Moduli Spaces for
Right Ideals of the Weyl Algebra

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Abstract
Using Beilinson's spectral sequence for the quantum space associated to the first Weyl algebra we reduce the study of classifying its right ideals up to isomorphism to a problem in linear algebra.

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Right Ideals, Weyl Algebra

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1 Introduction

In [21, p.625] J. T. Stafford noticed an intriguing analogy between the study of right ideals of the first Weyl algebra $A = A_1(\mathbb{C})$ and that of projective right ideals of a polynomial ring over a division algebra (or,even,any ring occuring in [22]).

The special case of projective right ideals of $I\!H[x,y]$ (where $I\!H$ is the quaternion algebra) has been worked out extensively in a series of papers by M. Knus,M. Ojanguren, R. Parimala and R. Sridharan,see a.o. [12],[13], [14] and [15]. Their approach is as follows. A projective (non-free) ideal P of $I\!H[x,y]$ is a free module of rank 2 over the subalgebra $\mathbb{C}[x,y]$. They show that P can be extended to a vectorbundle $\mathcal{P}$ of rank 2 over the projective plane $I\!P^2(\mathbb{C})$ with first Chern number $c_1 = 0$. Using Beilinson's spectral sequence [19, Ch.II §3] one can describe the moduli spaces of such bundles entirely by linear data,see a.o. [3], [4] or [10]. Translating these results

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back to $I\!H[x,y]$ they obtain a 'moduli space' description of the isoclasses of projective right ideals.

Recently,R.C. Cannings and M.P. Holland [7] showed that right ideals of A also come in families determined by certain 'moduli'.This raised the question whether there is a vector bundle like description of right ideals in A similar to the $I\!H[x,y]$ case.

A first attempt is inspired by the methods of [21]. One can use reduction modulo a prime p to associate to a right ideal of A a rank p vectorbundle over the projective plane over a field of characteristic p. However, this approach is not very helpful as there is not much known on moduli spaces of hight rank stable bundles over the projective plane,certainly not over a field of non-zero characteristic.

The approach of the present paper is to stick to characteristic zero but replace the complex projective plane $I\!P^2(\mathbb{C})$ by Artin's quantum plane $I\!P_q = Proj(H)$ associated to a certain graded Auslander regular domain H of global dimension 3. This quantum plane turns out to have a scheme-like structure with one of the affine open section algebras isomorphic to A. Although the other two section algebras have quite different properties (e.g. they are no longer almost commutative and have global dimension 2) they are still Auslander regular algebras. This property turns out to be sufficient to mimic the Knus et al. approach to this quantum situation.

We show that every right ideal P of A can be extended to a vectorbundle $\mathcal{P}$ on the quantum plane $I\!P_q$. We have a version of Beilinson's derived equivalence which enables us to show that $\mathcal{P}$ can be described via a monad determined by its 'cohomology' groups and even by a Kronecker module in case P is not principal.

That is, we associate to every non-principal right ideal P three $k \times l$ matrices with entries in $\mathbb{C}$ (the numbers k and l can be calculated from a generating set for P). Two right ideals are isomorphic if and only if the corresponding triples of matrices are equivalent under the natural $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$-action.

Hence, the study of isoclasses of right ideals of A reduces to the study of certain moduli spaces $M(A;k,l)$ which can be described entirely in linear data. As an example we show that the 'canonical' non principal right ideal $P_n = x^{n+1}A + (xy+n)A$ belongs to $M(A;n,n)$ and one can even associate to P_n a rank n bundle over $I\!P^2(\mathbb{C})$.

The precise connection between the moduli spaces $M(A;k,l)$ and mod-

2

uli spaces of vectorbundles on $I\!P^2(\mathbb{C})$ remains to be explicitated as is the connection with the 'moduli' of M.P. Holland and R.C. Cannings. Also, a similar study of vectorbundles on $I\!P_q$ of rank > 1 might be interesting to investigate.

2 The quantum plane $I\!P_q$

Let H be the graded Auslander-regular domain of global dimension 3 generated by X, Y and Z and defining quadratic relations

$$
\begin{aligned}
XY - YX &= Z^2 \\
XZ - ZX &= 0 \\
YZ - ZY &= 0
\end{aligned} \tag{1}
$$

Following M. Artin [1] we introduce the quantum plane $I\!P_q$ by defining the coherent sheaves over it to be

$$Coh(I\!P_q) := gr(H)/(torsion)$$

the quotient category of the category $gr(H)$ of all f.g. graded right H-modules modulo the Serre subcategory of the H_+-torsion modules.

