Stable Rationality of certain Moduli Spaces

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Abstract

In this paper we study the (stable) rationality problem of quotient varieties of a vector space under an almost free action of a reductive quotient of SL_n by reducing it to integral and modular representation theory of the symmetric group S_n. In particular, we prove stable rationality of PGL_5 and PGL_7 quotients.

Disclaimer

This is not the final version of the paper and it is intended for Oberwolfach–consumption only.

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1 Introduction

Let $\mathbb{C}$ be an algebraically closed field of characteristic zero and G a reductive linear algebraic group. Let X be a finite dimensional vectorspace on which G acts almost freely, i.e. the stabilizer of a generic point is trivial. We want to investigate stable rationality of the quotient variety X/G i.e. whether the fixed field $\mathbb{C}(X)^G$ is stable rational over $\mathbb{C}$. Bogomolov [6] has shown that this question is independent of the particular choice of X.

Moreover, Saltman [27] has indicated how this problem can be reduced to checking stable rationality of certain fields of twisted multiplicative invariants under the corresponding Weyl group. So, let T be a maximal torus in G with normalizer $N_G(T)$ then the Weyl group of G is the finite group $W = N_G(T)/T$. Then, consider the character group $X(T) = Hom_{alg}(T, \mathbb{C}^*)$ as a lattice over the integral groupring $\mathbb{Z}W$ and take a permutation $\mathbb{Z}W$-lattice P (i.e. a $\mathbb{Z}W$-lattice having a basis which is permuted under the action of W) fitting into a $\mathbb{Z}W$-exact sequence

$$0 \to M \to P \to X(T) \to 0 \qquad (1)$$

Saltman [27,Cor.2.7] shows that $\mathbb{C}(X)^G$ is stable equivalent to the field of twisted multiplicative invariants $\mathbb{C}_\alpha(M)^W$ where α is some extension of M by $\mathbb{C}^*$ and in many cases (as the ones we are interested in) the twisting by α can be dispensed with.

Let us now concentrate on the special case where G is a quotient of SL_n. Note that the center of SL_n is cyclic of order n and for each divisor r of n let C_r be the unique central subgroup of SL_n which is cyclic of order r. If we define $SL_n(r) = SL_n/C_r$ then clearly $SL_n(n) = PGL_n$ and we have epimorphisms

$$\begin{array}{ccccc} SL_n & & \longrightarrow & & PGL_n \\ & \searrow & & \nearrow & \\ & & SL_n(r) & & \end{array} \qquad (2)$$

Then, the symmetric group on n letters, S_n, is the Weyl group of all these groups and the character lattices of the corresponding maximal tori can be described as follows : let A_{n-1} be the classical root lattice consisting of all integer vectors $(x_1, ..., x_n) \in \mathbb{Z}^{\oplus n}$ such that $\sum_{i=1}^n x_i = 0$. Then A_{n-1} is the character lattice of a maximal torus in PGL_n whereas its dual A_{n-1}^* is that of a maximal torus in SL_n and the Coxeter lattice

$$A_{n-1}[s] = \cup_{i=0,r,2r,\ldots,n-r}([i] + A_{n-1}) \qquad (3)$$

where $r.s = n$ and for $i + j = n$

$$[i] = \frac{1}{n}(\underbrace{i, ..., i}_{j}, \underbrace{-j, ..., -j}_{i}) \tag{4}$$

is the character lattice of a maximal torus of $SL_n(r)$. Saltman realized these $\mathbb{Z}S_n$-lattices as epimorphic images of the following permutation lattices : Let S_{n-1} be embedded in S_n as the stabilizer of u_1 in the standard permutation representation $U_n = \mathbb{Z}u_1 \oplus ...\mathbb{Z}u_n$ of S_n. Let V_n be the $\mathbb{Z}$-lattice on the nondiagonal entries of an n by n matrix $V_n = \mathbb{Z}y_{12} \oplus \mathbb{Z}y_{13} \oplus ... \oplus \mathbb{Z}y_{n-1n}$ and turn it into a permutation $\mathbb{Z}S_n$-lattice via the action $\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. We now construct the following diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & G_{n,n} & \rightarrow & V_n & \xrightarrow{\phi} & A_{n-1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G_{n,r} & \rightarrow & V_n \oplus \mathbb{Z}S_n/S_{n-1} & \xrightarrow{\xi} & A_{n-1}[r] & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & G_{n,1} & \rightarrow & V_n \oplus \mathbb{Z}S_n/S_{n-1}^{\oplus 2} & \xrightarrow{\psi} & A_{n-1}^* & \rightarrow & 0
\end{array} \tag{5}$$

where ψ extends ξ extends ϕ and are defined by : $\phi(y_{ij}) = u_i - u_j$ (note that A_{n-1} is the kernel of the augmentation map on U_n); ξ sends the canonical generator of $\mathbb{Z}S_n/S_{n-1}$ to $\frac{1}{r}((u_2 - u_1) + (u_3 - u_1) + ... + (u_n - u_1))$ and ψ sends the canonical generator of the second $\mathbb{Z}S_n/S_{n-1}$ factor to $\frac{1}{n}((u_2 - u_1) + (u_3 - u_1) + ... + (u_n - u_1))$. Now, ψ restricted to this second factor is an epimorphism with kernel the trivial lattice showing that $G_{n,1}$ is the permutation $\mathbb{Z}S_n$-lattice $V_n \oplus \mathbb{Z}S_n/S_{n-1} \oplus \mathbb{Z}$. Using this terminology we have

Theorem 1 (Saltman,Prop.3.1) *Let X_r be a vectorspace on which $SL_n(r)$ acts almost freely. Then, $\mathbb{C}(X_r)^{SL_n(r)}$ is stable equivalent to the field of lattice invariants $\mathbb{C}(G_{n,r})^{S_n}$. Moreover, both fields are stable equivalent to the center of the generic division algebra of degree n and exponent r.*

In particular, if PGL_n acts almost freely on a vectorspace X, then $\mathbb{C}(X)^{PGL_n}$ is stable equivalent to $\mathbb{C}(G_{n,n})^{S_n}$ which has the center of the generic division algebra of degree n as a rational field extension. Rationality of this field of lattice invariants for $n = 2$ was known already in the last century and Formanek proved rationality if $n = 3$ or 4 cf. [14],[15]. The stable rationality problem for PGL_n quotients arises naturally in many seemingly unrelated problems. Let us give a few examples :

- *Merkurjev-Suslin result :* Using a result of Bloch [5], Procesi [24] showed that stable rationality of $\mathbb{C}(X)^{PGL_n}$ for all n would imply that the Brauer group of any $\mathbb{C}$-field is generated by cyclic algebras.
- *Vector bundles :* The functionfield of the moduli space of stable rank n vectorbundles over the projective plane $\mathbb{P}_2$ with Chern-numbers $c_1 = 0$ and $c_2 = n$ is a rational field over $\mathbb{C}(G_{n,n})^{S_n}$,cfr. [19]
- *m-Subspace problem :* Consider the action of GL_l on $Grass(k_1, l) \times \ldots \times Grass(k_m, l)$. If this action has a stable point then the functionfield of the quotient variety is stable equivalent to $\mathbb{C}(X)^{PGL_n}$ where n is the g.c.d. of the k_i and l,cfr. [20].

In case n is even, $\mathbb{C}(G_{n,2})^{S_n}$ is also of some importance. Let PO_n be the projective orthogonal group and PSP_n the projective symplectic group. Then, $\mathbb{C}(G_{n,2})^{S_n}$ is stable equivalent to the functionfields of the quotient varieties of an almost free vectorspace action of PO_n or PSP_n see [27,Cor. 3.4].

In this paper we aim to study stable rationality of the fields of lattice invariants $\mathbb{C}(G_{n,r})^{S_n}$ with special emphasis to the case $\mathbb{C}(G_{p,p})^{S_p}$ for p a prime number. The starting point is the theory of tori- invariants as developed in the early seventies by a.o. Endo and Miyata [13], Lenstra [21], Voskresenskii [29] and Colliot-Thélène and Sansuc [7].

Recall that a $\mathbb{Z}W$-lattice (W an arbitrary finite group) M is said to be flasque (resp. coflasque) if $\check{H}^{-1}(H, M) = 0$ (resp $\check{H}^1(H, M) = 0$) for all subgroups H of W. Every lattice M has a coflasque resolution

$$0 \to Q \to P \to M \to 0 \tag{6}$$

with Q coflasque and P permutation. An explicit resolution is obtained by taking

$$P = \oplus_{H<W} \mathbb{Z}W/H \otimes_{\mathbb{Z}} M^H \tag{7}$$

with M^H the H-invariant elements given trivial G-action and the sum taken over all conjugacy classes of subgroups H of G. Moreover, if Q_1 and Q_2 are end terms of coflasque resolutions of M then there exist permutation $\mathbb{Z}W$-lattices P_i s.t. $Q_1 \oplus P_2 \cong Q_2 \oplus P_1$. Hence, introducing the Abelian semigroup

$Cofl(W)$ of stable permutation classes of coflasque $\mathbb{Z}W$-lattices gives a well defined map

$$\kappa : \mathbb{Z}W - \text{lattices} \to Cofl(W) \tag{8}$$

assigning to a lattice the class of an end term of a coflasque resolution.

Dually, every $\mathbb{Z}W$-lattice M has a flasque resolution

$$0 \to M \to P \to F \to 0 \tag{9}$$

with P permutation and F flasque and introducing $Flas(W)$ to be the Abelian semigroup of stable permutation classes of flasque $\mathbb{Z}W$-lattices gives a well defined map

$$\phi : \mathbb{Z}W - \text{lattices} \to Flas(W) \tag{10}$$

assigning to a lattice the class of an end term of a flasque resolution.

