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Graded Orders

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CONTENTS

Introduction.

Tracing back the Nile to its origin must be about as difficult as tracing back the origins of our interest in the theory of orders. At many junctions one has to choose in an almost arbitrary way which is the Nile and which is the other river joining it, wondering whether in such problems one should stick to the wider or to the deeper stream. Perhaps a convenient solution is to recognize that there are many sources and then to list just a few. Those inspired by number theory will certainly think first about the theory of maximal orders over Dedekind domains in number fields, the representation theory-based algebraist will refer to integral group rings, an algebraic geometer will perhaps point to orders over normal domains, and the ring theorist might view orders in central simple algebras as his favorite class of P.I. rings. In these topics graded orders and orders over graded rings appear not only as natural examples, but also as important basic ingredients : crossed products for finite groups, group rings considered as graded rings, orders over projective varieties, rings of generic matrices, trace rings, etc. On these observations we founded our belief that the application of methods from the theory of graded rings to the special case of orders may lead to some interesting topics for research, new points of view, and results. The formulation of this intent alone creates several problems of choice. Do we consider graded orders or orders over graded commutative rings or orders satisfying both ? What restrictions on the grading groups are allowable and which conditions must be considered too restrictive ? Which conditions on the gradation do we allow ? Which are the central ground rings we consider ? Let us describe the program for this book by working through these questions starting with the last one. A simple dimension argument shows that apart from $k[t]$, $k[t,t^{-1}]$ where k is a field, there are not many Dedekind domains graded by the integers; similarly, it is hard to give an example of a group ring over a Dedekind domain which is again a Dedekind domain (in fact there is no nontrivial example). Therefore, allowing at least polynomial extensions of Dedekind domains, one should consider the class of Krull domains as a natural first candidate. Since the centre of a maximal order is a completely integrally closed domain, the class of Krull domains is also very close to the most general one for which the existence of maximal orders is still guaranteed. In Chapter I we provide a rather detailed treatment of graded Krull domains and in Section II.4. we recall some of L. Silver's work on tame orders over Krull domains, cf. [49], but in the generality allowed by R.M. Fossum's treatment of this theory in [20].

In order to decide what conditions are necessary on the gradation we have to point out that one of the main tools in the application of graded ring theory to orders is the construction of so called generalized Rees rings over orders. The gradations on group rings, twisted group rings, and crossed products all have the property that $R_\sigma R_\tau = R_{\sigma\tau}$ for every $\sigma,\tau \in G$ where $R = \oplus_{\sigma\in G} R_\sigma$ is a G-graded ring. Rings which are graded in this way are said to be strongly graded. In Section II.2. we show that it is much more natural, in view of the fact that we are dealing with orders over Krull domains and with reflexive modules on most occasions, to weaken the foregoing condition to $(R_\sigma R_\tau)^{**} = R_{\sigma\tau}$ where $**$ denotes the double dual (of $Z(R_e)$-lattices).
For these divisorially graded rings it is possible to relate the class group of R_e and R in terms of the image of the group morphism $G \to CCl(R_e)$ which derives from the existence of the divisorial gradation on R (e is the unit element of G). The vanishing of the class group of an order may in turn be related to the structural properties of the order : if the class group of the order equals the class group of the centre then the order is a reflexive Azumaya algebra, up to some exceptional cases that can easily be traced and excluded. Since both the structure of a reflexive Azumaya and the properties of divisorially graded rings are easily investigated this presents a method for studying orders that we

intend to focus on in this work. Evidently the divisorially graded rings will thus be of capital interest to us. In Section II.3. we investigate the properties obtained from a combination of P.I. theory and of the theory of divisorially graded rings; here some restrictions on the grading group are necessary.

Concerning restrictive conditions on the type of groups considered let us point out that the centre of a graded order is in general not graded, as when the group is finite, but it is the case when G is abelian. The fact that we are studying prime rings entails that two specific cases appear naturally : G is a finite group or G is a torsion free abelian group. This explains the splitting of part of Chapter III.

If we add that Section II.5. contains a few rather general methods that will be applied occasionally in the book, and that Section III.4. is an application of the theory of divisorially graded rings to the study of extensions of tame orders, we have completed a rough survey of the first three chapters.

