# Rational Identities of Matrices Revisited

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**Key Words**
Rational Identities, Azumaya Algebra

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# Rational Identities of Matrices Revisited

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**Abstract**

In this paper we give constructive proofs of Bergman's fundamental results on rational identities of matrices. We give an explicit method to find rational identities holding in $M_n(\mathbb{C})$ not holding in any $M_{m_i}(\mathbb{C})$ where the $m_i$ are integers s.t. the equation $n = \sum a_i m_i$ has no solution in positive integers $a_i$. Moreover, our examples are minimal with respect to the distortion number which is a measure of the complexity of rational expressions.

## 1 Introduction

One of the historic roots of p.i. theory is the foundations of geometry, in particular the problem of constructing Desarguian projective planes which do not satisfy the Pappus theorem, see e.g. [2, p.342-359], [3] or [14, Ch.8]. Some attempts in this direction have been made in the first half of this century by a.o. Dehn (1922) [10] who actually introduced polynomial identities and Wagner (1937) [16] who introduced generic matrices, proved the first polynomial identity for $n \times n$ matrices and introduced the symbolic method (which he attributes to Magnus) rediscovered later by several people. Wagner was

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also able to prove the first structural result in p.i.-theory : an ordered ring which satisfies a polynomial identity is commutative. This was a first step in proving that ordered Desarguian geometry satisfies the Pappus theorem in the presence of any other intersection theorem which does not follow from the Desargue theorem, a result which was finally proved in 1966 by Amitsur [2].

It is well known that the presence of the Desargue theorem enables the introduction of homogeneous coordinates from a division algebra $D$ and that universal theorems of intersection involving constructible configurations correspond to rational identities in $D$ (and not just polynomial identities). The theory of rational identities has some striking differences compared to that of polynomial identities. It was shown in [2] that a rational identity which holds for all $M_n(\mathbb{C})$ are precisely those which hold for all division algebras containing $\mathbb{C}$ and there are such non-trivial identities e.g. Hua's identity (compare this to the fact that there is no polynomial identity holding in all $M_n(\mathbb{C})$). In fact A. Schofield showed that one can bound the size of the matrices $n$ (in function of the complexity of the expression) in order to verify whether it is a universal rational identity for division algebras, [15]. Another (more surprising) difference between polynomial and rational identities is that whereas all polynomial identities of $M_n(\mathbb{C})$ also hold in $M_m(\mathbb{C})$ for all $m \leq n$, a similar statement for rational identities only holds when $m \mid n$, a theorem proved in 1976 by Bergman [7] thereby correcting an erroneous statement in [2, Th.12], see [7, p.259,footnote] for more details. This fact raises the question of constructing such 'exotic' rational identities (or,equivalently,intersection theorems which do not follow from Desargue nor the $S_n$-intersection theorem of [2]) holding in $M_n(\mathbb{C})$ but not holding in certain $M_m(\mathbb{C})$. My motivation for studying this problem arose because of similar divisibility results in the study of the rationality problem of the centers of universal division algebras.

Bergman's proof [7, Th.4.1] gives no clue for the construction and, in fact, only a few published examples are known to the author. In [7, §6] Bergman gave the following $(n,m) = (3,2)$ example : let $f' = f.x - x.f$ and $\theta(f) = \frac{(f'^2)'}{(f'^{-1})'}$ then

$$B(x,y) = 1 - \theta(y).\theta(y'').\theta(y''^{-1}).\theta(y'''^{-1}) \qquad (1)$$

represents 0 in $D_{2,3}$ and 1 in $D_{2,2}$. Here, $D_{k,n}$ is the universal division algebra on $k$ generic $n \times n$ matrices (see §2 below for the definition) to which any division algebra $D$ of degree $n$ is rationally equivalent (i.e. satisfy the same set of rational identities). In [14, §8.3] Rowen gave a more instructive example when $(n,m) = (4,3)$ : $D_{k,4}$ is a crossed-product with group $V_4$, Klein's Vierergruppe. This can be proved from the fact that

$$f(x_1,x_2,x_3) = [[x_1,x_2]^2,x_3][[x_1,x_2],x_3]^{-1} \quad (2)$$

is such that $f(X_1,X_2,X_3)^2$ has degree 2 over the center of $D_{k,4}$.But then,

$$C_5(1, f(x_1,x_2,x_3)^2, f(x_1,x_2,x_3)^4, x_4, x_5) \quad (3)$$

where $C_5$ is the 5-th Capelli polynomial (see e.g. [14, p.12]) represents 0 in $D_{5,4}$ but is not an identity in $D_{5,3}$ due to the fact that $f(x_1,x_2,x_3)^2$ does not lie in the center of $D_{5,3}$ and hence must have degree 3 over it. Unfortunately, we have no pleasant description for the corresponding element in $D_{5,3}$.