Now, if L is a $\mathbb{C}$-field with a faithful W-action we can define for every $\mathbb{Z}W$-lattice M the field of tori-invariants $L(M)^W$. Crucial for our purposes is the following characterization of stable equivalence classes of tori-invariants and its consequence for lattice-invariants :

Theorem 2 (Colliot-Thélène,Sansuc) *For $\mathbb{Z}W$-lattices M and N we have*

- *$L(M)^W$ is stable equivalent to $L(N)^W$ over L^W if and only if $\phi(M) = \phi(N)$ in $Flas(W)$*
- *If M and N are faithful with $\phi(M) = \phi(N)$ in $Flas(W)$ then $\mathbb{C}(M)^W$ is stable equivalent to $\mathbb{C}(N)^W$ over $\mathbb{C}$*

The main aim of this paper is therefore to study the place of $\phi(G_{n,r})$ in $Flas(S_n)$.

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2 Some known results.

In this section we summarize the present knowledge on $\mathbb{C}(G_{n,n})^{S_n}$ apart from rationality for $n \leq 4$ mentioned before.

The moduli space $M(n;0,n)$ of stable rank n-vectorbundles over $\mathbb{P}_2$ with Chern-numbers $c_1 = 0$ and $c_2 = n$ has as its functionfield a rational fieldextension of $\mathbb{C}(U_n \oplus G_{n,n})^{S_n}$ see e.g. [22,19]. So, the natural approach is to prove that the field of tori-invariants $\mathbb{C}(U_n)(G_{n,n})^{S_n}$ is stable rational over $\mathbb{C}(U_n)^{S_n}$ which is the rational field on the symmetric functions in the u_i. Barth [2] proved that stable vectorbundles over $\mathbb{P}_2$ with $c_1 = 0$ are classified by their curve of jumping lines in the dual plane $\mathbb{P}_2^*$ (i.e. those lines l s.t. $\mathcal{E} \mid l \neq \mathcal{O}_l^{\oplus n}$) and a theta divisor on this curve. If we take a point x in $\mathbb{P}_2$ then all lines through x form a $\mathbb{P}_1$ in $\mathbb{P}_2^*$. So, for a sufficiently general vectorbundle $\mathcal{E}$ this line will intersect the curve of jumping lines in n distinct (unordered) points defining a natural map

$$M(n;0,n) \rightarrow \overbrace{\mathbb{P}_1 \times \ldots \times \mathbb{P}_1}^{n} / S_n \tag{11}$$

Maruyama [22] claims that this map induces a stably rational field extension. Unfortunately, his proof breaks down because of the false alleged $PGL(n)$-invariance of the map in [22,p.87].

Using the Colliot-Thélène and Sansuc result mentioned above this approach is equivalent to establishing an exact $\mathbb{Z}S_n$-sequence

$$0 \rightarrow G_{n,n} \rightarrow P_1 \rightarrow P_2 \rightarrow 0 \tag{12}$$

with P_1 and P_2 permutation $\mathbb{Z}S_n$-lattices. For, stable rationality of the field of tori-invariants $\mathbb{C}(U_n)(G_{n,n})^{S_n}$ is equivalent to $\phi(G_{n,n}) = 0$ i.e. $\phi(G_{n,n})$ being stable permutation. By 1980 it was already common knowledge among ringtheorists studying the center of the generic division algebras,cfr. e.g. [15] and [24] that this could not be true in general. For the readers convenience we will recall here an amplification of this result and its cohomological proof due to Saltman [26] :

Theorem 3 (Snider,Saltman) *If n is not squarefree, then there does not exist an exact $\mathbb{Z}S_n$-sequence*

$$0 \rightarrow G_{n,n} \rightarrow P \rightarrow Q \rightarrow 0 \tag{13}$$

with P permutation and Q coflasque.

Proof : If $n = p^2.m$, then S_n contains a subgroup G which is the direct product of a cyclic group of order p and one of order $p.m$ such that the action on n letters is the product action. Restricting any permutation $\mathbb{Z}S_n$-lattice P down to G we can write it as $\oplus_i \mathbb{Z}G/H_i$ for some subgroups H_i of G. But then by Shapiro's lemma : $\check{H}^2(G,P) = \oplus_i Hom(H_i, \mathbb{Q}/\mathbb{Z})$. Thus from the existence of the required sequence we would have

$$0 \rightarrow \check{H}^2(G, G_{n,n}) \rightarrow \oplus_i Hom(H_i, \mathbb{Q}/\mathbb{Z}) \tag{14}$$

whence any element of $\check{H}^2(G, G_{n,n})$ must have order dividing $p.m < n$. However, using that V_n and U_n are free $\mathbb{Z}G$-lattices we have that $\check{H}^2(G, G_{n,n}) = \check{H}^1(G, A_{n-1}) = \mathbb{Z}/n\mathbb{Z}$, a contradiction. □

However, things change drastically if $n = p$ is a prime number. Motivated by the retract rationality result of Saltman [25], Colliot-Thélène and Sansuc proved that such a sequence does exist, even with Q an invertible $\mathbb{Z}S_p$-lattice. We will give a short proof of this result based on the following characterization of invertible lattices which can be found in [7] and [3]

Lemma 1 *For a $\mathbb{Z}W$-lattice M the following are equivalent*

- *M is an invertible $\mathbb{Z}W$-lattice*
- *$M \otimes \mathbb{Z}_p$ is an invertible $\mathbb{Z}_pW$-lattice for all primes p dividing the order of W*
- *$M \downarrow_S$ is an invertible $\mathbb{Z}S$-lattice for all Sylow subgroups S of W*
- *$M \downarrow_S \otimes \mathbb{Z}_p$ is a permutation $\mathbb{Z}_pS$-lattice for all Sylow p-subgroups S of W (for all p dividing the order of W)*

The following result will be the starting point for our further investigation :

Lemma 2 *For all prime numbers p we have*

1. *$\kappa(A_{p-1})$ is an invertible $\mathbb{Z}S_p$-lattice*
2. *In a coflasque resolution of A_{p-1} the permutation lattice can be taken to contain only factors $\mathbb{Z}S_p/H$ where p does not divide the order of H*

Proof : Consider a coflasque resolution of A_{p-1} as an S_p-lattice

$$0 \to \kappa(A_{p-1}) \to P \to A_{p-1} \to 0$$

For all primes $q < p$ the epimorphism $\mathbb{Z}_q \otimes U_p \to \mathbb{Z}_q$ splits entailing that $\mathbb{Z}_q \otimes A_{p-1}$ is an invertible $\mathbb{Z}_q S_p$-lattice. Then, by tensoring the flasque resolution above with $\mathbb{Z}_q$ also $\mathbb{Z}_q \otimes \kappa(A_{p-1})$ is invertible. Applying the foregoing result we have that $\kappa(A_{p-1})$ is invertible when restricted to a q-Sylow subgroup of S_p. Further, the p-Sylow subgroup of S_p is cyclic and thus the restriction of the coflasque lattice $\kappa(A_{p-1})$ is invertible. The foregoing result finishes the proof.

(2) : Consider the explicit description (7) of the middle term of a coflasque resolution. Now let H be a subgroup of S_p such that p divides its order. Then, H contains a p-cycle and we have that $U_p^H = \mathbb{Z}(\sum_{i=1}^p u_i)$ from which we deduce that $A_{p-1}^H = 0$, done. □

As an immediate consequence of this result we get a short proof of the Colliot-Thélène and Sansuc result [8] :

Theorem 4 (Colliot-Thélène,Sansuc) *For p a prime number, there does exist an exact sequence of $\mathbb{Z}S_p$-lattices*

$$0 \to G_{p,p} \to P \to I \to 0$$

with P a permutation $\mathbb{Z}S_p$-lattice and I an invertible $\mathbb{Z}S_n$-lattice (and hence in particular coflasque)

Proof : Consider the pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_{p,p} & = & G_{p,p} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \kappa(A_{p-1}) & \to & V_p \times_{A_{p-1}} P & \to & V_p & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \kappa(A_{p-1}) & \to & P & \to & A_{p-1} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Because V_p is permutation and $\kappa(A_{p-1})$ coflasque we get

$$P \times_{A_{p-1}} V_p \cong \kappa(A_{p-1}) \oplus V_p$$

giving rise to the exact $\mathbb{Z}S_p$-sequence

$$0 \to G_{p,p} \to \kappa(A_{p-1}) \oplus V_p \to P \to 0$$

and since $\kappa(A_{p-1})$ is invertible there exists an invertible lattice I and a permutation lattice P' such that $\kappa(A_{p-1}) \oplus I \cong P'$ leading to an exact sequence

$$0 \to G_{p,p} \to V_p \oplus P' \to I \oplus P \to 0$$

of the required type. □

3 Some general results

Given a finite group W and two $\mathbb{Z}W$-lattices M and N, we want to develop a method to decide whether or not $\phi(M) = \phi(N)$ in $Flas(W)$. There are basically three major obstructions :

- *local obstruction :* Are the local invariants $\phi_p(M)$ and $\phi_p(N)$ equal for all primes $p \mid \#W$?
- *Burnside obstruction :* If all the local invariants coincide can we find representatives in $\phi(M)$ and $\phi(N)$ which lie in the same genus ?
- *genus obstruction :* Given representatives in $\phi(M)$ and $\phi(N)$ which lie in the same genus, does this imply that $\phi(M) = \phi(N)$?

