GENERIC BRAUER ALGEBRAS I

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Abstract
We use the relativised Kimmerle equivalence of the Grünberg-Roggenkamp equivalence to construct Brauer algebras, having a generic property. We deduce rationality results for their centers.

Key Words
Generic Division Algebras, Rationality Problem.

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0. Introduction

Ever since Hamilton and Hilbert, people have tried to construct new examples of division algebras. The most frantic examples known today are the generic division algebras $UD(n)$ introduced in 1966 by Amitsur and Procesi and the generic crossed products, discovered in the late seventies by Snider and Rosset, independently.
Although these two constructions are totally different, their centers can be described in terms of lattice invariants. As the representation theory of finite groups recently has led to some remarkable results about these fields, it is our hope that a representation theoretic description of the division algebras themselves may shed some new light on such old problems as: when are division algebras crossed products ? are division algebras of prime degree cyclic ? etc. This paper can be seen as a first step in this program.
In this paper we introduce and study generic Brauer algebras, generalizing the generic examples described above. Roughly speaking generic Brauer algebras are division algebras generic w.r.t. having a maximal subfield which is the fixed field under H of its Galois splitting with Galois group G, where H is a subgroup of G.
Its construction is a relative version of Rosset's and Snider's. Consider a presentation of the group G

$$1 \to K \to F * H \to G \to 1 \qquad (1)$$

where F is a free group. Then its abelianized sequence

$$1 \to R = K/[K,K] \to E = F * H/[K,K] \to G \to 1$$

is such that R usually is a $\mathbb{Z}G$-lattice and E is a group containing no finite normal subgroups and whose maximal finite subgroups are conjugated to H. But then, by Moody's result the groupalgebra kE is prime p.i. having quotientring

$$Q(kE) = M_{\#H}(\Delta)$$

for some division algebra Δ having as center the field of lattice invariants $(kR)^G$. This Δ then is our generic Brauer algebra corresponding to the couple (G, H).

Instead of studying this problem using group extensions, one can also study it using representation theory of $\mathbb{Z}G$. For, there is a unique exact sequence of $\mathbb{Z}G$-lattices

$$0 \to R \to M \to I_{G/H} \to 0 \tag{2}$$

corresponding to (1) where $I_{G/H}$ is the kernel of the augmentation map on the permutation lattice $\mathbb{Z}G/H$. The free $\mathbb{Z}G$-lattice M can be given a ring theoretical interpretation by showing that its field of lattice invariants is the function field of the Brauer-Severi variety corresponding to Δ.
In the first section we will recall this natural equivalence of categories between relative group extensions and module extensions which is due to Kimmerle. We will interprete everything in terms of division algebras. It will turn out that we recover the known correspondence between division algebras and Brauer-Severi varieties.
In the second section we introduce and study the generic Brauer algebras in the same vein as Rosset's study in [8]. Moreover, it turns out that there are more such algebras deserving to be called "generic" than the ones coming from a sequence (1). We might start from a sequence (2) with M only an invertible lattice (i.e. a direct summand of a permutation lattice) and make a similar construction. Then, these algebras have an analoguous generic property. The centers of all these algebras turn out to be retract equivalent to one another.
In the final section we illustrate the above by giving a representation theoretic description of the generic division algebras. The starting point is the Procesi-Formanek description of the center :

$$0 \to \mathbb{Z}S_n/S_{n-1} \oplus G_n \to \mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z}S_n/S_{n-2} \to I_{S_n/S_{n-1}} \to 0$$

To this sequence corresponds a relative group extension

$$1 \to K \to F * S_{n-1} \to S_n \to 1$$

whose abelianized sequence is

$$1 \to \mathbb{Z}S_n/S_{n-1} \oplus G_n \to E \to S_n \to 1$$

The group algebra kE has a quotient ring

$$Q(kE) = M_{(n-1)!}(UD(n)).$$

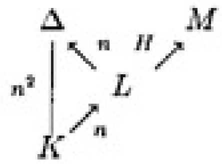
Moreover, in this monstrous simple algebra one can embed all Rosset-Snider generic crossed products w.r.t. finite groups of order n.
The relationship between these algebras deserves closer attention and we terminate this paper with a few remaining problems.

1. The Noether-Brauer-Severi equivalence

In this section we will show that the relativized Kimmerle equivalence of the Grünberg-Roggenkamp equivalence between group extensions and module extensions contains the classical equivalence between central simple algebras of degree n and their Brauer-Severi varieties of dimension $n-1$. Usually the later equivalence follows from the fact that both objects are classified by the Galois cohomology group $H^1(Gal(\overline{k}/k), PGl_n)$. Our integral representation theoretic interpretation has the advantage that it extends to relative crossed products having coefficients in lattices leading naturally to the generic objects treated in the next section.