The Rowen example seems to indicate that one requires insight in the more delicate structure of finite dimensional division algebras in order to construct 'exotic' $(n,m)$-examples for general $m$ not dividing $n$. While this is indeed the case we are able to constrain the difficulty to the following black-box problem : given an affine Azumaya-algebra, write the identity as a linear combination of evaluations of central polynomials. Provided one can effectively solve this problem, we will present an inductive procedure in this paper to construct rational identities holding in $M_n(\mathbb{C})$ not holding in any $M_{n_i}(\mathbb{C})$ where the equation $n = \sum a_i n_i$ has no solution in positive integers $a_i$. Moreover, our examples combine the good properties of the Bergman and Rowen examples : they are minimal with respect to a complexity measure called the distortion number of an expression (as is the case for the Rowen example) and they represent 1 in $D_{k,n_i}$ for all $n_i < n$ (as is the case in Bergman's example). Knowledge on the minimal distortion number $\Delta(m,n)$ of a rational identity holding in $M_n(\mathbb{C})$ not holding in $M_m(\mathbb{C})$ (which we prove to be the integer $a$ s.t. $a.m < n < (a+1).m$) enables us to prove a natural extension of the classical result on polynomial identities : if $r$ is a rational identity holding in $M_n(\mathbb{C})$ with distortion number $< \Delta(m,n)$ then $r$ holds (possibly degenerately) in $M_m(\mathbb{C})$ (meaning roughly that $r$ or a subexpression of $r$ is a rational identity for $M_m(\mathbb{C})$.

This paper is organized as follows : in section two we recall the definition of rational identities and introduce the distortion number of expressions. In section three we give a short proof of a version of Bergman-Small needed for the rest of this paper. We introduce the Bergman-Small diagrams of an affine p.i.-algebra as a conceptual tool in our constructions. In section 4 we single out two tricks which we use frequently.

The main idea to construct the desired 'exotic' rational identities is as follows : adjoin to the ring of $k$ generic $n \times n$ matrices $G_{k,n}$ suitable inverses of expressions which are well defined in all $D_{k,n_i}$ s.t. the large algebra is Azumaya. Then, write the identity as a combination of evaluations of central identities.The resulting expression (as a rational relation among $x_1, ..., x_k$ will then represent 0 in $D_{k,n}$ and 1 in all $D_{k,n_i}$ s.t. $n_i < n$ and we can slightly alter it to be non-zero for all $n_i > n$, too.

In section 5 we carry out this program of embedding $G_{k,n}$ into Azumaya algebras which is based on killing off all Bergman-Small diagrams except $(n)$ by tossing in inverses of rational Azumaya identities for suitable degrees $m < n$. In section 6 we show that one can considerably shorten this inductive procedure if one is willing to use some geometrical results from [11] on $n$-dimensional representations and their degenerations. In section 7 we give a constructive proof of the main result of Bergman on rational identities of matrices [7, Th.9.1 and Cor.9.3]. The final section gives a pun connection with the game of Sylver coinage.

Finally, note that our insistence on $\mathbb{C}$ as the base field is merely a bad habit of the author which is nowhere crucial for the construction in view of [6] (with the possible exception of the shortening section 6). Also it should be acknowledged that the seminal idea of this paper is taken from [7, §5].

# 2 The distortion number

In this section we will recall and introduce some definitions. Throughout we fix a number $k$ and all expressions will involve the non-commutative variables $x_1, ..., x_k$. Recall that any rational expression in the $x_i$ can be thought off as an element of an affine algebra $\mathbb{C} < x_1, ..., x_k; z_1, ..., z_l >$ modulo the ideal generated by the elements $1 - z_i.p_i(x_1, ..., x_k; z_1, ...z_{i-1})$ (all $i$) where the $p_i$ are non-commutative polynomials in the appropriate number of indeterminates. Sometimes, we will abbreviate this by denoting $r = r(x_1, ..., x_k, p_1^{-1}, ..., p_l^{-1})$. Our measure of the complexity of a rational identity will be the distortion number to be defined below :

**Definition 1** *Let $r$ be a rational expression in $x_1, ..., x_k$, then we define the* **distortion number** *of $r$ (denoted $\delta(r)$) inductively as follows :*

*$\delta(r) = 0$ iff $r$ is a non-commutative polynomial in the $x_i$ i.e. an element of the free algebra $\mathbb{C} < x_1, ..., x_k >$*

