

**Stable Rationality of certain  
 $PGL_n$  Quotients**

Christine Bessenrodt<sup>1</sup>

Lieven Le Bruyn<sup>2</sup>

July 1989      -      89-29

**Abstract**

Let  $V$  be a  $PGL_n$ -representation such that the stabilizer of a generic point is trivial. We study the stable rationality problem of the quotient variety  $V/PGL_n$ . In particular, one gets a positive solution when  $n = 2, 3, 4, 5$  and  $7$ . Moreover, fairly precise information is obtained when  $n = p$  a prime number.

**AMS-classification**

14M20, 14L30

**Key Words**

Rationality problems,  $PGL_n$ -quotients

---

<sup>1</sup> Fachbereich Mathematik, Universität Duisburg; partly supported by the DFG

<sup>2</sup> Departement Wiskunde, Universiteit Antwerpen; Research associate of the NFWO

# Stable Rationality of Certain $PGL_n$ -Quotients

Christine Bessenrodt\*  
Fachbereich Mathematik  
Universität Duisburg

Lieven Le Bruyn†  
Departement Wiskunde  
Universiteit Antwerpen

June 15, 1989

## Abstract

Let  $V$  be a  $PGL_n$ -representation such that the stabilizer of a generic point is trivial. We study the stable rationality problem of the quotient variety  $V/PGL_n$ . In particular, one gets a positive solution when  $n = 2, 3, 4, 5$  and  $7$ . Moreover, fairly precise information is obtained when  $n = p$  a prime number.

## 1 Introduction

In this paper we study the following problem : let  $V$  be a good  $PGL_n$ -representation, i.e.  $V$  is a finite dimensional vectorspace with  $PGL_n$ -action such that the stabilizer of a generic point is trivial. Then, there is an affine  $PGL_n$ -invariant open set  $U$  of  $V$  such that generic orbits are closed. We now ask whether the quotient variety  $U/PGL_n$  is stably rational, i.e. is a rational extension of the invariant field  $\mathbb{C}(V)^{PGL_n}$  rational over  $\mathbb{C}$  ?

Of course we can phrase the same problem for any reductive linear group  $G$ . Then, Bogomolov [7] has shown that the answer is independent of the particular choice of a good  $G$ -representation. In case of special groups ( $SL_n, Sp_n$  and products of them) one gets a positive answer using the fact that all principal  $G$ -bundles are locally trivial for the Zariski topology. For other groups (such as  $O_n$ ) one can construct a particular good  $G$ -representation and show (stable) rationality of the quotient variety by some ad hoc argument and use Bogomolov's result for the general case.

However, these methods cannot be applied in the  $PGL_n$ -case as it is well known that principal  $PGL_n$ -bundles cannot be trivial in the Zariski topology.

Geometers usually refer for this to the following argument by Haboush [21]: the conjugation action of  $PGL_n$  on  $M_n(\mathbb{C})$  gives a morphism  $PGL_n \rightarrow GL_n^2$

---

\*Partially supported by the DFG

†Research associate of the NFWO

and  $GL_{n^2}$  embeds naturally in  $V = \mathbb{C}^{n^4}$ . The left multiplication of  $GL_{n^2}$  by  $PGL_n$  extends to a linear action on  $V$ . Now, let  $P$  be the stabilizer of a point in the action of  $PGL_n$  on  $\mathbb{P}^{n^4-1}$  and consider the map  $\alpha : GL_{n^2}/P \rightarrow GL_{n^2}/PGL_n$ , this is generically a universal Brauer-Severi scheme. Since non-trivial Brauer-Severi schemes exist over function fields in any dimension, it follows that  $\alpha$  cannot be uniruled for else it would be generically trivial by Châtelet's result.

Ringtheorists will probably feel more at ease with the next argument : consider the good  $PGL_n$ -representation  $X_n = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$  with action given by componentswise conjugation. Then  $\mathbb{C}[X_n/PGL_n]$  is the center of the trace ring of two  $n \times n$  matrices [37] and taking  $U$  to be the inverse image under the quotient map of any affine open set of Azumaya points determines an element in  $H_{\text{ét}}^1(U/PGL_n, PGL_n)$  which is mapped to a nonzero element in  $H_{\text{ét}}^2(U/PGL_n, G_m) = Br(U/PGL_n) \hookrightarrow Br(\mathbb{C}(U/PGL_n))$  as it corresponds to the class of Amitsur's generic division algebra [1].

For this reason it is perhaps not too surprising that most noteworthy results on stable rationality of  $PGL_n$ -quotients were found by ringtheorists studying the center of the generic division algebras in an attempt to prove what is now known as the Merkurjev-Suslin result (Brauer group is generated by cyclic algebras provided we have enough roots of unity). Let us briefly recall some of these contributions :

In 1972 Procesi [36] proved that quotients of good  $PGL_2$ -representations are stably rational although this result can be traced back at least to an 1883 paper by Sylvester [47]. In 1979 and 1980 Formanek [19],[20] proved a similar result for  $PGL_3$  and  $PGL_4$ -representations. As Bogomolov's no-name lemma was not known at that time, they actually proved more namely that  $X_n/PGL_n$  is rational.

In view of the exponential growth in complexity of the proofs of these results it was commonly believed that a similar approach was not feasible for  $n = 5$ , see e.g. [39],[2]. For this reason, attention shifted to general (but weaker) results. In 1984 Saltman [40] proved that quotients of good  $PGL_n$ -representations are 'retract' rational (i.e. birational to a retract of a rational variety) for  $n = p$  a prime number. his result was refined by Colliot-Thélène and Sansuc in 1987 [10]. Moreover, Saltman [42] proved that the Brauer group of a smooth model of good  $PGL_n$ -quotients is trivial, killing the obvious approach to disprove stable rationality, e.g. [38]. Later, Bogomolov [8] extended this result to good quotients under any connected reductive algebraic group. Further, by 1980 it was common knowledge among ringtheorists (e.g. [20] or [38]) that the natural approach to prove rationality of  $\mathbb{C}(X_n)^{PGL_n}$  (i.e. extending the transcendence basis of  $\mathbb{C}(M_n(\mathbb{C}))^{PGL_n}$  given by the coefficients of the characteristic polynomial to one of  $\mathbb{C}(X_n)^{PGL_n}$ ) fails for  $n$  not squarefree. This argument due to Snider was unfortunately never published as it invalidates the 'proof' of Maruyama [31] for general  $n$  modulo the translations from moduli spaces of vectorbundles

to  $PGL_n$ -representations as implicit in the work of Hulek [22], see also [27] or [49]. In 1986, Saltman [44] provided a write up of a nice extension of Snider's argument.

Let us run through the contents : In the next section we develop the general strategy of attack. There is a general procedure [45] to pass from invariant fields of good representations under reductive groups to fields of (twisted) lattice invariants under the corresponding Weyl group. In our case, the Weyl group is  $S_n$  and the lattice in question  $G_n$  is called the generic lattice (it is a sort of permutation-syzygy of the root lattice  $A_{n-1}$ ). Then  $\mathcal{C}(V)^{PGL_n}$  is stable equivalent to the field of lattice invariants  $\mathcal{C}(G_n)^{S_n}$  reducing our problem to that of studying stable rationality of lattice invariants under finite groups. This theory started off in the fifties by work of Masuda [32] and was led to its present elegant form by work of a.o. Kuyk [26] Endo and Miyata [16], Voskresenskii [50], Lenstra [29] and Colliot-Thélène and Sansuc [9]. It reduces our problem to that of finding  $S_n$ -lattices of low rank having the same  $\phi$ -invariant as  $G_n$ . However, computing  $\phi$ -invariants is not an easy task mainly because representation theory of  $\mathbb{Z}S_n$  is only partially developed. For this reason we had to find a large detour via modular representation theory to obtain the local data, glue this information together using the theory of Burnside rings to obtain genus data which allows us finally to compute  $\phi$ -invariants using results on the classgroups of integral grouprings. Finally, rationality of the lattice invariants of these low rank lattices is then proved by an ad hoc method (at least for small values of  $n$ ).