As the powers of X (resp. Y,Z) form a twosided Ore set of homogeneous elements in H we can introduce the section algebra H_x (resp. H_y,H_z) to be the degree zero part of the corresponding localization i.e.

$$H_X = H_x[X, X^{-1}, \phi_X]$$

Taking the obvious generators one easily verifies that $H_z \simeq A$ whereas the defining relation for H_x (resp. H_y) is $[p, q] = p^3$ (resp. $[p, q] = q^3$).

Observe that these section algebras have quite different properties. Whereas $H_z = A$ is a typical example of an almost commutative algebra, H_x and H_y are archetypical examples of non almost commutative algebras [18]. Further, whereas all three section algebras are Auslander-regular domains they have different global dimensions

$$gldim(H_x) = gldim(H_y) = 2 \quad \text{and} \quad gldim(H_z) = 1$$

These section algebras also allow a scheme-like description of $I\!P_q$. That is, if $\mathcal{M} \in Coh(I\!P_q)$ is determined by $M \in gr(H)$, then $\mathcal{M}$ is determined via the

3

triple of section-modules (M_x, M_y, M_z) where e.g. M_x is the degree zero part of the localization M_X. This desciption allows us to extend classical notions to $I\!\!P_q$. For example, $\mathcal{M}$ will be called a vectorbundle on $I\!\!P_q$ iff each of the section-modules is a f.g. projective right module over the corresponding section algebra.

In order to have a form of Serre duality in $Coh(I\!\!P_q)$ we note that we have a canonical projective resolution of the trivial module

$$0 \to H(-3) \otimes \wedge^3(V) \to H(-2) \otimes \wedge^2(V) \to H(-1) \otimes V \to H \to \mathbb{C} \to 0$$

where $V = \mathbb{C}\,X + \mathbb{C}\,Y + \mathbb{C}\,Z$ and where the rightmost map is the augmentation, the previous one the sum, the middle one determined via

$$
\begin{aligned}
X \wedge Y &\to X \otimes Y - Y \otimes X - Z \otimes Z \\
X \wedge Z &\to X \otimes Z - Z \otimes X \\
Y \wedge Z &\to Y \otimes Z - Z \otimes Y
\end{aligned}
\tag{2}
$$

the kernel of which is generated by

$$X \otimes Y \wedge Z - Y \otimes X \wedge Z + Z \otimes X \wedge Y$$

If we denote by $\mathcal{O}(i)$ the object in $Coh(I\!\!P_q)$ determined by the twisted H-module $H(i)$, then the above resoltution gives the following Koszul sequence in $Coh(I\!\!P_q)$

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2)^{\oplus 3} \to \mathcal{O}(-1)^{\oplus 3} \to \mathcal{O} \to 0$$

3 From right ideals to vectorbundles

The following lemma is crucial for our purposes. Its elegant proof is due to Th. Levasseur.

Lemma 1 *Let A be an Auslander-regular algebra of $gldim(A) \leq 2$. If M is a f.g. right A-module then $M^* = Hom_A(M, A)$ is either zero or a projective left A-module.*

Proof : By Auslander regularity we have a spectral sequence

$$E_2^{p,-q}(M) = Ext_A^p(Ext_A^q(M, A), A) \Rightarrow M$$

4

see a.o. [16] or [17]. Using the fact that $gldim(A) \leq 2$ one obtains that

$$
\begin{aligned}
E_2^{1,0}(M) &= E_\infty^{1,0}(M) = 0 \\
E_2^{2,0}(M) &= E_\infty^{2,0}(M) = 0
\end{aligned} \tag{3}
$$

Therefore $Ext_A^i(M^*, A) = 0$ for $i = 1, 2$ and hence $pd(M^*) = 0$ finishing the proof. $\square$

We say that a projective right module P of $H_z = A$ extends to a vector-bundle on $I\!P_q$ if there is a vectorbundle $\mathcal{P} = \{P_x, P_y, P_z\}$ in $I\!P_q$ such that $P \simeq P_z$.

