Some Remarks on Solvable Lie Superalgebras

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Abstract

New invariants are associated to (solvable) Lie superalgebras. A classification is given of all solvable Lie superalgebras having two dimensional odd component.

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1 Introduction

A Lie superalgebra $\mathfrak{s}$ consists of a Lie algebra $\mathfrak{g}$, a finite dimensional $\mathfrak{g}$-representation V and a bilinear mapping $b : V \times V \to \mathfrak{g}$ which is $\mathfrak{g}$-invariant and satisfies a Jacobi superidentity (details will be recalled below).

It is an interesting but apparently hopeless problem to construct and classify all Lie superstructures over a fixed Lie algebra $\mathfrak{g}$. The main aim of this paper is to present a few observations which may be seen as the first steps in such a classification project.

In section 2 we study how the Lie superalgebra deforms if we deform the $\mathfrak{g}$-representation V towards its semi-simplification. We show that one cannot expect a full version of Artin's theorem [1] and prove a weak version of it. The result is particularly usefull if we have a flag of $\mathfrak{g}$-subrepresentations of V (as is always the case if $\mathfrak{g}$ is a solvable Lie algebra). In that case one can build up the symmetric matrix M_b describing the bilinear form b with respect to this flag from top to bottom and obtain fairly restrictive conditions on the possible entries.

In section 3 we no longer focuss on the representation but investigate what restrictions the G-invariance of the bilinear form gives us. We show that a Lie superalgebra $\mathfrak{s}$ determines a stratification of $\mathfrak{g}^*$ by G-stable subvarieties where G is the associated algebraic group. In particular, if b is non-degenerate, the open stratum is determined by the determinant of M_b whose symmetrization is a semi-invariant of $U(\mathfrak{g})$. Whereas it is not always possible to extend this semi-invariant to a semi-invariant of the super enveloping algebra $U(\mathfrak{s})$ we show that one can microlocalize $U(\mathfrak{s})$ at it and obtain a maximal order having finite global dimension. Recall that both ringtheoretical and homological properties of the enveloping algebra $U(\mathfrak{s})$ itself are rather poor [4].

In the final section we apply the obtained methods to classify all Lie superalgebras over a solvable Lie algebra $\mathfrak{g}$ with 2-dimensional representation V. Whereas one knows that solvable Lie superalgebras do not satisfy Lie's theorem in general (and hence are not necessarely completely solvable) we show that in this case nearly all of them are in fact completely solvable. We show that the only solvable not completely solvable Lie superalgebras are those such that $\mathfrak{g}$ contains the non-Abelian two-dimensional Lie algebra as the ideal generated by the entries of M_b and such that the restriction of V to this Lie ideal is isomorphic to its adjoint representation.

We hope that the methods and results of this paper will be usefull in obtaining similar classification results for solvable not completely solvable Lie algebras for higher dimensional V.

2 Degenerations of Lie superstructures

In this section we will study degenerations of Lie superstructures over a fixed finite dimensional Lie algebra $\mathbf{g}$ of dimension n with basis $x_1, \ldots, x_n$. We do this by investigating orbit closures of the natural GL_m-action on the variety $Sup_m(\mathbf{g})$ of all possible Lie superalgebras $\mathbf{s}$ with $\mathbf{s}_0 = \mathbf{g}$ and $\mathbf{s}_1$ an m-dimensional $\mathbf{g}$-representation.

Let us start by recalling some classical facts on $Rep_m(\mathbf{g})$ the variety of all m-dimensional representations of $\mathbf{g}$. After fixing once and for all a basis $e_1, \ldots, e_m$ in $\mathbb{C}^m$ an m-dimensional $\mathbf{g}$-representation is determined via the matrix $M_i \in M_m(\mathbb{C})$ describing the action of x_i. Hence, $Rep_m(\mathbf{g})$ is the closed subvariety of $M_m(\mathbb{C}) \times \cdots \times M_m(\mathbb{C})$ (n copies) consisting of all n-tuples of $m \times m$ matrices $(M_1, \ldots, M_n)$ satisfying the same commutation relations as the corresponding Lie brackets among the $x_i \in \mathbf{g}$.

Basechange in $\mathbb{C}^m$ induces a natural action of GL_m on $Rep_m(\mathbf{g})$ via

$$B \cdot (M_1, \ldots, M_n) = (B^{-1} \cdot M_1 \cdot B, \ldots, B^{-1} \cdot M_n \cdot B)$$

for all $B \in GL_m(\mathbb{C})$. The orbits of $Rep_m(\mathbf{g})$ under this action are the isomorphism classes of m-dimensional $\mathbf{g}$-representations, see e.g. [18] or [12] for more details.

M. Artin [1] proved that the closed orbits correspond to the semi-simple representations and that the semi-simplification of a representation V (i.e. the direct sum of the Jordan-Hölder factors of V) lies in the closure of the orbit $GL_m(\mathbb{C}) \cdot V$.

If $\mathbf{h}$ is a Lie subalgebra of $\mathbf{g}$ then the restriction map induces a morphism

$$r_{\mathbf{g},\mathbf{h}} : Rep_m(\mathbf{g}) \rightarrow Rep_m(\mathbf{h})$$

which is GL_m-equivariant.

Next, we bring in the second component of the Lie superstructure : the quadratic form. Let $Quad_m(\mathbf{g})$ be the variety of couples (V, b) where V is an m-dimensional $\mathbf{g}$-representation and $b : V \times V \rightarrow \mathbf{g}$ a symmetric bilinear mapping (determined by a symmetric $m \times m$ matrix M_b with entries in $\mathbf{g}$). Hence, $Quad_m(\mathbf{g})$ is the trivial vectorbundle over $Rep_m(\mathbf{g})$ with fiber $\mathbf{g}^{\frac{m(m+1)}{2}}$.

Again, basechange in $\mathbb{C}^m$ induces a GL_m-action on $Quad_m(\mathbf{g})$ given by

$$B \cdot (V, b) = (B^{-1} \cdot V \cdot B, B^{\tau} \cdot b \cdot B)$$

where B^{τ} is the transposed matrix and $B^{\tau} \cdot b \cdot B$ is the bilinear mapping corresponding to the matrix $B^{\tau} \cdot M_b \cdot B$. The projection morphism

$$p : Quad_m(\mathbf{g}) \rightarrow Rep_m(\mathbf{g})$$

(forgetting the bilinear mapping) is GL_m-equivariant.

We want to investigate the variety $Sup_m(\mathfrak{g})$ which is the closed subvariety of $Quad_m(\mathfrak{g})$ consisting of the points (V, b) satisfying the equations

$$[x_i, b(e_j, e_k)] = b(x_i \cdot e_j, e_k) + b(e_j, x_i \cdot e_k)$$

for all $1 \leq i \leq n$ and $1 \leq j, k \leq m$ (expressing the fact that the bilinear mapping is $\mathfrak{g}$-invariant) and

$$b(e_i, e_j) \cdot e_k + b(e_j, e_k) \cdot e_i + b(e_k, e_i) \cdot e_j = 0$$

for all $1 \leq i, j, k \leq m$ (expressing the Jacobi (super) identity).

