Homogenized $sl(2)$

L. Le Bruyn*
Dept. Wiskunde & Informatica
Universiteit Antwerpen (UIA), Belgium

S. P. Smith**
Dept. Mathematics, University of Washington
Seattle, USA 98195

July 1991 – 91-37

Abstract

This note relates the lines on the pencil of quadrics which are the conjugacy classes in $\mathbf{sl}(2, \mathbb{C})$ to the line-modules in Artin's projective geometry [1] of the homogenization of the enveloping algebra $U(\mathbf{sl}(2, \mathbb{C}))$.

AMS-Classification
16S30

Keywords
Enveloping Algebras, Quantum Spaces

* Research Associate of the NFWO
** Supported by NSF grant DMS-8901890

Homogenized $sl(2)$

lieven Le Bruyn[*]
Dept. Wiskunde
U.I.Antwerpen
B-2610 Wilrijk, Belgium

S.P. Smith[†]
Dept. Mathematics
University of Washington
Seattle, USA 98195

July 4, 1991

Abstract

This note relates the lines on the pencil of quadrics which are the conjugacy classes in $\mathbf{sl}(2,\mathbb{C})$ to the line-modules in Artin's projective geometry [1] of the homogenization of the enveloping algebra $U(\mathbf{sl}(2,\mathbb{C}))$.

1 Level quadrics for $\mathrm{sl}(2,\mathbb{C})$

Throughout we will write $\mathfrak{g} = \mathbb{C}\,e \oplus \mathbb{C}\,f \oplus \mathbb{C}\,h$ and define a vector space isomorphism $\mathbf{sl}(2,\mathbb{C}) \to \mathfrak{g}$ by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to h$$

We transfer the Lie bracket on $\mathbf{sl}(2,\mathbb{C})$ to $\mathfrak{g}$ giving

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

The cone of nilpotent elements in $\mathbf{sl}(2,\mathbb{C})$ is the variety defined by the quadratic relation $det = 0$ where det is the determinant function on $\mathbf{sl}(2,\mathbb{C})$. The conjugacy classes of semi-simple elements in $\mathbf{sl}(2,\mathbb{C})$ are the level surfaces $det = \lambda^2$ where $\lambda \in \mathbb{C}^*$; in particular this surface is the conjugacy class of the element $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$.

Transfering this to $\mathfrak{g}$ via the above isomorphism it follows that the determinant function on $\mathfrak{g}$ is given by $det = -h^{*2} - e^*f^*$ where e^*, f^*, h^* is the

[*]research associate of the NFWO

[†]supported by NSF grant DMS-8901890

dual basis to e, f, h. Hence the nilpotent cone (resp. the conjugacy class of the element λh) is given by the quadric surface $-h^{*2} - e^* f^* = 0$ (resp. $= \lambda^2$) in $\mathfrak{g}$.

It is a good tradition though to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ under the Killing form induced by the non-degenerate pairing

$$\mathbf{sl}(2, \mathbb{C}) \times \mathbf{sl}(2, \mathbb{C}) \to \mathbb{C}; \quad (x, y) \to Tr(x.y).$$

Transfering this to $\mathfrak{g}$, gives the identifications $e = f^*, f = e^*$ and $h = 2h^*$. Under this identification the nilpotent cone (resp. the conjugacy class of λh) is given by the equation $det = -\frac{1}{4}h^2 - ef = 0$ (resp. $= \lambda^2$).

We may homogenize the defining equations with respect to a new variable t and hence thus consider the following pencil of quadrics in $\mathbb{P}^3$:

$$Q(\delta) = \mathcal{V}(det + \delta^2 t^2) \quad \text{for all } \delta \in \mathbb{P}^1.$$

The base locus of this pencil is the conic $\mathcal{V}(t^2, -h^2 - 4ef)$ in the plane at infinity. The only singular quadrics in this pencil are $Q(0)$ and $Q(\infty)$ (the plane at infinity twice). If we identify $\mathfrak{g}$ with the affine open piece $t = 1$, then the intersection of $Q(0)$ with $\mathfrak{g}$ is the cone of nilpotent elements. If $\lambda \neq 0$ then $Q(\lambda)$ is smooth and its intersection with $\mathfrak{g}$ is a conjugacy class of semi-simple elements.

