On Quantum Spaces of Lie Algebras

Lieven le Bruyn*
Michel Van den Bergh
Dept. of Mathematics and Computer Science,
University of Antwerp, UIA
B-2610 Wilrijk (Belgium)

September 1991 – 91-45

Abstract

The homogenization $H(\mathfrak{g})$ of the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ is an Artin-Schelter regular algebra. We characterize d-dimensional linear subspaces in the corresponding quantum space $\mathbb{P}_q(\mathfrak{g})$ as homogenizations of induced representations from codimension d Lie subalgebras. Furthermore we prove that the point variety has an embedded component iff there is a line, not contained in this point variety.

Key Words

Enveloping Algebras, Quantum Spaces

AMS Subject Classification

16S30

---

* Research Associate of the NFWO (Belgium)

On quantum spaces of Lie algebras

Lieven Le Bruyn*and Michel Van den Bergh
Departement Wiskunde en Informatica
Universitaire Instelling Antwerpen
B-2610 Wilrijk (Belgium)

September 10, 1991

Abstract

The homogenization $H(\mathfrak{g})$ of the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ is an Artin-Schelter regular algebra. We characterize d-dimensional linear subspaces in the corresponding quantum space $\mathbb{P}_q(\mathfrak{g})$ as homogenizations of induced representations from codimension d Lie subalgebras. Furthermore we prove that the point variety has an embedded component iff there is a line, not contained in this point variety.

1 Introduction and notations

To any finite dimensional Lie algebra

$$\mathfrak{g} = \mathbb{C}X_1 \oplus \cdots \oplus \mathbb{C}X_n \quad \text{with} \quad [X_i, X_j] = \sum_{k=1}^{n} \alpha_{ij,k} X_k$$

one classically associates an associative algebra $U(\mathfrak{g})$, the so-called "enveloping algebra" of $\mathfrak{g}$. The enveloping algebra carries a natural filtration

$$U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \cdots \subset U_n(\mathfrak{g}) \subset \cdots$$

*Research associate of the NFWO (Belgium)

such that $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$. This is the famous Poincaré-Birkhoff-Witt-theorem.

By homogenizing the defining relations of $U(\mathfrak{g})$ one can also associate a regular algebra in the sense of Artin and Schelter [1] to $\mathfrak{g}$. This algebra, which will be denoted by $H(\mathfrak{g})$ in the sequel, is the positively graded quadratic algebra generated by $X_0, ..., X_n$ where X_0 is taken to be central and the remaining defining relations are

$$X_i X_j - X_j X_i - \sum_{i=1}^{n} \alpha_{ij,k} X_k X_0$$

Observe that $H(\mathfrak{g})/(X_0 - 1)H(\mathfrak{g}) \cong U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$ and that $H(\mathfrak{g})/X_0 H(\mathfrak{g}) \cong \mathrm{gr}(U(\mathfrak{g})) \cong S(\mathfrak{g})$. From these facts one derives as usual (see e.g. [8] or [9]) that $H(\mathfrak{g})$ is a Noetherian maximal order domain, it is Auslander-regular of dimension $n+1$ and satisfies the Cohen-Macaulay property.

Following M. Artin [3] we define the quantum space $\mathbb{P}_q(\mathfrak{g})$ of $\mathfrak{g}$ to be $\mathrm{Proj}(H(\mathfrak{g}))$ which is the quotient category of the category of all finitely generated graded left $H(\mathfrak{g})$-modules modulo the subcategory of finite length modules. Via the identification $S(\mathfrak{g}) = H(\mathfrak{g})/X_0 H(\mathfrak{g})$, $\mathbb{P}(\mathfrak{g}^*) = \mathrm{Proj}\, S(\mathfrak{g})$ will be thought of as the hyperplane at infinity of $\mathbb{P}_q(\mathfrak{g})$ with equation $X_0 = 0$.

As in [2] one can define points and lines in $\mathbb{P}_q(\mathfrak{g})$. This note originated from our attempts to understand the following example, taken from [7].