Chapter IV is of a more geometrical inspiration. Indeed, the study of regularity of orders may be thought of (for those readers who like to indulge in such thought-experiments) as the search for a good notion of regularity for "non-commutative varieties" in an algebraic geometry for P.I. Rings. Since regularity is a local condition we restrict ourselves to orders of finite type over a local Noetherian domain contained in the centre of the order. Using the concept of moderated Gorenstein algebras we define moderated regular algebras (by adding the condition that the global dimension is finite) and prove that such algebras have an integrally closed centre which is even a Cohen-Macauley ring in many cases. It turns out that a moderated regular order is a tame order. The generalized Rees ring constructions may then be used to construct moderated regular orders. Let us point out that tame orders of global dimension two are moderated regular. The centre of a generalized Rees ring turns out to be a so-called "scaled" Rees ring. Therefore we investigate in Section IV.3. how the regularity of a normal domain behaves under a scaled Rees ring extension.

Section IV.4 deals with moderated regular orders having a suitably nice ramification divisor; these smooth orders are defined, roughly stated, by the fact that a suitable generalized Rees ring extension becomes an Azumaya algebra with regular centre and we obtain a good picture of the structure of such orders in division rings. In this section we use the graded Brauer group of a graded ring without going too deeply into this theory; we refer to [12] for a detailed account of it.

The final section is concerned with a certain open subset of the Brauer-Severi scheme of a smooth maximal order.

In the final chapter, we continue considering orders of finite representation type, striving for a classification of these in terms of certain invariants. Theorem V.2.20 states that the representation theory of two dimensional tame orders is determined by a rational double point together with the action of a cyclic group on its module category. A structure theorem for tame orders of such two-dimensional orders of finite representation type is given and the Cohen-Macaulay modules are related to certain projective representations of a finite group. In a finite appendix we give an idea about the classification of two-dimensional orders of finite representation type in terms of generators and relators and we provide an outline of how the first classification (following recent work of M. Artin [3], [4]) relates to the latter one (following I. Reiten, M. Van den Bergh [46]). Chapter V is somewhat less self-contained than the other chapters, however we have provided adequate references where necessary.

I. Commutative Arithmetical Graded Rings.

The graded rings encountered in this chapter are the ones that will appear as the centres of the graded orders considered in this book. An order graded by an arbitrary group need not have a graded centre, but when the grading group is abelian this property does hold. Because we exclusively consider orders over domains it makes sense to restrict attention to commutative rings which are graded by torsion free abelian groups, in particular where Krull domains are concerned. On the other hand, constructions over gr-Dedekind rings appear as examples or in the constructive methods for studying class groups, hence it will be sufficient to develop the basic facts about gr-Dedekind rings and the related valuation theory in the $\mathbb{Z}$-graded case only.

I.1. Graded Krull Domains.

In this section G is always a torsion free abelian group and $\Gamma \subset G$ is a submonoid such that the group $< \Gamma >$ generated by Γ is exactly G. Since G is ordered the Γ considered usually will be the set of non-negative elements of G.

Let R be a commutative domain graded by Γ. The set $S = \{x \neq 0, x$ homogeneous in $R\}$ is multiplicatively closed and $S^{-1}R$ is G-graded such that $(S^{-1}R)_0$ is a field and every homogeneous element x of $S^{-1}R$, $x \neq 0$, is invertible. We write $Q^g = S^{-1}R$ and we sometimes say that Q^g is a gr-field; let us look more closely at the structure of a gr-field. We have an exact sequence :

$$1 \to ((Q^g)_0)^* \to (Q^g)^* \underset{\deg}{\longrightarrow} G \to 1$$

By chosing a representative $u_\sigma \in Q^g$ for $\sigma \in G$, we see that $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma\tau}$ for $\sigma, \tau \in G$, where $c : G \times G \to ((Q^g)_0)^*$ is a 2-cocycle. Consequently : $Q^g = (Q^g)_0 G^t$, the twisted group ring with respect to the cocycle c. If G is a free group then $Q^g \cong (Q^g)_0 G$. Since every finitely generated subgroup H of G is necessarily free it is easily seen that $Q^g = (Q^g)_0 G$ holds too whenever $(Q^g)_0$ is root-closed.

I.1.1. Proposition. 1. Q^g is completely integrally closed,
2. Q^g is a Krull domain if and only if it satisfies the ascending chain condition on principal ideals, if and only if Q^g is factorial.
3. If G satisfies the ascending chain condition on cyclic subgroups then Q^g is a Krull domain (the converse of this does not hold in general).