On each smooth quadric there are two families of lines. Our first objective is to characterize the lines in the pencil of quadrics $Q(\delta)$.

Any two-dimensional Lie subalgebra of $\mathfrak{g}$ is a Borel subalgebra. A *standard basis* for a Borel subalgebra is a basis E, H such that $[H, E] = 2E$. Any standard basis for a Borel subalgebra may be extended to a standard triple E, F, H which is a triple of linearly independent elements such that

$$[E, F] = H, \quad [H, E] = 2E \quad \text{and} \quad [H, F] = -2F$$

The standard triples form a single orbit under conjugation by $GL(2)$. If $\mathfrak{b}$ is a Borel subalgebra with standard basis E, H then $\lambda \in \mathbb{C}$ determines $f_\lambda \in \mathfrak{b}^*$ by requiring that $f_\lambda(E) = 0$ and $f_\lambda(H) = \lambda$. Notice that f_λ depends only on λ and not on the choice of the standard basis, since any other standard basis for $\mathfrak{b}$ is of the form $\nu E, H + \mu E$.

For each pair $(\mathfrak{b}, \lambda)$ we define a line in $\mathbb{P}^3$:

$$\ell_{\mathfrak{b},\lambda} = \mathcal{V}(E, H - \lambda t)$$

where E, H is a standard basis for $\mathfrak{b}$. If we identify the plane at infinity with $\mathbb{P}(\mathfrak{g})$ and the affine open piece as before with $\mathfrak{g}$ then the following two facts are easily verified :

1. $\ell_{\mathfrak{b},\lambda} \cap \mathbb{P}(\mathfrak{g}) = [\mathfrak{b}, \mathfrak{b}] = \mathbb{C}\, E$,

2. $\ell_{\mathfrak{b},\lambda} \cap \mathfrak{g} = \frac{1}{2}\lambda H + \mathbb{C}\, E$ where E, H is a standard basis for $\mathfrak{b}$.

In particular, $\ell_{\mathfrak{b},\lambda}$ is a line on the quadric $Q(\frac{1}{2}\lambda)$ (the factor $\frac{1}{2}$ comes from the identification via the Killing form). In fact,

Theorem 1 *The lines which lie on the quadrics $Q(\delta) = \mathcal{V}(det + \delta^2 t^2)$ for $\delta \in \mathbb{P}^1$ are*

1. *the lines at infinity, and*
2. *lines $\ell_{\mathfrak{b},\lambda}$ for a Borel subalgebra b and $\lambda \in \mathbb{C}$.*

Proof : The first case corresponds to lines on $Q(\infty)$. Suppose $\delta \neq 0$. Then $Q(\delta)$ is the conjugacy class of $\begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}$. If ℓ is a line lying on $Q(\delta)$ then we can choose a basis for $\mathbf{sl}(2, \mathbb{C})$ such that $\ell = \{x + \nu y \mid \nu \in \mathbb{C}\}$ where $x = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}$ and y is some other element of $\mathbf{sl}(2, \mathbb{C})$. Write $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $det(x + \nu y) = det(x)$ for all ν, a calculation shows that $det(y) = 0$ and $a = 0$. Thus y is (up to scalar multiples) either $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In the first case $\ell = \ell_{\mathfrak{b},2\delta}$ with $\mathfrak{b}$ having standard basis y, x. In the second case $\ell = \ell_{\mathfrak{b},-2\delta}$ with $\mathfrak{b}$ having standard basis $y, -x$.

Suppose that $\delta = 0$, so that $Q(\delta)$ is the nilpotent cone. We can choose a basis for $\mathbf{sl}(2, \mathbb{C})$ such that $\ell = \{x + \nu y \mid \nu \in \mathbb{C}\}$ where $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $det(x+\nu y) = 0$ for all ν, a calculation shows that $det(y) = 0$ and $c = 0$ (whence $a = 0$ also). Thus y is a scalar multiple of x, so the line is $\ell = \mathbb{C} x = \ell_{\mathfrak{b},\lambda}$ having standard basis x, H where H satisfies $[H, x] = 2x$ (such an H does exist since every non-zero nilpotent element belongs to some standard triple).