Proposition 1 *Any f.g. projective right A-module P extends to a vector-bundle $\mathcal{P}$ on $I\!P_q$.*

Proof : As P is finitely generated one can equip it with a good filtration. Let $h(P)$ be the homogenization w.r.t. this filtration i.e. $h(P) = \oplus_i P_i Z^i \subset P[Z, Z^{-1}]$. Clearly $h(P)$ is a graded H-module. Now, consider $\mathcal{P}$ in $I\!P_q^2$ represented by $h(M)^{**} = HOM_H(HOM_H(h(M), H), H)$. All sections are reflexive modules over an Auslander-regular algebra of $gldim$ at most 2, hence they are projective by the foregoing lemma. Hence, $\mathcal{P}$ is a vectorbundle on $I\!P_q$ extending P. $\square$

Remark 1 *In case P is a right ideal of A one can do without taking the double dual. Equip P with the induced filtration on A, then using the fact that H is a domain it is easily verified that $HOM_H(h(P), H) = h(Hom_A(P, A)) = \{q \in Q^g(H) : q.h(P) \subset H\}$.*

Proposition 2 *Let M be a graded reflexive right H-module.*

1. There is an exact sequence

$$
0 \rightarrow \oplus_i H(n_i) \rightarrow \oplus_j H(m_j) \rightarrow M \rightarrow 0
$$

for certain integers n_i, m_j

2. $Ext_H^1(M^*, H) \simeq Ext_H^3(Ext_H^1(M, H), H)$

5

Proof : Using the spectral sequence for the Auslander regular algebra H we have

$$Ext_H^2(M^*, H) = E_2^{2,0}(M) = E_\infty^{2,0}(M) = 0$$
$$Ext_H^3(M^*, H) = E_2^{3,0}(M) = E_\infty^{3,0}(M) = 0 \qquad (4)$$

As M is reflexive we therefore get

$$Ext_H^2(M, H) = Ext_H^3(M, H) = 0$$

whence $pd(M) \leq 1$. Hence we have a gradation preserving projective resolution of M

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_i a graded projective right H-module, hence free, finishing the proof of (1).

Part (2) follows from $pd(M) \leq 1$ which entails that $E_3^{p,-q}(M) = E_\infty^{p,-q}(M)$. $\qquad \square$

Remark 2 *If P is a non-principal right ideal of A, then the above result applies to $h(P)$ and hence we get an exact sequence in $\mathbb{P}_q$*

$$0 \to \oplus_i \mathcal{O}(n_i) \to \oplus_j \mathcal{O}(m_j) \to \mathcal{P} \to 0$$

where $\mathcal{P}$ is the vectorbundle extending P. Observe that the sequence for $h(P)$ does not split, i.e. $h(P)$ is not a graded projective H-module for then it would be isomorphic to $H(m)$ for some m yielding that P is principal. For a right ideal P of A the numbers n_i and m_j can be readily calculated from the theory of Galligo stairs [9, §2] if we are given generators of P.

4 The right ideals $P_n = x^{n+1}A + (xy + n)A$

Whereas all invariants introduced in this paper can be readily computed given a generating set for the right ideal, we will stick to the 'canonical' non-principal right ideals P_n as for them everything can be seen immediately from eigen-space calculations as in [8, §3],[11, §1] or [20].

For $t \in \mathbb{Z}$ define $A(t) = \{f \in A \mid [xy, f] = tf\}$. Then, $A = \oplus_{t=-\infty}^\infty A(t)$ with $A(t)$ equal to

$$y^t \mathbb{C}[xy] = \mathbb{C}[xy]y^t \quad for \quad t \geq 0$$
$$x^{-t} \mathbb{C}[xy] = \mathbb{C}[xy]x^{-t} \quad for \quad t < 0 \qquad (5)$$

Lemma 2 *If $P_n = x^{n+1}A + (xy + n)A$ then $P_n(t) = x^{n+1}A(t+n+1) + (xy + n)A(t)$ is equal to*

$$
\begin{array}{rcl}
(xy + n)\mathbb{C}[xy]y^t & for & t \geq 0 \\
(xy + n)\mathbb{C}[xy]x^{-t} & for & -n \leq t < 0 \\
\mathbb{C}[xy]x^{-t} & for & t < -n
\end{array}
\tag{6}
$$