Proof. 1. Since Q^g is the direct limit of the subrings $Q^g_{(H)} = \oplus_{\sigma \in H}(Q^g)_\sigma$, where H is a finitely generated subgroup of G, it suffices to look at the latter rings. Now H is a free group, hence each $Q^g_{(H)}$ is completely integrally closed. If $Q_{(H)}$ is the field of fractions of $Q^g_{(H)}$ then $Q^g \cap Q_{(H)} = Q^g_{(H)}$, for every finitely generated subgroup H of G. Under these conditions it is not hard to verify that the property of being completely integrally closed is preserved under taking direct limits over subgroups H of G as above.
2. If $x, y \in Q^g$, say $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$, $y = y_{\tau_1} + \ldots + y_{\tau_m}$ then $x, y \in Q^g_{(H)}$ where H is generated by $\{\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m\}$. Since $Q^g_{(H)}$ is factorial, $xQ^g_{(H)} \cap yQ^g_{(H)} = zQ^g_{(H)}$ for some $z \in Q^g_{(H)}$. But Q^g is a free $Q^g_{(H)}$-module, hence faithfully flat and therefore $zQ^g = xQ^g \cap yQ^g$.
3. Follows easily from 1. and 2.
4. The algebraic closure $\overline{k}$ of $k = (Q^g)_0$ is root closed, so $\overline{k} \otimes_k Q^g \cong \overline{k}G$. A result of Matsuda states that any group ring AG is a Krull domain if and only if A is a Krull domain and G satisfies the ascending chain condition on cyclic subgroups, cf. [32]. □

I.1.2. Corollaries. 1. Let $R = \oplus_{\sigma \in \Gamma} R_\sigma$ be a Γ-graded domain. If Γ satisfies the ascending chain condition on cyclic submonoids then $S^{-1}R = Q^g$ is a Krull domain and factorial.
2. If Q^g is factorial and H is any subgroup of G then $Q^g_{(H)}$ is again factorial.

Let K be the field of fractions of R; we will consider fractional R-ideals (in K) and those that are contained in R will be called **integral R-ideals** but if no confusion is possible we simply refer to all of these as "ideals of R". If I and J are nonzero fractional R-ideals, we define $(I : J) = \{x \in K, xJ \subset I\}$ and this is again a fractional R-ideal. We denote : $(R : I) = I^{-1}$, $R : (R : I) = I^{**} = I_\nu$. A fractional R-ideal I is commonly called a **divisorial R-ideal** (or a **ν-ideal**) if $I = I^{**}$. A fractional R-ideal J is graded if there is an $x \in S$ such that xJ is a graded ideal of R; consequently the graded fractional R-ideals are all contained in Q^g. Clearly, I^{**} and I^{-1} will be graded as soon as I graded. Let $D(R)$ be the monoid of divisorial R-ideals with respect to the multiplication $I * J = (IJ)^{**}$.
The **divisor class monoid** of R, denoted by $Cl(R)$, is $D(R)$ modulo

the group of principal fractional ideals of R. It is well-known that $Cl(R)$ is a group if and only if R is completely integrally closed and it is then called the **class group of** R. If R is a graded domain we define the **graded class monoid** and the **graded class group**, $Cl^g(R)$, as the submonoid of $Cl(K)$ consisting of the elements which are represented by graded divisorial R-ideals. If $x \in R$ is decomposed as $x = x_{\sigma_1}+\ldots+x_{\sigma_n}$; $\sigma_1, \ldots, \sigma_n \in \Gamma$ then we define the **content** of x to be $C(x) = Rx_{\sigma_1} + \ldots + Rx_{\sigma_n}$. Recall the following lemma due to D. G. Northcott, [40].

I.1.3. Lemma : Let the commutative domain R be graded by Γ. For every $x, y \in R$, $C(x)^n C(xy) = C(x)^{n+1} C(y)$, for some $n \in \mathbb{N}$.
If I is any ideal of R then I_g is the graded ideal of R generated by the homogeneous elements of I and I^g is the smallest graded ideal of R containing I i.e. the ideal generated by the homogeneous components of elements in I. Clearly if $I = Rx$, $x \in R$, then $I^g = C(x)$.
We now review some results of D.D. Anderson, D.F. Anderson, [1].