(1.1) The Noether-Brauer category

In general, division algebras are neither cyclic nor crossed products. The only positive property they have is that a certain matrixring over it becomes a crossed product. This classical fact due to Noether and Brauer is proved as follows. Take a division algebra Δ of dimension n^2 over its center K, then Δ contains a maximal commutative subfield L which is separable of dimension n over K. Let M be the splitting field of L which is Galois over K with Galois group $G = Gal(M/K)$ and let $H = Gal(M/L)$ then $[G : H] = n$ and $M^H = L$, i.e. we have a situation



Then, M is a subfield of $M_{\#H}(\Delta)$ and is clearly a maximal subfield. As M is Galois over K we can write $M_{\#H}(\Delta)$ as the crossed product represented by the group extension

$$1 \to M^* \to E \to G \to 1 \tag{3}$$

and the fact that L is a maximal subfield of Δ, where $L = M^H$ can be interpreted as follows : consider the pullback diagram of (3) over H

$$\begin{array}{ccccccccc} 1 & \to & M^* & \to & E' & \to & H & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & M^* & \to & E & \to & G & \to & 1 \end{array}$$

then the lower sequence splits or, equivalently, E has a subgroup U isomorphic to H. We will now formalize this situation in the following

(1.2) Definition

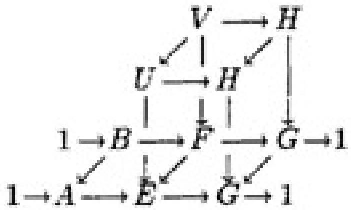
Let G be any finite group and H a subgroup of index n. Then, the Noether-Brauer category $\underline{(G/\!\!/H)}$ of G relative to H has as objects exact sequences of groups

$$1 \to A \to E \to G \to 1$$

with A an Abelian group such that the pullback

$$\begin{array}{ccccccccc} 1 & \rightarrow & A & \rightarrow & U & \rightarrow & H & \rightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & A & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \end{array}$$

splits, i.e. E has a subgroup isomorphic to H. Morphisms are morphisms of group extensions such that the following diagram is commutative



(1.3) The Brauer-Severi category

Let $\mathbb{Z}G/H$ be the permutation lattice over $\mathbb{Z}G$ on the cosets of H in G., then there is an augmentation sequence

$$0 \rightarrow I_{G/H} \rightarrow \mathbb{Z}G/H \rightarrow \mathbb{Z} \rightarrow 0$$

$$gH \mapsto 1$$

So, $I_{G/H}$ can be viewed as a relative augmentation ideal of G w.r.t. H. Now, consider a Galois extension M of K with $Gal(M/K) = G$ and view M^* as a $\mathbb{Z}G$-module in the usual way. Then any extension

$$0 \rightarrow M^* \rightarrow B \rightarrow I_{G/H} \rightarrow 0 \tag{4}$$

of $\mathbb{Z}G$-modules determines an element of

$$\ker\left(H^2(M^*, G) \rightarrow H^2(M^*, H)\right)$$

i.e. a Brauer class split by M^H. The function field of the Brauer-Severi variety of this class can be described as the field of invariants of a twisted-tori situation: consider a fixed $\mathbb{Z}$-splitting of (4) :

$$B \simeq M^* \oplus I_{G/H}$$

After fixing an embedding $I_{G/H} \hookrightarrow B$ we can define an isomorphism

$$\exp : B \rightarrow M\left[I_{G/H}\right]^* = M^*.I_{G/H}$$

where $M\left[I_{G/H}\right]$ is the groupalgebra over M of the abelian torsion free group of rank $n-1$: $I_{G/H}$. For each $g \in G$ we let it act on an $x \in I_{G/H}$ via its embedding in B and then each $g \in G$ gives an algebra morphism

$$\varphi_g : M\left[I_{G/H}\right] \rightarrow M\left[I_{G/H}\right]$$

Thus, G acts as a group of automorphisms on the group algebra $M[I_{G/H}]$ and on its field of fractions $M(I_{G/H})$. Denote the twisted action by $M_t[I_{G/H}]$. It follows from work of Roquette and Saltman that the function field of the Brauer-Severi variety of the Brauer class corresponding to the extension (4) is the field of invariants of the twisted tori field $M_t(I_{G/H})^G$. Again, let us formalize this situation in

(1.4) Definition

Let G be any finite group and H a subgroup of index n. Then, the Brauer-Severi category $(\underline{I_{G/H}})$ of G relative to H has as objects exact sequences of $\mathbb{Z}G$-modules

$$0 \to A \to B \to I_{G/H} \to 0$$

Morphisms in the category are simply morphisms of exact sequences with the identity map between the last terms.