In the special case of the symmetric groups S_n we can describe these obstructions fairly explicitly :

3.1 The local obstruction

For any domain R we define an RW-lattice M to be flasque iff $Ext^1_W(M, RW/H) = 0$ for all subgroups H of W. With $Flas(RW)$ we denote the semigroup of stable permutation equivalence classes of flasque RW-lattices.

A first test to distinguish $\phi(M)$ and $\phi(N)$ is to check whether they have the same image under the localization map

$$\text{loc} : Flas(W) \rightarrow \prod_{p|\#W} Flas(\mathbb{Z}_p W) \tag{15}$$

That is, we want to study the local invariants $\phi_p(M) = [\phi(M) \otimes \mathbb{Z}_p] \in Flas(\mathbb{Z}_p W)$. The local semigroups can be described using ordinary and modular representation theory of W as we have by descent-theory a pullback diagram

$$\begin{array}{ccc} Flas(\mathbb{Z}_p W) & \rightarrow & Flas(\widehat{\mathbb{Z}}_p W) \\ \downarrow & & \downarrow \\ Flas(\mathbb{Q} W) & \rightarrow & Flas(\widehat{\mathbb{Q}}_p W) \end{array} \tag{16}$$

In the S_n-case we get a first simplification because the permutation characters generate all :

Lemma 3 *For all n and all primes p we have $Flas(\mathbb{Z}_p S_n) \cong Flas(\widehat{\mathbb{Z}}_p S_n)$*

This reduces the computation of the local invariants to modular representation theory of S_n. In the special case of the lattices $G_{n,r}$ a lot of these local invariants can be extracted from the definition diagram (5):

Lemma 4 *For all primes p and $r \mid n$ we have :*

- *If $p \not| r$ then $\phi_p(G_{n,r}) = 0$ in $Flas(\widehat{\mathbb{Z}}_p S_n)$*
- *If $p \not| \frac{n}{r}$ then $\phi_p(G_{n,r}) = \phi_p(G_{n,n})$ in $Flas(\widehat{\mathbb{Z}}_p S_n)$*

Before we can describe the local invariants $\phi_p(G_{n,r})$ for $p \mid r$ we have to recall a result of G. James [17] on the indecomposable factors of permutation modules corresponding to Young subgroups.

Let $\alpha = (a_1, ..., a_k)$ and $\beta = (b_1, ..., b_l)$ be two partitions of n then we define $\alpha \trianglelefteq \beta$ iff for all i we have $\sum_{j=1}^{i} a_j \leq \sum_{j=1}^{i} b_j$. Further, we denote by P_α the permutation representation corresponding to the Young subgroup of α i.e.

$$P_\alpha = \widehat{\mathbb{Z}}_p S_n / S_{a_1} \times ... \times S_{a_k} \tag{17}$$

Using this terminology we have :

Proposition 1 (G.D. James) *For any partition α of n there exists an indecomposable $\widehat{\mathbb{Z}}_pS_n$- lattice I_α such that for all partitions β of n we have*

- $P_\beta \cong I_{\alpha_1}^{\oplus e_1} \oplus ... \oplus I_{\alpha_m}^{\oplus e_m}$ *with $\beta \trianglelefteq \alpha_i$ forr all i and I_β occurs with multiplicity one*
- *If $I_\alpha = I_\beta$ then $\alpha = \beta$*

Using this result we get by induction on the dominance order $\trianglelefteq$ the following

Lemma 5 *For all n and all primes p we have :*

- *All invertible $\widehat{\mathbb{Z}}_pS_n$-lattices of the form I_α are stable permutation lattices*
- *In particular, all projective $\widehat{\mathbb{Z}}_pS_n$-lattices are stable permutation lattices*

With $A^p_{n-1}[r]$ we will denote $A_{n-1}[r] \otimes \widehat{\mathbb{Z}}_p$. Notice that $A^p_{n-1}[r]$ is an indecomposable $\widehat{\mathbb{Z}}_pS_n$-lattice and let $\Omega(A^p_{n-1}[r])$ be its first syzygy :

Lemma 6 *For all primes p and all $r \mid n$ we have that $\phi_p(G_{n,r}) = \phi_p(\Omega(A^p_{n-1}[r]))$ in $Flas(\widehat{\mathbb{Z}}_pS_n)$*

Proof : From the defining sequence of $G_{n,r}$ we deduce that $\phi_p(G_{n,r}) = \phi_p(\kappa_p(A^p_{n-1}[r]))$. On the other hand, taking a projective cover gives a sequence

$$0 \to \Omega(A^p_{n-1}[r]) \oplus P' \to P \to A^p_{n-1}[r] \to 0 \qquad (18)$$

As the projective modules P is stable permutation we can add a permutation $\widehat{\mathbb{Z}}_pS_n$-lattice to the first two terms yielding a sequence

$$0 \to \Omega(A^p_{n-1}[r]) \oplus P_1 \to P_2 \to A^p_{n-1}[r] \to 0 \qquad (19)$$

with P_i permutation. But then, $\phi_p(\Omega(A^p_{n-1}[r])) = \phi_p(\kappa_p(A^p_{n-1}[r]))$,done. $\square$

In the special case that $n = r = p$ we even get :

Lemma 7 $\phi_p(G_{p,p}) = -[\Omega^2(\widehat{\mathbb{Z}}_p)] \in Flas(\widehat{\mathbb{Z}}_pS_p)$

Proof : $\widehat{\mathbb{Z}}_p S_p/S_{p-1}$ is a projective $\widehat{\mathbb{Z}}_p S_p$-lattice,so $A^p_{p-1} = \Omega(\widehat{\mathbb{Z}}_p)$. By the foregoing this gives us

$$\phi_p(G_{p,p}) = \phi_p(\Omega(A^p_{p-1})) = \phi(\Omega^2(\widehat{\mathbb{Z}}_p)) \tag{20}$$

But, $\Omega^2(\widehat{\mathbb{Z}}_p)$ is an invertible $\widehat{\mathbb{Z}}_p S_p$-lattice and thus $\phi(\Omega^2(\widehat{\mathbb{Z}}_p)) = -[\Omega^2(\widehat{\mathbb{Z}}_p)] \in Flas(\widehat{\mathbb{Z}}_p S_p)$. □

3.2 The Burnside obstruction

Suppose we have two $\mathbb{Z}W$-lattices such that for all primes $p \mid \#W$ we have $\phi_p(M) = \phi_p(N)$ i.e. we can find W-sets T_p and T'_p (depending on the prime p) such that

$$\phi_p(M) \oplus \widehat{\mathbb{Z}}_p T_p \cong \phi_p(N) \oplus \widehat{\mathbb{Z}}_p T'_p \tag{21}$$

We now ask whether one can globalize this to get W-sets T and T' (independent on the prime p) such that $\phi(M) \oplus \mathbb{Z}T$ lies in the same genus as $\phi(N) \oplus \mathbb{Z}T'$ i.e. such that for all primes $p \mid \#W$:

$$\phi_p(M) \oplus \widehat{\mathbb{Z}}_p T \cong \phi_p(N) \oplus \widehat{\mathbb{Z}}_p T' \tag{22}$$

The method to vizualize this obstruction is a slight variation on an idea of A. Dress [12] and is based on the description of $b(W)$ the Burnside ring of W. This is the Grothendieck ring constructed from the isomorphism classes of finite W-sets with addition induced by disjoint union and multiplication induced by the cartesian product with diagonal W-action, see e.g. [10] or [9].

If $C(W)$ denotes the set of conjugacy classes of subgroups H of W, then we have an injective ringmorphism

$$\beta = (\beta_H)_H : b(W) \to \prod_{(H)\in C(W)} \mathbb{Z} \tag{23}$$

determined by the Burnside-marks β_H assigning to a W-set S the number $\#S^H$ of H-fixed elements. Dress has shown that the image of $b(W)$ can be characterized as the set of those elements $\chi = (\chi(H)) \in \prod \mathbb{Z}$ satisfying the congruence relations

$$\sum_{(K)} n(H,K).\chi(H) \equiv 0 \bmod \#N_W(H)/H \tag{24}$$

where the sum is taken over all conjugacy classes of subgroups K of $N_W(K)$ such that $H \lhd K$ with cyclic quotient K/H and

$$n(H,K) = \#\frac{N_W(K)}{N_W(H) \cap N_W(K)} . \#(\frac{K}{H})^* \tag{25}$$

the last term being the number of generators of the cyclic group K/H. For more details on computing this Möbius-like function we refer to [18].