I.1.4. Theorem : Let R be a domain graded by Γ, then the following statements are equivalent :
1. For $s \in S$, $x \in R$, $(Rs : Rx)$ is graded.
2. If I is divisorial in R with $I_g \neq 0$ then $I = I_g$.
3. If I is divisorial of finite type in R with $I_g \neq 0$ then $I = I_g$.
4. For all $x, y \neq 0$ in R, $((xy)^{**} = (C(x)C(y))^{**}$
5. For each $x \neq 0$ in R, $Q^g x \cap R = xC(x)^{-1}$.
6. If I is divisorial in R and I is of finite type, then $I = qJ$ for some $q \in Q^g$ and some graded divisorial ideal J in R which is also of finite type.

Proof. Cf. Theorem 3.2. of [1]. □

Note that for $\Gamma = \mathbb{Z}$ or $\Gamma = Z_+$, the six conditions of Theorem I.1.4. are also equivalent to the following : each divisorial ideal in R is of the form xJ for some $x \in S^{-1}$ and some graded divisorial ideal of J in R.
The Γ-graded domain R is said to be **almost normal** if any homogeneous $x \in Q^g = S^{-1}R$ with $\deg(x) \neq 0$ which is integral over R is contained in R. In the torsion free abelian case it is easily verified that R is integrally closed if and only if it is gr-integrally closed in Q^g;

thus R is integrally closed if and only if R is almost normal and R_0 is integrally closed in $(Q^g)_0$. Note also that for an almost normal domain R, the integral closure $\overline{R}$ of R is given by $\overline{R}_o \oplus (\oplus_{\gamma\neq 0} R_\gamma)$ where $\overline{R}_0$ is the integral closure of R_0.

I.1.5. Theorem. Let R be a Γ-graded domain.
1. If $C(xy)^{**} = (C(x)C(y))^{**}$ for all $x, y \in R - \{0\}$ then R is almost normal.
2. If R is integrally closed then $C(xy)^{**} = (C(x)C(y))^{**}$ for $x, y \in R - \{0\}$. Consequently R is integrally closed if and only if R_0 is integrally closed in $(Q^g)_0$ and any one of the equivalent conditions of Theorem I.1.4. holds, in particular if $C(xy)^{**} = (C(x)C(y))^{**}$.
3. If R contains a unit of nonzero degree then R is integrally closed if and only if R is almost normal.

Proof. 1. Let $x \in (Q^g)_\gamma, \gamma \neq 0$, be integral over R. The monic polynomial satisfied by x may be taken to be : $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ where each a_j is homogeneous of degree γ^{n-j}. Over Q^g we may factorize $f(x) = (X - x)g(X)$ where the coefficients of $g(X)$ in Q^g are homogeneous of nonzero degree, say $g(X) = X^{n-1} + b_{n-2}X^{n-2} + \ldots + b_0$. Clearly, $C(1 - x)$ contains x and $C(g(1)) = R + Rb_{n-2} + \ldots + Rb_0$. Hence $x \in C(1-x)C(g(1)) \subset (C(1-x)C(g(1))^{**} = C((1-x)g(1))^{**} = C(f(1))^{**}$ is contained in R.

2. If R is integrally closed, take $z \in C(x)C(y) :_K C(xy)$. So $zC(xy)C(x)^n \subset C(x)C(y)C(x)^n = C(xy)C(x)^n$ by Lemma I.1.3.
Thus $z \in C(xy)C(x)^n xy)C(x)^n :_K C(xy)C(x)^n = R$ because R is integrally closed.
Consequently, $C(x)C(y) :_K C(xy)) :_K C(xy) = R$ and furthermore :

$$\begin{aligned} R \underset{K}{:} C(x)C(y) &= (C(x)C(y) \underset{K}{:} C(xy)) \underset{K}{:} C(x)C(y) \\ &= C(x)C(y) \underset{K}{:} C(xy)C(xy)C(x)C(y) \\ &= (C(x)C(y) \underset{K}{:} C(x)C(y)) \underset{K}{:} C(xy) = R \underset{K}{:} C(xy) \end{aligned}$$

It follows that $C(xy)^{**} = (C(x)C(y))^{**}$.