Proof : Let $t = -1$ then $P_n(-1)$ is $x^{n+1}\mathbb{C}[xy]y^n + (xy + n)\mathbb{C}[xy]x$ which, using $x^{n+1}y^{n+1} = xy(xy + 1)...(xy + n)$ is equal to

$$
xy(xy + 1)...(xy + n)\mathbb{C}[xy]y^{-1} + (xy + n)\mathbb{C}[xy]x
$$

The first factor is $(xy + 1)...(xy + n)\mathbb{C}[xy]x$ giving the desired result. All other calculations are similar. □

Proposition 3 *We have an exact sequence of graded right H-modules*

$$
0 \rightarrow H(-n - 2) \rightarrow H(-2) \oplus H(-n - 1) \rightarrow h(P_n) \rightarrow 0
$$

Proof : It is clear that $h(P_n)$ is generated by X^{n+1} and $XY + nZ^2$ and that there is a relation between these two generators namely $X^{n+1}Y = (XY + nZ^2)X^n$. This gives the required sequence. In order to verify that it is exact we have to compute the Hilbert series of $h(P_n)$. Using the foregoing lemma we see that it is equal to

$$
\frac{1}{(1 - s)}\left(\frac{s^2}{(1 - s)(1 - s^2)} + \frac{s^3\sum_{i=1}^{n-1} s^i}{(1 - s^2)} + \frac{s^{n+1}}{(1 - s)(1 - s^2)}\right)
$$

which simplifies to $\frac{s^2 + s^{n+1} - s^{n+2}}{(1-s)^3}$ which fits with exactness of the sequence. □

Next, we perform similar computations for the dual module

Lemma 3 *The left A-module $P_n^* = Ax^{-n-1} \cap A(xy + n)^{-1}$ has eigenspace decomposition $P_n^*(t) = A(t - n - 1)x^{-n-1} \cap A(t)(xy + n)^{-1}$ which is equal to*

$$
\begin{array}{rcl}
y^t\mathbb{C}[xy](xy + n)^{-1} & for & t \geq n + 1 \\
y^t\mathbb{C}[xy] & for & 0 \leq t \leq n \\
x^{-t}\mathbb{C}[xy] & for & t < 0
\end{array}
\tag{7}
$$

Proof : Let us illustrate the calculations with the case $t = n$. then $P_n^*(t) = x\mathbb{C}[xy]x^{-n-1} \cap y^n\mathbb{C}[xy](xy + n)^{-1}$ which equals

$$y^n\mathbb{C}[xy](xy + n - 1)^{-1}...(xy)^{-1} \cap y^n\mathbb{C}[xy](xy + n)^{-1}$$

giving the desired result. The other computations are similar. □

Proposition 4 *There is an exact sequence of graded left H-modules*

$$0 \to H(-n) \to H \oplus H(-n + 1) \to h(P_n^*) \to 0$$

Proof: Again, it is easy to see the generators (1 and $Y^{n+1}(XY + nZ^2)^{-1}$) and the relation between them. Therefore it is sufficient to show that the Hilbert series of $h(P_n^*)$ is of the required shape. Using the above lemma the series is

$$\frac{1}{(1 - s)}\left(\frac{s^{n-1}}{(1 - s)(1 - s^2)} + \frac{\sum_{i=1}^{n} s^i}{(1 - s^2)} + \frac{s}{(1 - s)(1 - s^2)}\right)$$

which simplifies (as required) to $\frac{1+s^{n-1}-s^n}{(1-s)^3}$. □

Observe that $h(P_n)$ is not projective as $h(P_n^*)h(P_n)$ does not have elements of degree 0 so it cannot be equal to H. The study of the bundle $\mathcal{P}_n$ associated to $h(P_n)$ will be continued in the last section.

5   Beilinson's derived equivalence

Beilinson [5] proved that there is an equuivalence between the derived category of bounded complexes of coherent sheaves on $I\!P^n$ and that of f.g. modules over a certain finite dimensional algebra of global dimension n. This equivalence allows one to study certain classes of coherent $I\!P^n$-modules via linear algebra.

In this section we will scetch the Beilinson procedure for the quantum plane $I\!P_q$. Note however that this is a very general argument and is applicable to other situations. The best result to my knowledge is due to A. Bondal [6].