In section three we begin our investigation of the  $S_n$ -lattice  $G_n$ . From the observation that  $G_n = A_{n-1} \otimes A_{n-1}$  it follows immediately that  $G_n$  is an invertible lattice (i.e. a direct factor of a permutation lattice) for  $n = p$  a prime number. Direct consequences are Saltman's retract rationality result as well as Colliot-Thélène's and Sansuc's result. On the other hand, for composite values of  $n$  the lattice structure of  $G_n$  is not so nice. We show that is not even coflasque and if  $n$  is not squarefree then even its  $\phi$ -invariant is not coflasque (a slight extension of the Snider-Saltman argument). Further, we reduce the problem to a Noether rationality problem. To be precise,  $\mathcal{C}(V)^{PGL_p}$  is stable equivalent to  $\mathcal{C}(G)^G$  where  $G$  is a semidirect product of  $S_p$  with  $A = \text{Hom}_{\mathbb{Z}}(\mathbb{F}_p A_{p-1}, \mu)$  where  $\mu$  is the group of roots of unity.

In section four we prove stable rationality of quotients of good  $PGL_n$ -representations (for small  $n$ ) using the above sketched method. The local  $\phi$ -invariants are determined for all prime values of  $n$  from which we deduce in particular that for  $p > 3$  the  $\phi$ -invariant of  $G_p$  cannot be trivial entailing that one cannot extend the coefficients of the characteristic polynomial of the first generic matrix to a rationality basis of  $\mathcal{C}(X_n/PGL_n)$ .

In the final section we give some applications to seemingly unrelated areas. Stable rationality of  $PGL_n$ -quotients implies stable rationality of  $M(n; 0, n)$  the moduli space of stable vectorbundles of rank  $n$  over  $\mathbb{P}^2$  having Chern-classes 0 and  $n$  as well as stable rationality of the generic Jacobian variety of plane curves of degree  $n$ , [30],[27],[49]. Moreover, if  $\alpha$  is a Schur root for a quiver

$Q$  (e.g. [24],[25]), then the variety parametrizing isoclasses of  $\alpha$ -dimensional representations is stable equivalent to  $U/PGL_n$  where  $n$  is the greatest common divisor of the component-dimensions of  $\alpha$ , [28]. In particular, our results entail stable rationality for many  $m$ -subspace problems cfr. [33].

Perhaps we should include a fairly pessimistic comment. As there does not seem to emerge a common theme from the cases where we can prove stable rationality, it may very well be that the answer is negative for large  $n$ . There are some reasons to believe that things might already go wrong for  $n = 8$ . It would be interesting to compute finer birational invariants than the Brauer group for a smooth model of  $\mathcal{C}(V)^{PGL_n}$ . The new invariants of Colliot-Thélène and Ojanguren [11] may very well be the right ones to disprove stable rationality of quotients of good  $PGL_8$ -representations.

**Acknowledgement :** We like to thank Michel Van den Bergh (UIA) for many discussions and some simplifications of the original arguments. The calculations for the  $PGL_7$ -case were done with CAYLEY (version 3.5) on the IBM 4381 at the University of Essen (BRD).

## 2 The strategy

In this section we explain the overall strategy to prove (or disprove) stable rationality of quotients of good  $PGL_n$ -representations for a particular  $n$ .

First, Saltman [45] has indicated how to problem of checking stable rationality of invariant fields of good representations of a reductive group  $G$  can be reduced to that of certain fields of (twisted) lattice invariants under the corresponding Weyl group. So, let  $T$  be a maximal torus in  $G$  with normalizer  $N_G(T)$  then the Weyl group of  $G$  is the finite quotient  $W(G) = N_G(T)/T$ . Consider the character group  $X(T) = Hom_{alg}(T, \mathbb{C}^*)$  as a lattice over the integral groupring  $\mathbb{Z}W(G)$  and take a permutation-syzygy

$$0 \rightarrow M \rightarrow P \rightarrow X(T) \rightarrow 0 \quad (1)$$

where  $P$  is a permutation  $W(G)$ -lattice i.e. a torsion free  $\mathbb{Z}W(G)$ -module with a finite basis which is permuted under the action of  $W(G)$ . Using this notation, Saltman [45, Cor.2.7] shows that the invariant field  $\mathbb{C}(V)^G$  of a good  $G$ -representation  $V$  is stable equivalent to the field of twisted lattice invariants  $\mathbb{C}_\alpha(M)^{W(G)}$  where  $\mathbb{C}(M)$  is the quotientfield of the groupalgebra of the Abelian group having the induced  $W(G)$ -action by automorphisms and  $\alpha$  is some extension of  $M$  by  $\mathbb{C}^*$ . In many cases (as the one we are interested in) the twisting by  $\alpha$  can be dispensed with.

Let us specialize to the  $PGL_n$ -case : the Weyl-group is clearly  $S_n$ , the symmetric group on  $n$  letters and the character lattice of a maximal torus is the classical root lattice  $A_{n-1}$  consisting of all integral vectors  $(x_1, \dots, x_n) \in \mathbb{Z}^n$

such that  $\sum_{i=1}^n x_i = 0$ . A permutation  $S_n$ -syzygy of  $A_{n-1}$  is given by :

$$0 \rightarrow G_n \rightarrow V_n \rightarrow A_{n-1} \rightarrow 0 \quad (2)$$

where  $V_n$  is the  $\mathbb{Z}S_n$ -lattice on the off-diagonal entries of an  $n \times n$  matrix  $V_n = \mathbb{Z}y_{12} \oplus \dots \oplus \mathbb{Z}y_{n-1n}$  which is a permutation lattice under the action  $\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$ . Note that  $V_n \simeq \mathbb{Z}S_n/S_{n-2}$ . The map  $V_n \rightarrow A_{n-1}$  is given by sending  $y_{ij}$  to  $e_i - e_j$  where the  $e_i$  are the standard basis of  $\mathbb{Z}^{\oplus n}$ . Consider the group algebra  $\mathbb{C}[G_n] = \mathbb{C}[x_1, x_1^{-1}, \dots, x_k, x_k^{-1}]$  (where  $k = n^2 - 2n + 1$ ) on which  $S_n$  acts as a group of automorphisms, whence also on the quotient field  $\mathbb{C}(G_n)$ . By Saltmans result we are now reduced to prove stable rationality of the field of lattice invariants  $\mathbb{C}(G_n)^{S_n}$ . Readers familiar with the theory of generic division algebras will remember the Procesi-Formanek description of the center [36],[19] as the field of lattice invariants of the lattice  $G_n \oplus U_n^{\oplus 2}$  where  $U_n$  is the standard rank  $n$  permutation representation  $\mathbb{Z}S_n/S_{n-1}$ .

The theory of lattice and tori-invariants, arising naturally from efforts to solve Noethers rationality problem (for which finite groups  $G$  is the invariant field  $\mathbb{C}(G)^G$  (stably) rational ?), was brought to its present elegant form by contributions of a.o. Masuda [32], Kuyk [26], Endo and Miyata [16], Voskresenskii [50] and Colliot-Thélène and Sansuc [9]. Although these results are valid for any finite group we will specialize here for convenience to  $S_n$  and basefield  $\mathbb{C}$  keeping in mind that the same results hold for any basefield.

Recall that a  $\mathbb{Z}S_n$ -lattice  $F$  is said to be flasque (resp.  $Q$  to be coflasque) iff  $\tilde{H}^{-1}(H, F) = 0$  (resp.  $\tilde{H}^1(H, Q) = 0$ ) for every subgroup  $H$  of  $S_n$  (here,  $\tilde{H}^i$  is Tate cohomology). Every lattice  $M$  has a flasque resolution

$$0 \rightarrow M \rightarrow P \rightarrow F \rightarrow 0 \quad (3)$$

with  $P$  permutation and  $F$  flasque. Moreover, end terms of flasque resolution are unique upto stable permutation. That is, let  $F_1$  and  $F_2$  be end terms of flasque resolutions of  $M$  then there exist permutation lattices  $P_1$  and  $P_2$  such that  $F_1 \oplus P_1 \simeq F_2 \oplus P_2$ . Hence, introducing the Abelian semigroup  $Flas(S_n)$  of stable permutation classes of flasque  $\mathbb{Z}S_n$ -lattices gives a well defined map

$$\phi : Latt(S_n) \rightarrow Flas(S_n) \quad (4)$$

assigning to a lattice the class of an end term of a flasque resolution.