(1.5) The equivalence

We will now recall the following fundamental result of Kimmerle [5] extending previous work of Roggenkamp [7] and Grünberg [4] in the absolute case (i.e. when $H = 1$).

(1.6) Theorem (Kimmerle)

There exists an equivalence of categories between $(\underline{G/\!/H})$ *and* $(\underline{I_{G/H}})$.

For a proof see [5].

For later use we recall the construction of this equivalence. Given an object in $(\underline{G/\!/H})$

$$\begin{array}{ccccccccc} & & & & U & \simeq & H & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & A & \to & E & \to & G & \to & 1 \end{array}$$

we consider E as being generated by A and symbols $u_\sigma, \sigma \in G$ so that $u_\sigma u_\tau = f(\sigma,\tau)u_{\sigma\tau}$ with $f(\sigma,\tau) = 1$ for all $\sigma, \tau \in H$. An element $e = \sum_x \left(\sum_{a \in A} \nu_a^x a\right) u_x$ belongs to $\ker \pi$, $\pi : \mathbb{Z}E/U \to \mathbb{Z}G/H$ if and only if e belongs to $I_A.E/U$, with I_A the augmentation ideal of A. We then obtain

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & I_A.E/U & \to & I_{E/U} & \to & I_{G/H} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & I_A.E/U & \to & \mathbb{Z}E/U & \to & \mathbb{Z}G/H & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbb{Z} & = & \mathbb{Z} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

From the top row we deduce

$$0 \to (I_A.E/U)/_{(I_A.I_{E/U})} \to (I_{E/U})/_{(I_{E/U}.I_A)} \to I_{G/H} \to 0$$

and thus

$$0 \to A \to (I_{E/U}/E)/_{(I_{E/U}.I_A)} \to I_{G/H} \to 0.$$

Conversely, given a sequence

$$0 \to A \to M \to I_{G/H} \to 0$$

we form the pullback diagram

$$\begin{array}{ccc} E & \overset{\psi}{\to} & G \\ \gamma\downarrow & & \downarrow j \\ M & \overset{\psi_1}{\to} & I_{G/H} \end{array}$$

j being given by $g \mapsto gH - H$. As a set E consists of pairs (m,g) such that $\psi_1(m) = gH - H$. ψ and γ are the obvious projections. E is made into a group by

$$(m,g).(m',g') = (m + gm', gg').$$

As M is a $\mathbb{Z}G/H$-module (the action of g only depends on the coset to which it belongs) one has $(m,h)(m',h') = (m+m',hh')$ for all $h,h' \in H$ entailing that $U = \{(0,h) | h \in H\}$ is a subgroup of E isomorphic to H.

From these functors Kimmerle esstablishes the equivalence of $(\underline{G/\!/H})$ and $(\underline{I_{G/H}})$. In the special case when the kernel of an object is M^* for a G-Galois extension $\overline{M/k}$ the foregoing equivalence gives the well known one-to-one correspondence between central simple algebras of degree n and their Brauer-Severi varieties.

2. Generic Brauer Algebras

In this section we use te previously described relativized Kimmerle equivalence to construct generic Brauer algebras, not only starting from free objects, but also from objects in $(\underline{I_{G/H}})$ with invertible middle term, called "almost generic Brauer algebras". It will be shown that such Brauer algebras, coming from objects with a coflasque left term in addition, can be considered as minimal. The importance of these constructions lies in the fact that the centers of almost generic Brauer algebras are retract equivalent, whereas the centers of the "ordinary" generic Brauer algebras are stably equivalent. Finally, we deduce results for Brauer-Severi varieties.

We would like to spot free objects in both categories and study their relationship under the equivalence.

(2.1) Definition

An object in $(\underline{G/\!/H})$ is called free if there exists a set $S \subseteq E$ such that E is generated by S and U and so that every application $\mu : S \to F$, making the following diagram exact

$$\begin{array}{ccccccccc} & & & & S & \to & G & & \\ & & & & \mu\downarrow & & \| & & \\ 1 & \to & B & \to & F & \to & G & \to & 1 \end{array}$$

extends to a morphism of objects of $(\underline{G/\!/H})$.

