On a Theorem of G. M. Bergman

L. Le Bruyn*
Dept. Wiskunde & Informatica
Universiteit Antwerpen (UIA), Belgium

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* Research Associate of the NFWO

On a theorem of G.M. Bergman

Lieven Le Bruyn*
Departement Wiskunde en Informatica
Universiteit Antwerpen (UIA),Belgium

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Abstract

A short proof is presented of a result of G. Bergman [1] on rational identities of matrices.

Example 1 (L.H. Rowen [3]) *Consider the rational expression in non-commutative variables*

$$f(x_1, x_2, x_3) = [[x_1, x_2]^2, x_3].[[x_1, x_2], x_3]^{-1} \tag{1}$$

If we specialize the x_i *to generic* 4×4 *generic matrices* X_i*, then* $F^2 = f(X_1, X_2, X_3)^2 \in UD(4)$ *has degree* 2 *over the center* $K(4)$ *of the generic division algebra* $UD(4)$*.So,*

$$c_5(1, F^2, F^4, X_4, X_5) = 0 \in UD(4) \tag{2}$$

where c_j *is the* j*-th Capelli polynomial [3, p.12]. On the other hand, if we specialize the* x_i *to generic* 3×3 *matrices, then* F^2 *can be shown to be noncentral in* $UD(3)$ *whence it has to be of degree* 3 *over the center. But then*

$$c_5(1, F^2, F^4, X_4, X_5) \neq 0 \in UD(3) \tag{3}$$

*Research associate of the NFWO

Hence, rational identities holding in $UD(n)$ do not have to hold in $UD(m)$ when $m < n$. This situation was studied by G. Bergman [1] who proved the following fundamental result :

Theorem 1 (Bergman) *Let n and $m_1, ..., m_k$ be positive integers. Then, there exists a rational identity for $UD(n)$ not holding in any of the $UD(m_i)$ if and only if*

$$n = \sum_{i=1}^{k} a_i.m_i \tag{4}$$

has no solution in integers $a_i \geq 0$.

In this note we aim to give a short elementary proof of this result.

Definition 1 *An affine algebra Λ is called* **special** *if it has a presentation in terms of generators and relation as*

$$\Lambda = k < x_1, ..., x_k; y_1, ..., y_l > /(y_i.p_i - 1; 1 \leq i \leq l) \tag{5}$$

where each p_i is a noncommutative polynomial in $x_1, ..., x_k, y_1, ..., y_{i-1}$. The **inversion depth** *of Λ , $idt(\Lambda)$, is the minimal number l required for a special presentation of Λ.*

From now on Λ will be special with a given minimal presentation as above. With $\mathbf{def}(\Lambda)$ we denote the set of positive integers n s.t. specializing x_i to a generic $n \times n$ matrix X_i induces a well defined algebra morphism $\pi_n : \Lambda \to UD(n)$. Equivalently, $def(\Lambda)$ consists of those integers n s.t. none of the p_i occuring in the given presentation of Λ is a rational identity for $UD(n)$ when viewed as a rational expression in the x_i.

Observation 1 *$def(\Lambda)$ is an additive sub-semigroup of $I\!N$.*

Proof : It is easy to see that $def(\Lambda)$ is precisely the set of integers n such that Λ has an n-dimensional representation. □

With $\mathbf{gen}(\Lambda)$ we will denote the set of semigroup generators of $def(\Lambda)$.

Observation 2 *If $n \in gen(\Lambda)$ then $\pi_n(\Lambda) \subset UD(n)$ is an Azumaya algebra.*

Proof : If $n \in gen(\Lambda)$, Λ has only irreducible n-dimensional representations. Now use the Artin-Procesi result. $\square$

Example 2 *Let $g_n(x_1,...,x_k)$ be a central identity for $n \times n$ matrices and consider the special affine algebra*

$$\Lambda = k < x_1,...,x_k,z > /(z.g_n(x_1,...,x_k) - 1) \tag{6}$$

Then, $def(\Lambda) = \{m \in I\!N : n \leq m\}$ and $gen(\Lambda) = \{m \in I\!N : n \leq m \leq 2n-1\}$.