Let  $L$  be a  $\mathbb{C}$ -field with faithful  $S_n$ -action, then we can define as in the case of lattice invariants for each  $\mathbb{Z}S_n$ -lattice  $M$  the field of tori-invariants  $L(M)^G$ . The importance of  $\phi$ -invariants is evident from the following crucial result :

**Theorem 1 (Colliot-Thélène, Sansuc 1977)** *For  $\mathbb{Z}S_n$ -lattices  $M$  and  $N$  we have the following*

- $L(M)^{S_n}$  is stable equivalent to  $L(N)^{S_n}$  over  $L^{S_n}$  if and only if  $\phi(M) = \phi(N)$  in  $Flas(S_n)$

- If  $M$  and  $N$  are faithful  $S_n$ -lattices with  $\phi(M) = \phi(N)$  in  $Flas(S_n)$ , then  $\mathbb{C}(M)^{S_n}$  is stable equivalent to  $\mathbb{C}(N)^{S_n}$  over  $\mathbb{C}$

In view of this result, we are reduced to finding a  $\mathbb{Z}S_n$ -lattice  $M$  of small rank (compared to  $rk(G_n) = n^2 - 2n + 1$ ) such that  $\phi(M) = \phi(G_n)$  and prove stable rationality of the field of lattice invariants  $\mathbb{C}(M)^{S_n}$ . Though it is fairly easy to write down a flasque resolution for a given lattice [9] it is more difficult to determine whether two flasques determine the same class in  $Flas(S_n)$  mainly because integral representation theory of  $S_n$  is a bit messy at this time.

To bypass this problem we will focuss on the three major obstructions that can arise, each of which is tractable because the necessary machinery for the subproblem (modular representation theory, Burnside rings, classgroups) is well developed.

So, assume we have two  $\mathbb{Z}S_n$ -lattices  $M$  and  $N$  having as end terms of a flasque resolution the flasque lattices  $F_M$  resp.  $F_N$ . How can we determine whether or not  $\phi(M) = \phi(N)$  i.e. whether  $[F_M] = [F_N]$  in  $Flas(S_n)$ ?

A first test is to see whether they lie in the same class locally. For any domain  $R$  we can define an  $RS_n$  lattice  $F$  to be flasque iff  $Ext_{RS_n}^1(F, RS_n/H) = 0$  for all subgroups  $H$  of  $S_n$ . With  $Flas(RS_n)$  we can then denote the Abelian semigroup of stable permutation classes of flasque  $RS_n$ -lattices. So, a first test whether  $[F_M] = [F_N]$  in  $Flas(S_n)$  is to verify whether they have the same image under the localization map

$$loc : Flas(S_n) \rightarrow \prod_{p \leq n} Flas(\mathbb{Z}_p S_n) \quad (5)$$

a problem which can be settled by modular representation theory as for all  $n$  and all primes  $p$  we have the canonical isomorphism

$$Flas(\mathbb{Z}_p S_n) \cong Flas(\widehat{\mathbb{Z}_p S_n}) \quad (6)$$

which follows immediately from descent and  $Flas(\mathbb{Q} S_n) = Flas(\widehat{\mathbb{Q}_p S_n}) = 0$  (permutation characters generate all).

Assume we survived the local obstruction, i.e.  $[F_M \otimes \widehat{\mathbb{Z}_p}] = [F_N \otimes \widehat{\mathbb{Z}_p}]$  in  $Flas(\widehat{\mathbb{Z}_p S_n})$  for all  $p$ , i.e. we can find  $S_n$ -sets  $T_p$  and  $T'_p$  such that

$$(F_M \otimes \widehat{\mathbb{Z}_p}) \oplus \widehat{\mathbb{Z}_p} T_p \simeq (F_N \otimes \widehat{\mathbb{Z}_p}) \oplus \widehat{\mathbb{Z}_p} T'_p \quad (7)$$

We now ask whether these  $S_n$ -sets can be taken independent of the prime  $p$ , i.e. do there exist  $S_n$ -sets  $T$  and  $T'$  such that  $F_M \oplus \mathbb{Z}T$  lies in the same genus as  $F_N \oplus \mathbb{Z}T'$ ?

The method to solve this problem is a slight variation on an idea of Dress [15] and is based on the description of  $b(S_n)$ , the Burnside ring of  $S_n$ . This is the Grothendieck ring constructed from the isomorphism classes of finite  $S_n$ -sets with addition induced by disjoint union and multiplication by Cartesian

product with diagonal action, see a.o. [12], [13]. By means of Burnside marks,  $b(S_n)$  can be identified as the subring of  $\prod_{(H) \in C(S_n)} \mathbb{Z}$  (the product being taken over  $C(S_n)$  the set of conjugacy classes of subgroups  $H$  of  $S_n$ ) obtained by rightmultiplication with the Burside matrix  $(a_{H,H'})_{H,H'} \in M_c(\mathbb{Z})$  where  $c$  is the number of conjugacy classes of subgroups,  $H$  and  $H'$  are representants of classes and  $a_{H,H'} = \#(S_n/H)^{H'}$  the number of  $H'$ -fixed elements of the transitive  $S_n$ -set  $S_n/H$ .

Starting from the local data (7) we can construct a partial function

$$\chi : \mathcal{H} = \cup_{p \leq n} Hyp_p(S_n) \rightarrow \mathbb{Z} \quad (8)$$

Here,  $Hyp_p(S_n)$  is the set of conjugacy classes of  $p$ -hypolementary subgroups of  $S_n$  (i.e. those subgroups  $H$  s.t.  $H/O_p(H)$  is cyclic where  $O_p(H)$  is the largest normal  $p$ -subgroup of  $H$ ). If  $H$  is a representant from  $Hyp_p(S_n)$ , we define  $\chi(H) = \#T_p^H - \#T'_p{}^H$ . This map is well defined as  $H$  is  $p$  and  $q$ -hypo iff  $H$  is cyclic. But then,  $\chi(H)$  is just the difference of the character values of a generator on  $F_M \otimes \mathbb{Q}$  and  $F_N \otimes \mathbb{Q}$ . The relevance of  $\chi$  is given by the next result

**Lemma 1** *Starting from a setting as in (7) we can find  $S_n$  sets  $T$  and  $T'$  s.t.  $F_M \oplus \mathbb{Z}T$  lies in the same genus as  $F_N \oplus \mathbb{Z}T'$  iff the partial function  $\chi$  can be extended to an element in the Burnside ring  $b(S_n)$ .*

**Proof :** Assume  $\chi$  extends to an element in  $b(S_n)$ , then there are  $S_n$ -sets  $T$  and  $T'$  s.t. for all  $H \in Hyp_p(S_n)$  we have

$$\chi(H) = \#T_p^H - \#T'_p{}^H = \#T^H - \#T'^H \quad (9)$$

which implies by a result of Dress [14] that

$$\widehat{\mathbb{Z}}_p T_p \oplus \widehat{\mathbb{Z}}_p T'_p \simeq \widehat{\mathbb{Z}}_p T'_p \oplus \mathbb{Z}T \quad (10)$$

Adding  $\widehat{\mathbb{Z}}_p T \oplus \widehat{\mathbb{Z}}_p T'$  to both sides of (7) we obtain from the above by cancellation that

$$(F_M \otimes \widehat{\mathbb{Z}}_p) \oplus \widehat{\mathbb{Z}}_p T \simeq (F_N \otimes \widehat{\mathbb{Z}}_p) \oplus \widehat{\mathbb{Z}}_p T' \quad (11)$$

which is independent of the prime  $p$ , done. The converse implication is trivial.  $\square$

Assume we also survived the Burnside obstruction, i.e. we have found  $S_n$ -sets  $T$  and  $T'$  such that  $F_M \oplus \mathbb{Z}T$  and  $F_N \oplus \mathbb{Z}T'$  lie in the same genus, does this imply that  $[F_M] = [F_N]$  in  $Flas(S_n)$ ? For arbitrary groups this is far from being true (large cyclic groups already produce counterexamples). However, in the  $S_n$ -case we do have :

**Lemma 2** *If  $F_M \oplus \mathbb{Z}T \setminus F_N \oplus \mathbb{Z}T'$  then  $[F_M] = [F_N]$  in  $Flas(S_n)$*

**Proof :** By Roiter's replacement lemma there exists a projective left ideal  $I$  of  $\mathbb{Z}S_n$  such that

$$F_M \oplus \mathbb{Z}T \oplus \mathbb{Z}S_n \simeq F_N \oplus \mathbb{Z}T' \oplus I \quad (12)$$

By the results of Endo and Miyata on the projective classgroups of  $\mathbb{Z}S_n$  [17,Th.3.3] (or work of Oliver [34],[35]) we know that there exists a finite  $S_n$ -set  $T''$  such that  $I \oplus \mathbb{Z}T'' \cong \mathbb{Z}S_n \oplus \mathbb{Z}T'$ , yielding that

$$F_M \oplus \mathbb{Z}T \oplus \mathbb{Z}T'' \simeq F_N \oplus \mathbb{Z}T' \oplus \mathbb{Z}T'' \quad (13)$$

and hence  $[F_M] = [F_N]$  in  $\text{Flas}(S_n)$ .  $\square$

### 3 The prime case and Noethers problem

In this section we will study the  $S_n$ -lattice structure of  $G_n$  and reduce the  $PGL_n$ -problem to a Noether rationality setting.