The existence of free objects is guaranteed by the following construction. Let Y be an arbitrary set and let $v : Y \to G$ be an application satisfying $< \mathrm{Im}v, H >= G$. Then we can form a unique homomorphism $\lambda : F(Y) * H \to G$ (with $F(Y)$ the free group on the set Y and $F(Y) * H$ the free product of both groups) so that $\lambda|_Y = v$ and $\lambda|_H$ is the canonical injection of H in G.
Denoting by R the kernel of this homomorphism, we obtain, after division by $R' = [R, R]$:

$$\begin{array}{ccccccccc} & & & & H.R'/R' & \to & H & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & R/R' & \to & (F(Y) * H)/R' & \to & G & \to & 1 \end{array}$$

We denote this object by $B(Y, v)$.

(2.2) Proposition

Every free object in $(\underline{G/\!/H})$ is isomorphic to a certain $B(Y, v)$.

For a proof see Kimmerle [5].

(2.3) Definition

An object in $(\underline{I_{G/H}})$ is called free if there exists a set $S \subseteq M$ for which $M =< S >$ so that for every object M_1 and every application $\mu : S \to M_1$ with

$$\begin{array}{ccc} S & \to & I_{G/H} \\ \downarrow & & \| \\ M_1 & \to & I_{G/H} \end{array}$$

commutative, extends uniquely to a morphism of objects.

(2.4) Proposition

Every free object in $(\underline{I_{G/H}})$ is isomorphic to an object $C(X,w)$:

$$0 \to \ker w \to F(X) \to I_{G/H} \to 0.$$

$F(X)$ denotes the free $\mathbb{Z}G$-module on X.

For a proof see Kimmerle [5].

(2.5) Lemma

Let $B(Y,v)$ be a free object in $(\underline{G/\!/H})$. Then

$$1 \to \overline{R} \to (F * H)/R' \to G \to 1 \qquad (5)$$

is a generator for

$$H^2(G,\overline{R}) = \mathbb{Z}/d\mathbb{Z}$$

where $d = \#(G/H)$ and $\overline{R}$ denotes R/R'.

Proof. From $(\mathbb{Z}G/H)^G \to \mathbb{Z}^G \to H^1(G, I_{G/H}) \to 0$ and taking into account that $(\mathbb{Z}G/H)^G = \mathbb{Z}(\sum_x xH)$, where x runs through a transversal for H in G, as G acts transversally, we obtain

$$H^1(G, I_{G/H}) = \mathbb{Z}/d\mathbb{Z}.$$

The equivalence for $B(Y,v)$ entails the sequence

$$0 \to \overline{R} \to F \to I_{G/H} \to 0$$

with F a free module. Thus

$$H^1(G, I_{G/H}) = H^2(G,\overline{R}).$$

Let β be an arbitrary generator for $H^2(G,\overline{R})$. Then we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \overline{R} & \to & (F(Y) * H)/R' & \to & G & \to & 1 \\ & & f\downarrow & & \downarrow & & \| & & \\ 1 & \to & \overline{R} & \to & E & \to & G & \to & 1 \end{array}$$

As $f_*(\alpha) = \beta$ we obtain that the order of α is equal to d, implying that α generates $H^2(G,\overline{R})$, thus finishing the proof.

The category $(\underline{G/\!/H})$ can also be used to describe division algebras. Contrary to Rosset's construction, it will no longer be possible to get crossed products that are division algebras. The crossed products arising from objects in $(\underline{G/\!/H})$ usually only are simple algebras.

Let $\binom{U}{A|E}$ be an object in $(G/\!\!/H)$, corresponding to $\alpha \in \ker(H^2(G,A) \to H^2(H,A))$. Again A is a free abelian normal subgroup of E and A is assumed to be a faithful $\mathbb{Z}G$-lattice.

Let $(t_i)_{1 \leq i \leq m}$ be a $\mathbb{Z}$-basis for A, then we define the group algebra $k[A]$ as $k[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}]$, with function field $M = k(A)$. Define $L = k(A)^H$, $K = k(A)^G$. The exponential map $\exp : A \hookrightarrow M^*$ sends α to $f = \exp \circ \alpha$, an element of $\ker(H^2(G, M^*) \to H^2(H, M^*))$.
As $G = Gal(M/K)$ we can form a crossed product $C_\alpha = \sum_{\sigma \in G} M v_\sigma$, unambiguously defined by α. As M is a Galois closure of L/K, we can form a division algebra Δ over K, containing L as a maximal subfield, so that C_α is a full matrix ring over Δ. The order of this matrix ring over Δ is given by Moody's Theorem.
It also follows that C_α is the quotient field of kE. We collect these results in the following