Clearly, a Lie superstructure is independent of the choice of basis in its odd component. Hence, the GL_m-action on $Quad_m(\mathfrak{g})$ restricts to an action on $Sup_m(\mathfrak{g})$ and the projection map

$$p : Sup_m(\mathfrak{g}) \rightarrow Rep_m(\mathfrak{g})$$

is GL_m-equivariant. The zero bilinear form gives a section to p and have to study the fibers of the projection map p.

2.1 Artin's theorem for Lie superalgebras

The degeneration result alluded to above is the following weak version of Artin's theorem :

Theorem 1 *Let $(V, b) \in Sup_m(\mathfrak{g})$ with $0 \rightarrow M \rightarrow V \rightarrow N \rightarrow 0$ an extension of $\mathfrak{g}$-representations. Choose a point on the orbit such that the symmetric matrix M_b can be written accordingly in block components*

$$M_b = \begin{pmatrix} D & E \\ E^\tau & F \end{pmatrix}$$

then we have :

1. *$(M \oplus N, b_D \perp 0)$ lies in the closure of the GL_m-orbit of (V, b) and hence determines a superstructure on $M \oplus N$. Consequently $(M, b_D) \in Sup_k(\mathfrak{g})$ with $k = dim(M) < m$.*

2. *If $D = 0$ then $(M \oplus N, b_{M'})$ lies in the closure of the GL_m-orbit of (V, b) where*
$$M' = \begin{pmatrix} 0 & E \\ E^\tau & 0 \end{pmatrix}$$
and hence determines a superstructure on $M \oplus N$.

3. *If $D = 0$ and $E = 0$, then $(M \oplus N, 0 \perp b_F)$ lies in the closure of the GL_m-orbit of (V, b) and hence determines a superstructure on $M \oplus N$. Consequently, $(N, b_F) \in Sup_l(\mathfrak{g})$ with $l = dim(N) < m$.*

Proof : Let the action of x_i on V be determined by the matrix

$$\begin{pmatrix} U_i & V_i \\ 0 & W_i \end{pmatrix}$$

where U_i (resp. W_i) is the matrix describing the action of x_i on M (resp. N).

(1): Consider the following point in the orbit

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \cdot \begin{pmatrix} U_i & V_i \\ 0 & W_i \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} U_i & \varepsilon V_i \\ 0 & W_i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \cdot \begin{pmatrix} D & E \\ E^\tau & F \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} D & \varepsilon E \\ \varepsilon E^\tau & \varepsilon^2 F \end{pmatrix}$$

then the statements follow by letting $\varepsilon \to 0$ and the fact that $Sup_m(\mathrm{g})$ is a GL_m-invariant closed subvariety of $Quad_m(\mathrm{g})$.

(2): This time consider the following point in the orbit

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \cdot \begin{pmatrix} U_i & V_i \\ 0 & W_i \end{pmatrix} \cdot \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} U_i & \varepsilon^2 V_i \\ 0 & W_i \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} \cdot \begin{pmatrix} 0 & E \\ E^\tau & D \end{pmatrix} \cdot \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} 0 & E \\ E^\tau & \varepsilon^2 F \end{pmatrix}$$

from which the statement follows.

(3): Consider the following point in the orbit

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} U_i & V_i \\ 0 & W_i \end{pmatrix} \cdot \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U_i & \varepsilon V_i \\ 0 & W_i \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \cdot \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$$

concluding the proof. □

Observe that we do not get a full substitute for Artin's result. That is, whereas $(M, b_D) \in Sup_k(\mathrm{g})$ it is not always true that (N, b_F) determines a superstructure as the example below illustrates :

Example 1 *Let* **b** *be the two-dimenesional non-Abelian Lie algebra* $[x,y] = y$ *(note* **b** *is the Borel subalgebra of* sl_2*). Consider the non-trivial extension*

$$0 \to \mathbb{C}\, e_1 \to V \to \mathbb{C}\, e_2 \to 0$$

where $\mathbb{C}\, e_i$ *is the trivial (augmentation) representation and* V *is determined via the matrices*

$$M_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then (V,b) is a Lie superalgebra over $\mathbf{b}$ *if*

$$M_b = \begin{pmatrix} 0 & -\frac{y}{2} \\ -\frac{y}{2} & x \end{pmatrix}$$

Whereas $(\mathbb{C}\,e_1, 0)$ is obviously a Lie superalgebra over $\mathbf{b}$*, this is not the case for $(\mathbb{C}\,e_2, x)$ as*

$$-y = [y, b(e_2, e_2)] \neq 2b(y \cdot e_2, e_2) = 0$$

Hence, a Lie superstructure (V,b) over $\mathbf{g}$ does not necessarily induce a Lie superstructure on the semi-simplification of V. Still, theorem 1 allows one to fill the matrix M_b from top left till bottom right provided we have a good flag of $\mathbf{g}$-subrepresentations on V.

2.2 V a semi-simple g-representation

In this section we want to describe the fiber of the projection morphism

$$p : Sup_m(\mathbf{g}) \longrightarrow Rep_m(\mathbf{g})$$

over a point of $Rep_m(\mathbf{g})$ corresponding to a semi-simple m-dimensional representation V having one-dimensional simple factors. Observe that when $\mathbf{g}$ is a solvable Lie algebra all simple representations are one-dimensional.

Hence, we assume

$$V = \bigoplus_{i=1}^{m} \mathbb{C}\,e_i$$

with

$$x_j \cdot e_i = \lambda_i(x_j) e_i$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Clearly, λ_i is a weight of $\mathbf{g}$, i.e. $\lambda_i \in \mathbf{g}^*$ such that $\lambda_i \mid [\mathbf{g}, \mathbf{g}] = 0$.

Proposition 1 *Let V the m-dimensional semi-simple* $\mathbf{g}$*-representation determined by $(\lambda_1, \ldots, \lambda_m) \in Rep_1(\mathbf{g})^m$. Then the fiber $p^{-1}(V)$ of the projection map*

$$p : Sup_m(\mathbf{g}) \longrightarrow Rep_m(\mathbf{g})$$

consists of all (V,b) with

$$b(e_i, e_j) \in \mathbf{g}^{\lambda_i + \lambda_j} \cap Ann_{\mathbf{g}}(V)$$

In particular, the fiber is a vectorspace of dimension

$$dim(p^{-1}(V)) = \sum_{i \leq j}^{m} dim(\mathbf{g}^{\lambda_i + \lambda_j} \cap Ann_{\mathbf{g}}(V))$$

Proof : From the $\mathbf{g}$-invariance of the bilinear mapping we deduce

$$[g, b(e_i, e_j)] = b(g \cdot e_i, e_j) + b(e_i, g \cdot e_j) = (\lambda_i(g) + \lambda_j(g))b(e_i, e_j)$$

for all $g \in \mathbf{g}$. Hence, $b(e_i, e_j) \in \mathbf{g}^{\lambda_i+\lambda_j}$. The Jacobi superidentity implies $b(e_i, e_j) \cdot e_k = 0$ for all $1 \leq k \leq n$ so $b(e_i, e_j) \in Ann_{\mathbf{g}}(V)$.