The finite dimensional algebra in question will be the incidence algebra of the following quiver

$$\bullet \quad \overset{X_1}{\underset{\underset{\longrightarrow}{\overset{Y_1}{\longrightarrow}}}{\longrightarrow}} \quad \bullet \quad \overset{X_2}{\underset{\underset{\longrightarrow}{\overset{Y_2}{\longrightarrow}}}{\longrightarrow}} \quad \bullet$$

with relations

$$
\begin{aligned}
X_1Y_2 - Y_1X_2 &= Z_1Z_2 \\
X_1Z_2 - Z_1X_2 &= 0 \\
Y_1Z_2 - Z_1Y_2 &= 0
\end{aligned}
\tag{8}
$$

We will call this algebra B. Observe that

$$B = End_{Coh(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$$

The version of Beilinson's derived equivalence which we need is then :

Proposition 5 *The functors*

$$F = Hom_{Coh(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), -) : Coh(\mathbb{P}_q^2) \to mod(B)$$

$$G = - \otimes_B (\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) : mod(B) \to \mathbb{P}_q^2$$

induces an equivalence of derived categories

$$\mathcal{D}^b(Coh(\mathbb{P}_q)) \simeq \mathcal{D}^b(mod(B))$$

Proof : This is just [6, Theorem 6.2] addapted to our situation. The required conditions follow from the Koszul sequence. □

Of course, we would prefer to be able to attach to an object in $Coh(\mathbb{P}_q)$ a right B-module rather than a bounded complex of such modules. This can be achieved for certain subclasses of objects.

With $\mathcal{X}_i$ we denote the set of all $\mathcal{M} \in Coh(\mathbb{P}_q)$ such that

$$Ext^j_{Coh(\mathbb{P}_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), \mathcal{M}) = 0 \text{ for all } j \neq i$$

Likewise, with $\mathcal{Y}_i$ we denote the set of all $M \in mod(B)$ such that

$$Tor^B_j(M, \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)) = 0 \text{ for all } j \neq i$$

Then one deduces precisely as in e.g. [2, §3.2]

9

Corollary 1 *The functors*

$$F^i = Ext^i_{Coh(I\!\!P_q)}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2), -)$$

$$G_i = Tor^B_i(-, \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2))$$

establish an equivalence

$$\mathcal{X}_i \simeq \mathcal{Y}_i$$

Hence, an object $\mathcal{M} \in \mathcal{X}_i$ is uniquely determined by a B-module and hence by linear data. Our next job will be to show that specific shifts of the vectorbundle extending a right ideal of A lie in $\mathcal{X}_1$.

6 Monadology

In $Coh(I\!\!P_q)$ we can introduce 'cohomology' groups as the derived functors of $Hom_{Coh(I\!\!P_q)}(\mathcal{O}, -)$ i.e.

$$H^i(I\!\!P_q, \mathcal{M}) = Ext^i_{Coh(I\!\!P_q)}(\mathcal{O}, \mathcal{M})$$

As we have that $Ext^i_{I\!\!P^2_q}(\mathcal{O}(k), \mathcal{M}) = H^i(I\!\!P^2_q, \mathcal{M}(-k))$ we can define the category $\mathcal{X}_i$ as the set of those objects $\mathcal{M} \in Coh(I\!\!P_q)$ such that

$$H^j(I\!\!P^2_q, \mathcal{M}(-k)) = 0$$

for all $j \neq i$ and all $k = 0, 1, 2$. That is, for $\mathcal{M} \in \mathcal{X}_1$ we have the following cohomology groups $H^j(I\!\!P_q, \mathcal{M}(i))$

	j				
	?	0	0	0	0
	?	*	*	*	?
	0	0	0	0	?
	-3	-2	-1	0	1	i

Let us return to the case of a vectorbundle $\mathcal{P}$ extending a right ideal P of A determined by $h(P)$. We have

$$h(P) = \oplus_k H^0(I\!\!P_q, \mathcal{P}(k))$$

and from the Koszul sequence we have a form of Serre duality for vectorbundles $\mathcal{M}$ in $Coh(I\!P_q)$ stating that

$$H^2(I\!P_q, \mathcal{M})^* \simeq H^0(I\!P_q, \mathcal{M}^*(-3))$$

where the first dual is of $\mathbb{C}$-vectorspaces and the second is the $Hom(\mathcal{O}, \mathcal{M})$. As $\mathcal{P}^*$ is the object in $Coh(I\!P_q)$ associated to the homogenization of $P^* = \{q \in D_1 : qP \subset A\}$ we can compute all H^0 and H^2 cohomology groups of $\mathcal{P}$ given a generating set for P. As a consequence we have