*If $r = r(x_1, ..., x_k, p_1^{-1}, ..., p_l^{-1})$ where the $p_i$ are rational expressions of distortion number $\delta(p_i) \leq j$, then $\delta(r) \leq j + 1$.*

Next, we recall the definition of rational identities of $M_n(\mathbb{C})$. For every $1 \leq i \leq k$ let $X_i$ be the generic $n \times n$ matrix

$$X_i = \begin{pmatrix} x_{11}(i) & \cdots & x_{1n}(i) \\ \vdots & & \vdots \\ x_{n1}(i) & \cdots & x_{nn}(i) \end{pmatrix} \tag{4}$$

Then, $r = r(x_1, ..., x_k)$ is said to be a polynomial identity of $M_n(\mathbb{C})$ iff $r(X_1, ..., X_k)$ is the zero matrix in $M_n(\mathbb{C}(x_{uv}(i) : 1 \leq u, v \leq n, 1 \leq i \leq k))$. Note that the quotient algebra of $\mathbb{C} < x_1, .., x_k >$ modulo the ideal of all polynomial identities of $M_n(\mathbb{C})$ is the ring of $k$ generic $n \times n$ matrices, $G_{k,n}$ which is the subalgebra of $M_n(\mathbb{C}(x_{uv}(i); u, v, i))$ generated by the matrices $X_i$ , $1 \leq i \leq k$. $G_{k,n}$ is known to be an Ore-domain [1] and hence has a classical division ring of fractions $D_{k,n}$ which is called the generic (or universal) division algebra of $k$ generic $n \times n$ matrices.

Now, let $r = r(x_1, ..., x_k; p_1^{-1}, ..., p_l^{-1})$ then we say that $r$ is well defined on $M_n(\mathbb{C})$ iff every $p_i(X_1, ..., X_k)$ represents an invertible matrix denoted $P_i$ in $M_n(\mathbb{C}(x_{uv}(i); u, v, i))$. If $r$ is well defined on $M_n(\mathbb{C})$, then $r(X_1, ..., X_k, P_1^{-1}, ..., P_l^{-1})$ defines an element $R$ in $D_{k,n}$. Alternatively, $R$ is an element of the affine subalgebra $G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ of $D_{k,n}$.

**Definition 2** *A rational expression $r = r(x_1, ..., x_k; p_1^{-1}, ..., p_l^{-1})$ is said to hold in $M_n(\mathbb{C})$ iff either $r$ is well defined on $M_n(\mathbb{C})$ and $r(X_1, ..., X_k; P_1^{-1}, ..., P_l^{-1})$ is the zero matrix in $M_n(\mathbb{C}(x_{uv}(i); u, v, i))$ (in this case we say that $r$ is a* **rational identity** *of $M_n(\mathbb{C})$) or $r$ is not well-defined on $M_n(\mathbb{C})$ i.e. one of the $p_i$ is a rational identity of $M_n(\mathbb{C})$ (in this case we say that $r$ holds* **degenerately** *in $M_n(\mathbb{C})$).*

Hence, $r$ holds in $M_n(\mathbb{C})$ iff $r$ or one of its subexpressions $p_i$ is a rational identity of $M_n(\mathbb{C})$.

# 3 Bergman-Small diagrams

A proper study of rational identities seems to be impossible without relying on the delicate Bergman-Small theorem which gives restrictions on the p.i.-degrees of prime quotients of p.i.-rings, see [8] or [14, §1.10]. We will give here a different formulation and an easy proof of it in the special (affine) case needed here. Probably this proof is the one attributed to Procesi in [6, p.433 bottom].

Let $R$ be a prime affine p.i.-algebra and let $TR$ its trace algebra (see e.g. [4]) which is a finite module over its center $C$ which is an affine commutative domain. Procesi [12, p.177-178] has shown that the maximal ideals of $C$ parametrize the isomorphism classes of semi-simple $n$-dimensional representations of $R$ where $n$ is the p.i.-degree of $R$.

To each maximal ideal $m$ of $C$ one can associate a Young diagram $Y(m) = (n_1^{a_1}, ..., n_l^{a_l})$ with $n$ boxes, where $n_1 > n_2 > ... > n_l$ and $a_i$ is the number of $n_i$-dimensional simple components (counting multiplicities if necessary) ocuring in the $n$-dimensional semi-simple representation of $R$ corresponding to $m$.