Example 1.1 *Let* $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}e \oplus \mathbb{C}f \oplus \mathbb{C}h$ *with* $[e,f] = h$, $[h,e] = 2e$ *and* $[h,f] = -2f$. *Then the lines in* $\mathbb{P}_q(\mathfrak{sl}(2,\mathbb{C}))$ *are precisely the lines lying on the pencil of quadrics in* $\mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[e,f,h,t])$

$$Q(\delta) = \mathcal{V}((h^2 + 4ef) + \delta^2 t^2)$$

In particular, all lines intersect the hyperplane $V(t)$ *at infinity in a point of the conic* $(t, h^2 + 4ef)$ *which is precisely the embedded component of the point-variety of* $\mathbb{P}_q(\mathfrak{sl}(2,\mathbb{C}))$.

Moreover all three dimensional Lie algebras behave in a similar way. That is, the point-variety has an embedded component lying in the plane at infinity and all lines not lying on the point-variety intersect with the hyperplane at infinity in a point of this component.

In this note we partially generalize Example 1.1 by showing that there always is a subvariety of the hyperplane at infinity in $\mathbb{P}_q(\mathfrak{g})$, closely related with a possible embedded component of the point variety, such that all lines, not lying in the hyperplane at infinity intersect this variety. In case $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ this variety *is* the embedded component of the point variety.

We also show that the existence of such an embedded component is caused by the fact that there may be lines in $\mathbb{P}_q(\mathfrak{g})$, not lying in the point variety of $\mathbb{P}_q(\mathfrak{g})$.

The proofs of these results, given in Section 3, rely on the classification of linear subspaces in $\mathbb{P}_q(\mathfrak{g})$, given in the next section.

2 Linear subspaces, Lie subalgebras and polarizations

We first state a few generalities. To any finitely generated filtered $U(\mathfrak{g})$-module

$$\cdots \subset M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset \bigcup_{i=-\infty}^{\infty} M_i = M$$

we can associate a graded $H(\mathfrak{g})$-module (and hence an object in $\mathbb{P}_q(\mathfrak{g})$) :

$$\tilde{M} = \oplus_{i=-\infty}^{\infty} M_i X_0^i \subset M[X_0, X_0^{-1}]$$

We observe that $\tilde{M}/(X_0 - 1)\tilde{M} \cong M$ and conversely, any X_0-torsion free graded left $H(\mathfrak{g})$-module arises in this way, see [4] or [6].

In the sequel there will be a few occasions where we use the notation $\tilde{M}$ for a graded $H(\mathfrak{g})$-module, even if it is not a priori obtained from some filtered $U(\mathfrak{g})$-module M. I.e. $\tilde{M}$ could have X_0-torsion.

In this note we concentrate on a specific class of graded left $H(\mathfrak{g})$-modules :

Definition 2.1 *A d-dimensional linear subspace of $\mathbb{P}_q(\mathfrak{g})$ is the object in $\mathbb{P}_q(\mathfrak{g})$ associated to a cyclic graded left $H(\mathfrak{g})$-module $\tilde{M} = \oplus_{i=0}^{\infty} M_i$ with Hilbert series $\mathcal{H}(\tilde{M}, t) = \sum_{i=0}^{\infty} \dim_{\mathbb{C}}(M_i) t^i = 1/(1-t)^{d+1}$.*

Theorem 2.2 *The d-dimensional linear subspaces of $\mathbb{P}_q(\mathfrak{g})$ are :*

1. *The d-dimensional subspaces in $\mathbb{P}^{n-1} = \mathrm{Proj}(S(\mathfrak{g}))$*

2. *The homogenizations $\tilde{M}$ of induced modules*

$$M = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f$$

where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ of codimension d and $\mathbb{C}_f$ is the one-dimensional representation of $\mathfrak{h}$ determined by $f \in \mathfrak{h}^*$ such that $f([\mathfrak{h},\mathfrak{h}]) = 0$. Here the pair $(\mathfrak{h}, f)$ is uniquely determined by $\tilde{M}$.

Proof Let $\tilde{M}$ be a graded $H(\mathfrak{g})$-module as in Definition 2.1. Then it is easy to see that $\tilde{M}$ is either annhilated by X_0 or X_0 acts without torsion. In the first case $\tilde{M}$ will be a quotient of $H(\mathfrak{g})/X_0H(\mathfrak{g}) = S(\mathfrak{g})$. I.e. $\tilde{M}$ will represent a linear subspace in $\operatorname{Proj} S(\mathfrak{g})$.