Tensoring the defining sequence of  $A_{n-1}$  (i.e.  $0 \rightarrow A_{n-1} \rightarrow U_n \rightarrow \mathbb{Z} \rightarrow 0$  where  $U_n = \mathbb{Z}S_n/S_{n-1}$  is the standard rank  $n$ -permutation representation) with  $A_{n-1}$  we obtain

$$0 \rightarrow A_{n-1} \otimes A_{n-1} \rightarrow A_{n-1} \otimes U_n \rightarrow A_{n-1} \rightarrow 0 \quad (14)$$

We can identify  $V_n$  with  $A_{n-1} \otimes U_n$  by sending  $y_{ij}$  to  $(e_i - e_j) \otimes e_i$  and note that the map to  $A_{n-1}$  coincides with that of the defining sequence for  $G_n$ . Therefore,

$$G_n \simeq A_{n-1} \otimes A_{n-1} \quad (15)$$

This easy observation has some direct consequences. Recall that a  $\mathbb{Z}S_n$ -lattice  $I$  is said to be invertible iff it is a direct factor of a permutation lattice. Note that invertible lattices are both flasque and coflasque and that there classes  $[I]$  in  $\text{Flas}(S_n)$  are precisely the invertible elements.

**Proposition 1** *For all prime numbers  $p$ ,  $G_p$  is an invertible  $\mathbb{Z}S_p$ -lattice*

**Proof :** For all  $q \neq p$  we have  $(A_{p-1} \otimes \widehat{\mathbb{Z}}_q) \oplus \widehat{\mathbb{Z}}_q \simeq \widehat{\mathbb{Z}}_q U_p$  yielding that  $G_p \otimes \widehat{\mathbb{Z}}_q$  is stable permutation. Further,  $A_{p-1} \otimes \widehat{\mathbb{Z}}_p \simeq \Omega(\widehat{\mathbb{Z}}_p)$  (here  $\Omega$  is the Heller operator) yielding that  $G_p \otimes \widehat{\mathbb{Z}}_p \simeq \Omega^2(\widehat{\mathbb{Z}}_p) \oplus \widehat{\mathbb{Z}}_p S_p / (S_2 \times S_{p-2})$ . As  $\Omega^2$  operates on the set of trivial source modules,  $G_p \otimes \widehat{\mathbb{Z}}_p$  is an invertible  $\widehat{\mathbb{Z}}_p S_p$ -lattice. So,  $G_p$  is locally invertible whence invertible.  $\square$

As an immediate consequence we obtain :

**Corollary 1** *For all prime numbers  $p$  we have :*

- (Saltman, 1984)  $\mathbb{C}(G_p)^{S_p}$  is retract rational over  $\mathbb{C}$
- (Colliot-Thélène, Sansuc, 1987)  $\phi(G_p)$  is invertible

The situation for composite  $n$  is totally different :

**Proposition 2** *If  $n$  is composite  $G_n$  is not a coflasque  $\mathbb{Z}S_n$ -lattice.*

**Proof :** Let  $n = m.k$  with  $m, k > 1$  and consider the subgroup  $G = S_m \times S_{(k-1)m}$  of  $S_n$  acting in the natural way on the  $n$  elements. From the defining sequence of  $G_n$  we get the exact sequence

$$V_n^G \xrightarrow{\pi^G} A_{n-1}^G \rightarrow H^1(G, G_n) \rightarrow 0 \quad (16)$$

Now, it is easy to see that

$$\underbrace{(k-1, \dots, k-1)}_m, \underbrace{(-1, \dots, -1)}_{(k-1)m} \in A_{n-1}^G \quad (17)$$

and that the image of  $V_n^G$  under  $\pi^G$  consists of the vectors

$$\mathbb{Z} \underbrace{(m(k-1), \dots, m(k-1))}_m, \underbrace{(-m, \dots, -m)}_{(k-1)m} \quad (18)$$

and therefore  $H^1(G, G_n) \neq 0$ , done.  $\square$

Though we believe that  $\phi(G_n)$  cannot be coflasque for  $n$  composite, this is only known for non-squarefree  $n$  as the following cohomological argument due to Saltman [44] shows :

**Proposition 3 (Snider, Saltman)** *If  $n$  is not squarefree, then there does not exist an exact  $\mathbb{Z}S_n$ -sequence*

$$0 \rightarrow G_n \rightarrow P \rightarrow Q \rightarrow 0 \quad (19)$$

with  $P$  permutation and  $Q$  coflasque.

**Proof :** If  $n = p^2.m$ , then  $S_n$  contains a subgroup  $G$  which is the direct product of a cyclic group of order  $p$  and one of order  $p.m$  such that the action on  $n$  letters is the product action. Restricting any permutation  $\mathbb{Z}S_n$ -lattice  $P$  down to  $G$  we can write it as  $\oplus_i \mathbb{Z}G/H_i$  for some subgroups  $H_i$  of  $G$ . But then by Shapiro's lemma :  $\check{H}^2(G, P) = \oplus_i \text{Hom}(H_i, \mathbb{Q}/\mathbb{Z})$ . Thus from the existence of the required sequence we would have

$$0 \rightarrow \check{H}^2(G, G_n) \rightarrow \oplus_i \text{Hom}(H_i, \mathbb{Q}/\mathbb{Z}) \quad (20)$$

whence any element of  $\check{H}^2(G, G_n)$  must have order dividing  $p.m < n$ . However, using that  $V_n$  and  $U_n$  are free  $\mathbb{Z}G$ -lattices we have that  $\check{H}^2(G, G_n) = \check{H}^1(G, A_{n-1}) = \mathbb{Z}/n\mathbb{Z}$ , a contradiction.  $\square$

Recall that Noethers rationality problem asks for which finite groups  $G$  the fixed field  $\mathbb{C}(G)^G$  is (stably) rational, see [48],[29] or [41] for motivation and counterexamples. In [43] Saltman developed a method to find counterexamples starting from lattice invariants of a lattice which is of finite index in a permutation lattice. As an easy variation on his idea we get :

**Proposition 4** *Let  $M$  be any  $\mathbb{Z}S_n$ -lattice. Then,  $\mathbb{C}(M)^{S_n}$  is stable equivalent to a Noether setting  $\mathbb{C}(G)^G$  where  $G$  is a semidirect product of  $S_n$  with a finite Abelian group.*

**Proof :** Apply [43,cor.3.3] noting that all  $\mathbb{Q}S_n$ -lattices are stable permutation (which suffices for the proof of [43,3.3]).  $\square$

Barge [3] proved that for any finite group  $G$  the Brauer groups of smooth models for all lattice-invariants  $\mathbb{C}(M)^G$  are trivial if and only if  $G$  has all its Sylow subgroups Abelian bicyclic. Thus, for  $n \geq 4$ , the foregoing result gives a vast resource of counterexamples to the Noether problem. It also shows that lattice-invariants of  $\mathbb{Z}S_n$ -lattices are rarely (stable) rational. However, in view of Saltmans result [42] that the Brauer group of a smooth model for the center of the generic division algebra is trivial we cannot use this approach to disprove (stable) rationality of  $\mathbb{C}(G_n)^{S_n}$ . Still, we have a very explicit description of the group  $G$  in case our lattice is  $G_p$  for  $p$  a prime number :

**Theorem 2** *For  $p$  prime, the field of lattice invariants  $\mathbb{C}(G_p)^{S_p}$  is stable equivalent to the Noether setting  $\mathbb{C}(G)^G$  where  $G$  is the semidirect product of  $S_p$  with  $A = \text{Hom}_{\mathbb{Z}}(\mathbb{F}_p A_{p-1}, \mathbb{F}_p)$ .*

**Proof :** For any  $n$  we have the exact  $S_n$ -sequence :

$$0 \rightarrow A_{n-1} \otimes A_{n-1} \rightarrow A_{n-1}^* \otimes A_{n-1} \rightarrow \mathbb{Z}/(n) \otimes A_{n-1} \rightarrow 0 \quad (21)$$

On the other hand, consider the pullback-diagram :