(2.6) Proposition

1. *kE is a prime p.i. ring.*
2. *$\exp$ as defined above induces $\exp_* : H^2(G,A) \to H^2(G,M^*)$, a part of $Br(K)$, such that the class of $k(G)$ in $Br(K)$ is determined by $\exp_*(\alpha)$.*
3. *The index of $[k(E)]$ in $Br(K)$ equals the order of α in $H^2(G,A)$.*

We will only give a sketch of the **proof**, as this resembles Rosset's argument in [8]. As the set $\{v_\sigma | \sigma \in G\}$ generates a subring of C_α which is isomorphic with kE, one easily deduces the first and the second asssertion.
To prove 3. we only have to show that $\exp_*$ is order-preserving. Consider the diagram

$$\begin{array}{ccc} & A.k^* & \\ {}^{\exp_1}\nearrow & & \searrow^{i} \\ A & \xrightarrow{\exp} & M^* \end{array}$$

$\exp_1$ is injective, so we merely have to show that

$$i_* : H^2(G, A.k^*) \to H^2(G, M^*)$$

is injective. This can be done by the classical Lenstra argument, asserting that

$$H^1(G, M^*/(A.k^*)) = 0,$$

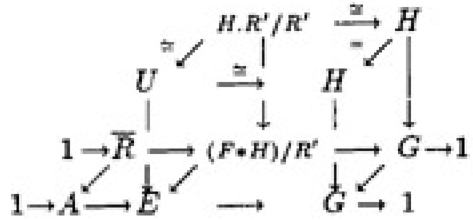
thus finishing the proof.

Consider the free object

$$\alpha : \qquad 1 \to \overline{R} \to (F * H)/[R,R] \to G \to 1$$

We know α to be a generator for $H^2(G, \overline{R}) = \mathbb{Z}/e\mathbb{Z}$ where $e = \#G/H$. We thus get C_α, a crossed product of order e and degreee d, which is a full matrix ring over a division ring of degree e. Counting degrees, we get $C_\alpha = \mathrm{M}_{\#H}(\Delta)$.

This situation can be considered generic in the following sense. For an arbitrary object in $\underline{(G/\!/H)}$ (with the usual hypotheses upon the kernel) we can form a free covering, i.e. a commutative diagram



Denote $(F*H)/R'$ by $\overline{F}$. The homomorphism $\phi : \overline{F} \to E$ extends to a homomorphism $\phi : k[\overline{F}] \to k[E]$, which in turn gives rise to $C_\alpha \to C_\beta$ where α (β) denotes the first (second) object.
We can extend the foregoing generic construction by starting from module extensions having middle term an invertible module, i.e. a direct summand of a permutation module. Consider an extension

$$0 \to R \to I \to I_{G/H} \to 0$$

with I an invertible module. This gives rise to a group extension in $\underline{(G/\!/H)}$

$$1 \to R \to E_I \to G \to 1$$

If we now take an object

$$1 \to Q \to E \to G \to 1$$

with Q coflasque(*) (e.g. $Q = M^*$ when $G = Gal(M/K)$) then we can form a map $kE_I \to kE$. This can be shown as follows.

(2.7) Lemma

For objects $1 \to R \to E_I \to G \to 1$ and $1 \to Q \to E \to G \to 1$ with Q coflasque, there exists a map $kE_I \to kE$.

Proof. By the equivalence of categories the second sequence is transformed into a sequence $0 \to Q \to M_E \to I_{G/H} \to 0$. As Q is coflasque we have that $0 \to Q^K \to (M_E)^K \to (I_{G/H})^K \to 0$ is exact for alle subgroups K of G. Arnold shows in [1] that in this case the map $I \to I_{G/H}$ lifts to M_E, giving the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Q & \to & M_E & \to & I_{G/H} & \to & 0 \\ & & \uparrow & & \uparrow & \nearrow & \| & & \\ 0 & \to & R & \to & I & \to & I_{G/H} & \to & 0 \end{array}$$

(*) A $\mathbb{Z}G$-module Q is called *coflasque* if $TH^1(K,Q) = 0$ for every subgroup K of G. A $\mathbb{Z}G$-module is called *flasque* if $TH^1(K,F) = 0$ for every subgroup K of G. TH^n denotes the n-th Tate cohomology group.

Again by the equivalence of categories the middle map is transformed into a map between the corresponding modules. So we get the desired map.

This means that objects in $\underline{(G/\!/H)}$, coming from objects in $\overline{(I_{G/H})}$ with invertible middle term can also be considered generic. Let us formalize this in the following definition.

