamma X_0$ commutes with the X_i and multiply on both sides by f^{-1}. Now it is obvious that an appropriate choice of basis will simplify the aimed equations. So choose a basis $\{y_1, \ldots, y_n{=}x\}$ of $\mathfrak{g}$ such that the matrix of $\mathrm{ad}x$ with respect to this basis is in Jordan normal form. More precisely, there exists a subset I of $\{1, \ldots, n\}$ and a function $\alpha : I \to E$ such that

$$\mathrm{ad}x(y_i) = \begin{cases} \alpha(i)y_i & \text{if } i \in I \\ \alpha(j)y_i + y_{i+1} & \text{if } i \notin I \end{cases} \qquad (1)$$

where $j = \min(\{i+1, \ldots, n\} \cap I)$. For convenience, we extend the function α to $\{1, \ldots, n\}$ by letting $\alpha(i) = \alpha(j)$ if $j = \min(\{i, \ldots, n\} \cap I)$. Let Y_i be the degree one elements of $H(\mathfrak{g})$ corresponding to y_i. Set $Z_i = Y_i(X + \gamma X_0)^{-1}$ $\forall i = 1, \ldots, n$, $T = X_0(X + \gamma X_0)^{-1} = (X + \gamma X_0)^{-1}X_0$. Thus $Z_n + \gamma T = 1$. With these notations, we have :

Theorem 3 *The sectionalgebra $\Gamma(f)$ is the algebra generated by $Z_1, \ldots, Z_{n-1}$ over $\mathbb{C}[T]_{\{1+\alpha T \mid \alpha \in \mathbb{Z}E\}}$ with relations :*

- $\forall i, j \in I \setminus \{n\}$:

$$Z_i Z_j = Z_j Z_i (1+\alpha(i)T)(1+\alpha(j)T)^{-1} + \sum_{k=1}^{n} \beta_{ij,k} Z_k T (1+\alpha(j)T)^{-1} \quad (2)$$

- $\forall i \in I \setminus \{n\}, \forall j \in \{1, \ldots, n\} \setminus I$:

$$\begin{aligned} Z_i Z_j &= Z_j Z_i (1+\alpha(i)T)(1+\alpha(j)T)^{-1} + \sum_{k=1}^{n} \beta_{ij,k} Z_k T (1+\alpha(j)T)^{-1} \\ &\quad - Z_i Z_{j+1} T (1+\alpha(j)T)^{-1} \end{aligned} \quad (3)$$

- $\forall i, j \in \{1, \ldots, n\} \setminus I$:

$$\begin{aligned} Z_i Z_j &= Z_j Z_i (1+\alpha(i)T)(1+\alpha(j)T)^{-1} + \sum_{k=1}^{n} \beta_{ij,k} Z_k T (1+\alpha(j)T)^{-1} \\ &\quad - Z_i Z_{j+1} T (1+\alpha(j)T)^{-1} + Z_j Z_{i+1} T (1+\alpha(j)T)^{-1} \end{aligned} \quad (4)$$

- $\forall i \in I \setminus \{n\}$:

$$T Z_i = Z_i T (1+\alpha(i)T)^{-1} \quad (5)$$

- $\forall i \in \{1, \ldots, n\} \setminus I$:

$$T Z_i = Z_i T (1+\alpha(i)T)^{-1} - T Z_{i+1} T (1+\alpha(i)T)^{-1} \quad (6)$$

Proof : The relations (1) may be written in $U(\mathfrak{g})$ as

$$\begin{aligned} (x-\alpha(i))y_i &= y_i x \quad i \in I \\ (x-\alpha(i))y_i &= y_i x + y_{i+1} \quad i \notin I \end{aligned}$$

In $H(\mathfrak{g})$, these relations become :

$$\begin{aligned} Y_i(X+\gamma X_0) &= (X+\gamma X_0)Y_i - \alpha(i) Y_i X_0 \quad i \in I \\ Y_i(X+\gamma X_0) &= (X+\gamma X_0)Y_i - \alpha(i) Y_i X_0 - Y_{i+1} X_0 \quad i \notin I \end{aligned}$$

By multiplying these on both sides by $(X+\gamma X_0)^{-1}$, we get :

$$\begin{aligned} (X+\gamma X_0)^{-1} Y_i &= (Y_i - \alpha(i)(X+\gamma X_0)^{-1} Y_i X_0)(X+\gamma X_0)^{-1} \quad i \in I \quad (7) \\ (X+\gamma X_0)^{-1} Y_i &= (Y_i - \alpha(i)(X+\gamma X_0)^{-1} Y_i X_0)(X+\gamma X_0)^{-1} \\ &\quad - (X+\gamma X_0)^{-1} Y_{i+1} X_0 (X+\gamma X_0)^{-1} \quad i \notin I \end{aligned}$$

Consequently, $\forall i,j \in I \setminus \{n\}$:

$$\begin{aligned}
[Z_i, Z_j] &= Y_i(X+\gamma X_0)^{-1}Y_j(X+\gamma X_0)^{-1} - Y_j(X+\gamma X_0)^{-1}Y_i(X+\gamma X_0)^{-1} \\
&\overset{(7)}{=} Y_i(Y_j - \alpha(j)(X+\gamma X_0)^{-1}Y_j X_0)(X+\gamma X_0)^{-2} \\
&\quad - Y_j(Y_i - \alpha(i)(X+\gamma X_0)^{-1}Y_i X_0)(X+\gamma X_0)^{-2} \\
&= \sum_{k=1}^{n} \beta_{ij,k} Z_k T + \alpha(i) Z_j Z_i T - \alpha(j) Z_i Z_j T \qquad (8)
\end{aligned}$$