Conversely, it is easily verified that such a symmetric matrix determines a superstructure. The $\mathbf{g}$-invariance of the bilinear mapping is clear. The Jacobi superidentity follows from the fact that $b(e_i, e_j) \in Ann_{\mathbf{g}}(V)$. □

As there are only finitely many $\lambda \in \mathbf{g}^*$ s.t. $\mathbf{g}^{\lambda} \neq 0$ the above result states that over allmost all semi-simple representations of $\mathbf{g}$ (having one-dimensional simple components) the fiber consists only of the trivial superstructure obtained by $b = 0$.

2.3 V a triangular g-representation

In this section we investigate the fiber of the projection morphism

$$p : Sup_m(\mathbf{g}) \rightarrow Rep_m(\mathbf{g})$$

over a point $V \in Rep_m(\mathbf{g})$ having all its Jordan-Hölder components one-dimensional. Observe again that this is the general case if $\mathbf{g}$ is a solvable Lie algebra.

Hence, we may assume that

$$V = \sum_{i=1}^{m} \mathbb{C}\, e_i$$

and the action of $\mathbf{g}$ on V is triangular. That is, we have $\lambda_i \in Rep_1(\mathbf{g}) = \{\lambda \in \mathbf{g}^* \mid \lambda([\mathbf{g}, \mathbf{g}]) = 0\}$ such that for all $1 \leq i \leq m$ and all $1 \leq j \leq n$:

$$x_j \cdot e_i = \lambda_i(x_j)e_i + \sum_{k=1}^{i-1} r_{ik}(x_j)e_k$$

with $r_{ij} \in \mathbf{g}^*$ making V into a $\mathbf{g}$-representation.

This time the information we can obtain about the matrix elements $b(e_i, e_j)$ is much weaker in general, as we have no real control on the functions r_{ij}. However, we do have the following result which in practice limits the possibilities for the entries of M_b drastically.

Proposition 2 *With notations as above. For each $i \leq j$ let $\mathbf{b}_{ij}$ be the following sub vectorspace of* $\mathbf{g}$

$$\mathbf{b}_{ij} = \sum_{k \leq i, l \leq j, (i,j) \neq (k,l)} \mathbb{C}\, b(e_k, e_l)$$

Then we have :

1. $\mathbf{b}_{ij}$ *is a Lie ideal of* $\mathbf{g}$

2. *The image of* $b(e_i, e_j)$ *in the quotient Lie-algebra* $\mathbf{g}/\mathbf{b}_{ij}$ *lies in* $(\mathbf{g}/\mathbf{b}_{ij})^{\lambda_i+\lambda_j}$

Proof : Both statements follow from the following computation which holds for every $1 \leq i \leq j \leq m$ and $1 \leq h \leq n$

$$\begin{aligned} [x_h, b(e_i, e_j)] &= b(x_h \cdot e_i, e_j) + b(e_i, x_h \cdot e_j) \\ &= (\lambda_i(x_h) + \lambda_j(x_h))b(e_i, e_j) + \sum_{k \leq i, l \leq j, (k,l) \neq (i,j)} \alpha_{kl}(x_h) b(e_k, e_l) \end{aligned}$$

where the $\alpha_{kl} \in \mathbf{g}^*$ can be easily computed from the $r'_{ij}s$. □

3 The rank varieties of Lie superalgebras

In the foregoing section we have investigated the influence of the representation V (and its degenerations) on the possible bilinear forms b. In this section we will show that the $\mathbf{g}$-invariance of the bilinear form puts equally strong restrictions on the possible Lie superstructures over $\mathbf{g}$.

We will also give an application to super enveloping algebras. The super enveloping algebra $U(\mathbf{s})$ of a Lie superalgebra $\mathbf{s} = (\mathbf{g}, V, b)$ is the quotient of the tensor algebra on $\mathbf{g} \oplus V$ modulo the ideal generated by the following relations :

$$\begin{gathered} x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j] \\ x_i \otimes e_j - e_j \otimes x_i - x_i \cdot e_j \\ e_i \otimes e_j + e_j \otimes e_i - b(e_i, e_j) \end{gathered}$$

From work of E.J. Behr [4] we know that $U(\mathbf{s})$ has some properties in common with enveloping algebras of Lie algebras, e.g. it is a Noetherian Jacobson algebra satisfying the Nullstellensatz [4, Prop.1,Prop.2]. However, there are some noticable differences too. For example, $U(\mathbf{s})$ does not have to be semi-prime [4, Ex.2.2] and even when it is prime it may fail to have finite global dimension [4, Prop.5] or even to be a maximal order.

We will show however that the microlocalization of $U(\mathbf{s})$ at one element (the symmetrization of $det(M_b)$) is a maximal order having finite global dimension.

3.1 The rank varieties

If $\mathbf{s} = (V, b)$ is a Lie superalgebra over $\mathbf{g}$, $M_b = (b(e_i, e_j))_{i,j}$ is a symmetric $m \times m$ matrix with linear entries in $S(\mathbf{g})$ (the symmetric algebra over $\mathbf{g}$ or equivalently, the coordinate ring of $\mathbf{g}^*$).

Clearly, not all symmetric matrices can arise from a Lie superalgebra. For example, assume that $\mathbf{g}$ is solvable, then it follows from proposition 2 that the entries must form a chain of Lie ideals in $\mathbf{g}$.

In general, an immediate consequence of g-invariance of the bilinear form i.e. the restrictions

$$[x_i, b(e_j, e_k)] = b(x_i \cdot e_j, e_k) + b(e_j, x_i \cdot e_k)$$

for all $1 \leq i \leq n$ and all $1 \leq j, k \leq m$ is the following

Proposition 3 *If $1 \leq k \leq m$ let I_k denote the ideal of $S(g)$ generated by all $k \times k$-minors of M_b for some Lie superalgebra $s = (V, b)$ over g. Then, I_k is a homogenous G-invariant ideal of $S(g)$ where G is the algebraic group associated to* **g**.

In particular, if we restrict to the two extreme cases (i.e. when $k = 1$ or $k = m$) we obtain

Corollary 1 *With notations as above we have*

1. *The determinant $\delta = det(M_b)$ is a g-semi-invariant of $S(g)$. Hence, its symmetrization $\Delta \in U(g)$ is a normalizing element of $U(g)$*
2. *The entries of M_b generate a Lie ideal in* g

These observations alone may already restrict the possible dimensions m of the g-representations V

Example 2 *Let* g *be a simple Lie algebra.*

The first part states that m must be the degree of a central element of $U(g)$. These degrees can be calculated from the action of the Weyl group on the root system (see e.g. [10, 7.3.8]).

The second part states that m must satisfy the inequality $\frac{m(m+1)}{2} \geq n = dim(g)$.

To a Lie superalgebra s one can associate the invariants

$$\{(p_1, m_1, n_1), (p_2, m_2, n_2), \ldots, (p_k, m_k, n_k)\}$$

where the p_i are the G-invariant height one prime ideals of $S(g)$ containing $\delta = p_1^{n_1} \cdots p_k^{n_k}$ and m_i is the dimension of the non-singular part of the induced quadratic form over the function-field of $S(g)/p_i$. It would be interesting to know which sets of such triples can arise from a Lie superalgebra. Even the problem of deciding which g (or G) semi-invariants δ arise from Lie super algebras is not settled. The next example shows that not every semi-invariant is obtained in this way.