Now, assume M and N are two $\mathbb{Z}W$-lattices satisfying (refeq:local). Recall that a group H is said to be p-hypoelementary if $H/O_p(H)$ is cyclic where $O_p(H)$ is the largest normal p-subgroup of H. With $Hyp_p(W)$ we will denote the set of all p-hypoelementary subgroups of W and let $\mathcal{H} = \bigcup_{p|\#W} Hyp_p(W)$. We will now define a mapping

$$\chi : \mathcal{H} \to \mathbb{Z} \tag{26}$$

assigning to a subgroup $H \in Hyp_p(W)$ the value $\#T_p^H - \#S_p^H$ defined by the p-local data. Of course, we have to verify that this map is well-defined. If $H \in Hyp_p(W) \cap Hyp_q(W)$ then H is cyclic with generator say w. But then,

$$\#T_p^H - \#S_p^H = \chi_{\phi(M}(w) - \chi_{\phi(N)}(w) = \#T_q^H - \#S_q^H \tag{27}$$

where χ_V denotes the rational character of V. Clearly, χ is invariant on conjugacyclasses, thereby defining a partial function

$$\chi : C(\mathcal{H}) \subset C(W) \to \mathbb{Z} \tag{28}$$

Using this terminology we have :

Lemma 8 *If M and N have the same local invariants then the Burnside obstruction vanishes iff the partial function χ defined above can be extended to an element in $\prod_{(H)\in C(W)} \mathbb{Z}$ belonging to the image of β.*

Proof : Assume that χ can be extended to an element in the image of β. Then, there exist W-sets S and T such that for all $H \in Hyp_p(W)$

$$\chi(H) = \#T_p^H - \#S_p^H = \#T^H - \#S^H \tag{29}$$

which entails by [11] that

$$\widehat{\mathbb{Z}}_p T_p \oplus \widehat{\mathbb{Z}}_p S \cong \widehat{\mathbb{Z}}_p T \oplus \widehat{\mathbb{Z}}_p S_p \tag{30}$$

But then by adding $\widehat{\mathbb{Z}}_pS \oplus \widehat{\mathbb{Z}}_pT$ to both sides of (21) we obtain

$$\phi_p(M) \oplus \widehat{\mathbb{Z}}_pS_p \oplus \widehat{\mathbb{Z}}_pS \oplus \widehat{\mathbb{Z}}_pT \cong \phi_p(N) \oplus \widehat{\mathbb{Z}}_pT_p \oplus \widehat{\mathbb{Z}}_pS \oplus \widehat{\mathbb{Z}}_pT \tag{31}$$

But then,by cancellation and the above isomorphism we get

$$\phi_p(M) \oplus \widehat{\mathbb{Z}}_pS \cong \phi_p(N) \oplus \widehat{\mathbb{Z}}_pT \tag{32}$$

which is independent of the prime p. Thus, $\phi(M) \oplus \mathbb{Z}S$ and $\phi(N) \oplus \mathbb{Z}T$ belong to the same genus.

The other implication is obvious. □

We are not able to get any substantial simplifications in the case of the symmetric groups. Clearly, James' result (Proposition 1) will be very useful in computing the function χ.

3.3 The genus obstruction

Assume we can find representatives $\Phi_M \in \phi(M)$ and $\Phi_N \in \phi(N)$ which lie in the same genus. Does this imply that $\phi(M) = \phi(N)$? In general, this cannot be true as large cyclic groups already produce counterexamples. For non-cyclic groups this obstruction is related to cancellative properties of $Flas(W)$:

Lemma 9 *If W is non-cyclic and $Flas(W)$ is a cancellative semigroup, then the genus obstruction vanishes.*

Proof : From [16] we recall that the relation modules for a non-cyclic group form a complete genus-class and that they all have the same ϕ-invariant. If Φ_M and Φ_N lie in the same genus and if R is a relation module, then by Roiter's replacement lemma there is a relation module R' such that

$$\Phi_M \oplus R \cong \Phi_N \oplus R' \tag{33}$$

Taking the ϕ-invariants on both sides and using cancellation in $Flas(W)$ gives $\phi(M) = \phi(N)$,done. □

Unfortunately, $Flas(W)$ is seldom cancellative so we have to find another approach.

Let $K_0(\mathbb{Z}W)$ be the Grothendieck group of finitely generated projective $\mathbb{Z}W$-modules. By a result of R. Swan we know that all projective $\mathbb{Z}W$-modules are locally free entailing that

$$K_0(\mathbb{Z}W) \cong \mathbb{Z} \times Cl(\mathbb{Z}W) \tag{34}$$

where for any $\mathbb{Z}$-order Λ in a finite dimensional semisimple $\mathbb{Q}$-algebra Δ the locally free classgroup $Cl(\Lambda) \subset K_0(\Lambda)$ is defined to be

$$Cl(\Lambda) = \{[P] - [P'] \in K_0(\Lambda) \mid P_p \cong P'_p \text{ for all primes } p\} \tag{35}$$

Now, let Λ be a maximal order in $\mathbb{Q}W$ containing $\mathbb{Z}W$ and define

$$D(\mathbb{Z}W) = Ker[Cl(\mathbb{Z}W) \to Cl(\Lambda)] \tag{36}$$

induced by the inclusion map. Then, $D(\mathbb{Z}W)$ is known to be independent of the choice of Λ and $Cl(\Lambda) = Cl(\mathbb{Z}W)/D(\mathbb{Z}W)$ is isomorphic to a product of ray classgroups of certain cyclotomic numberfields see e.g. [9]. Oliver [23,Th 7] has shown that

$$D(\mathbb{Z}W) \subset Cl^q(\mathbb{Z}W) \tag{37}$$

where $Cl^q(\mathbb{Z}W)$ is the subgroup of $Cl(\mathbb{Z}W)$ generated by the projective left ideals of $\mathbb{Z}W$ with trivial ϕ-invariant. Using this terminology we have

Lemma 10 *If $Cl^q(\mathbb{Z}W) = Cl(\mathbb{Z}W)$ then the genus obstruction disappears and any f.g. projective $\mathbb{Z}W$-lattice is stable permutation.*

Proof : Let Φ_M and Φ_N be in the same genus, then by Roiter's replacement lemma there is a projective left ideal I of $\mathbb{Z}W$ such that

$$\Phi_M \oplus \mathbb{Z}W \cong \Phi_N \oplus I \tag{38}$$

and taking the ϕ-invariants on both sides gives $\phi(M) = \phi(N)$. □

In particular, the lemma applies if $D(\mathbb{Z}W) = Cl(\mathbb{Z}W)$ as is the case for example if $\mathbb{Q}$ is a splitting field for W :

Lemma 11 *For S_n the genus obstruction vanishes and all f.g. projective $\mathbb{Z}S_n$- modules are stable permutation.*

4 Some Negative Results

In this section we will show that the local obstruction can be used to prove that $\phi(G_{p,p}) \neq 0$ if p is a prime larger than 3. We have seen before (lemma 7) that

$$\phi_p(G_{p,p}) = -[\Omega^2(\widehat{\mathbb{Z}_p})] \in Flas(\widehat{\mathbb{Z}_p}S_p) \tag{39}$$

Therefore, we have to show that $\Omega^2(\widehat{\mathbb{Z}_p})$ cannot be a stable permutation $\widehat{\mathbb{Z}_p}S_p$-lattice. The vertex of $\Omega^2(\widehat{\mathbb{Z}_p})$ is equal to the cyclic p-Sylow subgroup $G_p =< x = (1,...,p) >$ of S_p. The normalizer of G_p is a p-hypoelementary subgroup

$$N_p = N_{S_p}(G_p) =< x, y : x^p = y^{p-1} = 1, y.x.y^{-1} = x^a > \tag{40}$$

where a is a generator of the cyclic group $\mathbb{F}_p^*$.

Since the p-Sylow subgroup is cyclic of order p, non-projective indecomposable $\widehat{\mathbb{Z}_p}S_p$-lattices and $\widehat{\mathbb{Z}_p}N_p$-lattice behave very weel with respect to Green correspondence (see e.g. [1] or Feit III,5) :

Lemma 12 *There is a one-to-one correspondence between isomorphism classes of indecomposable non-projective $\widehat{\mathbb{Z}_p}S_p$-lattices M and indecomposable non-projective $\widehat{\mathbb{Z}_p}N_p$-lattices N such that*

$$M \downarrow_{N_p} = N \oplus P$$

$$N \uparrow^{S_p} = M \oplus P'$$

where P' (resp. P) is a projective $\widehat{\mathbb{Z}_p}S_p$- (resp. $\widehat{\mathbb{Z}_p}N_p$-) lattice.

In particular, the Green correspondent of the $\widehat{\mathbb{Z}_p}S_p$-lattice $\Omega^2(\widehat{\mathbb{Z}_p})$ is the $\widehat{\mathbb{Z}_p}N_p$-lattice $\Omega^2(\widehat{\mathbb{Z}_p})$. Now, let $X_1(\zeta^j)$ be the $\widehat{\mathbb{Z}_p}N_p$-lattice of rank one given by the action $x \to 1$ and $y \to \zeta^j$ where ζ is a primitive $(p-1)$-th root of unity reducing to a mod p. Then, it is easy to see (e.g. using [4,p.189] that we have :

Lemma 13 *The $\widehat{\mathbb{Z}_p}N_p$-lattice $\Omega^2(\widehat{\mathbb{Z}_p})$ is of rank one determined by the action $x \to 1$ and $y \to \zeta^{p-2}$ where ζ is a primitive $p-1$-th root of unity.*

This gives our first negative result :

Theorem 5 *$\phi(G_{p,p}) \neq 0$ in $Flas(S_p)$ for primes $p \geq 5$*

Proof : Suppose there exist S_p-sets S and T such that

$$\Omega^2(\widehat{\mathbb{Z}_p}) \oplus \widehat{\mathbb{Z}_p}S \cong \widehat{\mathbb{Z}_p}T \tag{41}$$

as $\widehat{\mathbb{Z}_p}S_p$-lattices. Restricting down to N_p gives us by Green correspondence an isomorphism of $\widehat{\mathbb{Z}_p}N_p$-lattices

$$X_1(\zeta^{p-2}) \oplus (\oplus_i \widehat{\mathbb{Z}_p}N_p/H_i) \oplus \mathbb{P} \cong (\oplus_j \widehat{\mathbb{Z}_p}N_p/K_j) \oplus \mathbb{P}' \tag{42}$$

where $\mathbb{P}$ and $\mathbb{P}'$ are the projective parts and H_i and K_j are subgroups of N_p containing C_p. Clearly, the projective parts can be cancelled showing that $X_1(\zeta^{p-2})$ must be a stable permutation $\widehat{\mathbb{Z}_p}N_p$-lattice which by the above lemma is impossible for $p \geq 5$ by looking at the character value on y which is not an integer. □

This result extends the Snider-Saltman vindication of Maruyama's argument to prime values of n.