In the second case $M = \tilde{M}/(X_0 - 1)\tilde{M}$ is a filtered $H(\mathfrak{g})/(X_0 - 1)H(\mathfrak{g}) = U(\mathfrak{g})$-module. I.e. there is map $\phi : U(\mathfrak{g}) \to M$ such that

$$\dim \phi(U_i(\mathfrak{g})) = \binom{d+i}{i}$$

The result now follows from the following proposition ∎

Proposition 2.3 *Assume that M is a left $U(\mathfrak{g})$-module and there is a surjective map $\phi : U(\mathfrak{g}) \to M$ such that*

$$\dim \phi(U_i(\mathfrak{g})) = \binom{d+i}{i} \qquad \text{for } i \in \mathbb{N}. \tag{1}$$

Then there is a unique pair $(\mathfrak{h}, f)$ such that

$$M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f \tag{2}$$

with notation as in Theorem 2.2.

Furthermore the filtration induced on M via ϕ makes (2) into a filtered isomorphism.

Proof The uniqueness of the pair $(\mathfrak{h}, f)$ is easy to see. Therefore we concentrate on its existence. First define

$$\mathfrak{h}' = U_1(\mathfrak{g}) \cap \ker \phi$$

Clearly $\dim \mathfrak{h}' = n - d$.

Using the fact that $U(\mathfrak{g})$ has a PBW-basis, it is easy to see that

$$U_1(\mathfrak{g})\mathfrak{h}' \cap U_1(\mathfrak{g}) = [\mathfrak{h}', \mathfrak{h}'] + \mathfrak{h}'$$

Furthermore if V is the image of $U_1(\mathfrak{g})\mathfrak{h}'$ in $U_2(\mathfrak{g})/U_1(\mathfrak{g}) = S_2(\mathfrak{g})$ then we compute that

$$\dim V = \binom{n+1}{2} - \binom{d+1}{2}$$

Hence we obtain

$$\begin{aligned} \dim U_1(\mathfrak{g})\mathfrak{h}' &= \dim V + \dim U_1(\mathfrak{g})\mathfrak{h}' \cap U_1(\mathfrak{g}) \\ &= \binom{n+1}{2} - \binom{d+1}{2} + \dim([\mathfrak{h}', \mathfrak{h}'] + \mathfrak{h}') \end{aligned} \tag{3}$$

Also, using (1) and the fact that $U_1(\mathfrak{g})\mathfrak{h}' \subset \ker \phi$ we find that

$$\dim U_1(\mathfrak{g})\mathfrak{h}' + \binom{d+2}{2} \leq \binom{n+2}{2} \tag{4}$$

Combining (3) and (4) we find

$$\dim([\mathfrak{h}', \mathfrak{h}'] + \mathfrak{h}') \leq n - d = \dim \mathfrak{h}'$$

Hence

$$[\mathfrak{h}', \mathfrak{h}'] \subset \mathfrak{h}'$$

and thus we find that $\mathfrak{h}'$ is a Lie subalgebra of $U_1(\mathfrak{g})$.

Let $\mathfrak{h}$ be the image of $\mathfrak{h}'$ in $\mathfrak{g}$ under the canonical decomposition $U_1(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g}$. Since $1 \notin \mathfrak{h}'$, there will be a linear map $f \in \mathfrak{h}^*$ such that

$$\mathfrak{h}' = \{u - f(u) \mid u \in \mathfrak{h}\}$$

Let $u, v \in \mathfrak{h}$. Then

$$[u - f(u), v - f(v)] = [u, v] \in \mathfrak{h}'$$

and therefore $f([u, v]) = 0$.

We have now a surjective map

$$U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}' \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f \to M$$

Applying (1) yields that this map must be a filtered isomorphism. ∎

To any d-dimensional linear subspace in $\mathbb{P}_q(\mathfrak{g})$ we can associate a unique d-dimensional linear subspace in $\mathbb{P}^n = \mathbb{P}(\mathfrak{g}^* \oplus \mathbb{C}X_0^*)$ as follows. Let $\tilde{M}$ be a graded quotient $H(\mathfrak{g}) \to \tilde{M}$ Then restriction to degree 1 defines a linear map $\mathfrak{g} \oplus \mathbb{C}X_0 \to \tilde{M}_1$, hence a linear subspace in $\mathbb{P}^n$. Conversely, a linear subspace in $\mathbb{P}^n$ can at most correspond to one linear subspace in Proj $H(\mathfrak{g})$.