**Definition 3** *The* **Bergman-Small diagrams** *of $R$ form the set $BS(R)$ of all Young diagrams obtained from maximal ideals of $C = Z(TR)$.*

**Proposition 1 (Bergman-Small revisited)** *Let $R$ be a prime affine p.i.-algebra of p.i.-degree $n$. The set of p.i.-degrees of prime quotients of $R$ is precisely the set of row lengths ocuring among the Bergman-Small diagrams $BS(R)$.*

**Proof :** Let $P$ be a prime ideal of $R$ then there is a maximal ideal $M$ of $R$ containing $P$ s.t. $p.i.deg(R/P) = p.i.deg(R/M)$ (lift an Azumaya point of $R/P$). As $R/M$ is finite dimensional, $M$ lifts to a maximal ideal of $TR$ and hence determines a maximal ideal $m$ of $C = Z(TR)$. Then, $R/M$ is the endomorphism ring of one of the simple components ocuring in the representation corresponding to $m$. Thus, $p.i.deg(R/P)$ occurs as a row length in a Bergman-Small diagram of $R$. The converse is entirely trivial. □

If $G_{k,n}$ is the ring of $k$ generic $n \times n$ matrices, then it is well known and easy from the above mentioned result) that $BS(G_{k,n})$ is the set of all Young diagrams having $n$ boxes.

Also note that $R$ is an Azumaya algebra of p.i.-degree $n$ iff $BS(R) = \{(n)\}$ by the Artin-Procesi result, see [5, Th.8.3] or [14, §1.8].

# 4 Two tricks

In order to shorten the proofs below we single out the two main tricks here

## 4.1 Surgery on expressions

**Proposition 2** *Let $s_1 < ... < s_l < n < t_1 < ... < t_m$ be an ascending chain of integers and let $r$ be a rational expression in $x_1, ..., x_k$ with $\delta(r) = a$ which is well-defined on $M_{s_j}(\mathbb{C})$ for $1 \leq j \leq l$ and on $M_n(\mathbb{C})$. Then, there is a rational expression $r'$ with $\delta(r') = a$ which is well-defined on $M_n(\mathbb{C})$, $M_{s_j}(\mathbb{C})$ $1 \leq j \leq l$ and $M_{t_j}(\mathbb{C})$ $1 \leq j \leq m$ such that $R' = R$ in $D_{k,s_j}$ (all $j$) and in $D_{k,n}$ and $R' \neq 0$ in $D_{k,t_j}$ (all $j$).*

**Proof :** Let $r = r(x_1, .., x_k; p_1^{-1}, ..., p_l^{-1})$. Let $j$ be minimal s.t. either $r$ is not well-defined on $M_{t_j}(\mathbb{C})$ (meaning that some $P_i = 0$ in $D_{k,t_j}$) or that $R = 0$ in $D_{k,t_j}$. Then, we can cure this difficulty by modifying $r$ into $r + p$ or $p_i$ into $p_i + p$ where $p$ is a polynomial identity for $t_j - 1 \times t_j - 1$ matrices not holding in $M_{t_j}(\mathbb{C})$. Now, iterate this process. □

**Corollary 1** *Let $s_1 < ... < s_l < n < t_1 < ... < t_m$ be an ascending chain of integers. Assume there is a rational identity holding in $M_n(\mathbb{C})$ but not in any of the $M_{s_i}(\mathbb{C})$ $1 \leq i \leq l$. Then, there is a rational identity (with equal distortion number) holding in $M_n(\mathbb{C})$ but not in any of the $M_{s_i}(\mathbb{C})$ $1 \leq i \leq l$ nor in any of the $M_{t_j}(\mathbb{C})$ for $1 \leq j \leq m$.*

## 4.2 Azumaya's black box

**Proposition 3** *Let $p_1, ..., p_l$ be rational expressions with $\delta(p_i) < a$ which are well defined and do not hold in $M_n(\mathbb{C})$ nor in $M_m(\mathbb{C})$ where $m < n$. If the affine algebra $G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ is an Azumaya algebra, then there is a rational expression $r$ with $\delta(r) \leq r$ such that $R = 0$ in $D_{k,n}$ (and hence is a rational identity) and $R = 1$ in $D_{k,m}$ (and hence $r$ does not hold in $M_m(\mathbb{C})$).*

**Proof :** By [14, Th.1.8.48] we can find elements $A_{ij}, B_i \in G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ such that

$$\sum_{i=1}^{t} g_n(A_{i1}, ..., A_{id}).B_i = 1 \quad (5)$$

where $g_n(y_1, ..., y_d)$ is the central polynomial for $M_n(\mathbb{C})$ coming from the Capelli polynomial [14, p.26].But then it is clear that

$$r(x_1, ..., x_k; p_1, ..., p_l) = 1 - \sum_{i=1}^{t} g_n(a_{i1}, ..., a_{id}).b_i \quad (6)$$

has the required properties. □

A rational expression $r$ as in the above proof will be called a **rational Azumaya identity** of degree $n$.