3. Since any unit in a domain graded by a torsion free abelian group is homogeneous (also for $\Gamma \subset G$) we may assume that u is a unit of

nonzero degree γ of R. If $x \in (Q^g)_0$ is integral over R then $ux \in (Q^g)_\gamma$ is integral over R. If R is almost normal then $ux \in R$ and $x \in R$ follows, so R is then also integrally closed. The converse follows from 2. and 1. □

I.1.6. Proposition. Let Γ be the set of non-negative elements in an ordering of the torsion free abelian group G and let R be a Γ-graded domain. The conditions of Theorem I.1.4. are equivalent to R being almost normal.

Proof. We have to prove the converse to part 1 in the foregoing theorem. Assuming that R is almost normal its integral closure equals $\overline{R}_0 \oplus (\bigoplus_{0\neq\gamma\in\Gamma} R_\gamma)$ where $\overline{R}_0$ is the integral closure of R_0 in $(Q^g)_0$.

If I is divisorial in R with $I_g \neq 0$ then $(\overline{R}I)^{**}$ is divisorial ($R \hookrightarrow \overline{R}$ satisfies PDE) and by Theorem I.1.5. (2) and Theorem I.1.4. applied to $\overline{R}$ it follows that $(\overline{R}I)^{**}$ is graded. Obviously I is the intersection of $(\overline{R}I)^{**}$ with R and as such I is graded too; hence we established (2) of Theorem I.1.4. for R. □

I.1.7. Corollary. Let Γ be the set of non-negative elements of G and let R_+ be the non-negative part of a twisted group ring RG^t. Then R_+ satisfies one of the equivalent conditions of Theorem I.1.4. if and only if R_+ is integrally closed.

Proof. By the foregoing proposition R_+ is almost mormal. Suppose $\lambda \in K$ is integral over R. Consider $0 \neq \gamma \in \Gamma$. Then $\lambda u_\gamma (u_\gamma \in RG^t)$ is integral over R_+ and $\deg(\lambda_\gamma) = \gamma \neq 0$, hence $\lambda u_\gamma \in R_+$ i.e. $\lambda \in R$. This proves that R is integrally closed and combined with the fact that $\overline{R}_+ = \overline{R} \oplus (\oplus_{0\neq\gamma\in\Gamma} Ru_\gamma) = R \oplus (\oplus_{0\neq\gamma\in\Gamma} Ru_\gamma) = R_+$ it follows that $\overline{R}_+$ is integrally closed. □

I.1.8. Example. Let Q^g be a gr-field graded by the torsion free abelian group G and let Γ be a set of non-negative elements in an ordering of G. Write $Q^g_+ = \oplus_{\gamma\in\Gamma}(Q^g)_\gamma$. It is clear that Q^g is the graded ring of fractions of Q^g_+ and if $x_\gamma \in Q^g$ with $\gamma \neq 0$ is integral over Q^g_+, say $x^n_\gamma + a_{n-1}x^{n-1}_\gamma + \ldots + a_0 = 0$, $a_j \in Q^g_+$ then we may assume that the

a_i are homogeneous and then $\deg a_{n-1} = \gamma, \ldots, \deg(a_0) = \gamma^n$. If $\gamma < 0$ then $a_{n-1}, \ldots, a_0$ cannot be in Q^g_+, hence $\gamma > 0$ and $x_\gamma \in Q^g_+$ follows. We established that Q^g_+ is almost normal. By Proposition I.1.6. it follows that Q^g_+ satisfies the conditions of Theorem I.1.4. By Theorem I.1.5. (2) it follows that Q^g_+ is integrally closed because $(Q^g_+)_0 = Q_0$ is certainly integrally closed in $(Q^g)_0 = Q_0$.

I.1.9. Note. Corollary I.1.7. is in [1] in case R_+ is a semigroupring. The slight generalization we included here is motivated by the example given above, i.e. in order to check whether certain subrings of Q^g are integrally closed it suffices to check whether they are almost normal and the latter is an exercise of purely graded nature.

Recall that the Picard group of a commutative ring R is obtained by taking the isomorphism classes of invertible R-modules together with the product induced by the tensorproduct. The graded Picardgroup of R, $\mathrm{Pic}^g(R)$ is the subgroup of $\mathrm{Pic}(R)$ consisting of the elements represented by graded modules. We write $\mathrm{Pic}_g(R)$ for the group of isomorphism classes in degree zero of graded invertible R-modules; i.e. we have canonical morphisms $\mathrm{Pic}_g(R) \to \mathrm{Pic}^g(R) \hookrightarrow \mathrm{Pic}(R)$. Since an invertible R-module is projective of rank one it is not hard to see that $\mathrm{Pic}(R)$ may also be described as the subgroup of $Cl(R)$ obtained by considering the invertible ideals modulo the principal fractional ideals. Similarly $\mathrm{Pic}^g(R)$ is a subgroup of $Cl^g(R)$ but obviously we do not have an embedding of $\mathrm{Pic}_g(R)$ in any of these groups.