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & \mathbb{Z} & = & \mathbb{Z} & \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & A_{n-1}^* \otimes A_{n-1} & \rightarrow & U_n \times_{A_{n-1}^*} A_{n-1}^* \otimes U_n & \rightarrow & U_n \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & A_{n-1}^* \otimes A_{n-1} & \rightarrow & A_{n-1}^* \otimes U_n & \rightarrow & A_{n-1}^* \rightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Now,  $A_{n-1}^* \otimes U_n \simeq (A_{n-1} \otimes U_n)^* \simeq V_n$  whence  $U_n \times_{A_{n-1}^*} (A_{n-1}^* \otimes U_n) \cong V_n \oplus \mathbb{Z}$ , giving us a sequence

$$0 \rightarrow A_{n-1}^* \otimes A_{n-1} \rightarrow V_n \oplus \mathbb{Z} \rightarrow U_n \rightarrow 0 \quad (22)$$

Now, let us restrict to  $n = p$  a prime number. Then,  $A_{p-1}^* \otimes A_{p-1}$  is invertible by an argument similar to that of proposition 1 whence  $(A_{p-1}^* \otimes A_{p-1}) \oplus U_p \simeq V_p \oplus \mathbb{Z}$ . Then, we can add  $U_p$  to the first two terms in (21) and apply [43, Th. 3.1].  
□

## 4 Stable rationality for small primes

In this section we will prove stable rationality of quotients of good  $PGL_n$ -quotients for small  $n$  using the method explained in section two. Let us concentrate on the case when  $n = p$  a prime number :

From the proof of proposition 1 we obtain that if  $q \neq p$  we have

$$(G_p \otimes \widehat{\mathbb{Z}}_q) \oplus \widehat{\mathbb{Z}}_q U_p \simeq \widehat{\mathbb{Z}}_q V_p \oplus \widehat{\mathbb{Z}}_q \quad (23)$$

and at  $p$  we have locally

$$G_p \otimes \widehat{\mathbb{Z}}_p \simeq \Omega^2(\widehat{\mathbb{Z}}_p) \oplus \widehat{\mathbb{Z}}_p S_p / (S_2 \times S_{p-2}) \quad (24)$$

Our first task is to find a  $\mathbb{Z}S_p$ -lattice  $M$  having the same local  $\phi$ -invariants as  $G_p$  i.e. such that  $[F_M \otimes \widehat{\mathbb{Z}}_q] = 0$  in  $Flas(\widehat{\mathbb{Z}}_q S_p)$  and  $[F_M \otimes \widehat{\mathbb{Z}}_p] = -[\Omega^2(\widehat{\mathbb{Z}}_p)]$  in  $Flas(\widehat{\mathbb{Z}}_p S_p)$ .

Now,  $\Omega^2(\widehat{\mathbb{Z}}_p)$  is an indecomposable  $\widehat{\mathbb{Z}}_p S_p$ -lattice with vertex equal to the cyclic  $p$ -Sylow subgroup  $C_p = \langle x = (1, \dots, p) \rangle$  of  $S_p$ . The normalizer of  $C_p$  is a  $p$ -hypo-elementary subgroup

$$N_p = N_{S_p}(C_p) = \langle x, y : x^p = y^{p-1} = 1, y.x.y^{-1} = x^a \rangle \quad (25)$$

where  $a$  is a generator of the cyclic group  $\mathbb{F}_p^*$ .

Since the  $p$ -Sylow subgroup is cyclic of order  $p$ , non-projective indecomposable  $\widehat{\mathbb{Z}}_p S_p$ -lattices and  $\widehat{\mathbb{Z}}_p N_p$ -lattice behave very well with respect to Green correspondence (see e.g. [18 III, 5]) :

**Lemma 3** *There is a one-to-one correspondence between isomorphism classes of indecomposable non-projective  $\widehat{\mathbb{Z}}_p S_p$ -lattices  $M$  and indecomposable non-projective  $\widehat{\mathbb{Z}}_p N_p$ -lattices  $N$  such that*

$$M \downarrow_{N_p} = N \oplus P$$

$$N \uparrow^{S_p} = M \oplus P'$$

where  $P'$  (resp.  $P$ ) is a projective  $\widehat{\mathbb{Z}}_p S_p$ - (resp.  $\widehat{\mathbb{Z}}_p N_p$ -) lattice.

In particular, the Green correspondent of the  $\widehat{\mathbb{Z}}_p S_p$ -lattice  $\Omega^2(\widehat{\mathbb{Z}}_p)$  is the  $\widehat{\mathbb{Z}}_p N_p$ -lattice  $\Omega^2(\widehat{\mathbb{Z}}_p)$ . Now, let  $X_1(\zeta^j)$  be the  $\widehat{\mathbb{Z}}_p N_p$ -lattice of rank one given by the action  $x \rightarrow 1$  and  $y \rightarrow \zeta^j$  where  $\zeta$  is a primitive  $(p-1)$ -th root of unity reducing to  $a \bmod p$ . Then, it is easy to see (e.g. using [5, p.189]) that we have :

**Lemma 4** *The  $\widehat{\mathbb{Z}}_p N_p$ -lattice  $\Omega^2(\widehat{\mathbb{Z}}_p)$  is of rank one determined by the action  $x \rightarrow 1$  and  $y \rightarrow \zeta^{p-2}$  where  $\zeta$  is a primitive  $p-1$ -th root of unity.*

Hence, if  $\Omega^2(\widehat{\mathbb{Z}}_p)$  is a factor of a permutation  $\widehat{\mathbb{Z}}_p S_p$ -lattice, then by restriction so is  $\Omega^2(\widehat{\mathbb{Z}}_p) = X_1(\zeta^{p-2})$  a direct summand of a permutation  $\widehat{\mathbb{Z}}_p N_p$ -lattice. But then, all lattices of the form  $X_1(\zeta^a) = \Omega^{2c}(\widehat{\mathbb{Z}}_p)$  with  $a$  a primitive  $p-1$ -th root of unity and  $c = p-1-a$  are direct summands too. Inducing this information to the  $S_p$ -level and taking into account that all projective  $\widehat{\mathbb{Z}}_p S_p$ -lattices are stable permutation (follows by induction on the dominance order from [23]) we obtain :

**Proposition 5**  $\sum_{(i,p-1)=1}^{p-2} [\Omega^{2i}(\widehat{\mathbb{Z}}_p)] = 0$  in  $\text{Flas}(\widehat{\mathbb{Z}}_p S_p)$

Immediate consequences are :

**Proposition 6** 1. For  $p = 2, 3$ ,  $G_p$  is locally stable permutation.

2. For  $p > 3$ ,  $\phi(G_p) \neq 0$  in  $\text{Flas}(S_p)$ .

**Proof :** (1) : Follows from the general formula. (2) : For  $p \geq 5$  there are at least two primitive  $p-1$ -th roots of unity so by the argument given above  $X_1(\zeta^{p-2})$  cannot be stable permutation.  $\square$

**Corollary 2** For  $p \geq 5$  the coefficients of the characteristic polynomial of the first generic matrix cannot be extended to a rationality basis for  $\mathbb{C}(X_p)^{PGL_p}$ , giving a prime analogue to the Snider-Saltman result.

**Proposition 7** 1. For  $p = 5, 7$ ,  $G_p$  has the same local classes as  $A_{p-1}^*$ .

2. For  $p > 7$ ,  $\phi(G_p) \neq \phi(A_{p-1}^*)$  in  $\text{Flas}(S_p)$ .

**Proof :** (1) : By the above formula :  $-\Omega^2(\widehat{\mathbb{Z}}_p) = [\Omega^{-2}(\widehat{\mathbb{Z}}_p)]$  yielding that locally  $-[G_p] = [G_p^*]$ . Now, dualizing the defining sequence for  $G_p$  yields that  $\phi(A_{p-1}^*) = [G_p^*]$ , done.

(2) : There are more than two primitive  $p-1$ -th roots of unity and any proper subsum of  $X_1(\zeta^a)$  cannot be stable permutation.  $\square$

So, for  $p > 7$  a lattice having the same  $\phi$ -invariant as  $G_p$  needs to have rank at least  $p$ . Anyway, for  $p = 2$  or  $3$  we can now guess that  $\phi(G_p) = 0$  (i.e.  $G_p$  is stable permutation) and for  $p = 5$  or  $7$  the guess is that  $\phi(G_p) = \phi(A_{p-1}^*)$  (which

would follow from the above if  $G_p \oplus G_p^*$  were stable permutation). The most elegant way to prove these results (or more generally, that the local invariants separate) would be to show that there is no torsion in  $Flas(S_p)$  (or even in  $p - Flas(S_p)$  consisting of those classes with all  $q \neq p$ -classes trivial). However, we were only able to show that  $Flas(\widehat{\mathbb{Z}}_p S_p)$  has no torsion. So, there is no escape from the cumbersome Burnside-computations.