# 5 Fifty ways to get an Azumaya algebra

In view of the foregoing proposition, we have to find sensible ways to obtain an Azumaya algebra by adjoining to $G_{k,n}$ inverses of suitable rational expressions (everything inside $D_{k,n}$). Usually, one embeds $G_{k,n}$ into an Azumaya algebra by adjoining to it the inverse of a central element (i.e. a central identity of $M_n(\mathbb{C})$). However, this procedure is not suitable for our purposes as a central identity for $M_n(\mathbb{C})$ is a polynomial identity of $M_m(\mathbb{C})$ and hence its inverse is not well defined on $M_m(\mathbb{C})$. Still, there are plenty of ways to get an Azumaya algebra (see also [8, Prop.8.1 and p.458 bottom]):

**Proposition 4** *Let $d$ be the largest integer $\leq \frac{n}{2}$ and $p(x_1, ..., x_k)$ a polynomial identity for $M_d(\mathbb{C})$ not holding for $M_{d+1}(\mathbb{C})$, then $G_{k,n}\{P^{-1}\} \subset D_{k,n}$ is an Azumaya algebra.*

**Proof :** The image of $P$ has to remain a unit in every quotient of $G_{k,n}\{P^{-1}\}$ so none of its Bergman-Small diagrams can have a row of length $\leq d$. But then we have killed off all diagrams of $G_{k,n}$ except $(n)$ finishing the proof. □

At this point we can ask the following :

**Question 1** *What is the minimal degree of a homogeneous element $P \in G_{k,n}$ s.t. $G_{k,n}\{P^{-1}\}$ is an Azumaya algebra ? In particular, is it equal to the minimal degree of a polynomial identity of degree $d$ in $k$ variables where $d$ is the largest integer $\leq \frac{n}{2}$ ?*

For any pair of integers $m < n$ with $m$ not dividing $n$ let us denote by $\mathbf{\Delta(m,n)}$ the minimal distortion number of a rational identity of $M_n(\mathbb{C})$ not holding in $M_m(\mathbb{C})$. Knowledge of $\Delta(m,n)$ has two applications : first, it gives us a measure of the difficulty to produce explicit rational identities of $M_n(\mathbb{C})$ not holding in $M_m(\mathbb{C})$. Secondly, it allows us to extend the classical result on polynomial identities as follows : let $r$ be a rational identity of $M_n(\mathbb{C})$ with $\delta(r) < \Delta(m,n)$, then $r$ holds (possibly degenerately) in $M_m(\mathbb{C})$.

**Corollary 2** *If $\frac{n}{2} < m < n$ then $\Delta(m,n) = 1$. Moreover, there is a rational expression $r$ with $\delta(r) = 1$ s.t. $R = 0$ on $M_n(\mathbb{C})$ and $R = 1$ on $M_m(\mathbb{C})$.*

**Proof :** The black box result gives a required rational identity of distortion $\leq 1$ for all $m$ s.t. $p^{-1}$ is well-defined on $M_m(\mathbb{C})$. Clearly, this holds precisely for the indicated $m$. □

Note that Bergman's rational identity has distortion number 3 whereas $\Delta(2,3) = 1$ and that Rowen's example has distortion number $1 = \Delta(3,4)$.

Before we treat the general case we need a universal property of the overrings $G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ which can be considered as universal localizations in the category of rings embedable in $n \times n$ matrices over a commutative algebra. To this end, we give the following description of their trace rings and centers.

Let $R = G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ and assume as above that each $p_i$ is a polynomial in $x_1, ..., x_k, p_1, ..., p_{i-1}$. Then, $R$ can be viewed as the quotient of $G_{k+l,n} = k\{X_1, ..., X_k, Z_1, ..., Z_l\}$ modulo the ideal generated by the relations $Z_i.p_i(X_1, ..., X_k, P_1^{-1}, ..., P_{i-1}^{-1}) = 1$.

These relations give rise to $l.n^2$ relations among the entries of the generic matrices $x_{uv}(i), z_{uv}(i)$ and we will denote by $I$ the ideal in $P = \mathbb{C}[x_{uv}(i), z_{uv}(j) \mid 1 \leq u,v \leq n, 1 \leq i \leq k, 1 \leq j \leq l]$ and let $A$ be the quotient algebra $P/I$.