Proposition 6 *If P is a non-principal right ideal of A and if $\mathcal{P}$ is the vectorbundle on $I\!P_q$ extending P corresponding to $h(P)$, then $\mathcal{P}(d-1) \in \mathcal{X}_1$ where d is the minimal filtration degree for elements of P.*

Proof : As H and hence $Q^g(H)$ is a domain it is clear that the filtration degree of any element of P^* is $\geq -d$. Therefore $H^2(I\!P_q, \mathcal{P}(k)) = 0$ for all $k \geq d - 2$. Now assume that $H^2(I\!P_q, \mathcal{P}(d-3)) \neq 0$ then $h(P^*)$ has an element of degree $-d$ meaning that $h(P^*)h(P) = H$ whence $h(P)$ is a graded projective right H-module. But then $h(P)$ has to be principal and so does P, a contradiction. $\square$

Hence, the right ideal P is completely determined by the B-module with representation

$$
\begin{array}{ccccc}
 & \rightrightarrows & & \rightrightarrows & \\
V_1 & \rightarrow & V_2 & \rightarrow & V_3 \\
 & \rightrightarrows & & \dashrightarrow &
\end{array}
$$

where $V_i = H^1(I\!P_q, \mathcal{P}(d - 4 + i))$ and where the maps are induced by multiplication with X (resp. Y and Z). Observe that the only effect of taking another representant in the isomorphism class is having to take a different twist d (and perhaps a different choice of basis in the V_i).

Conversely, one can recover $\mathcal{P}$ as the 'cohomology-bundle' of the monad

$$V_1 \otimes \mathcal{O}(-1) \hookrightarrow V_2 \otimes \Omega^1(1) \mapsto V_3 \otimes \mathcal{O}$$

11

That is we have an exact diagram in $Coh(\mathbb{P}_q)$

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \to & V_1 \otimes \mathcal{O}(-1) & \to & \mathcal{K} & \to & \mathcal{P}(d-1) & \to & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \to & V_1 \otimes \mathcal{O}(-1) & \to & V_2 \otimes \Omega^1(1) & \to & \mathcal{C} & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & V_3 \otimes \mathcal{O} & = & V_3 \otimes \mathcal{O} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

where Ω^1 is the first syzygy bundle i.e. the kernel of the rightmost map in the Koszul sequence.

7 Kronecker modules

Whereas the above reduces the study of right ideals of A to the study of certain finite dimensional modules of B (and hence to linear algebra) we want to do better, i.e. we want to associate to a right ideal a point of a specific Grassmannian. In module theoretic terms, we want the finite dimensional B-module corresponding to P to be uniquely determined by its three rightmost maps. The importance of this being that we no longer have to care for the defining relations of B but are reduced to a Kronecker situation.

We can repeat the argument of D. Baer given in [2, Cor.7.2]. Then we get that $\mathcal{M} \in Coh(\mathbb{P}_q)$ is uniquely determined by a Kronecker module if both $\mathcal{M}$ and $\mathcal{M}(1)$ belong to $\mathcal{X}_1$. That is, the cohomology groups $H^j(\mathbb{P}_q, \mathcal{M}(i))$ have following shape

?	0	0	0	0	0
?	*	*	*	*	?
0	0	0	0	0	?
-3	-2	-1	0	1	

Fortunately, we have

12

Proposition 7 *Let P be a non-principal right ideal of A with minimal filtration degree d. Then, both $\mathcal{P}(d-2)$ and $\mathcal{P}(d-1)$ belong to $\mathcal{X}_1$. As a consequence, $\mathcal{P}(d-2)$ (and hence P) is uniquely determined by a Kronecker module.*