For  $p = 2$  there is nothing to prove as it is clear that  $G_2 = \mathbb{Z}(y_{12} + y_{21})$  so let us consider the case  $p = 3$  : information on conjugacy classes of subgroups of  $S_3$  is summarized in the following table

class	representative	order	hypo
(1)	1	1	all
(2)	(12)	2	all
(3)	(123)	3	all
(4)	$S_3$	6	3

With respect to this ordering it is also easy to compute the Burnside matrix

$$\begin{pmatrix} 6 & & & \\ 3 & 1 & & \\ 2 & 0 & 2 & \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

In order to show that  $G_3$  lies in the same genus as a stable permutation lattice we have to show that the  $\chi$ -function of  $G_3$  on the hypoelementary subgroups (which are all) extends to (i.e. is) an element of  $b(S_3)$ .  $\chi$  on the cyclics are just the character values so

$H$	(1)	(2)	(3)
$\chi(H)$	4	0	1

To compute  $\chi$  on the 3-hypoclass (4) we have to go to the  $N_3$ -level (here, of course, this is trivial) and obtain  $\widehat{\mathbb{Z}}_3[3] \simeq \widehat{\mathbb{Z}}_3 \oplus \Omega^2(\widehat{\mathbb{Z}}_3)$  which combined with (24) that  $G_3 \otimes \widehat{\mathbb{Z}}_3 \oplus \widehat{\mathbb{Z}}_3 \simeq \widehat{\mathbb{Z}}_3[3] \oplus \widehat{\mathbb{Z}}_3[2]$  (in such expressions we denote  $[i]$  for the  $S_n$ -set  $S_n/(i)$ ) allows us to compute that  $\chi([4]) = -1$ . This  $\chi$  is an element of  $b(S_3)$  as it is obtained from the vector  $(0, 1, 1, -1)$  by rightmultiplication with the Burnside matrix. Thus we obtain :

$$G_3 \oplus \mathbb{Z} \bigvee \mathbb{Z}[2] \oplus \mathbb{Z}[3] \quad (26)$$

which by vanishing of the genus-obstruction does imply that  $\phi(G_3) = 0$ .

Let us concentrate now on the case  $p = 5$ . We need to have fairly precise information on the conjugacy classes of subgroups of  $S_5$  which we summarize in the following table :

class	representative	order	hypo
(1)	1	1	all
(2)	(12)	2	all
(3)	(12)(34)	2	all
(4)	(123)	3	all
(5)	(12),(34)	4	2
(6)	(1234)	4	all
(7)	(12)(34),(13)(24)	4	2
(8)	(12345)	5	all
(9)	(12),(345)	6	all
(10)	(123),(12)	6	3
(11)	(123),(12)(45)	6	3
(12)	(1234),(12)(34)	8	2
(13)	(12345),(25)(34)	10	5
(14)	$S_3 \times C_2$	12	none
(15)	$A_4$	12	2
(16)	(12345),(2354)	20	5
(17)	$S_4$	24	none
(18)	$A_5$	60	none
(19)	$S_5$	120	none

Using this information one can now describe the Burnside ring of  $S_5$  as the image of  $\mathbb{Z}^{\oplus 19}$  under multiplication on the right by the matrix :

$$\begin{pmatrix}
 120 & \\
 60 & 6 & & & & & & & & & & & & & & & & & & & \\
 60 & 0 & 4 & & & & & & & & & & & & & & & & & & \\
 40 & 0 & 0 & 4 & & & & & & & & & & & & & & & & \\
 30 & 6 & 2 & 0 & 2 & & & & & & & & & & & & & & & \\
 30 & 0 & 2 & 0 & 0 & 2 & & & & & & & & & & & & & & \\
 30 & 0 & 6 & 0 & 0 & 0 & 6 & & & & & & & & & & & & & \\
 24 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & & & & & & & & & & & & \\
 20 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & & & & & & & & & & & \\
 20 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & & & & & & & & & & \\
 20 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & & & & & & & & & \\
 15 & 3 & 3 & 0 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 1 & & & & & & & & \\
 12 & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & & & & & & & \\
 10 & 4 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & & & & & & \\
 10 & 0 & 2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & & & & & \\
 6 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & & & & \\
 5 & 3 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & & & \\
 2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & & \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 &
 \end{pmatrix}$$

Next, we have to compute  $\chi$  of  $G_5 \oplus G_5^*$  on hypoelementary subgroups. If

$H$  is 2- or 3-hypoelementary we can use (23) to obtain that

$$\chi(H) = 2 + 2\#V_5^H - 2\#U_5^H \quad (27)$$

$H$	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$\chi(H)$	32	8	0	2	0	0	0

  

$H$	(8)	(9)	(10)	(11)	(12)	(15)
$\chi(H)$	2	2	2	2	0	0

Moreover, we have over the 5-adic integers that

$$\Omega^2(\widehat{\mathbb{Z}}_5) \oplus \Omega^{-2}(\widehat{\mathbb{Z}}_5) \oplus \widehat{\mathbb{Z}}_5[13] \simeq \widehat{\mathbb{Z}}_5[8] \quad (28)$$

yielding the isomorphism

$$(G_5 \oplus G_5^*) \otimes \widehat{\mathbb{Z}}_5 \oplus \widehat{\mathbb{Z}}_5[13] \simeq \widehat{\mathbb{Z}}_5[8] \oplus (\widehat{\mathbb{Z}}_5[14])^{\oplus 2} \quad (29)$$

allowing us to compute  $\chi$  on 5-hypoelementary subgroups :

$H$	(13)	(16)
$\chi(H)$	-2	0

We can extend this partial function  $\chi$  by unknowns  $\chi((14)) = a_1$ ,  $\chi((17)) = a_2$ ,  $\chi((18)) = a_3$  and  $\chi((19)) = a_4$ . Then, multiplying this  $\chi$ -vector by the inverse of the Burnside matrix we get an integer valued vector (and hence the Burnside obstruction vanishes) provided we have that  $a_1$  and  $a_2$  are even and  $a_3 \equiv a_4$  modulo 2. Hence, we can take all  $a_i$  to be zero and then we obtain from the above computations :

**Lemma 5**  $G_5 \oplus G_5^* \oplus \mathbb{Z}[4] \oplus \mathbb{Z}[13]$  lies in the same genus as  $\mathbb{Z}[8] \oplus \mathbb{Z}[9] \oplus \mathbb{Z}[10] \oplus \mathbb{Z}[11]$ .

Now, let us turn our attention to the case when  $p = 7$  : Again, we need precise information on the conjugacy classes of subgroups of  $S_7$ . This information was obtained using CAYLEY (version 3.5) running on the IBM 4381 of the University of Essen. In the sequel we use the canonical Cayley-ordering of the subgroups. We hope that the reader can guess the structure of these subgroups from the Burnside matrix given in the appendix. More suspicious readers may consult the full list of representatives given in the first preprint version [6] of this paper. There are 96 conjugacy classes of subgroups of  $S_7$  out of which 55 are hypoelementary subgroups.

As mentioned before it is easy to compute  $\chi(H)$  of a  $q$ -hypoelementary subgroup with  $q \neq 7$  using

$$\chi(H) = 2 + 2\#V_7^H - 2\#U_7^H \quad (30)$$

As  $V_7$  (resp.  $U_7$ ) is the permutation representation corresponding to the subgroup of class (88) (resp (94)) the values of  $\chi$  are easily deduced from the Burnside matrix. We obtain :

$H$	(1)	(2)	(3)	(4)	(5)	(6)	(9)	(10)	(11)	(12)
$\chi(H)$	72	32	8	0	0	18	8	0	0	8
$H$	(13)	(14)	(15)	(7)	(16)	(17)	(18)	(19)	(20)	(21)
$\chi(H)$	8	0	0	2	2	0	0	0	2	2
$H$	(22)	(23)	(8)	(30)	(31)	(32)	(33)	(34)	(35)	(36)
$\chi(H)$	2	18	2	8	0	0	0	0	0	0
$H$	(24)	(25)	(26)	(27)	(37)	(43)	(44)	(45)	(47)	(48)
$\chi(H)$	0	2	2	2	2	2	2	2	2	8
$H$	(49)	(57)	(50)	(51)	(52)	(54)	(55)	(59)	(64)	(65)
$\chi(H)$	0	0	0	0	0	2	2	2	0	0
$H$	(69)	(74)								
$\chi(H)$	0	0								