(2.8) Definition

An algebra, coming from a free object in $(G/\!/H)$ is called a *generic Brauer algebra*. An algebra, constructed from an object in $\overline{(G/\!/H)}$ with invertible middle term, is called an *almost generic Brauer algebra*. When an almost generic Brauer algebra is given by an object which also has a coflasque left term, we call it a *minimal Brauer algebra*.

The minimality can be seen as follows. Consider a coflasque resolution(**) for $I_{G/H}$. We now have a diagram

$$\begin{array}{ccccccccc} 0 & \to & Q & \to & P & \to & I_{G/H} & \to & 0 \\ & & \uparrow & & \uparrow & \nearrow & \| & & \\ 0 & \to & R & \to & I & \to & I_{G/H} & \to & 0 \end{array}$$

This leads to mappings

$$kE_I \to kE_P \to kE.$$

We are now interested in the relation between all these generic Brauer algebras. Let us first relate the centers of the "truly" generic Brauer algebras.

(2.9) Definition

The kernel $\overline{R} = R/[R,R]$ of a free object $B(Y,v)$ is called *relative relation module*.

Relative relation modules are closely related to each other.

(2.10) Proposition

Let $B(Y_i, v_i)$ $(i = 1,2)$ be free objects in $\underline{(G/\!/H)}$. Then

$$\overline{R_1} \oplus \mathbb{Z}G^{\#Y_2} \simeq \overline{R_2} \oplus \mathbb{Z}G^{\#Y_1}.$$

Proof. Under the equivalence $B(Y_i, v_i)$ goes to

$$0 \to R_i \to \mathbb{Z}G^{|Y_i|} \to I_{G/H} \to 0.$$

(**) We call an exact sequence of modules $0 \to Q \to P \to M \to 0$ a coflasque resolution for the module M if P is a permutation module and Q is a coflasque module. Remark that if $0 \to Q_i \to P_i \to I_{G/H} \to 0$ for $i = 1,2$ are two coflasque resolutions then one has that $Q_1 \oplus P_2 \simeq Q_2 \oplus P_1$ and thus we may say that a coflasque resolution is esentially unique. An exact sequence $0 \to M \to P \to F \to 0$ with P permutation and F flasque is called a flasque resolution for M. There is an analoguous statement about essential uniqueness of flasque resolutions.

This implies that R_i is a $\mathbb{Z}G$-lattice. We now can apply Schanuel's Lemma, from which the result follows.

Remark

In the classical case (i.e. when $H = 1$) generic Brauer algebras are also minimal. By proposition 2.10 it suffices to prove coflasqueness of the left term for only one object. Now, I_G is generated by the elements $g - 1$ for all $g \in G - 1$. Consider a free $\mathbb{Z}G$-lattice $F = \oplus_{g \in G-1} \mathbb{Z}G\alpha_g$ where α_g is sent to $g - 1$. An easy computation shows that $F^H \to (I_G)^H$ is surjective for all subgroups H of G. This is equivalent to the coflasqueness of the kernel.

Before continuing our investigation we need to recall some definitions and terminology.

(2.11) Definition

A field extension l/k is called *rational* if there exists an isomorphism of k-algebras $l \simeq k(x_1, \ldots, x_n)$.
Two field extensions l_1/k and l_2/k are called *stably equivalent over* k if there exists a k-algebra isomorphisms $l_1(x_1, \ldots, x_n) \simeq l_2(y_1, \ldots, y_m)$.
A field extension l/k is said to be *retract rational* if there exists an affine k-algebra R with field of fractions l, together with nonzero elements $f \in k[x_1, \ldots, x_n]$ and $r \in R$ such that we can form a commutative diagram

$$\begin{array}{ccccc} & & k[x_1, \ldots, x_n][\frac{1}{f}] & & \\ & \nearrow & & \searrow & \\ R & & \hookrightarrow & & R[\frac{1}{r}] \end{array}$$

The definition of retract rationality does not depend upon the particular choice of the affine algebra R.
If two $\mathbb{Z}G$-lattices M_1 and M_2 have got essentially the same flasque resolution (we denote this fact by $\phi(M_1) = \phi(M_2)$), then $(kM_1)^G$ and $(kM_2)^G$ are stably equivalent (see e.g. [6]). We thus immediately deduce from proposition 2.10 :

(2.12) Proposition

The centers of generic Brauer algebras are stably equivalent to one another.

Now, let us study the more general construction. We also have an important relationship between the kernels.