It follows from [13, Th.2.6] that there is a natural action of $PGL_n$ an $A$ and $M_n(A)$ such that $TR = M_n(A)^{PGL_n}$ and $C = Z(TR) = A^{PGL_n}$. From this description we deduce the following universal property of $R$ :

**Proposition 5** *Let $B$ be a commutative $\mathbb{C}$-algebra and $\beta_j \in M_n(B)$ such that $p_i(\beta_1, ..., \beta_k)$ is an invertible matrix in $M_n(B)$ for all $1 \leq i \leq l$. Then, there is a well-defined morphism $R = G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\} \to M_n(B)$ sending $X_i$ to $\beta_i$.*

**Proof :** The entries of the matrices $\beta_j$ and $p_i(\beta_1, ..., \beta_k)$ satisfy all relations defining $A$, so we get $\mathbb{C}$-algebra morphisms $A \to B$ and $M_n(A) \to M_n(B)$ which map $X_i \in M_n(A)$ to $\beta_i$. The required morphism is then the composition

$$R \hookrightarrow TR \hookrightarrow M_n(A) \to M_n(B) \tag{7}$$

finishing the proof. □

This result allows us to find a lower bound for $\Delta(m, n)$ :

**Proposition 6** *If $a.m < n$ then every rational identity $r$ of $M_n(\mathbb{C})$ with $\delta(r) < a$ holds (possibly degenerately) in $M_m(\mathbb{C})$. Equivalently, $a \leq \Delta(m, n)$.*

**Proof :** We use induction on $a$. If $a = 1$ it is the classical result on polynomial identities. So, let $a > 1$ and assume the result holds for $a - 1$. Let $r = r(x_1, ..., x_k; p_1^{-1}, ..., p_l^{-1})$ with $\delta(p_i) \leq a - 1$.

If one of the $p_i$ holds (possibly degenerately) in $M_m(\mathbb{C})$, then so does $r$ and there is nothing to prove. Hence we may assume that none of the $p_i$ holds in $M_m(\mathbb{C})$ so we can form the algebra $G_{k,m}\{P_1^{-1}, ..., P_l^{-1}\} \subset D_{k,m}$. Now, as $(a-1)m < n - m$ we may assume by induction that none of the $p_i$ holds in $M_{n-m}(\mathbb{C})$ and so we can similarly form the algebra $G_{k,n-m}\{P_1^{-1}, ..., P_l^{-1}\} \subset D_{k,n-m}$ (somewhat misusing notation). From the universal property we obtain a well defined morphism

$$G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\} \to \begin{pmatrix} G_{k,m}\{P_1^{-1}, ..., P_l^{-1}\} & 0 \\ 0 & G_{k,n-m}\{P_1^{-1}, ..., P_l^{-1}\} \end{pmatrix} \tag{8}$$

(mapping to the diagonal). But then if $R(X_1, ..., X_k; P_1^{-1}, ..., P_l^{-1}) = 0$ in $D_{k,n}$ it has to be zero in $D_{k,m}$ (and in $D_{k,n-m}$) too. □

We can now state and prove our first main result :

**Theorem 1** *Let $m, n$ and $a$ be positive integers determined by $a.m < n < (a+1).m$. Then, there exist rational expressions $p_i$ with $\delta(p_i) < a$ not holding* *in $M_n(\mathbb{C})$ nor in $M_m(\mathbb{C})$ s.t. $p_1$ is a polynomial identity of $M_{m-1}(\mathbb{C})$ and $p_i$ for $2 \leq i \leq l$ are rational Azumaya identities of rank $d$ for some $m < d < n$ s.t.*

$$G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\} \tag{9}$$

*is an Azumaya algebra. Hence we can find a rational Azumaya identity $r$ of degree $n$ s.t. $\delta(r) \leq a$ and $R = 1$ in $D_{k,m}$. In particular, $\Delta(m,n) = a$.*

**Proof:** Again, the proof goes by induction on $a$. If $a = 1$ the statement follows from proposition 4 above. Hence, assume $a > 1$ and that the theorem holds for all $a' < a$.

By tossing in the inverse of a polynomial identity $p_1$ holding in $M_{m-1}(\mathbb{C})$ not holding in $M_m(\mathbb{C})$ we have killed off all Bergman-Small diagrams containing a row of length $< m$. Consider one of the remaining diagrams $Y = (n_1^{a_1}, ..., n_h^{a_h})$ where $\sum a_i n_i = n$ and $n_1 > ... > n_h \geq m$. As $m$ does not divide $n$ there is a maximal $i$ s.t. $m$ does not divide $n_i$. For this $n_i$ we have $m < n_i < n - m < a.m$ but then there is an integer $a' < a$ s.t. $a'.m < n_i < (a' + 1).m$.