If we identity the open affine piece $X_0 = 1$ with $\mathfrak{g}^*$ we would like to know the minimal dimension $d(f)$ of a linear subspace in $\mathbb{P}_q(\mathfrak{g})$ containing $f \in \mathfrak{g}^*$.

If $f \in \mathfrak{g}^*$ then we denote by B_f the bilinear form $(x, y) \to f([x, y])$ and with $\mathfrak{g}^f = \{x \in \mathfrak{g} \mid f([x, \mathfrak{g}]) = 0\}$ the radical which is a Lie subalgebra of $\mathfrak{g}$ and as B_f induces a non-degenerate alternating bilinear form on $\mathfrak{g}/\mathfrak{g}^f$ we have that $n - \dim(\mathfrak{g}^f)$ is even.

In view of Theorem 2.2 we see that a d-dimensional linear subspace of $\mathbb{P}_q(\mathfrak{g})$ containing f is the homogenization of an induced one-dimensional representation of a codimension d Lie subalgebra $\mathfrak{h}$ such that $f([\mathfrak{h}, \mathfrak{h}]) = 0$. In the terminology of [5, Ch I, §12] $\mathfrak{h}$ is subordinate to f. One knows that the maximal dimension of such a Lie subalgebra is $\frac{1}{2}(n + \dim(\mathfrak{g}^f))$ meaning that

$$d(f) \geq \frac{1}{2}(n - \dim(\mathfrak{g}^f))$$

and if equality occurs, $\mathfrak{h}$ is said to be a polarization of $\mathfrak{g}$ at f. Note that polarizations exist for every f if $\mathfrak{g}$ is solvable or $\mathfrak{g} = \mathfrak{sl}(n)$ and for arbitrary $\mathfrak{g}$ a (solvable) polarization exists whenever f is a regular form, cfr. [5, Ch I, §11].

Corollary 2.4 *The minimal dimension d such that every point in $\mathbb{P}(\mathfrak{g}^* \oplus \mathbb{C}X_0^*)$ is contained in a d-dimensional linear subspace of $\mathbb{P}_q(\mathfrak{g})$ is equal to $\frac{1}{2}(n - r)$ where r is the index of $\mathfrak{g}$.*

If $\mathfrak{g}$ is semi-simple, the index of $\mathfrak{g}$ is equal to its rank. Moreover, in this case all linear subspaces in $\mathbb{P}_q(\mathfrak{g})$, not lying in the hyperplane at infinity and of minimal dimension, are obtained by parabolic induction.

3 Points, lines and embedded components

Recall from [2] that the bihomogeneous equations for the point-variety of $H(\mathfrak{g})$ are given by

$$x_i^{(1)} x_j^{(2)} - x_j^{(1)} x_i^{(2)} = (\sum \alpha_{ij,k} x_k^{(1)}) x_0^{(2)}$$

$$x_0^{(1)}x_i^{(2)} = x_i^{(1)}x_0^{(2)} \tag{5}$$

where $1 \leq i, j, k \leq n$.

To simplify some of the notations in the sequel, let us make the following conventions. If $U = \sum u_i X_i \in \mathfrak{g}$ then $u = \sum u_1 x_1$, $u^{(1)} = \sum u_i x_i^{(1)}$, etc.... Similarly, if U, $V \in \mathfrak{g}$ and $[U, V] = \sum w_i X_i$ then $[u, v] = \sum w_i x_i$. As above, U and V may be equipped with the superscripts (1) and (2).

Using this notation we may rewrite (5) more succinctly as

$$x_i^{(1)}x_j^{(2)} - x_j^{(1)}x_i^{(2)} = [x_i^{(1)}, x_j^{(1)}]x_0^{(2)} \tag{6}$$

$$x_0^{(1)}x_i^{(2)} = x_i^{(1)}x_0^{(2)} \tag{7}$$

These equations determine a variety $X \subset \mathbb{P}(\mathfrak{g}^* \oplus \mathbb{C}x_0^*) \times \mathbb{P}(\mathfrak{g}^* \oplus \mathbb{C}x_0^*)$.