(2.13) Proposition

If we have two exact sequences

$$0 \to R \to P \to I_{G/H} \to 0$$

and

$$0 \to R_I \to I \to I_{G/H} \to 0$$

with I invertible and P permutation (in particular, we may choose P to be a free $\mathbb{Z}G$-module), then $(kR_I)^G$ and kR^G are retract equivalent.

Proof. There exists an invertible module J such that $I \oplus J = P_1$ a permutation module. The second sequence then transforms into

$$0 \to R_I \oplus J \to P_1 \to I_{G/H} \to 0.$$

As $I_{G/H}$ is invertible, it is both flasque and coflasque. So, the modules $R_I \oplus J$ and R essentially have the same flasque resolution. Saltman [10] proved this to be equivalent with the stated retract equivalence.

Remarks

1. Even in the classical case of Rosset's (i.e. when $H = 1$), definition 2.8 provides us with an extended class of crossed products with a generic property.
2. From work of Saltman's it follows that the centra do not have to be (stably) rational over the basefield, even in the special case of crossed products.

Similarly one may study rationality of the Brauer-Severi variety over the basefield. Recall that Noether's problem asks for which groups G the field $k(x_1, \ldots, x_m)^G$ is (stably) rational over k, where G acts faithfully on $x_1, \ldots, x_m$.

(2.14) Proposition

1. *The function field of almost generic Brauer algebras, coming from an object in $(\underline{I_{G/H}})$ with middle term permutation, is stably rational if and only if the Noether problem is satisfied for G.*
2. *If the condition in 1. is satisfied, all Brauer-Severi varieties are retract rational over the basefield.*

Proof. Saltman proves in [9] that the function field $K(A)$ of the Brauer-Severi variety of the Brauer class corresponding to the extension (4) is the field of invariants of the twisted tori field $M_t(I_{G/H})^G$. Now, (4) corresponds to (3) under the equivalence and the middle term E of (3) is permutation in this case. To prove stable rationality, we may replace the field of invariants by the field occuring in Noether's problem. This yields the first assertion. The second statement then easily follows from the definition of retract rationality.

3. A monstrous simple algebra

Let us illustrate the foregoing by considering the main motivating example : the generic division algebra $UD(n)$. Let $\mathbb{G}_n$ be the ring of 2 generic n by n matrices over a field k, i.e. the subring of

$$M_n(k[x_{ij}, y_{ij} : 1 \leq i,j \leq n])$$

generated by the two generic n by n matrices

$$X = (x_{ij})_{1 \leq i,j \leq n} \qquad \text{and} \qquad Y = (y_{ij})_{1 \leq i,j \leq n}$$

This ring is a domain and its classical ring of fractions is $UD(n)$. It is well known that properties of $UD(n)$ (e.g. being a crossed product, being cyclic, being similar to a tensor product of cyclic algebras in the Brauer group etc.) are inherited by all division algebras over a field containing k as a subfield. All major results concerning the centers were obtained using the representation theoretical description which we will briefly recall.

Let $\mathbb{Z}S_n/S_{n-1} = \mathbb{Z}x_1 \oplus \ldots \oplus \mathbb{Z}x_n$ be the standard rank n permutation representation of the symmetric group S_n, with $\sigma.x_i = x_{\sigma(i)}$. Similarly $\mathbb{Z}S_n/S_{n-2} = \mathbb{Z}y_{12} \oplus \ldots \oplus \mathbb{Z}y_{(n-1)n}$ is the permutation representation of rank $n(n-1)$ given by $\sigma.y_{ij} = y_{\sigma(i)\sigma(j)}$ for all $\sigma \in S_n$. There is a canonical sequence

$$0 \to G_n \xrightarrow{\gamma} \mathbb{Z}S_n/S_{n-2} \xrightarrow{\beta} \mathbb{Z}S_n/S_{n-1} \xrightarrow{\alpha} \mathbb{Z} \to 0 \qquad (6)$$

where α is the augmentation map defined by $\alpha(x_i) = 1$ for all i. β is determined by $\beta(y_{ij}) = x_i - x_j$ for all i and j. G_n is the kernel of γ and is of $\mathbb{Z}$-rank $n^2 - n + 1$. Then, Procesi and Formanek proved that the center of $UD(n)$ can be written as a field of lattice invariants

$$Z(UD(n)) = k(G_n \oplus \mathbb{Z}S_n/S_{n-1})^{S_n}$$

It would be useful to have a similar representation theoretic description of $UD(n)$. From (6) we deduce

$$0 \to G_n \oplus \mathbb{Z}S_n/S_{n-1} \xrightarrow{\gamma \oplus id} \mathbb{Z}S_n/S_{n-2} \oplus \mathbb{Z}S_n/S_{n-1} \xrightarrow{\beta} I_{S_n/S_{n-1}} \to 0$$