Example 3 *Let* g *be the 5-dimensional nilpotent Lie algebra with nonzero bracket relations :*

$$[x_1, x_2] = x_4$$

$$[x_1, x_3] = x_5$$

Then, $x_2x_5 - x_3x_4$ is a quadratic $\mathbf{g}$ *semi-invariant. However, it is clear that there is no Lie ideal* $\mathbf{h}$ *of* $\mathbf{g}$ *with $dim(\mathbf{h}) \leq 3$ having this element as a semi-invariant. Hence, there is no Lie superalgebra over* $\mathbf{g}$ *with $dim(V) = 2$ having $x_2x_5 - x_3x_4$ as $det(M_b)$.*

The subvariety V_k of $\mathbf{g}^*$ determined by the ideal I_k will be called the k-th rank variety of $\mathbf{s}$. If its irreducible components are $Y_{k,1}, \ldots, Y_{k,i_k}$ then these components are G-stable and so corresponding to them (in case $\mathbf{g}$ is solvable) are prime ideals $P_{k,1}, \ldots, P_{k,i_k}$ of $U(\mathbf{g})$. It is clear that these prime ideals should be of interest in studying prime quotient factors of $U(\mathbf{s})$.

3.2 The good microlocalization of $U(\mathbf{s})$

A sufficient condition for $U(\mathbf{s})$ to be prime was found by A.D. Bell [5]. If we define $M_b = (b(e_i, e_j))_{i,j} \in M_m(S(\mathbf{g}))$ to be the product matrix of $\mathbf{s}$ then its determinant $det(M_b)$ is either zero or a homogeneous polynomial of degree m in the symmetric algebra $S(\mathbf{g})$. Bell proved that if $det(M_b) \neq 0$, then $U(\mathbf{s})$ is prime [5, Prop. 1.4].

He proves this by endowing $U(\mathbf{s})$ with the Aubry-Lemaire filtration [2] i.e. giving e_i filtration degree 1 and x_i filtration degree 2. Then, the associated graded algebra $gr(U(\mathbf{s})) = \oplus_{j=0}^{\infty} U(\mathbf{s})_j / U(\mathbf{s})_{j-1}$ has defining relations

$$\begin{aligned} x_i \cdot x_j - x_j \cdot x_i &= 0 \\ x_i \cdot e_j - e_j \cdot x_i &= 0 \\ e_i \cdot e_j + e_j \cdot e_i &= b(e_i, e_j) \end{aligned}$$

and hence is the Clifford algebra of a (degenerate) quadratic form over $S(\mathbf{g})$ determined by the product matrix of $\mathbf{s}$.

The structure of Clifford algebras is well known (cfr. [3] or [13, Ch.2]) provided the form is non-degenerate i.e. if the determinant is invertible.

Proposition 4 *Let* $\mathbf{s}$ *be a Lie superalgebra s.t. the product matrix $M_b = (b(e_i, e_j))_{i,j}$ has non-zero determinant δ in $S(\mathbf{g})$. Then,*

$$gr(U(\mathbf{s})) \otimes_{S(\mathbf{g})} S(\mathbf{g})_\delta$$

is a maximal order Azumaya algebra with global dimension n. Here, $U(\mathbf{s})$ is equiped with the Aubry-Lemaire filtration and $S(\mathbf{g})_\delta$ is the localization of $S(\mathbf{g})$ at $\{1, \delta, \delta^2, \ldots\}$.

Proof : As all entries of the product matrix are linear terms in $\mathbf{g}$, δ is a homogeneous form of degree m among the x_i. Hence, if m is odd the square root of δ is not contained in $S(\mathbf{g})$ or its localization $S(\mathbf{g})_\delta$.

As $gr(U(\mathrm{s})) \otimes S(\mathrm{g})_\delta$ is the Clifford algebra of a non-degenerate quadratic form over $S(\mathrm{g})_\delta$ we recall from [3] that it is an Azumaya algebra over its center Z which is equal to $S(\mathrm{g})_\delta$ if m is even or to $S(\mathrm{g})_\delta\sqrt{\delta}$ if m is odd.

One verifies easily that Z is a regular domain of dimension n. As an Azumaya algebra over a regular (in particular normal) domain, $gr(U(\mathrm{s})) \otimes S(\mathrm{g})_\delta$ is a maximal order with $gldim$ equal to $gldim(Z) = n$. □

Let $\Delta \in U(\mathrm{g})$ be the symmetrization of δ. It follows from the foregoing subsection that Δ is a semi-invariant (hence normalizing element) in $U(\mathrm{g})$. Unfortunately, Δ does not have to be a semi-invariant in $U(\mathrm{s})$ or even (at least if the basefield is not algebraically closed) the leading term in the Aubry-Lemaire filtration of a semi-invariant as the following example shows.

Example 4 *Consider the solvable Lie superalgebra* $\mathrm{s} = (\mathrm{g}, V, b)$ *over* $\mathbb{R}$ *where* g *is the 2-dimensional non-trivial Lie algebra with bracket relations* $[x_1, x_2] = x_2$, $V = \mathbb{C}\, e_1 + \mathbb{C}\, e_2$ *is the 2-dimensional* g*-representation determined by*

$$x_1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, x_2 \longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the product matrix of s *is*

$$M_b = \begin{pmatrix} 0 & \frac{x_2}{2} \\ \frac{x_2}{2} & 0 \end{pmatrix}$$

Then $U(\mathrm{s})$ *has no semi-invariants. If* s *is defined over* $\mathbb{C}$, *the semi-invariants of* $U(\mathrm{s})$ *are (up to nonzero scalars)* $x_2^j \pm i\sqrt{2j}\, x_2^{j-1} e_1$ *with* $j \geq 1$.

So, it might not always be possible to localize at the multiplicative set $\{1, \Delta, \Delta^2, \ldots\}$. Still, we can microlocalize at it. As the principal symbol of Δ w.r.t. the Aubry-Lemaire filtration is δ and $\{1, \delta, \delta^2, \ldots\}$ is an Ore-set in $gr(U(\mathrm{s}))$ it follows that the saturation

$$S = \{u \in U(\mathrm{s}) : \sigma(u) = \delta^j \text{ for some } j\}$$

is an Ore-set in $U(\mathrm{s})$ (where $\sigma(x)$ is the image of x in $gr(U(\mathrm{s}))_n$ for $x \in U(\mathrm{s})_n - U(\mathrm{s})_{n-1}$). The localization $U(\mathrm{s})_S$ has an extended Aubry-Lemaire filtration such that the associated graded ring $gr(U(\mathrm{s})_S)$ is isomorphic to $gr(U(\mathrm{s}))\otimes S(\mathrm{g})_\delta$. We call $U(\mathrm{s})_S$ the microlocalization of $U(\mathrm{s})$ at the determinant of the product matrix. Actually, the usual microlocalization is the completion of this ring with respect to the topology induced by the filtration.