Proof : Assume that $\mathcal{P}(d-2) \notin \mathcal{X}_1$, then $h(P^*)$ would have an element of degree $-d+1$. Observe first that the statements are preserved under isomorphism. Hence we can take a representant in the isomorphism class such that $P \cap \mathbb{C}[x] \neq 0$ (this can be done by [21]). Now look at the following picture
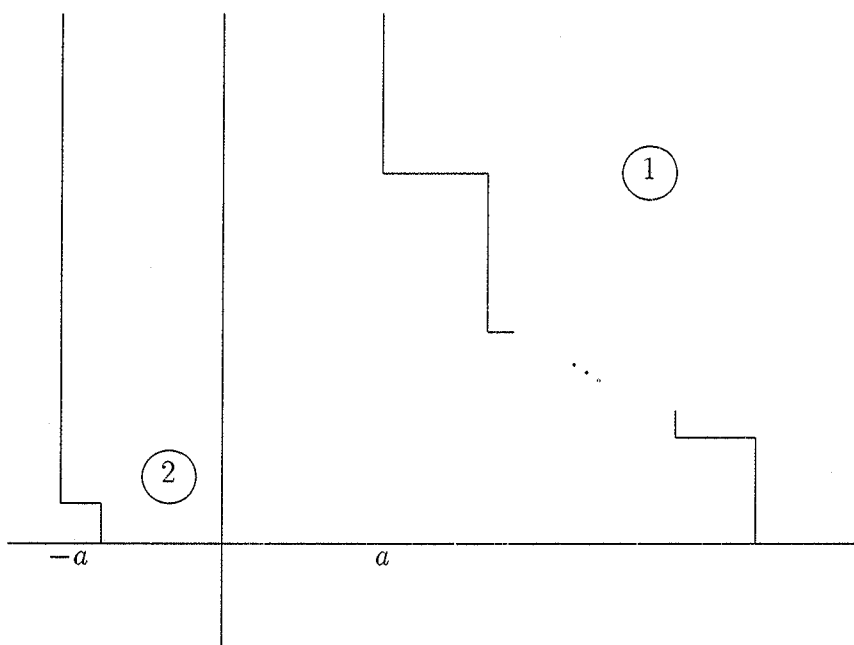
Here the top-right corner region (marked 1) is the set of couples (k_1, k_2) occuring as degrees of elements of P in the multi-filtration on A (and its ring of fractions D) by giving x degree $(1,0)$ and y degree $(0,1)$. By our assumption, the region is bounded on the bottom-right by the horizontal axis and on the top-left by the axis $j = a$. As the ideal is non-principal, the point $(a,0)$ cannot belong to this region, hence the (total) degree of an element from $h(P)$ is $\geq a+1$ (i.e. $d \geq a+1$).

Similarly, the region determined by the degrees of elements of P^* in the

13

multi-filtration must lie in the region marked 2 (bounded on the left by the line $j = -a$). Observe that $(-a, 0)$ cannot lie in this region as otherwise P^* (and hence P) would be principal. Hence the degree of any element of $h(P^*)$ has to be $\geq -a + 1$. So, $h(P^*)$ does not contain elements of degree $-d + 1$, done. $\square$

Hence we see that the isomorphism class of the right A-module P is fully determined by the three linear maps (given by multiplication with X, Y and Z) from V_1 to V_2 (with notation as in the foregoing section).

Therefore we can associate to any isomorphism class of right A-ideals 2 integers $dim(V_1)$ and $dim(V_2)$. If we are given generators for a representant right ideal there is an effective procedure to compute these numbers. It would be interesting to determine the couples (n_1, n_2) s.t. the moduli-space $M(A; n_1, n_2)$ describing the isoclasses of right A-ideals with the corresponding numerical invariants is non-empty and to determine the geometrical structure of these moduli spaces (e.g. are they rational ?). Further, it will be interesting to determine how the 'moduli' of Cannings and Holland are related to these numerical moduli.

8 The bundles $\mathcal{P}_n$

In this section we will continue our study of the bundles $\mathcal{P}_n$ corresponding to the right ideals $P_n = x^{n+1}A + (xy + n)A$. As the minimal filtration degree for elements of P_n is 2 we have to calculate the dimensions of

$$V_1 = H^1(I\!\!P_q, \mathcal{P}(-1)) \quad \text{and} \quad V_2 = H^1(I\!\!P_q, \mathcal{P})$$

and the three maps between them induced by multiplication with the variables.

Proposition 8 *The vectorbundle $\mathcal{P}_n$ is determined by the Kronecker module with dimension vector (n, n) and where the linear maps corresponding to multiplication with X (resp. Y and Z) can be represented by the matrices*

$$\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \end{pmatrix}$$

14

(resp.

$$\begin{pmatrix} 0 & 1-n & 0 & & & \\ & 0 & 2-n & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & -2 & 0 \\ & & & & 0 & -1 \end{pmatrix}$$

and I_n).