Now, the more challenging job to determine  $\chi(H)$  for the three 7- hypoelementary subgroups (28),(29) and (56). The starting point is the description of the stable permutation  $\widehat{\mathbb{Z}}_7(56)$ -lattice  $\Omega_{(56)}^2(\widehat{\mathbb{Z}}_7) \oplus \Omega_{(56)}^{-2}(\widehat{\mathbb{Z}}_7) = W$  induced up to the  $S_7$ -level giving :

$$W \uparrow^{S_7} \oplus \widehat{\mathbb{Z}}_7[28] \oplus \widehat{\mathbb{Z}}_7[29] \simeq \widehat{\mathbb{Z}}_7[8] \oplus \widehat{\mathbb{Z}}_7[56] \quad (31)$$

By Green correspondence we know that

$$W \uparrow^{S_7} = \Omega^2(\widehat{\mathbb{Z}}_7) \oplus \Omega^{-2}(\widehat{\mathbb{Z}}_7) \oplus IP^{\oplus 2} \quad (32)$$

where  $IP$  is a projective  $\mathbb{Z}_7 S_7$  lattice. By James' result we know that  $IP$  is stable permutation with all transitive permutation factors corresponding to Young subgroups. Hence, the description of  $IP$  as a stable permutation character coincides with the description of the corresponding character as a linear combination of the Young-subgroup permutation characters. We obtain the following description of the projective  $IP$  as a stable permutation lattice

$$\begin{aligned} IP \oplus \widehat{\mathbb{Z}}_7[32]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[23] \oplus \widehat{\mathbb{Z}}_7[73] \oplus \widehat{\mathbb{Z}}_7[91] \oplus \widehat{\mathbb{Z}}_7[88] \\ \simeq \widehat{\mathbb{Z}}_7[9] \oplus \widehat{\mathbb{Z}}_7[41]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[58] \oplus \widehat{\mathbb{Z}}_7[81] \oplus \widehat{\mathbb{Z}}_7[94] \end{aligned}$$

Next, we use the fact that

$$(G_7 \oplus G_7^*) \otimes \widehat{\mathbb{Z}}_7 \simeq \Omega^2(\widehat{\mathbb{Z}}_7) \oplus \Omega^{-2}(\widehat{\mathbb{Z}}_7) \oplus \widehat{\mathbb{Z}}_7[92]^{\oplus 2} \quad (33)$$

So,  $V = (G_7 \oplus G_7^*) \otimes \widehat{\mathbb{Z}}_7$  is stable permutation as :

$$V \oplus \widehat{\mathbb{Z}}_7[9]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[41]^{\oplus 4} \oplus \widehat{\mathbb{Z}}_7[58]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[81]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[94]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[28] \oplus \widehat{\mathbb{Z}}_7[29]$$

$$\simeq \widehat{\mathbb{Z}}_7[56] \oplus \widehat{\mathbb{Z}}_7[8] \oplus \widehat{\mathbb{Z}}_7[92]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[32]^{\oplus 4} \oplus \widehat{\mathbb{Z}}_7[23]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[73]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[91]^{\oplus 2} \oplus \widehat{\mathbb{Z}}_7[88]^{\oplus 2}$$

This allows us to compute the values for  $\chi(H)$  for the three 7- hypoelementary subgroups of  $S_7$  :

$H$	(28)	(29)	(56)
$\chi(H)$	-2	-1	1

This partial function  $\chi$  can be shown to extend to an element in the Burnside ring of  $S_7$  and we obtain :

**Lemma 6** *The lattice*

$$G_7 \oplus G_7^* \oplus \mathbb{Z}[3] \oplus \mathbb{Z}[6] \oplus \mathbb{Z}[18] \oplus \mathbb{Z}[28] \\ \oplus \mathbb{Z}[29] \oplus \mathbb{Z}[31]^{\oplus 2} \oplus \mathbb{Z}[34] \oplus \mathbb{Z}[52] \oplus \mathbb{Z}[58]$$

*lies in the same genus as the permutation  $\mathbb{Z}S_7$ -lattices*

$$\mathbb{Z}[4] \oplus \mathbb{Z}[5] \oplus \mathbb{Z}[8] \oplus \mathbb{Z}[16] \oplus \mathbb{Z}[20] \oplus \mathbb{Z}[32] \\ \oplus \mathbb{Z}[56] \oplus \mathbb{Z}[60] \oplus \mathbb{Z}[63] \oplus \mathbb{Z}[76]^{\oplus 2} \oplus \mathbb{Z}[83]$$

And therefore  $\phi(G_7) = \phi(A_6^*)$ . It is a bit surprising to note that in order to prove that the rank 36 lattice  $G_7$  has the same  $\phi$ -invariant as the rank 6 lattice  $A_6^*$ , we need to show that two lattices of rank 8092 lie in the same genus!

Concluding, we obtain the main result of this paper :

**Theorem 3** *For  $p = 2, 3, 5$  and 7 quotients of good  $PGL_p$ -representations are stably rational.*

**Proof :** For  $p = 2$  and 3 we have  $\phi(G_p) = 0$  , so  $\mathbb{C}(G_p)^{S_p}$  is stable equivalent to  $\mathbb{C}(U_p)^{S_p}$  which is rational on the elementary symmetric functions. For  $p = 5$  and 7 we have  $\phi(G_p) = \phi(A_{p-1}^*)$ , so  $\mathbb{C}(G_p)^{S_p}$  is stable equivalent to  $\mathbb{C}(A_{p-1}^*)^{S_p}$ . From the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow U_p \rightarrow A_{p-1}^* \rightarrow 0 \quad (34)$$

we know that  $\mathbb{C}(A_{p-1}^*)$  is the field of fractions of  $\mathbb{C}[U_p]/(e_1 \dots e_p - 1)$  with  $S_p$ -action induced by that on  $\mathbb{C}(U_p)$ . Hence,  $\mathbb{C}(A_{p-1}^*)$  is rational on the first  $p-1$  elementary symmetric functions on the  $e_i$ .  $\square$

## 5 Some applications

In this section we give a few applications of our main result to other areas. These connections between  $PGL_n$ -representations and vector bundles,

representation theory of quivers and Brauer groups are well documented (e.g. [22],[30],[27],[50],[28],[38]) and we refer to these papers for more details.

For [22],[30] or [27] we recall that  $X_n/PGL_n$  is birational to  $M(n;0,n)$  the moduli space of stable rank  $n$  vectorbundles on  $\mathbb{P}_2$  with Chernclasses  $c_1 = 0$  and  $c_2 = n$ . Barth [4] proved that stable vectorbundles on  $\mathbb{P}_2$  are classified by their curve of jumping lines in the dual plane  $\mathbb{P}_2^*$  (i.e. those lines  $l$  s.t.  $\mathcal{E}|_l \neq \mathcal{O}_l^{\oplus n}$ ) and a theta divisor on this curve. If we fix a point  $x$  in  $\mathbb{P}_2$  then all lines through  $x$  forms a  $\mathbb{P}_1$  in the dual plane. For a sufficiently general bundle  $\mathcal{E}$  this line intersects the curve of jumping lines in  $n$  distinct unordered points defining a map

$$M(n;0,n) \rightarrow \overbrace{\mathbb{P}_1 \times \dots \times \mathbb{P}_1}^n / S_n \quad (35)$$

Maruyama [31] claims that this map induces a stable rational field extension. Unfortunately, his proof breaks down because of the false alleged  $PGL_n$ -invariance of the map in [31,p 87]. Translating this claim to lattice invariants it says that  $\mathcal{C}(U_n^{\oplus 2} \oplus G_n)^{S_n}$  is stable rational over  $\mathcal{C}(U_n)^{S_n}$  which by Colliot-Thélès and Sansuc's theorem is equivalent to  $\phi(G_n) = 0$ . So, it holds for  $n = 2, 3$  but fails for  $n \geq 4$  (at least for  $n$  non-squarefree or  $n$  prime but probably for all  $n$ ). On the positive side we do have :

**Theorem 4** *The moduli space  $M(n;0,n)$  of stable rank  $n$  vectorbundles on  $\mathbb{P}_2$  with Chern-classes  $c_1 = 0$  and  $c_2 = n$  is rational for  $n \leq 4$  and is stably rational for  $n = 5$  and  $7$*

Now, let us turn to the representation theory of finite dimensional hereditary algebras, or equivalently, of quivers [24],[25]. Kac [24] has proved that the dimension vectors of indecomposable representations form the root system of a certain infinite dimensional Lie algebra. If the representation space  $R(Q, \alpha)$  contains an open set of indecomposables, he calls  $\alpha$  a Schur root. Further, Kac conjectures [25] that the scheme parametrizing isoclasses of indecomposable representations admits a cellular decomposition (which would immediately imply stable rationality of good  $PGL_n$ -representations). In [28] it was shown by using Bernstein-Gelfand-Ponomarev reflection functors and Bogomolov's lemma that for  $\alpha$  a Schur root, the field of quiver invariants  $\mathcal{C}(R(Q, \alpha))^{PGL(\alpha)}$  is stable equivalent to  $\mathcal{C}(X_n/PGL_n)$  where  $n = \gcd(\alpha_i)$ . Several geometrical moduli problems such as [33,4.4] or [33,p 163-167] can be translated into quiver-terms. For example, for the  $m$ -subspace problem  $\alpha$  is a Schur root iff  $R(Q, \alpha)$  contains a stable point, see [46].