I.1.10. Proposition. Let R be a domain graded by Γ as before. The following statements are equivalent :

1. If I is an invertible ideal in R such that $I_g \neq 0$ then I is a graded ideal.
2. If I is an invertible ideal in R then $I = qJ$ for some $q \in Q^g$ and some graded invertible ideal J in R.
3. $\mathrm{Pic}(R) = \mathrm{Pic}^g(R)$.

Proof. 1. $\Rightarrow$ 2. An easy version of the implication 3. $\Rightarrow$ 6. in Theorem I.1.4. by restricting to invertible ideals.
2. $\Rightarrow$ 1. Let I be invertible in R with $I_g \neq 0$. Write $I = qJ$ where

$q \in Q^g$ and J is a graded invertible ideal in R. If $q = xs^{-1}$ with $s \in S$ then $sI = xJ$. Since there is an $i \in I$ which is nonzero and homogeneous it follows that x is homogeneous hence sI is homogeneous and I is a graded ideal too (because R is a domain !).

2. $\Leftrightarrow$ 3. It is clear that $\mathrm{Pic}(R) = \mathrm{Pic}^g(R)$ if and only if each invertible ideal in R, I say, is of the form qJ with $q \in Q^g$ and $[J] \in \mathrm{Pic}^g(R)$ □

If one considers the subgroups $FCl(R)$ and $FCl^g(R)$ consisting of the elements of $Cl(R)$ and $Cl^g(R)$, respectively, which are represented by finitely generated divisorial ideals then it is clear that the equivalent conditions of Theorem I.1.4. are all equivalent to $FCl(R) = FCl^g(R)$. If we assume that R is an integrally closed domain then $Cl(R) = Cl^g(R)$ if and only if each divisorial ideal I in R is such that Q^gI becomes a principal ideal in Q^g. Consequently $Cl^g(R) = Cl(R)$ holds e.g. in case R is a Krull domain, or in case $G = \mathbb{Z}, \Gamma = \mathbb{Z}_+$.

We now pay particular attention to graded Krull domains. First let us point out that it is well-possible to define gr-Krull domains intrinsically in graded terms. From a purely ring theoretical point of view this may be of some interest, however for a torsion free abelian G one easily checks that a gr-Krull domain R such that Q^g is a Krull domain is also a (graded) Krull domain. Since the latter situation is the one which also yields $Cl(R) = Cl^g(R)$ it is clear why we restrict to graded Krull domains whereas for gr-Dedekind rings a totally different picture will be obtained later on.

If $R \subset S$ are Krull domains then one may try to define group morphisms $Cl(R) \to Cl(S)$ in two different ways. A first possibility is to send the class of I in $Cl(R)$ to the class of $(S : (S : SI))$. A second possibility would be to map $p \in X^1(R)$ to $q_1^{e_1} * \ldots * q_n^{e_n}$, where $q_i \in X^1(S)$ lies over p and e_i is the ramification of p in q_i, and extend this multiplicatively. In general these maps need not be well-defined and even if they were, the maps would not have to be equal. If the extension $R \subset S$ satisfies condition PDE (if $P \in X^1(S), ht(P \cap R) \leq 1$) then both maps are well-defined and they coincide; moreover this map $\Psi : Cl(R) \to Cl(S)$

makes the following diagram commutative :

$$\begin{tikzcd} \mathrm{Pic}(R) \arrow[r] \arrow[d] & \mathrm{Pic}(S) \arrow[d] \\ Cl(R) \arrow[r, "\Psi"'] & Cl(S) \end{tikzcd}$$

The following lemma which is due to C. Weibel may be useful

I.1.11. Lemma. Let R be a commutative ring graded by an abelian cancellative monoid Γ with only the trivial unit. If F is a functor from **Rings** to **Groups** such that $R \to R[\Gamma]$ induces an isomorphism $F(R) \simeq F(R[\Gamma])$ then it also induces an isomorphism $F(R_0) \simeq F(R)$.