Conversely, we have already seen that $\ell_{\mathfrak{b},\lambda}$ lies on the quadric $Q(\frac{1}{2}\lambda)$. □

If $\delta \neq 0, \infty$ then the two rulings on $Q(\delta)$ are given by $\{\ell_{\mathfrak{b},2\delta} \mid \mathfrak{b} \text{ is a Borel}\}$ and $\{\ell_{\mathfrak{b},-2\delta} \mid \mathfrak{b} \text{ is a Borel}\}$.

2 The quantum space of $\mathbf{sl}(2, \mathbb{C})$

Let A denote the homogenization of the enveloping algebra of $\mathfrak{g}$ with respect to a central variable t. That is,

$$A = \mathbb{C}[e, f, h, t]$$

with defining equations

$$ef - fe = ht, \quad he - eh = 2et, \quad hf - fh = -2ft$$

$$et - te = ft - tf = ht - th = 0$$

Note that $A/A(t-1) \simeq U(\mathfrak{g})$ and $A/At \simeq \mathbb{C}[e,f,h] = grU(\mathfrak{g})$. From these facts one deduces that A has Hilbert series $(1-t)^{-4}$, is a positively graded Noetherian domain, a maximal order, Auslander-regular of dimension 4 and satisfies the Cohen-Macaulay property, see e.g. [6],[4] and [5]. Moreover, the center of A is $\mathbb{C}[\Omega, t]$ where $\Omega = h^2 + 2ef + 2fe$ is the 'Casimir' element.

M. Artin [1] associates to any regular algebra R its quantum space $Proj(R)$ which is by definition the quotient-category of all finitely generated graded left R-modules by the full Serre subcategory of the finite length modules. We will denote the quantum space associated to A by $Q(\mathfrak{g})$. We want to characterize the linear subspaces in $Q(\mathfrak{g})$. There are three types to consider :

1. A *plane module* $\mathcal{S}$ is a cyclic module with Hilbert series $(1-t)^{-3}$.

2. A *line module* $\mathcal{L}$ is a cyclic module with Hilbert series $(1-t)^{-2}$.

3. A *point module* $\mathcal{P}$ is a cyclic module with Hilbert series $(1-t)^{-1}$.

As in [5] one can characterize these modules (up to shifts) as the Cohen-Macaulay modules with multiplicity one (there are two other obvious such modules, namely A and the trivial module $A/Ae + Af + Ah + At$). We will associate to each of them a linear subspace in ordinary $\mathbb{P}^3$.

As $dim(A_1) = 4$ and the homogeneous degree 1 component of a plane (resp. line, point) module is 3 (resp. 2, 1) one can find $a \in A_1$ (resp. $a, b \in A_1$, resp. $a, b, c \in A_1$) and surjections

$$A/Aa \to \mathcal{S} \quad \text{resp. } A/(Aa + Ab) \to \mathcal{L}, \quad \text{resp. } A/(Aa + Ab + Ac) \to \mathcal{P}$$

(in fact we will see shortly that these maps have to be isomorphisms). Hence to each plane (resp. line, point) module we can associate a plane (resp. line, point) in $\mathbb{P}^3 = \mathbb{P}(A_1^*)$ namely $\mathcal{V}(a)$ (resp. $\mathcal{V}(a,b)$, resp. $\mathcal{V}(a,b,c)$). We will now determine which linear subspaces of $\mathbb{P}^3$ arise in this way.

We observe that there is a dichotomy in the problem. As a linear subspace module is critical (because it is of multiplicity 1) it follows from [5] that t either kills the module or acts faithfully. The first case gives a linear subspace module over the commutative polynomial ring $A/At \simeq \mathbb{C}[e,f,h]$ i.e. we get a linear subspace in $\mathbb{P}^2 = \mathcal{V}(t)$. In the latter case we can form $\overline{M} = M/(t-1)M$ which is a filtered $U(\mathfrak{g})$-module $\overline{M}_0 \subset \overline{M}_1 \subset \ldots$ where $dim(\overline{M}_i)$ is equal to 1 (if M is a point module), $i+1$ (if M is a line module) or $\frac{1}{2}(i+1)(i+2)$ (if M is a plane module). Moreover, this process can be reversed, namely $M \simeq \bigoplus \overline{M}_i t^i$ (see [3] or [6]).