The next best strategy is to find a lattice M of minimal rank such that $\phi(M) = \phi(G_{p,p})$. A natural candidate for M would be A^*_{p-1} the dual of the root lattice for the following reason : for any $\mathbb{Z}W$-lattice M it is easy to see that $\phi(M^*) = (\kappa(M))^*$ and from the previous discussions we have obtained that

$$\phi(G_{p,p}) = \phi(\kappa(A_{p-1})) = -[\kappa(A_{p-1})] \in Flas(S_p) \tag{43}$$

Thus, if we can prove that $[\kappa A_{p-1}]^* = -[\kappa A_{p-1}]$ then we would indeed have that $\phi(A^*_{p-1}) = \phi(G_{p,p})$. However, we have

Theorem 6 *$\phi(G_{p,p}) \neq \phi(A^*_{p-1})$ in $Flas(S_p)$ for all primes $p \geq 11$*

Proof : We know that $\kappa_p(A_{p-1}) = [\Omega^2(\widehat{\mathbb{Z}_p})]$ and thus $\kappa_p(A_{p-1})^* = [\Omega^{-2}(\widehat{\mathbb{Z}_p})]$. Under Green correspondence $\Omega^{-2}(\widehat{\mathbb{Z}_p})$ corresponds to the rank one $\widehat{\mathbb{Z}_p}N_p$-lattice $X_1(\zeta)$ and as in the above proof we have to exclude that $X_1(\zeta) \oplus X_1(\zeta^{p-2})$ can be a stable permutation $\widehat{\mathbb{Z}_p}N_p$-lattice. This is easy as the only permutation $\widehat{\mathbb{Z}_p}N_p$-lattice containing a rank one lattice corresponding to a primitive $p-1$-th root of unity must contain all. And, for $p \geq 11$ there are more than two primitive $p-1$-th roots of unity. □

What's the upshot of all this ? We do have :

Proposition 2 *For $p = 5$ or 7 all the local invariants ϕ_q of A^*_{p-1} and $G_{p,p}$ coincide*

Proof : In these cases we do have that

$$X_1(\zeta) \oplus X_1(\zeta^{p-2}) \tag{44}$$

is a stable permutation $\widehat{\mathbb{Z}}_p N_p$-lattice. By Green correspondence this implies that

$$\Omega^2(\widehat{\mathbb{Z}}_p) \oplus \Omega^{-2}(\widehat{\mathbb{Z}}_p) \oplus \mathbb{P} \tag{45}$$

is a stable permutation $\widehat{\mathbb{Z}}_p S_p$-lattice where $\mathbb{P}$ is a projective $\widehat{\mathbb{Z}}_p S_p$-module. Using that all projectives are stable permutation gives us the wanted $[(\kappa_p A_{p-1})^*] = -[\kappa_p A_{p-1}]$ in $Flas(\widehat{\mathbb{Z}}_p S_p)$ whence $\phi_p(G_{p,p}) = \phi_p(A^*_{p-1})$. Equality for the q-invariants for $p \neq q$ is easy. □

Hence, at least for $p \leq 7$ there is some hope to prove stable rationality by reducing the problem to the lattice invariants of the rank $p-1$ lattice A^*_{p-1}. However, for $p > 7$ things become uniformly bad.

5 Some Positive Results

There is a remarkable difference in the $\mathbb{Z}S_n$-structure of $G_{n,n}$ when n is prime or composite :

Proposition 3 *1. If n is composite $G_{n,n}$ cannot be coflasque*

2. For p prime $G_{p,p}$ is an invertible $\mathbb{Z}S_p$-lattice

Proof : (1) : Let $n = m.k$ with $m, k > 1$ and consider the subgroup $G = S_m \times S_{(k-1)m}$ of S_n acting in the natural way on the n elements. From the defining sequence of $G_{n,n}$ we get the exact sequence

$$V_n^G \xrightarrow{\pi^G} A_{n-1}^G \rightarrow H^1(G, G_{n,n}) \rightarrow 0 \tag{46}$$

Now, it is easy to see that

$$(\underbrace{k-1, ..., k-1}_{m}, \underbrace{-1, ..., -1}_{(k-1)m}) \in A_{n-1}^G \tag{47}$$

and that the image of V_n^G under π^G consists of the vectors

$$\mathbb{Z}(\underbrace{m(k-1),...,m(k-1)}_{m},\underbrace{-m,...,-m}_{(k-1)m}) \tag{48}$$

and therefore $H^1(G, G_{n,n}) \neq 0$,done.

(2) : To verify coflasque-ness it is enough to check that $H^1(G, G_{p,p}) = 0$ for each conjugacy class of a subgroup G of a q-Sylow subgroup of S_p. Now, if $q \neq p$ we can take G to be a subgroup of S_{p-1} and hence G fixes at least one element of U_p. But then it is easy to see that the sequence

$$0 \to G_{p,p} \to V_p \to U_p \to \mathbb{Z} \to 0 \tag{49}$$

splits everywhere as $\mathbb{Z}G$-lattices. So, $G_{p,p} \downarrow_G$ is an invertible $\mathbb{Z}G$-lattice and hence $H^1(G, G_{p,p}) = 0$. If $q = p$ then $G = C_p$ can be chosen to permute the base of U_p and so $A_{p-1}^G = 0$ and then the cohomology sequence entails that $H^1(C_p, G_{p,p}) = 0$.

So, $G_{p,p}$ is coflasque and thus $[\kappa(A_{p-1})] = [G_{p,p}] \in Flas(S_p)$. Using that $\kappa(A_{p-1})$ is an invertible $\mathbb{Z}S_p$-lattice finishes the proof. □

Therefore, in order to prove that $\phi(G_{p,p}) = \phi(A_{p-1}^*)$ for $p = 5$ or 7 it suffices to verify that

$$G_{p,p} \oplus G_{p,p}^* \tag{50}$$

is a stable permutation $\mathbb{Z}S_p$-lattice. For, we have the sequence

$$0 \to A_{p-1}^* \to V_p \to G_{p,p}^* \to 0 \tag{51}$$

and if we have that $G_{p,p} \oplus G_{p,p}^* \oplus \mathbb{Z}S \simeq \mathbb{Z}T$ for some S_p-sets S and T we can add $G_{p,p} \oplus \mathbb{Z}S$ to the last two terms and obtain the sequence

$$0 \to A_{p-1}^* \to G_{p,p} \oplus V_p \oplus \mathbb{Z}S \to \mathbb{Z}T \to 0 \tag{52}$$

yielding that $\phi(G_{p,p}) = \phi(A_{p-1}^*)$.

In the previous section we have seen that all local invariants of $G_{p,p} \oplus G_{p,p}^*$ vanish, so we have to check whether the Burnside obstruction vanishes. Thus, we have to compute the partial function $\chi : C(\mathcal{H}) \to \mathbb{Z}$ and verify that it lies in the image of the Burnside ring.

Now, let us first compute $\chi(H)$ for q-hypoelementary subgroups H of S_p if $q \neq p$. Working over $\widehat{\mathbb{Z}}_q$ we have by invertibility of $G_{p,p}$ that

$$G^q_{p,p} \oplus G^{*q}_{p,p} \oplus (U^q_p)^{\oplus 2} \simeq (V^q_p)^{\oplus 2} \oplus \widehat{\mathbb{Z}}_q \oplus 2 \tag{53}$$

and hence for a q-hypo H we can compute $\chi(H)$ by

$$\chi(H) = 2 + 2\#V_p^H - 2\#U_p^H \tag{54}$$

On the other hand, if $q = p$ we can use the fact that over $\widehat{\mathbb{Z}}_p$ we have

$$G^p_{p,p} \simeq \Omega^2(\widehat{\mathbb{Z}}_p) \oplus \widehat{\mathbb{Z}}_p S_p/(S_{p-2} \times S_2) \tag{55}$$

and use the explicit description of the stable permutation $\widehat{\mathbb{Z}}_p S_p$- lattice $\Omega^2(\widehat{\mathbb{Z}}_p) \oplus \Omega^{-2}(\widehat{\mathbb{Z}}_p)$ to compute $\chi(H)$ for p-hypoelementary subgroups.

Let us concentrate now on the case $p = 5$. We need to have fairly precise information on the conjugacy classes of subgroups of S_5 which we summarize in the following table :

class	representative	order	length	normalizer	hypo
A	1	1	1	S	2,3,5
B	$\langle (12) \rangle$	2	10	N	2,3,5
C	$\langle (12)(34) \rangle$	2	15	L	2,3,5
D	$\langle (123) \rangle$	3	10	N	2,3,5
E	$\langle (12),(34) \rangle$	4	15	L	2
F	$\langle (1234) \rangle$	4	15	L	2,3,5
G	$\langle (12)(34),(13)(24) \rangle$	4	5	Q	2
H	$\langle (12345) \rangle$	5	6	P	2,3,5
I	$\langle (12),(345) \rangle$	6	10	N	2,3,5
J	$\langle (123),(12) \rangle$	6	10	N	3
K	$\langle (123),(12)(45) \rangle$	6	10	N	3
L	$\langle (1234),(12)(34) \rangle$	8	15	L	2
M	$\langle (12345),(25)(34) \rangle$	10	6	P	5
N	$S_3 \times C_2$	12	10	N	none
O	A_4	12	5	Q	2
P	$\langle (12345),(2354) \rangle$	20	6	P	5
Q	S_4	24	5	Q	none
R	A_5	60	1	S	none
S	S_5	120	1	S	none