Proof. Consider the homomorphisms (!) :

$$\begin{tikzcd} & & R \\ R \arrow[r, "i"] & R[\Gamma] \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & R \end{tikzcd}$$

where $\pi_1(\Sigma r_\gamma \gamma) = \Sigma r_\gamma$ and $\pi_2(\Sigma r_\gamma \gamma) = r_0$. Then $\pi_1 i = \pi_2 i = 1_R$ combined with the fact that $F(i)$ is an isomorphism yields that $F(\pi_1)$ and $F(\pi_2)$ are isomorphisms. Define $f : R \to R[\Gamma]$, $f(\Sigma r_\gamma) = \Sigma r_\gamma \gamma$. Then $\pi_1 f = 1_R$, hence $F(f)$ is an isomorphism, furthermore $\pi_2 f = \epsilon$ is the usual augmentation map $R \to R_0$ and $F(\epsilon)$ is an isomorphism. Finally, from the fact that $R_0 \to R \xrightarrow{\epsilon} R_0$ yields 1_{R_0} we may derive that $F(R_0) \to F(R)$ is an isomorphism.

I.1.12. Corollaries. 1. Let R be a Krull domain graded by Γ as above then $\mathrm{Pic}(R_0) \cong \mathrm{Pic}(R)$.

2. Let R be the positive part of $R_0G^t = S$, i.e. Γ is the set of non-negative elements in G and R is a "twisted" semigroup ring. Recall that a ring A is said to be seminormal if A is reduced and if $b^2 = c^3$ with $b, c \in A$ then $a^2 = c$, $a^3 = b$ for some $a \in A$. If R is seminormal then Γ is seminormal in the sense that for $g \in G$ such that $2g, 3g \in \Gamma$ we have $g \in \Gamma$. Indeed, take $x_g \in (S)_g$, then $(x_g)^2$ and $(x_g)^3$ are in the

positive part R of S hence $x_g \in R$ by seminormality of R, i.e. $g \in \Gamma$. Since R and Γ are seminormal $R\Gamma$ is seminormal too.
Then $\mathrm{Pic}(R\Gamma) = \mathrm{Pic}(R)$ implies that $\mathrm{Pic}(R) = \mathrm{Pic}(R_0)$, and this allows to derive the twisted versions of this result directly from the semigroup ring result.

3. If A is a regular ring affine over a field then projective $A[\Gamma]$-modules are extended from projective A-modules. If A is moreover positively graded then it follows that projective A-modules extend from projective A_0-modules (well-known because such a ring A has to be a symmetric algebra of a finitely generated projective A_0-module).

Let us round off this point of a more general nature by mentioning some further consequences and modifications of the results obtained so far.

I.1.13. Properties. 1. If R is a positively graded (for $\Gamma \geq 0$ in G) integrally closed domain then $\mathrm{Pic}(R_0) = \mathrm{Pic}(R)$.
2. If $R \oplus_{\gamma \in \Gamma} R_\gamma$ is factorial then R_0 is factorial.
3. If R is a Γ-graded Krull domain such that Γ contains only trivial units (e.g. when $\Gamma \geq 0$ in G) then $Cl(R_0) \hookrightarrow Cl(R)$, but this set-map needs not be a homomorphism unless $R_o \to R$ satisfies PDE
4. If $R_0 \to R$ satisfies PDE and R is a Γ-graded Krull domain such that for every $p \in \mathrm{Spec}(R_0)$ we have that $(Rp)^{**}$ is again a prime ideal of R then the following sequence is exact :

$$1 \to Cl(R_0) \to Cl(R) \to Cl(Q^g) \to 1$$

5. If R is a Krull domain with field of fractions K and if we assume that $R\Gamma$ is a Krull domain too (e.g. $\Gamma \subset \Gamma >= G$ satisfies the ascending chain conditions on cyclic subgroups) then we have :

$$Cl(R\Gamma) = Cl(R) \oplus Cl(K\Gamma)$$

Moreover $Cl(K\Gamma) = Cl(K'\Gamma)$ for every extension K'/K.