Proposition 1 *1. Every plane $\mathcal{V}(a)$ in $\mathbb{P}^3$ determines a plane module $\mathcal{S}$.*

2. The points at infinity and the origin $(0,0,0,1)$ are the only points in $\mathbb{P}^3$ which determine a point module $\mathcal{P}$.

Proof : Since A is a domain, A/Aa has Hilbert series $(1-t)^{-3}$ for every non-zero $a \in A_1$. Hence the surjection $A/Aa \to \mathcal{S}$ given above is an isomorphism.

¿From the dichotomy remark it follows that point-modules either correspond to points on the plane at infinity or to the one-dimensional representation of $\mathfrak{g}$ which corresponds to the origin in $\mathfrak{g}$ which is being identified with the complement to the plane at infinity. In the first case we can take $a = t$ and let b, c determine the point in $Proj(A/At) = \mathbb{P}^2$. Hence the Hilbert series of $A/At + Ab + Ac$ is $(1-t)^{-1}$ so $\mathcal{P} = A/At + Ab + Ac$. In the later case $A/Ae + Af + Ah \simeq \mathbb{C}[t]$ which also has the right Hilbert series. □

Still, the situation concerning point-modules is slightly more subtle. To describe the point-variety we will use the multilinearization trick as in [2] or [7]. That is, the point-variety is the zero set of the 4×4 minors of either of the following two matrices :

$$\begin{pmatrix} -f & e & 0 & -h \\ h & 0 & -e & -2e \\ 0 & h & -f & 2f \\ -t & 0 & 0 & e \\ 0 & -t & 0 & f \\ 0 & 0 & -t & h \end{pmatrix}$$

or

$$\begin{pmatrix} f & -h-2t & 0 & t & 0 & 0 \\ -e & 0 & -h+2t & 0 & t & 0 \\ -t & e & f & 0 & 0 & t \\ 0 & 0 & 0 & -e & -f & -h \end{pmatrix}.$$

Proposition 2 *The ideal determining the point-modules of A in $\mathbb{P}^3$ is*

$$t\langle (h^2+4ef)(e,f,h), te, tf, th \rangle$$

Hence the conic at infinity $\mathcal{V}(t, h^2+4ef)$ is an embedded component of the point-variety.

So the base locus of the pencil of quadrics described in Section 1 appears here as the embedded component of the point-variety for A. Let us now describe the line-modules of $Q(\mathfrak{g})$:

Theorem 2 *The lines in $\mathbb{P}^3$ determining line modules are precisely the lines in the pencil of quadrics $Q(\delta) = \mathcal{V}(det + \delta^2 t^2)$ for $\delta \in \mathbb{P}^1$.*

Proof : As in the case of point-modules, the lines in the plane at infinity are already accounted for. Hence we have to prove that any line module of A is of the form

$$\mathcal{L} \simeq A/(AE + A(H - \lambda t))$$

where E, H is a standard basis for a Borel subalgebra $\mathfrak{b}$ and $\lambda \in \mathbb{C}$.

There is a surjection $A/Aa + Ab \to \mathcal{L}$ for some $a, b \in A_1$ since $dim\mathcal{L}_1 = 2$. Clearly, we can change a, b if necessary so that $a = y$, $b = z + \lambda t$ with $y, z \in \mathfrak{g}$. Let x, y, z be a basis of $\mathfrak{g}$. Then, the following seven linearly independent elements in A_2 belong to $Aa + Ab$:

$$xy, yy, ty = yt, x(z + \lambda t), y(z + \lambda t), z(z + \lambda t), t(z + \lambda t) = (z + \lambda t)t$$

As $dim\mathcal{L}_2 = 3$ and $dim(A_2) = 10$ these elements must be a basis for $(Aa+Ab)_2$. Note that both $yz = y(z + \lambda t) - \lambda yt$ and zy belong to this space. Hence $yz - zy$ can be written as a linear combination of the seven elements. If $\lambda \neq 0$ then only ty can occur with non-zero coefficient and if $\lambda = 0$ so might tz. At any rate $\mathfrak{b} = \mathbb{C}\, y \oplus \mathbb{C}\, z$ is a two dimensional Lie subalgebra of $\mathfrak{g}$ and hence is a Borel subalgebra.