Let $\sigma : Y = pr_1(X) \to Y' = pr_2(Y)$ be the induced automorphism, which is given by shifting point modules one place to the right (we are working with left modules, instead of with right modules as in [2]). From the description (Theorem 2.2) of points as one-dimensional representations of $U(\mathfrak{g})$ or points at infinity it follows that σ is the identity on Y_{red}.

Theorem 3.1 *If we set $x_i = x_i^{(1)}$, then the defining relations for Y are*

$$(x_i[x_j, x_k] + x_j[x_k, x_i] + x_k[x_i, x_j])x_0 = 0$$

$$[x_i, x_j]x_0^2 = 0 \tag{8}$$

Proof We will cover Y by affine opens. First assume that $x_0 = x_0^{(1)} = 1$. Then from $\sigma = \text{id}$ on Y_{red} we may assume that $x_0^{(2)} = 1$. Then from (7) we obtain that $x_i^{(1)} = x_i^{(2)}$. Substituting this in (6) yields that $[x_i, x_j] = 0$ for $i, j \in \{1, \ldots, n\}$. This set of equations clearly defines the same variety as (8) when $x_0 = 1$.

Assume now that $x_s = x_s^{(1)} = 1$ for some $s \in \{1, \ldots, n\}$. Again we may assume that $x_s^{(2)} = 1$. Put $\epsilon_j = x_s^{(1)} - x_s^{(2)}$. Clearly $\epsilon_s = 0$, and from (7) it follows that $\epsilon_0 = 0$.

(6) and (7) may now be rewritten as

$$x_j\epsilon_k - x_k\epsilon_j = [x_j, x_k]x_0 \tag{9}$$

$$x_0\epsilon_j = 0 \tag{10}$$

$j, k \in \{1, \ldots, n\}$.

Applying (9) with $j = s$ yields

$$\epsilon_k = [x_s, x_k]x_0$$

Resubstituting yields the equations

$$\begin{aligned} (x_s[x_j, x_k] + x_k[x_s, x_j] + x_j[x_k, x_s])x_0 &= 0 \\ [x_s, x_j]x_0^2 &= 0 \end{aligned} \tag{11}$$

where $j, k \in \{1, \ldots, n\}$. Using the fact that $x_s = 1$, we find that the equations (11) are equivalent with the equations (8). ∎

Now we give an interpretation for the two sets of equations occuring in (8).

Lemma 3.2 *1. $A = V([x_i, x_j]; 1 \leq i, j \leq n) \subset \mathbb{P}(\mathfrak{g}^*)$ is the linear space of one-dimensional representations of $\mathfrak{g}$*

2. $B = V(x_i[x_j, x_k] + x_j[x_k, x_i] + x_k[x_i, x_j]; 1 \leq i, j, k \leq n) \subset \mathbb{P}(\mathfrak{g}^*)$ classifies codimension one Lie subalgebras of $\mathfrak{g}$

Proof (1) is clear. A point of B is a surjective map $\phi : \mathfrak{g} \to \mathbb{C}$ satisfying $\phi(u)\phi([v, w]) + \phi(v)\phi([w, u]) + \phi(w)\phi([u, v]) = 0$. Let $\mathfrak{h} = \ker(\phi)$ and take $u, v \in \mathfrak{h}$ and $w \notin \mathfrak{h}$, it follows that $\phi([u, v]) = 0$ i.e. $\mathfrak{h}$ is a codimension one Lie subalgebra. Conversely the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \cong \mathbb{C}$ for any codimension one Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ gives a point of B. ∎

Corollary 3.3 *Every line in $\mathbb{P}_q(\mathfrak{g})$, not lying in the hyperplane at infinity, intersects B.*

Proof Such a line is given by a module of the form

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f)^\sim$$

where $\mathfrak{h}$ is a codimension one Lie subalgebra of $\mathfrak{g}$, and $f \in \mathfrak{h}^*$ is a linear form such that $f([\mathfrak{h}, \mathfrak{h}]) = 0$.

The intersection with the hyperplane at infinity is represented by

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{C}_f)^\sim \otimes H(\mathfrak{g})/X_0 H(\mathfrak{g}) \cong S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}$$

The corresponding point in $\mathbb{P}(\mathfrak{g}^*)$ is given by the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \cong \mathbb{C}$. Hence it must be a point of B. ∎

Corollary 3.4 *If* $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ *then the point variety of* $\mathbb{P}_q(\mathfrak{g})$ *consists of*

1. *The origin.*
2. *The hyperplane at infinity.*
3. *An embedded component at infinity given by* B.