Using this information one can now describe the Burnside ring of S_5 as the image of $\mathbb{Z}^{\oplus 19}$ under multiplication on the right by the matrix :

$$\begin{pmatrix}
120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
60 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
40 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 6 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
24 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
20 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 3 & 3 & 0 & 1 & 1 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 4 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 4 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 2 & 4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
6 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 3 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Next, we compute the values of $\chi(H)$ for H a 2- or 3-hypoelementary subgroup of S_5. Using the above formula we obtain

H	$\chi(H)$
A	32
B	8
C	0
D	2
E	0
F	0
G	0
H	2
I	2
J	2
K	2
L	0
O	0

Moreover, we have over the 5-adic integers that

$$\Omega^2(\mathbb{Z}_5) \oplus \Omega^{-2}(\mathbb{Z}_5) \oplus \mathbb{Z}_5 S_5/M \simeq \mathbb{Z}_5 S_5/H \tag{56}$$

yielding the isomorphism

$$G_{5,5}^5 \oplus G_{5,5}^{*5} \oplus \mathbb{Z}_5 S_5/M \simeq \mathbb{Z}_5 S_5/H \oplus (\mathbb{Z}_5 S_5/N)^{\oplus 2} \tag{57}$$

allowing us to compute χ on 5-hypoelementary subgroups :

G	$\chi(G)$
M	-2
P	0

We can extend this partial function χ by unknowns $\chi(N) = a_1$, $\chi(Q) = a_2$, $\chi(R) = a_3$ and $\chi(S) = a_4$. Then, multiplying this χ-vector by the inverse of the Burnside matrix we get an integer valued vector (and hence the Burnside obstruction vanishes) provided we have that a_1 and a_2 are even and $a_3 \equiv a_4$ modulo 2. Hence, we can take all a_i to be zero and then we obtain from the above computations :

Lemma 14 *$G_{5,5} \oplus G_{5,5}^* \oplus \mathbb{Z}S_5/D \oplus \mathbb{Z}S_5/M$ lies in the same genus as $\mathbb{Z}S_5/H \oplus \mathbb{Z}S_5/I \oplus \mathbb{Z}S_5/J \oplus \mathbb{Z}S_5/K$. Moreover, $\phi(G_{5,5}) = \phi(A_4^*)$.*

Proof : The first statement follows from the above computations. Hence, because the genus obstruction always vanishes we have that $G_{5,5} \oplus G^*_{5,5}$ lies in the permutation tree. But then, since it is an invertible lattice it has to be stable permutation from which the last statement follows. □

Next, we have to find an ad hoc argument to prove (stable) rationality of the lattice invariants $\mathbb{C}(A_4^*)^{S_5}$. Clearly, we have the exact sequence

$$0 \to \mathbb{Z} \to U_5 \to A_4^* \to 0 \tag{58}$$

entailing that $\mathbb{C}(A_4^*)$ is the quotient field of the algebra $\mathbb{C}[U_5]/(u_1u_2u_3u_4u_5 - 1)$ and the S_5-action is induced by that on $\mathbb{C}(U_5)$. Clearly, the invariant field $\mathbb{C}(U_5)^{S_5}$ is rational on the elementary symmetric functions σ_i on the u_j. But then, $\mathbb{C}(A_4)^{S_5}$ is rational on the images of the first four elementary symmetric functions. One can also give an explicit trancendence basis of $\mathbb{C}(A_4^*)^{S_5}$ when $\mathbb{C}(A_4^*)$ is viewed as a quintic extension (obtained by adding $\sqrt[5]{a_1a_2a_3a_4}$) of $\mathbb{C}(A_4) = \mathbb{C}(a_1, a_2, a_3, a_4)$ with the S_5-action as in the following diagram

a	a_1	a_2	a_3	a_4
(12).a	$1/a_1$	a_2/a_1	a_3/a_1	a_4/a_1
(12345).a	a_2/a_1	a_3/a_1	a_4/a_1	$1/a_1$

Using this presentation we obtain that :

$$\mathbb{C}(A_4^*)^{S_5} = \mathbb{C}\left(\frac{\sum a_i + 1}{b}, \frac{\sum a_i^2 + 1}{b^2}, \frac{\sum a_i^3 + 1}{b^3}, \frac{\sum a_i^4 + 1}{b^4}\right) \tag{59}$$

Summarizing the above discussions we have proved :

Theorem 7 *Let X be a vectorspace with an almost free action of PGL_5. Then, the quotient variety X/PGL_5 is stably rational.*

In particular, this gives us :

Corollary 1 *The moduli space of stable rank 5 bundles over $\mathbb{P}_2$ with $c_1 = 0$ is stably rational.*

Now, let us turn our attention to the case when $p = 7$: Again, we need rather precise information on the conjugacy classes of subgroups of S_7. In this case there are 96 different classes out of which 55 are hypoelementary subgroups. In the appendix we have collected all the necessary information on these conjugacy classes as well as the Burnside matrix for S_7. In the rest of this section we follow the subgroup notations of the appendix.

As mentioned before it is easy to compute $\chi(H)$ of a q-hypoelementary subgroup with $q \neq 7$ using

$$\chi(H) = 2 + 2\#V_7^H - 2\#U_7^H \tag{60}$$

As V_7 (resp. U_7) is the permutation representation corresponding to the subgroup of class [88] (resp [94]) the values of χ are easily deduced from the Burnside matrix. We obtain :

H	[1]	[2]	[3]	[4]	[5]	[6]	[9]	[10]	[11]	[12]
$\chi(H)$	72	32	8	0	0	18	8	0	0	8
H	[13]	[14]	[15]	[7]	[16]	[17]	[18]	[19]	[20]	[21]
$\chi(H)$	8	0	0	2	2	0	0	0	2	2
H	[22]	[23]	[8]	[30]	[31]	[32]	[33]	[34]	[35]	[36]
$\chi(H)$	2	18	2	8	0	0	0	0	0	0
H	[24]	[25]	[26]	[27]	[37]	[43]	[44]	[45]	[47]	[48]
$\chi(H)$	0	2	2	2	2	2	2	2	2	8
H	[49]	[57]	[50]	[51]	[52]	[54]	[55]	[59]	[64]	[65]
$\chi(H)$	0	0	0	0	0	2	2	2	0	0
H	[69]	[74]								
$\chi(H)$	0	0								

Now, the more chalenging job to determine $\chi(H)$ for the three 7- hypoelementary subgroups [28],[29] and [56]. The starting point is the description of the stable permutation $\mathbb{Z}_7[56]$-lattice $\Omega^2_{[56]}(\mathbb{Z}_7) \oplus \Omega^{-2}_{[56]}(\mathbb{Z}_7) = W$ induced up to the S_7-level giving :

$$W \uparrow^{S_7} \oplus \mathbb{Z}_7[28] \oplus \mathbb{Z}_7[29] \simeq \mathbb{Z}_7[8] \oplus \mathbb{Z}_7[56] \tag{61}$$

where for each subgroup $[i]$ we denote by $\mathbb{Z}_7[i]$ the permutation lattice $\mathbb{Z}_7 S_7/[i]$. By Green correspondence we know that

$$W \uparrow^{S_7} = \Omega^2(\mathbb{Z}_7) \oplus \Omega^{-2}(\mathbb{Z}_7) \oplus \mathbb{P}^{\oplus 2} \tag{62}$$

where $\mathbb{P}$ is a projective $\mathbb{Z}_7 S_7$ lattice with characters

cyclic	[1]	[2]	[3]	[4]	[6]	[16]	[21]	[5]	[12]	[11]	[44]	[7]	[25]	[18]	[8]
charac	105	-5	1	-5	-3	1	1	-3	-1	1	-1	0	0	1	0

By James' result we know that $I\!P$ is stable permutation with all transitive permutation factors corresponding to Young subgroups. Hence, the description of $I\!P$ as a stable permutation character coincides with the description of the corresponding character as a linear combination of the Young-subgroup permutation characters. So, we have to multiply the character of $I\!P$ with the inverse of the permutation-character matrix :

$$\begin{pmatrix}
5040 \\
2520 & 120 \\
1260 & 120 & 12 \\
630 & 90 & 18 & 6 \\
840 & 120 & 0 & 0 & 24 \\
420 & 80 & 12 & 0 & 12 & 2 \\
210 & 50 & 14 & 6 & 6 & 2 & 2 \\
140 & 40 & 12 & 0 & 8 & 4 & 0 & 2 \\
210 & 60 & 6 & 0 & 24 & 0 & 0 & 0 & 6 \\
105 & 35 & 9 & 3 & 12 & 2 & 0 & 0 & 3 & 1 \\
35 & 15 & 7 & 3 & 5 & 3 & 1 & 2 & 1 & 1 & 1 \\
42 & 20 & 6 & 0 & 12 & 2 & 0 & 0 & 6 & 0 & 0 & 2 \\
21 & 11 & 5 & 3 & 6 & 2 & 2 & 0 & 3 & 1 & 0 & 1 & 1 \\
7 & 5 & 3 & 1 & 4 & 2 & 0 & 1 & 3 & 1 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}$$

Here, the columns correspond to the cyclics in the above mentioned ordering and the rows correspond to the Young-subgroups in the ordering : [1],[2],[9],[32],[23],[41],[58],[73],[60],[81],[91],[88],[92],[94],[96]. The resulting vector being

$$(0,0,1,-2,-1,2,1,-1,0,1,-1,-1,0,1,0) \tag{63}$$

we obtain the following description of the projective $I\!P$ as a stable permutation lattice

$$I\!P \oplus \mathbb{Z}_7[32]^{\oplus 2} \oplus \mathbb{Z}_7[23] \oplus \mathbb{Z}_7 \oplus [73] \oplus \mathbb{Z}_7[91] \oplus \mathbb{Z}_7[88]$$

$$\simeq \mathbb{Z}_7[9] \oplus \mathbb{Z}_7[41]^{\oplus 2} \oplus \mathbb{Z}_7[58] \oplus \mathbb{Z}_7[81] \oplus \mathbb{Z}_7[94]$$