But then $A/(Aa + Ab)$ is the homogenization of $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{f|_{\mathfrak{b}}}$ for some $f \in \mathfrak{g}^*$ such that $f \mid_{[\mathfrak{b},\mathfrak{b}]} = 0$ and hence has as its Hilbert series $(1 - t)^{-2}$. Therefore, $A/Aa + Ab = \mathcal{L}$ and $\mathcal{V}(a, b) = \ell_{\mathfrak{b},\lambda}$ as claimed. □

3 Some Comments

The line module associated to $\ell_{\mathfrak{b},\lambda}$ will be denoted by $M(\mathfrak{b}, \lambda)$. By Theorem 2 $M(\mathfrak{b}, \lambda) \cong A/AE + A(H - \lambda t)$, and hence $M(\mathfrak{b}, \lambda)$ is the homogenization of the Verma module $M_{\mathfrak{b}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}$ of highest weight λ. The $(n+1)$-dImensional simple $U(\mathbf{sl}(2, \mathbb{C}))$-modules will be denoted $V(n)$. For each Borel $\mathfrak{b}$ there is a short exact sequence $0 \to M_{\mathfrak{b}}(-n-2) \to M_{\mathfrak{b}}(n) \to V(n) \to 0$. Taking homogenized modules, there is a corresponding short exact sequence $0 \to M(\mathfrak{b}, -n-2) \to M(\mathfrak{b}, n) \to F(n) \to 0$ where $F(n)$ is a certain fat point module of multiplicity $n + 1$.

This is reminiscent of some of the results on the Sklyanin algebra in [8]. Homogenized $\mathbf{sl}(2, \mathbb{C})$ shares some other common features with the Sklyanin algebra: for example annihilators of line modules behave in a similar way. We leave the details to the interested reader.

The quantum space of any 3-dimensional Lie algebra has similar properties. Let us briefly scetch the case of the 3-dimensional Lie algebra $\mathfrak{h} = \mathbb{C}\, x \oplus \mathbb{C}\, y \oplus \mathbb{C}\, z$ with z central and $[x, y] = z$. The corresponding algebra $H(\mathfrak{h})$ is $\mathbb{C}[x, y, z, t]$ with relations

$$xy - yx = zt \text{ and } z \text{ and } t \text{ centra;}$$

The point variety in $Q(\mathfrak{h}) = Proj(H(\mathfrak{h}))$ is determined by the ideal

$$tz(z,t)$$

that is, it consists of the plane $\mathcal{V}(z)$, the plane at infinity $\mathcal{V}(t)$ and their intersection is an embedded component.

The line-modules of $H(\mathfrak{h})$ are precisely the lines in the pencil of planes

$$P(\delta) = \mathcal{V}(z + \delta t) \text{ for } \delta \in \mathbb{P}^1$$

which has as its base locus the embedded component (z, t).

References

[1] M. Artin, Geometry of quantum planes, Preprint MIT (1990)

[2] M.Artin, J.Tate, M.Van den Bergh, Some algebras associated to automorphisms of elliptic curves,The Grothendieck Festschrift Vol I ,Birkhauser (1990) 33-85

[3] V. Ginsburg, Characteristic varieties and vanishing cycles,Inv.Math 84 (1986) 327-402

[4] T. Levasseur, Some properties of non-commutative regular graded rings, Proc. Edinburgh Math. Soc. (to appear)

[5] T. Levasseur, S.P. Smith, Modules over the 4-dimensional Sklyanin algebra, Preprint Seattle (1991)

[6] Li Huishi, F. Van Oystaeyen, Dehomogenization of gradings to Zariskian filtrations and applications to invertible ideals, Proc. AMS (to appear)

[7] S.P. Smith, J.T. Stafford, Regularity of the four dimensional Sklyanin algebra, Compos. Math. (to appear)

[8] S.P. Smith, J. Staniszkis, Representations of the four dimensional Sklyanin algebra at points of infinite order, Preprint (1991).