Proof Clear from the equations (8). ∎

Theorem 3.5 *The following are equivalent*

1. *The point variety of* $H(\mathfrak{g})$ *is reduced, i.e.* $Y=Y_{\text{red}}=V([x_i,x_j]x_0;1\leq i,j\leq n)$
2. *All lines in* $\mathbb{P}_q(\mathfrak{g})$ *lie on the point variety*
3. *For every codimension one Lie subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *holds* $[\mathfrak{h},\mathfrak{h}]=[\mathfrak{g},\mathfrak{g}]$

Proof From the results of the foregoing section it is clear that (2) and (3) are equivalent. Let us prove that (1) implies (3) :

First, there can be no codimension one Lie subalgebra $\mathfrak{h}$ such that $[\mathfrak{g},\mathfrak{g}]$ is not contained in $\mathfrak{h}$ for otherwise there is a point in A not lying on B meaning that there would be an embedded point at infinity. Hence, assume there is a decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{C}e$ where $\mathfrak{h}$ is a codimension one Lie algebra and $[\mathfrak{h},\mathfrak{h}]$ is properly contained in $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{h}$. Let $\chi_{\mathfrak{h}}:\mathfrak{g}\to\mathbb{C}$ be the projection on the second factor. Then, there exists a $\phi\in\mathfrak{g}^*$ s.t. $\phi([\mathfrak{h},\mathfrak{h}])=0$ whereas $\phi([\mathfrak{g},\mathfrak{g}])\neq 0$. But then $\psi=\chi_{\mathfrak{h}}+\epsilon.\phi$ is a point of B over the dual ring $\mathbb{C}[\epsilon]$ which is not a point of A over $\mathbb{C}[\epsilon]$ meaning again that the point variety has an embedded point.

Finally, let us prove that (3) implies (1) : we have to show that $A=B$ and as A is smooth it suffices to prove that any point of B over $\mathbb{C}[\epsilon]$ lies in A. So, take a surjection $\psi:\mathfrak{g}\to\mathbb{C}[\epsilon]$ with the property that

$$\psi(u)\psi([v,w])+\psi(v)\psi([w,u])+\psi(w)\psi([u,v])=0$$

Decompose $\psi=\psi_1+\epsilon\psi_2$. Then the kernel of ψ_1 is a codimension one Lie subalgebra $\mathfrak{h}$. Put $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{C}e$ where $\psi(e)=1$. Substitute $u,v\in\mathfrak{h}$ and $w=e$ in the above relation and as $[\mathfrak{g},\mathfrak{g}]=[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$ we obtain $\psi([u,v])=0$, i.e. $\psi([\mathfrak{h},\mathfrak{h}])=\psi([\mathfrak{g},\mathfrak{g}])=0$ so it belongs to B. ∎

With this result in mind, we pose the question whether d-dimensional linear subspaces (for $d>1$) of $\mathbb{P}_q(\mathfrak{g})$ arise in a similar way from embedded points in varieties describing lower-dimensional subspaces.

References

[1] Artin M., Schelter W., Graded algebras of global dimension 3, Adv.Math. 66 (1987) 171-216

[2] Artin M., Tate J., Van den Bergh M., Some algebras associated to automorphisms of elliptic curves, Grothendieck Festschrift, Vol. 1, 33-85, Birkhäuser (1990)

[3] Artin M., Geometry of quantum planes, MIT preprint (1990)

[4] Ascensio M., Van den Bergh M., Van Oystaeyen F., A new algebraic approach to microlocalization, Trans. Amer. Math. Soc., 316, 2, 537-553, (1989).

[5] Dixmier J., "Enveloping algebras", North-Holland (1977)

[6] Ginsburg V., Characteristic varieties and vanishing cycles, Inv.Math. 84 (1986) 327-402

[7] Le Bruyn L., Smith S.P., Homogenized $\mathfrak{sl}(2)$, UIA-preprint (1991)

[8] Li Huishi,Van Oystaeyen F., Zariskian filtrations, monograph, to appear.

[9] Levasseur Th., Some properties of non-commutative regular graded rings, Proc. Edinb. Math. Soc., to appear.