Next, we use the fact that

$$G^7_{7,7} \oplus G^{7*}_{7,7} \simeq \Omega^2(\mathbb{Z}_7) \oplus \Omega^{-2}(\mathbb{Z}_7) \oplus \mathbb{Z}_7[92]^{\oplus 2} \qquad (64)$$

which allows us in view of the above mentioned facts to obtain a description of $V = G^7_{7,7} \oplus G^{7*}_{7,7}$ as a stable permutation lattice :

$$V \oplus \mathbb{Z}_7[9]^{\oplus 2} \oplus \mathbb{Z}_7[41]^{\oplus 4} \oplus \mathbb{Z}_7[58]^{\oplus 2} \oplus \mathbb{Z}_7[81]^{\oplus 2} \oplus \mathbb{Z}_7[94]^{\oplus 2} \oplus \mathbb{Z}_7[28] \oplus \mathbb{Z}_7[29]$$

$$\simeq \mathbb{Z}_7[56] \oplus \mathbb{Z}_7[8] \oplus \mathbb{Z}_7[92]^{\oplus 2} \oplus \mathbb{Z}_7[32]^{\oplus 4} \oplus \mathbb{Z}_7[23]^{\oplus 2} \oplus \mathbb{Z}_7[73]^{\oplus 2} \oplus \mathbb{Z}_7[91]^{\oplus 2} \oplus \mathbb{Z}_7[88]^{\oplus 2}$$

This allows us to compute the values for $\chi(H)$ for the three 7- hypoelementary subgroups of S_7 :

H	[28]	[29]	[56]
$\chi(H)$	-2	-1	1

This partial function χ can be shown to extend to an element in the Burnside ring of S_7. More precisely, the following combination of S_7- sets extends χ :

$$[1]-[2]-[3]+[4]-2.[6]+[8]-[10]-[13]-[18]+[20]+[23][24]-[28]-[29]+[30]$$

$$+[32]-[34]+[47]+[48]+[53]+[56]-[58]+[59]+[63]-[75]$$

Therefore, we obtain from the foregoing computations :

Lemma 15 *The lattice*

$$G_{7,7} \oplus G^*_{7,7} \oplus \mathbb{Z}[2] \oplus \mathbb{Z}[3] \oplus \mathbb{Z}[6]^{\oplus 2} \oplus \mathbb{Z}[10] \oplus \mathbb{Z}[13]$$

$$\oplus \mathbb{Z}[18] \oplus \mathbb{Z}[28] \oplus \mathbb{Z}[29] \oplus \mathbb{Z}[34] \oplus \mathbb{Z}[58] \oplus \mathbb{Z}[75]$$

lies in the same genus as the permutation $\mathbb{Z}S_7$-lattice

$$\mathbb{Z}[1] \oplus \mathbb{Z}[4] \oplus \mathbb{Z}[8] \oplus \mathbb{Z}[20] \oplus \mathbb{Z}[23] \oplus \mathbb{Z}[24] \oplus \mathbb{Z}[30]$$

$$\oplus \mathbb{Z}[32] \oplus \mathbb{Z}[47] \oplus \mathbb{Z}[48] \oplus \mathbb{Z}[53] \oplus \mathbb{Z}[56] \oplus \mathbb{Z}[59] \oplus \mathbb{Z}[63]$$

Moreover, $\phi(G_{7,7}) = \phi(A^*_6)$.

It is a bit surprising to note that in order to prove that the rank 36 lattice $G_{7,7}$ has the same ϕ-invariant as the rank 6 lattice A_6^*, we need to show that two lattices of rank 13412 lie in the same genus!

As in the 5×5 case it is easy to verify that $\mathbb{C}(A_6^*)^{S_7}$ is rational over $\mathbb{C}$ and therefore we obtain :

Theorem 8 *Let X be a vectorspace with an almost free action of PGL_7. Then, the quotient variety X/PGL_7 is stably rational.*

In particular, this gives us :

Corollary 2 *The moduli space of stable rank 7 bundles over $\mathbb{P}_2$ with $c_1 = 0$ is stably rational.*

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Appendix

A Subgroups of S_7

In this appendix we give the relevant information about conjugacy classes of subgroups of S_7. In the following list we give for each conjugacy class the order of a representative, a list of generators for a representative (note however that these generators are often not minimal nor the most elegant ones) and the primes for which the representative is hypoelementary :

```
SUBGROUPS OF S7
---------------
     [ 1 ] ORDER 1 ,GENERATORS :  IDENTITY                 ALL HYPO

     [ 2 ] ORDER 2 ,GENERATORS :  (6,7)                    ALL HYPO

     [ 3 ] ORDER 2 ,GENERATORS :  (1,2)(3,4)               ALL HYPO

     [ 4 ] ORDER 2 ,GENERATORS :  (1,5)(3,6)(4,7)          ALL HYPO

     [ 5 ] ORDER 3 ,GENERATORS :  (1,5,2)(3,4,7)           ALL HYPO

     [ 6 ] ORDER 3 ,GENERATORS :  (1,6,4)                  ALL HYPO
```