Theorem 2 *Let* S *be the saturated Ore-set associated to the multiplicative set* $\{1, \Delta, \Delta^2, \ldots\}$ *where* Δ *is the symmetrization of* $\delta = det(B)$. *Then, the microlocalization* $U(\mathrm{s})_S$ *of the super enveloping algebra at* δ *is a maximal order having finite global dimension.*

Proof : As we recalled above,

$$gr(U(\mathbf{s})_S) = gr(U(\mathbf{s})) \otimes S(\mathbf{g})_\delta$$

By proposition 4 we know that the right hand side is a maximal order having finite global dimension. Hence, both properties can be lifted to $U(\mathbf{s})_S$ by general filtered results, see e.g. [15] for the global dimension and [20] for the maximal order property. □

Observe that the same holds for any Ore-set containing Δ. So, by microinverting one element the super enveloping algebra has all the nice ringtheoretical properties of enveloping algebras.

However, the procedure of taking the microlocalization at an element is often too crude e.g. it may very well be that powers of the element already form an Ore-set. Hence one would like to have a general procedure to find a small Ore-set containing the powers of the element contained in the micro-Ore set.

In case $\mathbf{g}$ is solvable (i.e. when the Lie superalgebra is solvable) we have such a general result. As the method does apply to any Ore-set we state the result in that generality.

Consider the adjoint action of $\mathbf{g}$ on $\mathbf{g} \oplus V$ and let $E_{\mathbf{s}}$ be the additive sub semigroup of $\mathbf{g}^*$ generated by the set of its Jordan-Hölder values (i.e the weights of $\mathbf{g}$ corresponding to the Jordan-Hölder factors of $\mathbf{g} \oplus V$). Recall that each weight α induces an automorphism τ_α on $U(\mathbf{g})$ by putting $\tau_\alpha(x) = x + \alpha(x)$ for all $x \in \mathbf{g}$ (see e.g. [7, §10.2]). With this notation we have

Proposition 5 *Let $\mathbf{s} = (\mathbf{g}, V, b)$ be a solvable Lie superalgebra such that its enveloping algebra $U(\mathbf{s})$ is prime. A left (resp. right) Ore set S of $U(\mathbf{g})$ extends to a left (resp. right) Ore set of $U(\mathbf{s})$ if $\tau_{-\alpha}(S) \subset S$ (resp. $\tau_\alpha(S) \subset S$) for all $\alpha \in E_{\mathbf{s}}$.*

Proof : This is just [8, Satz 4.5] adjusted to our situation. Observe that the adjoint action of $\mathbf{g}$ on $U(\mathbf{s})$ is locally triangularizable as $\mathbf{g}$ is a solvable Lie algebra and that the set of Jordan-Hölder values is contained in $E_{\mathbf{s}}$ as $\mathbf{g} \oplus V$ generates $U(\mathbf{s})$. □

In particular, if δ is a semi-invariant of $U(\mathbf{g})$ we see that the multiplicative set generated by δ and all $\tau_{\pm\alpha}(\delta)$ ($\alpha \in E_{\mathbf{s}}$) is an Ore-set in $U(\mathbf{s})$. Denote this set with $\delta(\mathbf{s})$.

Corollary 2 *Let $\Delta \neq 0$ be the symmetrization of the determinant of the product matrix of the solvable Lie superalgebra $\mathbf{s} = (\mathbf{g}, V, b)$, then $\Delta(\mathbf{s})$ is an Ore-set in $U(\mathbf{s})$ and $U(\mathbf{s})_{\Delta(\mathbf{s})}$ is a maximal order having finite global dimension.*

4 Solvable versus completely solvable

One of the intruiging differences between Lie and super Lie theory is that a solvable Lie superalgebra s over g (i.e. g a solvable Lie algebra) does not have to satify Lie's theorem (i.e. there may be irreducible finite dimensional representations of dimension greater than 1). In particular, a solvable Lie superalgebra does not need to have a flag of Lie superideals. Solvable Lie superalgebras posesssing this extra structure are called completely solvable Lie superalgebras and their enveloping algebras are more manageable see e.g. [6] and [14].

One of the initial motivations for the present paper was to construct lots of solvable but not completely solvable Lie superalgebras. However, there seem to be embarrassingly few of them around. In this section we will show that if $dim(V) = 2$ such Lie algebras must have a Lie ideal isomorphic to $\mathbf{b}$ (generated by the entries of the matrix M_b) and that the restriction of V to $\mathbf{b}$ is isomorphic to the adjointrepresentation.

4.1 Some reductions

From now on we adopt the following notation: If $\mathsf{s} = (V, b)$ is a Lie superalgebra over g and if $\mathbf{h}$ denotes the Lie ideal of g generated by the entries of the symmetric matrix M_b, then with $\mathsf{s}_\downarrow$ we will denote the Lie superalgebra $(V_\downarrow, b)$ over $\mathbf{h}$ where $V_\downarrow$ is the restricted $\mathbf{h}$-representation.

Proposition 6 *With notations as above,*

1. *If* s *is completely solvable, then so is* $\mathsf{s}_\downarrow$
2. *If* s *is solvable and* $V_\downarrow$ *is the trivial* $\mathbf{h}$*-representation (e.g. in case* V *is a semi-simple* g*-representation), then* s *is completely solvable*

Proof : (1): Intersecting the chain of Lie superideals of s with $\mathsf{s}_\downarrow$ gives a chain of Lie superideals.

(2): Let $\{v_1, \ldots, v_m\}$ be a triangular g-basis for V, and $\{h_1, \ldots, h_k\}$ a triangular g-basis for $\mathbf{h}$ (note that g is solvable), then the ordering

$$h_1 < h_2 < \ldots < h_k < v_1 < v_2 < \ldots < v_m$$

gives a chain of Lie superideals for $s_\downarrow$ which can be completed via a triangular basis of $\mathsf{g}/\mathbf{h}$. □

Note that in general, it is not clear that s should be completely solvable if $\mathsf{s}_\downarrow$ is completely solvable. The reason being that a triangular $\mathbf{h}$-basis of $V_\downarrow$ induced by a chain of Lie superideals of $\mathsf{s}_\downarrow$ does not necessarily have to be a triangular g-basis for V. We will see below that the implication is valid though if $dim(V) = 2$. Observe that any solvable Lie superalgebra $\mathsf{s} = (V, b)$

with $dim(V) = 1$ is completely solvable, just start the chain of ideals with $b(e_1, e_1) < e_1 < \dots$

Applying the results of the forgoing sections, one could calculate all superstructures over a given solvable Lie algebra. In this section, we will give a coarse classification of all Lie superalgebras with $dim(V) = 2$. To begin, we can in this case reformulate the information obtained in propositions 1 and 2 as follows:

Lemma 1 *Let $\mu_1, \dots, \mu_k \in \mathbf{g}^*$ s.t. $\mathbf{g}^{\mu_i} \neq 0$. Let V be a 2-dimensional* **g***-representation with Jordan-Hölder factors $\alpha, \beta \in Rep_1(\mathbf{g}) \times Rep_1(\mathbf{g})$. Then, in order to have non trivial fiber under the map $p : Sup_2(\mathbf{g}) \longrightarrow Rep_2(\mathbf{g})$, V has to lie on one of the following lines:*

1. *the vertical lines: $\alpha = \mu_i/2$*
2. *the horizontal lines: $\beta = \mu_i/2$*
3. *the diagonal lines: $\alpha + \beta = \mu_i$*

In the first case, $b(e_1, e_1)$ may be nonzero, in the second case $b(e_2, e_2)$ and in the last case $b(e_1, e_2)$. So we see that $\delta = det(M_b)$ can only be nonzero if (α, β) lies on the diagonals or is an intersection point of an horizontal and a vertical line. According to Bell [5, 2.2] these are precisely the situations where the enveloping algebra $U(\mathbf{s})$ is prime. Note that for higher dimensional V it is still an open problem whether primeness of $U(\mathbf{s})$ forces b to be non-degenerate.