**Theorem 5** *Let  $\alpha$  be a Schur root of a quiver  $Q$  and let  $n = \gcd(\alpha_i)$ . Then, if  $n = 2, 3, 4, 5$  or  $7$  the field of rational quiver invariants  $\mathcal{C}(Q, \alpha)^{PGL(\alpha)}$  is stably rational*

Finally, let us look at the original motivation for studying centers of generic division algebras. Using a result of Bloch, Procesi proved that if the centers of

the generic division algebras are stably rational, then the Brauer group of any field containing enough roots of unity is generated by cyclic algebras. Of course this result is now known to be true by the celebrated Merkurjev-Suslin result.

Our results do not contribute directly to this problem. However, we think that they may be useful in investigating a major open problem in Brauer groups : is every division algebra of prime degree cyclic ? Weddenburn proved that this is the case when the degree is 2 or 3 but already for 5 it is still open. Clearly, if one could show that the generic division algebra of prime degree is cyclic then the result would follow. For this reason we phrase our main results in this setting under minimal assumptions on the basefield :

**Theorem 6** 1. *For any field  $k$ , the centers of the generic division algebras over  $k$  of degree 5 and 7 are stably rational*

2. *If  $k$  contains a primitive  $p$ -th root of unity, then the center of the generic division algebra over  $k$  of degree  $p$  is stable equivalent to the Noether setting  $k(G)^G$  where  $G$  is the semidirect product of  $S_p$  and  $\text{Hom}_{\mathbb{Z}}(\mathbb{F}_p A_{p-1}, \mathbb{F}_p)$*

## References

- [1] S. Amitsur, On central division algebras, Israel J. Math. 12 (1972) 408-420
- [2] S. Amitsur, in "Algebraists' Hommage" Contemp. Math. Vol 13, AMS
- [3] J. Barge, Cohomologie des groupes et corps d'invariants multiplicatifs, Math. Ann. 283 (1989) 519-528
- [4] W. Barth, Moduli of vector bundles over the projective plane, Inv. Math. 42 (1977) 63-91
- [5] D. Benson : "modular Representation Theory" LNM 1081 (1984)
- [6] C. Bessenrodt, L. Le Bruyn, Stable rationality of certain moduli spaces, preprint (1989)
- [7] F. Bogomolov, Stable rationality of factor spaces for simply connected groups, Mat. Sbornik 130 (1986) 3-17
- [8] F. Bogomolov, The Brauer group of quotient spaces by linear group actions, Izv. Akad. Nauk. SSSR Ser. mat. 51 (1987) 485-516
- [9] J-L. Colliot-Thélène, J-J. Sansuc, La  $R$ -équivalence sur les tores, Ann. scient. E.N.S. 10 (1977) 175-229
- [10] J-L. Colliot-Thélène, J-J. Sansuc, Principal homogeneous spaces under flasque tori: applications, J. Alg. 106 (1987) 145-205

- [11] J-L. Colliot-Thélène, M. Ojanguren, Variétés unirationnelles non rationnelles: au delà de l'exemple d'Artin et Mumford, preprint (1988)
- [12] C.W. Curtis, I. Reiner, "Methods of Representation Theory II"
- [13] T. tom Dieck, "Transformation groups and representation theory", LNM 766 (1979)
- [14] A. Dress, Contributions to the theory of induced representations, in "K-Theory II", LNM 342 (1973)
- [15] A. Dress, The permutation classgroup of a finite group, JPAA 6 (1975) 1-12
- [16] S. Endo, T. Miyata, On a classification of the function fields of algebraic tori, Nagoya Math. J. 56 (1974) 85-104
- [17] S. Endo, T. Miyata, On the projective classgroup of finite groups, Osaka J. Math. 13 (1976) 109-122
- [18] W. Feit, "Representation theory of finite groups"
- [19] E. Formanek, The center of  $3 \times 3$  generic matrices, Lin. Mult. Alg. 7 (1979) 203-212
- [20] E. Formanek, The center of  $4 \times 4$  generic matrices, J. Alg. 62 (1980) 304-319
- [21] W. Haboush, Brauer groups of homogeneous spaces I in "Proc. NATO summer school, Antwerp" (1983)
- [22] K. Hulek, On the classification of stable rank  $r$  vector bundles over the projective plane, Progr. Math. 7 (1980) 113-144
- [23] G. James, Trivial source modules for symmetric groups, Arch. Math. 41 (1983) 294-300
- [24] V. Kac, Infinite root systems, representations of graphs and invariant theory, Inv. Math. 56 (1980) 57-92
- [25] V. Kac, Root systems, representations of quivers and invariant theory, "Montecatini 1982" LNM 996, 74-108
- [26] W. Kuyk, On a theorem of E. Noether, Ndl. Akad. Wet. Proc. Ser. A 67 (1964) 32-39
- [27] L. Le Bruyn, Some remarks on rational matrix invariants, J. Alg. 118 (1988) 487-493
- [28] L. Le Bruyn, A. Schofield, Rational invariants of quivers and the ring of matrix invariants, NATO-ASI C-233 (1988) 21-30

- [29] H. Lenstra, Rational functions invariant under a finite group, *Inv. Math.* 25 (1974) 299-325
- [30] M. Maruyama, The equations of plane curves and the moduli spaces of vectorbundles over  $\mathbb{P}_2$ , "Algebraic and topological theories, to the memory of Miyata" (1985) 430-466
- [31] M. Maruyama, Stable rationality of some moduli spaces of vectorbundles over  $\mathbb{P}_2$ , *LNM* 1194 (1986) 80-89
- [32] K. Masuda, On a problem of Chevalley, *Nagoya Math. J.* 8 (1955) 59-63
- [33] D. Mumford, J. Fogarty : "geometric Invariant Theory" (2nd edition)
- [34] R. Oliver ,  $G$ -actions on disks and permutation representations, *Math. Z.* 157 (1977) 237-263
- [35] R. Oliver , in "Orders and their applications" *LNM* (1984)
- [36] C. Procesi : "Rings with polynomial identities" Dekker (1973)
- [37] C. Procesi , The invariant theory of  $n \times n$  matrices, *Adv. Math.* 19 (1976) 306-381
- [38] C. Procesi , Relazioni tra geometria algebrica ed algebra non commutativa, *Algebre cicliche e problema di Lüroth*, *Boll. Un. Math. Ital.* 18-A (1981) 1-10
- [39] L. Rowen : "Ring Theory II" Academic Press (1988)
- [40] D. Saltman, Retract rational fields and cyclic Galois extensions, *Israel J. Math.* 47 (1984) 165-215
- [41] D. Saltman, Noether's problem over an algebraically closed field, *Inv. Math.* 77 (1984) 71-84
- [42] D. Saltman, The Brauer group and the center of generic matrices, *Alg.* 97 (1985) 53-67
- [43] D. Saltman, Multiplicative field invariants, *J. Alg.* 106 (1987) 221-238
- [44] D. Saltman, letter (1986)
- [45] D. Saltman, Invariant fields of linear groups and division algebras, *NATO-ASI C-233* (1988) 279-298
- [46] A. Schofield, Generic representations of quivers, preprint
- [47] J. Sylvester, On the involution of two matrices of the second order, *British Association Report, Southport* (1883) 430-432

- [48] R. Swan, Galois theory in "Emmy Noether in Bryn Mawr" Springer (1983)
- [49] M. Van den Bergh, Centers of generic division algebras, J. Alg. to appear
- [50] V. Voskresenskii, Rationality of certain tori, Math. USSR-Izv 5 (1971) 1049-1056

1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36	24	25	26	27	37	38	39	40	41	42	43	44	45	46	47	48	49	28	57	50	51	52	53	54	55	29	58	59	60
1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22																																						