[7] ORDER 5 ,GENERATORS : (1,4,2,3,5) ALL HYPO

[8] ORDER 7 ,GENERATORS : (1,4,5,3,2,7,6) ALL HYPO

[9] ORDER 4 ,GENERATORS : (2,4), (6,7) 2 HYPO

[10] ORDER 4,GENERATORS : (1,2)(3,4), (6,7) 2 HYPO

[11] ORDER 4,GENERATORS : (1,3,2,4)(5,6) ALL HYPO

[12] ORDER 4,GENERATORS : (1,4,2,3) ALL HYPO

[13] ORDER 4,GENERATORS : (1,2)(3,4), (1,4)(2,3) 2 HYPO

[14] ORDER 4,GENERATORS : (1,2)(3,4), (3,4)(6,7) 2 HYPO

[15] ORDER 4,GENERATORS : (1,2)(3,4), (1,4)(2,3)(5,6) 2 HYPO

[16] ORDER 6,GENERATORS : (1,5,3), (6,7) ALL HYPO

[17] ORDER 6,GENERATORS : (1,5,2)(3,4,7), (1,2)(3,4) 3 HYPO

[18] ORDER 6,GENERATORS : (1,5,2)(3,4,7), (1,7)(2,4)(3,5)ALL HYPO

[19] ORDER 6,GENERATORS : (1,5,2)(3,4,7), (1,4)(2,7)(3,5) 3 HYPO

[20] ORDER 6,GENERATORS : (1,6,4), (1,4)(2,3) 3 HYPO

[21] ORDER 6,GENERATORS : (1,6,4), (2,3)(5,7) ALL HYPO

[22] ORDER 6,GENERATORS : (1,6,4), (1,4)(2,5)(3,7) 3 HYPO

[23] ORDER 6,GENERATORS : (1,6,4), (4,6) 3 HYPO

[24] ORDER 9,GENERATORS : (1,5,2), (3,4,7) 3 HYPO

[25] ORDER 10,GENERATORS : (1,4,2,3,5), (6,7) ALL HYPO

[26] ORDER 10,GENERATORS : (1,4,2,3,5), (1,3)(2,4) 5 HYPO

[27] ORDER 10,GENERATORS : (1,4,2,3,5), (1,5)(3,4)(6,7) 5 HYPO

[28] ORDER 14,GENERATORS :(1,4,5,3,2,7,6), (1,3)(2,6)(4,5) 7 HYPO

[29] ORDER 21,GENERATORS :(1,4,5,3,2,7,6), (1,4,7)(2,3,6) 7 HYPO

[30] ORDER 8 ,GENERATORS : (2,7,5,3), (2,3)(5,7) 2 HYPO

[31] ORDER 8 ,GENERATORS : (5,6), (1,4,2,3) 2 HYPO

[32] ORDER 8 ,GENERATORS : (1,2), (3,4), (6,7) 2 HYPO

[33] ORDER 8 ,GENERATORS : (1,2)(3,4), (1,4)(2,3), (5,6) 2 HYPO

[34] ORDER 8 ,GENERATORS : (2,7,4,6)(3,5), (2,6)(3,5)(4,7) 2 HYPO

[35] ORDER 8 ,GENERATORS : (1,3,2,4)(5,6), (1,4)(2,3) 2 HYPO

[36] ORDER 8 ,GENERATORS : (1,4,2,3), (3,4)(6,7) 2 HYPO

[37] ORDER 12,GENERATORS : (1,5,3), (2,4), (6,7) 2 HYPO

[38] ORDER 12,GENERATORS : (1,5,2)(3,4,7), (1,2)(3,4),
(1,4)(2,7)(3,5) NOT

[39] ORDER 12,GENERATORS : (1,6,4), (2,5)(3,7), (1,4) NOT

[40] ORDER 12,GENERATORS : (1,5,3), (1,3)(2,4), (6,7) NOT

[41] ORDER 12,GENERATORS : (1,5,3), (3,5), (6,7) NOT

[42] ORDER 12,GENERATORS : (1,6,4), (1,4)(2,3), (2,3)(5,7) NOT

[43] ORDER 12,GENERATORS : (1,6)(2,7,3,5), (2,7,3,5)(4,6) 3 HYPO

[44] ORDER 12,GENERATORS : (2,7,3,5), (1,6,4) ALL HYPO

[45] ORDER 12,GENERATORS : (1,6,4), (2,3)(5,7), (2,7)(3,5) 2 HYPO

[46] ORDER 12,GENERATORS :(1,6,4),(2,3)(5,7),(1,4)(2,5)(3,7) NOT

[47] ORDER 12,GENERATORS : (1,4,2)(5,6,7), (1,2,3)(5,6,7) 2 HYPO

[48] ORDER 12,GENERATORS : (1,4,2), (1,4,3) 2 HYPO

[49] ORDER 12,GENERATORS : (1,4,7)(2,3,6), (1,3,6)(2,4,7) 2 HYPO

[50] ORDER 18,GENERATORS :(1,5,2)(3,4,7), (1,2,5)(3,4,7), (1,7)(2,4)(3,5) 3 HYPO

[51] ORDER 18,GENERATORS : (1,5,2), (3,4,7), (1,2)(3,4) 3 HYPO

[52] ORDER 18,GENERATORS : (1,2,5), (3,4,7), (3,4) 3 HYPO

[53] ORDER 20,GENERATORS : (1,4,2,3,5), (1,3)(2,4), (1,5)(3,4)(6,7) NOT

[54] ORDER 20,GENERATORS : (2,4,3,5)(6,7), (1,2,3,4)(6,7) 5 HYPO

[55] ORDER 20,GENERATORS : (1,4,3,2), (2,5,3,4) 5 HYPO

[56] ORDER 42,GENERATORS : (1,4,5,3,2,7,6),(1,4,7)(2,3,6), (1,3)(2,6)(4,5) 7 HYPO

[57] ORDER 16,GENERATORS : (1,3,2,4)(6,7), (1,4,2,3), (1,4)(2,3) 2 HYPO

[58] ORDER 24,GENERATORS : (1,5,3), (1,3), (2,4), (6,7) NOT

[59] ORDER 24,GENERATORS : (2,7,3,5), (1,6,4), (2,7)(3,5) 2 HYPO

[60] ORDER 24,GENERATORS : (2,7,3,5), (2,3,7,5) NOT

[61] ORDER 24,GENERATORS : (5,6), (5,7), (1,4,2,3) NOT

[62] ORDER 24,GENERATORS : (5,7,6),(1,2)(3,4),(1,4)(2,3), NOT
(5,6)

[63] ORDER 24,GENERATORS : (2,7,4,6)(3,5), (1,3)(2,6,4,7), NOT
(2,6)(3,5)(4,7)

[64] ORDER 24,GENERATORS : (1,4,7)(2,3,6),(1,3,6)(2,4,7), 2 HYPO
(1,2)(3,4)(6,7)

[65] ORDER 24,GENERATORS :(1,4,2),(1,4,3),(1,2)(3,4)(5,6) 2 HYPO

[66] ORDER 24,GENERATORS :(1,6)(2,7,3,5), (2,7,3,5)(4,6), NOT
(1,4)(2,3)

[67] ORDER 24,GENERATORS : (2,7,3,5), (1,6,4), (1,4)(2,3) NOT

[68] ORDER 24,GENERATORS : (1,3,2,4)(5,6), (1,3,4,2)(6,7) NOT

[69] ORDER 24,GENERATORS : (1,3,2,4)(5,6), (1,4,3,2)(5,6) 2 HYPO

[70] ORDER 24,GENERATORS : (1,6,2,7)(3,4), (1,3,2,4)(6,7) NOT

[71] ORDER 24,GENERATORS : (1,4,2,3), (3,7,4,6) NOT

[72] ORDER 36,GENERATORS : (1,2,5), (3,4,7), (1,2)(3,4), NOT
(1,7)(2,4)(3,5)

[73] ORDER 36,GENERATORS : (1,2,5), (3,4,7), (1,2), (3,4) NOT

[74] ORDER 36,GENERATORS : (1,7)(2,3,5,4),(1,7,5,4)(2,3) 3 HYPO

[75] ORDER 36,GENERATORS : (1,6,4)(2,5,7), (1,6,4)(3,5,7) NOT

[76] ORDER 40,GENERATORS : (2,4,3,5), (6,7), (1,4,3,2) NOT

[77] ORDER 60,GENERATORS : (1,3,6,4,5), (1,5,3,4,6) NOT

[78] ORDER 60,GENERATORS : (1,5,7,4,6), (3,4,6,5,7) NOT

[79] ORDER 48,GENERATORS : (5,6), (5,7), (1,4,2,3), (1,4)(2,3) NOT

[80] ORDER 48,GENERATORS : (1,7,5,4)(3,6), (4,7), (1,6,5,3) NOT

[81] ORDER 48,GENERATORS : (5,6), (1,4,3,2), (1,4,2,3) NOT

[82] ORDER 72,GENERATORS : (1,7)(2,3,5,4), (1,7,5,4)(2,3), (1,7)(2,4)(3,5) NOT

[83] ORDER 72,GENERATORS :(2,7,3,5),(2,3,7,5),(1,6,4)(2,5,7) NOT

[84] ORDER 72,GENERATORS : (1,3,2,4)(5,6), (1,4,3,2)(5,6), (1,3,4,2)(6,7) NOT

[85] ORDER 72,GENERATORS : (1,4,2)(5,6,7), (1,2,4)(5,6,7), (1,3,2)(5,6,7), (1,2)(3,4)(5,6) NOT

[86] ORDER 120,GENERATORS : (3,7,4,6), (1,6,5,7), (1,6,3,5) NOT

[87] ORDER 120,GENERATORS : (1,3,5,4)(2,7), (1,3,4,5)(2,7), (1,6,4,3)(2,7) NOT

[88] ORDER 120,GENERATORS : (3,6,4,5), (1,6,3,5) NOT

[89] ORDER 120,GENERATORS : (1,3,6,4,5), (1,5,3,4,6), (2,7)(3,4)(5,6) NOT

[90] ORDER 168,GENERATORS : (1,4,3,2)(5,6), (1,6,2,7)(3,4) NOT

```
[ 91 ] ORDER 144,GENERATORS : (1,4),(2,7,5,3),(1,6),(2,7,3,5)  NOT

[ 92 ] ORDER 240,GENERATORS :  (1,3,5,4)(2,7),(1,3,4,5)(2,7),  NOT
                               (1,6,4,3)(2,7), (3,6,4,5)

[ 93 ] ORDER 360,GENERATORS :  (1,3,4,2)(6,7), (1,6,2,7)(3,4)  NOT

[ 94 ] ORDER 720,GENERATORS : (1,3,4,2)(6,7),(1,6,2,7)(3,4),   NOT
                                       (1,4,2,3)

[ 95 ] ORDER 2520,GENERATORS :(1,3,2,4)(5,6), (1,2)(3,4,6,5),  NOT
                                        (1,3,7,2)(4,5)

[ 96 ] ORDER 5040,GENERATORS :  (5,6), (1,2)(3,4,6,5),         NOT
                                (1,3,7,2)(4,5), (1,4,2,3)
```

B Burnside matrix of S_7

The following figure is the Burnside matrix for S_7 using the notations of the foregoing list. Note that this time the subgroups are ordered in ascending size :

	1	2	3	4	5	6	9	10	11	12	13	14	15	7	16	17	18	19	20	21	22	23	8	30	31	32	33	34	35	36
1	5040																													
2	2520	120																												
3	2520		24																											
4	2520			24																										
5	1680				12																									
6	1680					48																								
9	1260	120	12				12																							
10	1260	60	12	12				4																						
11	1260		12						4																					
12	1260		12							12																				
13	1260		36								36																			
14	1260		36									12																		
15	1260		12	24									4																	
7	1008													8																
16	840	40				24									4															
17	840		24		6											2														
18	840			8	6												2													
19	840			24	6													6												
20	840		24			24													4											
21	840		8			24														8										
22	840			24		24															8									
23	840	120				24																24								
8	720																						6							
30	630	60	18				6			6	18													6						
31	630	30	6	6				2	2	6															2					
32	630	90	18	6			18	6				6														6				
33	630	30	18	18				6			18		6														6			
34	630	60	6	12			6		2				2															2		
35	630		30						2		18	6																	2	
36	630		18	12						6		6	2																	2
24	560				8	32																								
25	504	24												4																
26	504		24											4																
27	504			24										4																
37	420	40	4			12	4								4					4										
38	420		12	16	3								4			1	1	3												
39	420	60	4	12		12		4												4	4	12								
40	420	20	12	12		12		4							2				2		4									
41	420	80	12			12	12								2				2			12								
42	420		28			12						12							4	4										
43	420		4			12			4											4										
44	420		4			12				4										4										
45	420		12			12					12									12										
46	420		4	24		12							4							4	8									
47	420		12		12						12																			
48	420		12			48					12																			
49	420		12		12							4																		
28	360			24																			3							
57	315	45	15	9			9	5	1	3	9	3	3											3	1	3	3	1	1	1
50	280			8	4	16											2	2												
51	280		24		4	16										4			8											
52	280	40			4	16									4							8								
53	252	12	12	12				4						2																
54	252		12						4					2																
55	252		12							12				2																
29	240				12																		2							
58	210	50	14	6		6	14	6				6			2				2	2	2	6				6				
59	210	20	6			6	2			2	6				2					6				2						
60	210	60	6			24	6			6	6											24		6						
61	210	30	2	6		6		2	2	2										2	2	6			2					
62	210	30	6	18		6		6			6		6							6	6	6					6			
63	210	20	2	12		6	2		2				2		2					2	4							2		
64	210	30	6	2	6		6	2				2					2									2				
65	210	10	6	6		24		2			6		2		4												2			
66	210		18			6			2		6	6							2	6									2	
67	210		14	12		6				2		6	2						2	2	4									2
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56	120			8	6												2						1							
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