In most points, the representation has to be semi-simple. This is a consequence of a result of M. Loupias [16] (or see [9, Th.2.2] for a massive generalization of her result):

Let $\mu_1, \dots, \mu_k$ be the Jordan-Hölder factors of the adjoint representation of **g** and let $\mathbb{C}e_1$ and $\mathbb{C}e_2$ be determined by $\alpha, \beta \in Rep_1(\mathbf{g})$, then there exists a non trivial extension of **g**-representations

$$0 \longrightarrow \mathbb{C}e_1 \longrightarrow V \longrightarrow \mathbb{C}e_2 \longrightarrow 0$$

iff $\alpha - \beta = \mu_i$ for some $i = 1, \dots, k$ or $\alpha - \beta = 0$.

Moreover, the extensions are easy to calculate explicitly:

Lemma 2 *If $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are represented by $\lambda + \mu, \lambda \in Rep_1(\mathbf{g})$ (μ a Jordan-Hölder factor of $ad(\mathbf{g})$) then the extensions of* **g***-representations*

$$0 \longrightarrow \mathbb{C}e_1 \longrightarrow V \longrightarrow \mathbb{C}e_2 \longrightarrow 0$$

are represented by

$$\begin{pmatrix} \lambda + \mu & \gamma \\ 0 & \lambda \end{pmatrix}$$

with

$$\gamma \in (\mu(x_i)x_j - \mu(x_j)x_i - [x_i, x_j])^{\perp} \subset \mathbf{g}^*$$

Combining this fact with lemma 1 above, we obtain the following determination of Lie superstructures s over g with $dim(V) = 2$ s.t. V contains a one-dimensional g-subrepresentation:

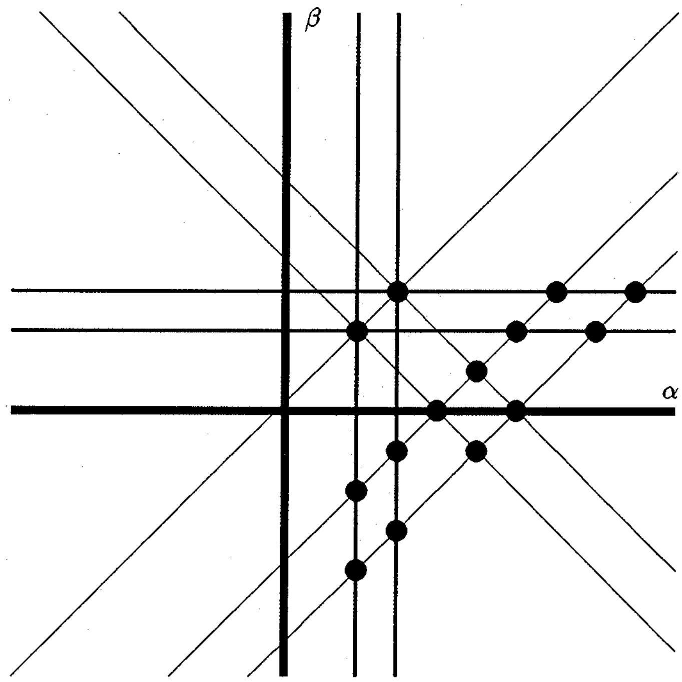
Proposition 7 *Let $\mu_1, ..., \mu_k \in g^*$ be such that $g^{\mu_i} \neq 0$. Let V be a 2-dimensional g-representation with Jordan-Hölder factors $(\alpha, \beta) \in Rep_1(g) \times Rep_1(g)$. If the fiber $p^{-1}(V)$ where*

$$p : Sup_2(g) \rightarrow Rep_2(g)$$

contains a non-trivial superstructure, then (α, β) lies on one of the following lines :

1. *the vertical lines: $\alpha = \mu_i/2$*
2. *the horizontal lines: $\beta = \mu_i/2$*
3. *the diagonal lines: $\alpha + \beta = \mu_i$*

The only points (α, β) where V is not necessarily semi-simple are the intersection points with the diagonals $\alpha = \beta$ or $\alpha = \beta + \mu_i$. Pictorially,



where non semi-simple V can only occur in the dotted places.

By proposition 1 we know the fibers of p at semi-simple V precisely. Therefore, classifying all Lie superalgebras $s = (V, b)$ over a solvable Lie algebra g amounts to discussing the finitely many non semi-simple possibilities.

4.2 The classification

As an application of our general results we now want to give a rather coarse classification of all Lie superalgebras $\mathbf{s} = (V, b)$ with $dim(V) = 2$ by describing precisely which $\mathbf{s}_{\downarrow}$ can occur. We only have to consider a few cases

Proposition 8 *Let* $\mathbf{s} = (V, b)$ *be a Lie superalgebra over* $\mathbf{g}$ *with* $dim(V) = 2$. *Let* $\mathbf{h}$ *be the Lie ideal of* $\mathbf{g}$ *generated by the entries of* M_b*, then* $\mathbf{h}$ *must be one of the following Lie algebras*

1. *Abelian of dimension* ≤ 3
2. $\mathbf{b}$ *the two-dimensional non-Abelian*
3. $\mathbf{sl_2}$

Proof : Obviously, $dim(\mathbf{h}) \leq 3$. If $dim(\mathbf{h}) = 3$ then we may assume that

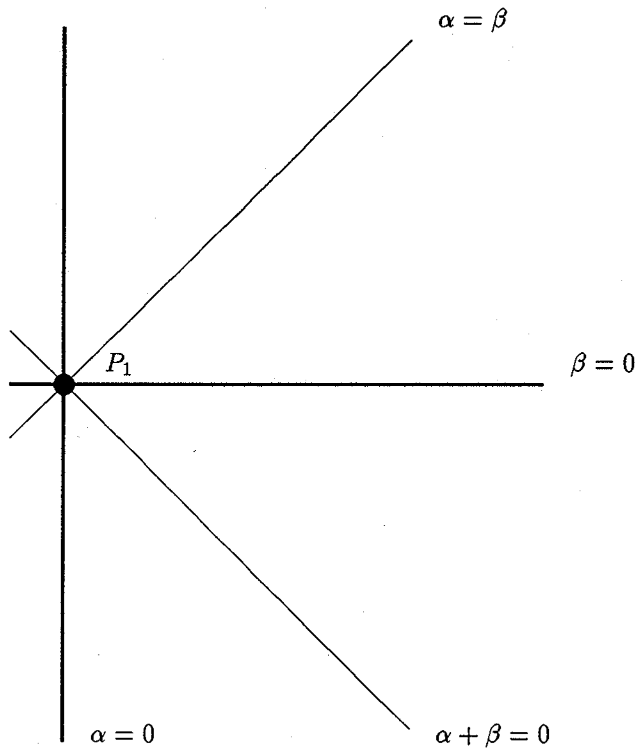
$$M_b = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$$

where x, y and z are a basis for $\mathbf{h}$. Hence, the symmetrization of $\delta = det(M_b) = xz - y^2$ is an indecomposable quadratic semi-invariant of $\mathbf{h}$. From the list of 3-dimensional Lie algebras it is clear that this can only happen if $\mathbf{h}$ is Abelian or if $\mathbf{h} = \mathbf{sl_2}$. Otherwise, $dim(\mathbf{h}) \leq 2$ concluding the proof. □

So we limit the classification to describing all Lie superalgebras over the Abelian Lie algebras of dimension ≤ 3, over the 2-dimensional non-Abelian Lie algebra $\mathbf{b}$ and over $\mathbf{sl_2}$.