Proof : We have discussed the exact gradation preserving sequence of right H-modules

$$0 \to H(-n-2) \quad \to \quad H(-n-1) \oplus H(-2) \quad \to \quad h(P_n) \to 0$$
$$(Y, -X^n). \qquad\qquad\qquad\qquad \begin{pmatrix} X^{n+1} \\ XY + nZ^2 \end{pmatrix}.$$

After dualization, this induces an exact sequence in $Coh^l(I\!P_q)$ (of **left** modules!)

$$0 \to \mathcal{P}^* \quad \to \quad \mathcal{O}(2) \oplus \mathcal{O}(n+1) \quad \to \quad \mathcal{O}(n+2) \to 0$$
$$.(XY + nZ^2, X^{n+1}) \qquad\qquad\qquad .\begin{pmatrix} -X^n \\ Y \end{pmatrix}$$

By Serre duality, the maps $H^1(I\!P_q, \mathcal{P}(-1)) \to H^1(I\!P_q, \mathcal{P})$ induced by right multiplication by a variable V are the transposed of the maps $H^1(I\!P_q, \mathcal{P}^*(-3)) \to H^1(I\!P_q, \mathcal{P}^*(-2))$ induced by left multiplication with V. Cohomology gives us the diagram with exact rows

$$\begin{pmatrix} -X^n \\ Y \end{pmatrix}.$$
$$\begin{array}{ccccccccc} 0 & \to & 0 \oplus H_{n-2} & \to & H_{n-1} & \to & H^1(I\!P_q, \mathcal{P}^*(-3)) & \to & 0 \\ & & V.\downarrow & & V.\downarrow & & V.\downarrow & & \\ 0 & \to & \mathbb{C} \oplus H_{n-1} & \to & H_n & \to & H^1(I\!P_q, \mathcal{P}^*(-2)) & \to & 0 \end{array}$$
$$\begin{pmatrix} -X^n \\ Y \end{pmatrix}.$$

From this one deduces that $H^1(I\!P_q, \mathcal{P}^*(-3))$ has dimension n and basis the images of $X^{n-1}, X^{n-2}Z, ..., Z^{n-1}$ and that $H^1(I\!P_q, \mathcal{P}^*(-2))$ has dimension n with basis the images of $X^{n-1}Z, X^{n-2}Z^2, ..., Z^n$. With respect to these basisses one verifies that the matrices representing left multiplication with

15

X (resp. Y and Z) are the transposed matrices of those in the statement of the proposition, finishing the proof. □

In fact, we can easily compute the dimensions of all cohomology groups. Those of H^0 or H^2 follow from knowing the Hilbert series of $h(P)$ and $h(P^*)$ and for H^1 we have

$$dim_{\mathbb{C}}\left(H^1(I\!\!P_q, \mathcal{P}_n(i))\right) = \begin{cases} n-i & \text{if } 0 \leq i \leq n \\ n+i+1 & \text{if } -n+1 \leq i \leq -1 \end{cases}$$

Recall from [10] that one can associate to a triple of $n \times n$ matrices (M_1, M_2, M_3) a stable vectorbundle on the usual projective plane $I\!\!P^2(\mathbb{C})$ provided $dim_{\mathbb{C}}\left(\mathbb{C}.M_1.v + \mathbb{C}.M_2.v + \mathbb{C}.M_3.v\right) \geq 2$ for all $0 \neq v \in \mathbb{C}^n$. If one of these matrices is the identity matrix, the rank of the vectorbundle is equal to the rank of the commutator matrix of the other two. Applying these facts to the above computations we see that there is a stable rank n vectorbundle on $I\!\!P_2(\mathbb{C})$ with Chern-numbers $c_1 = 0$ and $c_2 = n$ associated to the right ideal P_n. It would be interesting to determine the image of the rational map $M(A; n, n) \rightarrow M(n; 0, n)$ where $M(n; 0, n)$ is the moduli space of rank n vectorbundles on $I\!\!P^2(\mathbb{C})$ with $c_1 = 0, c_2 = n$. We hope to return to some of the questions raised in a forthcoming paper

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