This is an object in $(I_{G/H})$ with middle term a permutation lattice. We can form the corresponding object in $(G /\!\!/ H)$:

$$\begin{array}{ccccccccc} & & & & U & \overset{\simeq}{\to} & S_{n-1} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & G_n \oplus \mathbb{Z}S_n/S_{n-1} & \to & E & \to & S_n & \to & 0 \end{array}$$

and we know that the group algebra kE is a prime Azumaya algebra having classical ring of fractions $M_{(n-1)!}(\Delta)$ where Δ is a division algebra of degree n having center

$$k(G_n \oplus \mathbb{Z}S_n/S_{n-1})^{S_n}.$$

It is easy to verify that :

(3.1) Proposition

$kE \simeq M_{(n-1)!}(UD(n))$

Giving the desired representation theoretic description of $UD(n)$. In fact, more is true

(3.2) Proposition

If G is a finite group of order n, then there exists a Rosset-Snider generic crossed product with group G, $\mathcal{C}(G)$ say, embedded in kE. More precisely

$$\mathcal{C}(G) = \text{centralizer}_{kE}\left(k(G_n \oplus \mathbb{Z}S_n/S_{n-1})^G\right)$$

where G is represented as a subgroup of S_n via its natural permutation representation on $\mathbb{Z}x_1 \oplus \ldots \oplus \mathbb{Z}x_n$.

Proof. Let $G = \{id = x_1, \ldots, x_n\}$. G acts by left translation on this set. Then $G \hookrightarrow S_n$ such that $G \cap S_{n-1} = id$. Now, we can restrict the S_n-sequence

$$0 \to G_n \oplus \mathbb{Z}S_n/S_{n-1} \to \mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z}S_n/S_{n-2} \to I_{S_n/S_{n-1}} \to 0$$

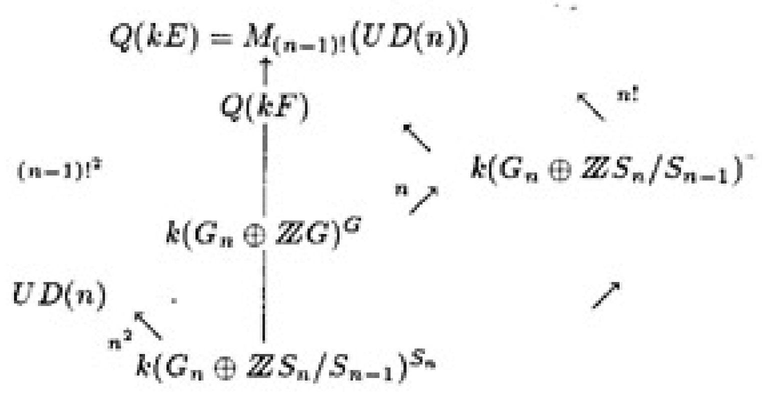
to G. Note that

1. $(\mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z}S_n/S_{n-2}) \downarrow_G$ is a free $\mathbb{Z}G$-lattice;
2. $(I_{S_n/S_{n-1}}) \downarrow_G \simeq I_G$;
3. $(G_n \oplus \mathbb{Z}S_n/S_{n-1}) \downarrow_G \simeq G_n \oplus \mathbb{Z}G$.

So this sequence gives rise to a group extension

$$1 \to G_n \oplus \mathbb{Z}G \to F \to G \to 1$$

and the group ring $kF = \mathcal{C}_\alpha$ is a Rosset-Snider generic crossed product with group G having center $k(G_n \oplus \mathbb{Z}G)^G$. But $kF \hookrightarrow kE$ and can be interpreted as the centralizer of $k(G_n \oplus \mathbb{Z}G)^G$ in kE. I.e. we have the following situation :



Now, consider the problem of when $UD(n)$ is a crossed product with group G. The generic property would then imply the following k-algebra maps

$$kF \to UD(n) \hookrightarrow M_{(n-1)!}(UD(n)) = Q(kE)$$

and we always have the inclusion maps

$$kF \hookrightarrow kE \hookrightarrow M_{(n-1)!}(UD(n))$$

It would be interesting to compute the behaviour of various invariants of the group algebras, such as cyclic cohomology groups, in order to get restrictions on the groups G for which $UD(n)$ can be a crossed product. We will come back to this problem in part II of this paper.

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