Observation 3 *Let $n \in gen(\Lambda)$ and $m_1 < ... < m_s \in def(\Lambda) - \{n\}$. Then,there exists $r \in \Lambda$ s.t. $\pi_n(r) = 0 \in UD(n)$ and $\pi_{m_j}(r) \neq 0 \in UD(m_j)$ for all $1 \leq j \leq s$.*

Proof : As $\pi_n(\Lambda)$ is an azumaya algebra we can find $R_{ij}, S_i \in \pi_n(\Lambda)$ and central identities for $n \times n$ matrices g_i such that

$$1 = \sum_i g_i(R_{ij})S_i \in UD(n) \tag{7}$$

see [2, Ch. 8]. Lifting R_{ij} (resp. S_i) to elements r_{ij} (resp. s_i) of Λ we can take

$$r_0 = 1 - \sum_i g_i(r_{ij})s_i \in \Lambda \tag{8}$$

Then, $\pi_n(r_0) = 0 \in UD(n)$ and $\pi_{m_i}(r_o) = 1 \neq 0 \in UD(m_i)$ for all $m_i < n$. Let m_t be the smallest m_i such that $\pi_{m_t}(r_0) = 0 \in UD(m_t)$ then we can take $r_1 = r_0 + ci_t$ where ci_t is a central identity for $m_t \times m_t$ matrices. Repeat this process until one reaches m_s. $\square$

Observation 4 *Let $n_1 < n_2 < ... < n_s$ be generators of a sub semigroup of $I\!N_+$. For every $a \geq 1$ there exists a special affine algebra Λ with $idt(\Lambda) \leq a$ such that $S \subset def(\Lambda)$ and $S \cup [0, an_1] = def(\Lambda) \cup [0, an_1]$.*

Proof : We use induction on a,the case $a = 1$ being accounted for by example 2 if $n = n_1$. Assume the statement holds for $a-1$ i.e. we have a special affine algebra Λ' with $idt(\Lambda') \leq a-1$ s.t. $S \subset def(\Lambda')$ and $def(\Lambda') \cup [0,(a-1)n_1] = S \cup [0,(a-1)n_1]$.

Let $\{m_1, ..., m_z\} = ([(a-1)n_1, an_1] \cup def(\Lambda)) - gen(S)$. Each m_i is seen to be in $gen(\Lambda)$ so by the trick of the foregoing proof we can find for each m_i an element $r_i \in \Lambda'$ s.t. $\pi_{m_i}(r_i) = 0$ and $\pi_{m_j}(r_i) \neq 0$ if $j \neq i$. Take

$$\Lambda = \Lambda < z > /(z.r_1...r_z - 1) \tag{9}$$

and check that this algebra has the properties required. □

In particular, if S is a sub semigroup of $I\!N_+$ with $gcd(S) = 1$,then there is a special affine algebra Λ with $def(\Lambda) = S$.

We will now show that the bound given on the inversion depth of Λ is the best possible.

Observation 5 *Let* $idt(\Lambda) = a$, $(a+1)m < n$ *and* $r \in \Lambda$ *such that* $\pi_n(r) = 0 \in UD(n)$. *Then, either* $m \notin def(\Lambda)$ *or* $\pi_m(r) = 0 \in UD(m)$.

Proof : We use induction on a, the case $a = 0$ is the classical result on polynomial identities. Hence assume the result holds for $b < a$.

Suppose we have $idt(\Lambda) = a, r \in \Lambda$ s.t. $\pi_n(r) = 0$, $m \in def(\Lambda)$ and $\pi_m(r) \neq 0$. We claim that $n - m \in def(\Lambda)$. If not, one of the p_i in the presentation of Λ would be a rational identity for $UD(n-m)$ and $p_i \in \Lambda'$ where $idt(\Lambda) < a$. By induction, $\pi_m(p_i) = 0 \in UD(m)$ contradicting the assumption that $m \in def(\Lambda)$.

If m and $n - m \in def(\Lambda)$ we can specialize each x_i to a block matrix with top left corner a generic $m \times m$ matrix and bottom right corner a generic $n - m \times n - m$ matrix and the other entries zero. This algebra map from Λ factors through $\pi_n(\Lambda)$ so the image of r has to be zero. Looking at the top left corner this implies that $\pi_m(r) = 0$, a contradiction. □

The last argument shows that if $n = \sum_i a_i m_i$ for integers $a_i \geq 0$ and $r \in \Lambda$ with $\{n, m_1, ..., m_s\} \subset def(\Lambda)$, then $\pi_n(r) = 0$ implies $\pi_{m_i}(r) = 0$ for all i.

As a consequence, we obtain the following extension of the classical p.i.-result

Corollary 1 *Let* $r \in \Lambda$ *with* $idt(\Lambda) \leq a$ *be a rational identity of* $UD(n)$. *If* $(a+1).m < n$, *then* r *or one of its rational subexpressions is a rational identity for* $UD(m)$.

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