4.2.1 The Abelian cases

Let $\mathbf{a}_n$ be the Abelian Lie algebra of dimension n. As $0 = \mu \in \mathbf{a}_n^*$ is the only weight s.t. $\mathbf{a}_n^\mu \neq 0$ we get the following picture



The Lie superalgebras with semi-simple V (not in P_1) are easy to determine

type	line	V	M_b	cond
s_1	$\alpha = 0$	$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$	$\lambda(g) = 0$
s_2	$\beta = 0$	$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$	$\lambda(g) = 0$
s_3	$\alpha = -\beta$	$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$	$\begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$	$\lambda(g) = 0$

where $\lambda \in \mathbf{a}_n^*$ and $g \in \mathbf{a}_n$. The condition $\lambda(g) = 0$ entails that for $\mathbf{a}_1$ there are no Lie superalgebras outside P_1, for $\mathbf{a}_2$ each of these points has a 1-dimensional fibre and for $\mathbf{a}_3$ the fibre is 2-dimensional.

Remains the study of the fibre in P_1. If V is not the trivial representation, it must be nilpotent i.e.

$$V = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

with $0 \neq \lambda \in \mathbf{a}_n^*$. If $\lambda(x) = a \neq 0$ then

$$0 = [x, b(e_1, e_2)] = b(x \cdot e_1, e_2) + b(e_1, x \cdot e_2) = ab(e_1, e_1)$$

and hence $b(e_1, e_1) = 0$. Similarly, $b(e_1, e_2) = 0$ leaving

$$M_b = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$$

with $g \in \mathbf{a}_n$. From the Jacobi superidentity one deduces that $b(e_1, e_1) \cdot e_1 = 0$ whence $\lambda(g) = 0$. Again, for $\mathbf{a}_1$ there is no such structure, for $\mathbf{a}_2$ a 1-dimensional family and for $\mathbf{a}_3$ a 2-dimensional one.

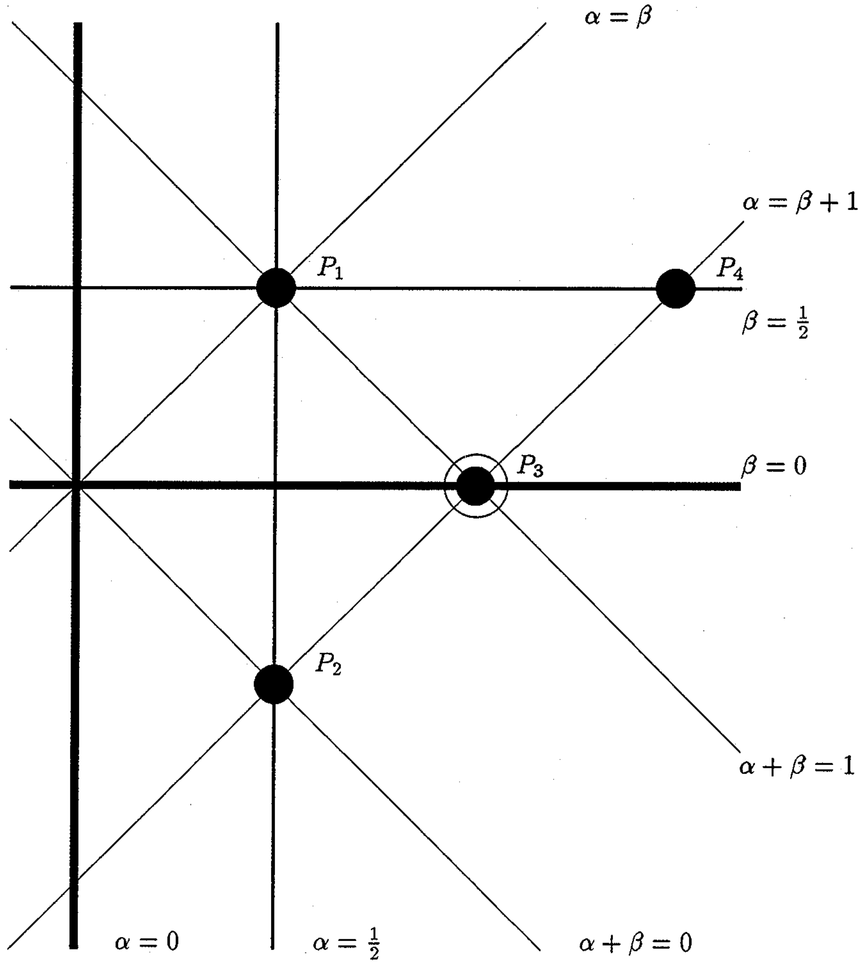
Finally, if V is the trivial representation then any symmetric matrix with entries from $\mathbf{a}_n$ will give rise to a Lie superalgebra giving a $3n$ dimensional family. Concluding, the remaining Lie superalgebras are classified as

type	point	V	M_b	cond
n_1	P_1	$\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$	$\lambda(g) = 0$
s_4	P_1	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_1 & g_2 \\ g_2 & g_3 \end{pmatrix}$	

For $\mathbf{a}_1$ and $\mathbf{a}_2$ it is trivial that every quadratic form can arise as the determinant of a Lie superstructure. For $\mathbf{a}_3$ this is also the case. Probably a much too roundabout proof of this fact goes as follows: every form of degree n in 3 variables occurs as $det(M_x x + M_y y + M_z z)$ for $M_x, M_y, M_z \in M_n(\mathbb{C})$ see e.g. [11] and for the smooth forms there corresponds a vectorbundle over $\mathbb{P}_2$ with rank n and Chern numbers $c_1 = 0$ and $c_2 = 2$. For $n = 2$ every such bundle has a quadratic form which translates back into the fact that one can choose the three matrices M_x, M_y and M_z to be symmetric 2×2 matrices over $\mathbb{C}$ [11] and hence one gets every degree 2 form as the determinant of M_b for a Lie superalgebra over $\mathbf{a}_3$.