By induction we have rational Azumaya identities $p_1(Y), ..., p_y(Y)$ of certain degree $d$ s.t. $m < d < n_i < n$ which do not hold in $M_m(\mathbb{C})$ nor in $M_{n_i}(\mathbb{C})$ and with $\delta(p_i) < a' < a$ s.t. $G_{k,n_i}\{P_1(Y)^{-1}, ..., P_y(Y)^{-1}\}$ is an Azumaya algebra giving rise to a rational Azumaya identity $p(Y)$ of degree $n_i$ s.t. $P(Y) = 1$ in $D_{k,m}$ and with $\delta(p(Y)) \leq a' < a$.

We do not have to perform surgery on the expressions $p_j(Y)$ or $p(Y)$ to make them not holding in $M_n(\mathbb{C})$ as this is a consequence of the foregoing result.

Therefore, tossing in $P_j(Y) \in D_{k,n}$ for all $1 \leq j \leq y$ and the inverse of $P(Y)$ we have killed off the Bergman-Small diagram $Y$. Clearly, one can iterate this argument to get rid of all diagrams different from $(n)$. So, ultimately we get an algebra $G_{k,n}\{P_1^{-1}, ..., P_l^{-1}\}$ which is Azumaya and with all $\delta(p_i) < a$. The black box result and the foregoing result now finish the proof. □

Note that the above proof provides us with an inductive procedure to find explicit rational expressions $r$ of minimal distortion number such that $R = 0$ in $D_{k,n}$ and $R = 1$ in $D_{k,m}$ for $m$ not dividing $n$.

## 6 Rationalizing the argument

If one is willing to use an extra bit of information, one can shorten the foregoing argument drastically. Recall from [11, §II.1] that the space $X_{k,n} = M_n(\mathbb{C})^{\oplus k}$ of $n$-dimensional representations of $G_{k,n}$ and its quotient variety $V_{k,n}$ under the group action of $PGL_n$ mentioned before, admit a nice stratification according to representation type. Moreover, one can define a notion of refinement among representation types [11, p.155] such that one stratum belongs to the Zariski closure of another one if and only if the first representation type is a refinement of the second.

Translating these facts to the language of Bergman-Small diagrams we see that any $n$-dimensional representation of $G_{k,n}$ with associated diagram $(n_1^{a_1}, ..., n_h^{a_h})$ is a degeneration of a representation with associated diagram $(n - n_h, n_h)$.

Further, note that the process of tossing in extra elements $P_1^{-1}, ..., P_l^{-1}$ of $D_{k,n}$ translates on the level of $n$-dimensional representations to restricting to a Zariski open subset determined by the invariant elements $Det(P_i)$ for $1 \leq i \leq l$.

Concluding, we see that it suffices to kill off the diagrams $(n, n - n_1)$ with $n_1 \geq m$ to obtain an Azumaya algebra. Note that either $n_1$ or $n - n_1$ is not divisible by $m$.

## 7 The main result

If $m$ does not divide $n$ we were able to embed $G_{k,n}$ into an Azumaya algebra by tossing in inverses of elements which are well defined in $D_{k,m}$. We want to generalize this now to any finite set of integers $\alpha = (n_1, ..., n_l)$ with $n_1 < n_2 < ... < n_l$. So, we are only allowed to adjoin inverses of elements to $G_{k,n}$ if they represent well defined elements in $D_{k,n_i}$ for all $i$. Such expressions will be called **$\alpha$-admissible**.

Again, we are interested in finding $\alpha$-admissible rational identities holding for $n \times n$ matrices which do not hold for $n_i \times n_i$-matrices for all $i$ (if they exist). The minimal distortion number of such an expression will be denoted by $\Delta(\alpha, n)$. By the corollary to the surgery trick we have :

**Lemma 1** *Let $\alpha = (n_1, ..., n_l)$ with $n_1 < ... < n_k < n < n_{k+1} < ... < n_l$ then $\Delta(\alpha, n) = \Delta(\beta, n)$ where $\beta = (n_1, ..., n_k)$.*

**Definition 4** *An integer n is said to be α-***reachable** *where* $\alpha = (n_1, ..., n_l)$ *if there is a Young diagram with n boxes s.t. all row lenghts belong to* $\alpha$. *Equivalently, there is a solution in positive integers to the equation* $n = \sum_{i=1}^{l} a_i n_i$ *(note that some of the* $a_i$ *may be zero).*

This notion gives us a necessary condition for the existence of the required expressions :

**Proposition 7** *If n is α-reachable where* $\alpha = (n_1, ..., n_l)$*, then every rational identity holding in* $D_{k,n}$ *holds (possibly degenerately) in some* $D_{k,n_i}$.