Since $\forall k \in \{1,\ldots,n\}$,

$$1 + \alpha(k)X_0(X+\gamma X_0)^{-1} = (X + \gamma X_0 + \alpha(k)X_0)(X+\gamma X_0)^{-1}$$

is invertible in $(H(\mathfrak{g})_{S_f})_0$, we can rewrite the equations (8) as follows : $\forall i,j \in I \setminus \{n\}$:

$$Z_i Z_j = Z_j Z_i (1+\alpha(i)T)(1+\alpha(j)T)^{-1} + \sum_{k=1}^{n} \beta_{ij,k} Z_k T (1+\alpha(j)T)^{-1}$$

The other equations are derived in a similar way. □

Corollary 1 *$\Gamma(X)$ is affine $\Leftrightarrow$ x is nilpotent.*

In case x is a semisimple element of $\mathfrak{g}$, $I = \{1,\ldots,n\}$ and we only have relations of type (2) and (5). If x is a semi-invariant (i.e. $\exists \lambda \in \mathfrak{g}^*$ such that $[x,y] = \lambda(y)x \quad \forall y \in \mathfrak{g}$) then α is identically zero because $E = 0$. Thus $\Gamma(f)$ is generated by $Z_1,\ldots,Z_{n-1}$ over T.

3.2 Two easy examples

Example 4 *Let $\mathfrak{g} = \mathbb{C}\,x \oplus \mathbb{C}\,y$ be the two-dimensional Lie-algebra determined by letting $[x,y] = x$.*

- *On the complement of the line X_0, we put*

$$(H(\mathfrak{g})_{\{1,X_0,X_0^2,\ldots\}})_0 \cong (U(\mathfrak{g})[X_0, X_0^{-1}])_0 \cong U(\mathfrak{g})$$

- *With respect to the ordered basis $\{y,x\}$, the matrix of adx is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so is in Jordan form. Now $I = \{2\}, \alpha = 0$ and $S_X = \{X^n \mid n \in \mathbb{N}\}$. Thus $\Gamma(X)$ is generated by Z_1 and T, and they obey the following relation : $TZ_1 = Z_1 T - T^2$*

- *To invert y, we have to localise at the set $\{Y - nX_0 \mid n \in \mathbb{Z}\}$. Consider the ordered basis $\{x,y\}$, then ady has the following Jordan form : $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, so on the complement of the lines $Y - nX_0$ we put the (non-affine) algebra $\Gamma(Y)$ generated by Z_1 over $\mathbb{C}[T]_{\{1+nT\}}$ with relation $TZ_1 = Z_1 T(1-T)^{-1}$.*

Example 5 *As an application, we rederive the result of [4]. We first have to compute the section-algebras of* $\mathfrak{g}$ *being* $\mathbf{sl}(2,\mathbb{C}) = \mathbb{C}\,e \oplus \mathbb{C}\,f \oplus \mathbb{C}\,h$ *with*

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

- $X = X_0$*: then as before :* $\Gamma(X_0) \cong U(\mathfrak{g})$
- $x = e$ *: Set* $y_1 = -\frac{1}{2}f$, $y_2 = -\frac{1}{2}h$, $y_3 = e$, *then adx has Jordan form*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Using the equations (4) and (6), we obtain :

$$\begin{aligned} Z_1 Z_2 &= Z_2 Z_1 - 2Z_1 T + Z_2^2 T \\ T Z_1 &= Z_1 T - Z_2 T^2 + T^3 \\ T Z_2 &= Z_2 T - T^2 \end{aligned}$$

We can write this as an iterated Ore-extension :

$$\Gamma(E) = \mathbb{C}\,[T][Z_2, \delta][Z_1, \sigma, \rho]$$

- $x = f$*: then we find the same ring as in the previous case.*
- $x = h$ *: Take* $\{e,f,h\}$ *as ordered basis, then* I=$\{1,2,3\}$, $\alpha(1)$=$2, \alpha(2)$=$-2, \alpha(3)$=0 *and these relations result :*

$$\begin{aligned} Z_1 Z_2 &= Z_2 Z_1 (1+2T)(1-2T)^{-1} + T(1-2T)^{-1} \\ T Z_1 &= Z_1 T (1+2T)^{-1} \\ T Z_2 &= Z_2 T (1-2T)^{-1} \end{aligned}$$

This section is also an iterated Ore-extension :

$$\Gamma(H) = \mathbb{C}\,[T]_{\{1+2nT \,|\, n \in \mathbb{Z}\}}[Z_1, \sigma]$$

The variety of one-dimensional representations of $\Gamma(H)$ *(after homogenization and correct inbedding in* $\mathbb{P}^4 = \mathbb{P}(t,e,f,h)$ *) is* $\mathcal{V}(t(2te, -2tf, h^2 + 4ef))$. *Similarly, we find that the corresponding variety for both* $\Gamma(E)$ *and* $\Gamma(F)$ *is* $\mathcal{V}(t(t, h^2 + 4ef))$, *for* $\Gamma(t)$ $\mathcal{V}(e,f,h)$. *Glued together, these give the variety* $\mathcal{V}(t((h^2 + 4ef)(e,f,h), t(e,f,h)))$, *which is exactly the point-variety of* $H(\mathfrak{g})$ *(see [4] for a proof).*

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