4.2.2 The Borel case

If $\mathbf{b}$ is the 2-dimensional non-Abelian Lie algebra $[x, y] = y$ then $x^* \in \mathbf{b}^*$ is the only weight s.t. $\mathbf{b}^{x^*} = \mathbb{C}y \neq 0$. Therefore, we get the following picture



Again, the cases where V is semi-simple (and not one of the points P_i) are easy to work out

type	line	V	M_b
s_1	$\alpha = \frac{1}{2}$	$\begin{pmatrix} \frac{x^*}{2} & 0 \\ 0 & \lambda x^* \end{pmatrix}$	$\begin{pmatrix} \mathbb{C}y & 0 \\ 0 & 0 \end{pmatrix}$
s_2	$\beta = \frac{1}{2}$	$\begin{pmatrix} \lambda x^* & 0 \\ 0 & \frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C}y \end{pmatrix}$
s_3	$\alpha + \beta = 1$	$\begin{pmatrix} \lambda x^* & 0 \\ 0 & (1-\lambda)x^* \end{pmatrix}$	$\begin{pmatrix} 0 & ay \\ ay & 0 \end{pmatrix}$

Next, let us consider the non semi-simple cases in the points P_i, i.e. V is determined via

$$x \longrightarrow \begin{pmatrix} \mu_1 & \gamma_1 \\ 0 & \mu_2 \end{pmatrix} \text{ and } y \longrightarrow \begin{pmatrix} 0 & \gamma_2 \\ 0 & 0 \end{pmatrix}$$

Observe that by lemma 2 we know that $\gamma_2 = 0$ in P_1.

As $b(e_1, e_1)$ has to be a semi-invariant we can take $b(e_1, e_1) = ay$ if $\mu_1 = \frac{1}{2}$ (i.e. in P_1 and P_2) and it must be zero otherwise. If $b(e_1, e_2) = cx + dy$ then we have the relations

$$\begin{aligned} [x, b(e_1, e_2)] &= (\mu_1 + \mu_2)b(e_1, e_2) + \gamma_1 b(e_1, e_1) \\ [y, b(e_1, e_2)] &= \gamma_2 b(e_1, e_1) \end{aligned}$$

Giving the conditions : $c+\gamma_2 a = 0, \gamma_2 b = 0$ and $(\mu_1+\mu_2)c = (\mu_1+\mu_2-1)d+\gamma_1 a = 0$. Finally, $b(e_2, e_2)$ has to be a solution of

$$\begin{aligned} [x, b(e_2, e_2)] &= 2\mu_2 b(e_2, e_2) + 2\gamma_1 b(e_1, e_2) \\ [y, b(e_2, e_2)] &= 2\gamma_2 b(e_1, e_2) \end{aligned}$$

giving $0 = \gamma_2 c, f + 2\gamma_2 d = 0$ and $0 = \gamma_1 c + \mu_2 f = 2\gamma_1 d + (2\mu_2 - 1)g$.

On top, we have to satisfy the Jacobi superidentities

$$\begin{aligned} b(e_i, e_i) \cdot e_i &= 0 \\ 2b(e_1, e_2) \cdot e_1 + b(e_1, e_1) \cdot e_2 &= 0 \\ 2b(e_1, e_2) \cdot e_2 + b(e_2, e_2) \cdot e_1 &= 0 \end{aligned}$$

giving the additional restrictions

$$\begin{aligned} f\mu_2 = f\gamma_1 + g\gamma_2 &= 0 \\ (2\mu_1 - 1)c &= 0 \\ c\mu_2 = 2(c\gamma - 1 + d\gamma_2) + f\mu_1 &= 0 \end{aligned}$$

Which gives us the following list of possible superstructures in the points P_i

type	point	V	M_b
s_4	P_1	$\begin{pmatrix} \frac{x^*}{2} & 0 \\ 0 & \frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} ay & dy \\ dy & gy \end{pmatrix}$
n_1	P_1	$\begin{pmatrix} \frac{x^*}{2} & \lambda_1 x^* \\ 0 & \frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & gy \end{pmatrix}$
s_5	P_2	$\begin{pmatrix} \frac{x^*}{2} & 0 \\ 0 & -\frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} ay & 0 \\ 0 & 0 \end{pmatrix}$
n_2	P_2	$\begin{pmatrix} \frac{x^*}{2} & \gamma_1 x^* \\ 0 & -\frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} ay & \gamma_1 ay \\ \gamma_1 ay & \gamma_1^2 ay \end{pmatrix}$
s_6	P_3	$\begin{pmatrix} x^* & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & dy \\ dy & 0 \end{pmatrix}$
n_3	P_3	$\begin{pmatrix} x^* & \gamma_1 x^* + \gamma_2 y^* \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & dy \\ dy & -2\gamma_2 dx + 2\gamma_1 dy \end{pmatrix}$
s_7	P_4	$\begin{pmatrix} \frac{3x^*}{2} & 0 \\ 0 & \frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & gy \end{pmatrix}$
n_4	P_4	$\begin{pmatrix} \frac{3x^*}{2} & \gamma_1 x^* \\ 0 & \frac{x^*}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & gy \end{pmatrix}$

Remark 1 *Only in the point P_3 are there Lie superalgebras $\mathrm{s} = (V, b)$ which are not completely solvable,namely when y acts non-trivially on V and $b(e_1, e_2) = dy \neq 0$. Observe that all these representations are isomorphic to the adjoint representation.*

4.2.3 The $\mathbf{sl_2}$-case

To complete the classification of superstructures (V, b) with $dim(V) = 2$, one should consider the non-solvable ones:

Proposition 9 *The only non trivial superstructure over $\mathbf{sl_2}$ with $dim(V) = 2$ is $osp(1, 2)$.*

Proof : Consider first the non-trivial 2-dimensional $\mathbf{sl_2}$-representation

$$e \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Put $b(e_1, e_1) = ke + lf + mh$. The $\mathfrak{g}$-invariance of b implies $l = m = 0$ or

$$b(e_1, e_1) = ke$$

and

$$b(e_1, e_2) = -kh/2$$

Checking the $\mathfrak{g}$-invariance for $b(e_1, e_2)$ yields

$$b(e_2, e_2) = -kf$$

The other relations to check give no more restrictions. So we have

$$M_b = \begin{pmatrix} ke & -\frac{kh}{2} \\ -\frac{kh}{2} & -kf \end{pmatrix}$$

which defines the Lie superalgebra $osp(1,2)$ (see [17]). If V is the trivial 2-dimensional representation, all $b(e_i, e_j)$ have to be central elements of $\mathbf{sl_2}$ so $M_b = 0$. □

4.3 The exceptional solvable Lie superalgebras

Concluding, we get the following characterization of solvable not completely solvable Lie superalgebras in case $dim(V) = 2$:

Theorem 3 *Let $\mathfrak{s} = (V, b)$ be a Lie superalgebra over $\mathfrak{g}$ with $dim(V) = 2$, then*

1. *$\mathfrak{s}$ is completely solvable iff $\mathfrak{s}_\downarrow$ is completely solvable*
2. *$\mathfrak{s}$ is solvable but not completely solvable iff $\mathfrak{s}_\downarrow = \mathbf{b}$ and $V_\downarrow$ is isomorphic to the adjoint representation.*

Proof : (1): Follows from proposition 6 and the above classification, as if we look at the possibilities for $\mathfrak{s}_\downarrow$ we can see that for all completely solvable ones, $V_\downarrow$ is the trivial representation.
(2): Follows from (1) and the classification. □

It would be very interesting to know a similar characterization of 'minimal' solvable not completely solvable Lie superalgebras when $dim(V) = 3$ or higher.
The methods of this paper can be viewed as a first step towards this (probably hard) classification project.

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