**Proof :** Let $r = r(x_1, ..., x_k, f_1^{-1}, ..., f_m^{-1})$ be a rational identity holding in $D_{k,n}$. We may assume that every $f_j$ is well defined in all $D_{k,n_i}$ for otherwise $r$ would hold degenerately in some $D_{k,n_i}$ and we are done. Hence we can form the rings $B_{k,n} = G_{k,n}\{f_1^{-1}, ..., f_m^{-1}\}$ and $B_{k,n_i} = G_{k,n_i}\{f_1^{-1}, ..., f_m^{-1}\}$.

Assume that $n = \sum a_i n_i$, them we have by the universal property a morphism

$$B_{k,n} \to \begin{pmatrix} I_{a_1}(B_{k,n_1}) & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & I_{a_l}(B_{k,n_l}) \end{pmatrix} \tag{10}$$

where $I_m(R)$ represents $R$ embedded diagonally in $M_m(R)$. But then we know that if $r = 0$ in $B_{k,n}$ it must be zero in every $B_{k,n_i}$,too. □

Actually, the proof admits a slight strengthening of the formulation : if $n = \sum a_i n_i$ and $r$ is a rational identity of $D_{k,n}$ which is defined in every $D_{k,n_i}$ s.t. $i \neq 0$, then $r$ is a rational identity for all $D_{k,n_i}$ s.t. $a_i \neq 0$.

We are now in a position to state and prove the main theorem of this paper

**Theorem 2** *Let* $\alpha = (n_1, ...n_l)$ *then the following statements are equivalent*

1. *There is a rational identity for* $D_{k,n}$ *which does not hold in any* $D_{k,n_i}$ $(n_i \in \alpha)$ *and* $\Delta(\alpha, n) = \Delta(n_1, n)$
2. *n is not α-reachable*
3. *We can embed* $G_{k,n}$ *into an Azumaya algebra by tossing in inverses of α-admissible elements (actually Azumaya rational identities) with distortion number* $< \Delta(n_1, n)$

**Proof :** (1) implies (2) by the foregoing proposition.

(2) implies (3) : We use induction on $n$. This is allowed as the statement holds for all $n < n_1$ and if $n_1 = 2$ it holds for $n = 3$. So, the only case not covered by the foregoing sections is when $n_1 = 2$ and $n_2 = 3$,but then every $n$ is $\alpha$-reachable. So, we may assume that the statement holds for every $m < n$. Now, take a Bergman-Small diagram for $G_{k,n}$. Then, by the assumption there is at least one row-length $m$ which does not belong to $\alpha$. As $m < n$ there exists by induction a rational identity for $D_{k,m}$ which does not hold in any $D_{k,n_i}$ (nor in $D_{k,n}$ eventually after performing surgery) and we may assume that $\delta(r) \leq \Delta(n_1, m)$. So, we can kill off this diagram by tossing in the inverse of this expression. The worst case possible will be the diagram $(n - n_1, n_1)$ which requires an expression with distortion $\Delta(n_1, n-n_1) < \Delta(n_1, n)$. Hence we can get rid of all diagrams except $(n)$ as required.

(3) implies (1) : Take an Azumaya rational identity in the ring constructed. Then we have a rational expression $r$ with $\delta(r) \leq \Delta(n_1, n)$ s.t. $r = 0$ in $D_{k,n}$ and $r = 1$ in $D_{k,n_i}$ for all $n_i \in \alpha$ s.t. $n_i < n$ and we can perform surgery on $r$ if required to have $r \neq 0$ in $D_{k,n_j}$ for all $n_j \in \alpha$ s.t. $n < n_j$. □

Note that we actually proved the existence of a rational identity $r$ in $D_{k,n}$ s.t. $r = 1$ in $D_{k,n_i}$ for all $n_i \in \alpha$ s.t. $n_i < n$. Unfortunately, we lose control on the expression in $D_{k,n_j}$ for $n < n_j$.

# 8 The Amitsur-Bergman game

The main result has an amusing interpretation in terms of the following two person game : assume two players A(mitsur) and B(ergman) alternatively name positive integers $n_1, n_2, ...$ subject to the condition that there is a rational identity holding in $n_k \times n_k$ matrices not holding in $n_i \times n_i$ matrices for all $i < k$. Of course, the first player forced to name a commutative identity is declared the loser.

In view of the theorem, this game coincides with Conway's Sylver Coinage game [9, Ch.18] from which we recall that Amitsur can win by starting with any prime number $n_1 = p \geq 5$. Unfortunately, it is not clear whether there is a natural ringtheoretic interpretation of a winning position.

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