Now we turn to the study of domains strongly graded by a torsion free abelian group G. An arbitrary ring A is said to be **strongly graded** by a group G if $A = \oplus_{\sigma \in G} A_\sigma$ and $A_\sigma A_\tau = A_{\sigma\tau}$ for all $\sigma, \tau \in G$. From

$A_\sigma \otimes_{A_e} A_{\sigma^{-1}} = A_e = A_\sigma A_{\sigma^{-1}}$ for all $\sigma \in G$, it follows that each A_σ is an invertible A_e-bimodule. In particular each A_σ is a finitely generated projective A_e-module on the left and on the right (we denote by $e \in G$ the neutral element of G). The morphism $A_0 \to A$ is faithfully flat. The functors $A \otimes_{A_0} -$ and $(\quad)_e$ define a category equivalence, $A_e - \mathrm{mod} \approx A - \mathrm{gr}$, where $A - \mathrm{gr}$ stands for the category of left graded A-modules.
If we consider a commutative domain R strongly graded by G then $\mathrm{Ker}(Cl(R_e) \to Cl(R)) = \mathrm{Ker}(\mathrm{Pic}(R_e) \to \mathrm{Pic}(R))$. To every strong G-gradation of R there corresponds a group morphism $\Phi : G \to \mathrm{Pic}(R_e)$. Together with a factor set $\{f_{\sigma,\tau} : R_\sigma \otimes_{R_e} R_\tau \longrightarrow R_{\sigma\tau}; \sigma, \tau \in G\}$ consisting of R_e-bimodule isomorphisms. If $g_{\sigma,\tau}$ is another factor set associated to Φ (note : these factor sets satisfy the obvious associativity condition $f_{\sigma\tau,\gamma}(f_{\sigma,\tau} \times 1_{R_\gamma}) = f_{\sigma,\tau\gamma}(1_{R_\sigma} \times f_{\tau,\gamma})$) then $g_{\sigma,\tau} = q_{\sigma,\tau} f_{\sigma,\tau}$ where $q : G \times G \to UZ(R_e)$ determines an element of $H^2(G, UZ(R_e))$.
If G is torsion free abelian then $Cl(R) = Cl^g(R), \mathrm{Pic}(R) = \mathrm{Pic}^g(R)$ So if R is strongly graded by G we may derive :

I.1.14. Lemma. Let the commutative Krull domain R be strongly graded by the torsion free abelian group G, then we have exact sequences of abelian groups

$$1 \to Im(\Phi) \to \mathrm{Pic}(R_e) \to \mathrm{Pic}(R) \to 1$$
$$1 \to Im(\Phi) \to Cl(R_e) \xrightarrow{\pi} Cl(R) \to 1$$

Proof. Since R is strongly graded, every graded (fractional) R-ideal I is of the form RI_0 and the extension $R_0 \to R$ is faithfully flat. If $[I] \in Cl(R_e)$ maps to 1 under π then $RI = Ra$ for some $a \in RI$. Since RI is graded and G is ordered we may show that A is homogeneous, say $\deg(a) = \tau$. Consequently $I = R_{\tau^{-1}} a$, hence $[I] = [R_{\tau^{-1}}] \in Im(\Phi)$. Conversely if $[I] \in Im(\Phi)$ then say $[I] = [R_\tau]$ for some $\tau \in G$. It is easily seen that $[R_{\tau^{-1}} \otimes_{R_e} I] = [R_{\tau^{-1}} I] = [R_e] = 1$ and since $RI = R(R_{\tau^{-1}} I)$ we obtain $[RI] = 1$ or $[I] \in \mathrm{Ker}\pi$. The surjectivity of π and $\pi|\mathrm{Pic}(R_e)$ follows easily from $Cl(R) = Cl^g(R), \mathrm{Pic}(R) = \mathrm{Pic}^g(R)$ combined with the fact that graded fractional R-ideals are extended from zero. $\square$

Although the Lemma I.1.14. may be formulated in the general setting of graded domains it makes most sense if R_e is completely integrally closed. The natural question is to find out whether this property lifts to R ? Before we solve this problem, we extend the definition of a strongly graded ring in order to obtain a notion wich is intrinsically linked to the class group. Indeed it is to restrictive to have $\mathrm{Ker}(Cl(R_e) \to Cl(R)) = \mathrm{Ker}(\mathrm{Pic}(R_e) \to \mathrm{Pic}(R))$ if we aim to study Krull domains which are not locally factorial (in the locally factorial case we do have Pic $= Cl$ and then there is nothing to say). At the moment we restrict attention to the commutative situation because a general non-commutative definition (which does exist !) requires more technicalities than can be introduced in a commutative chapter without destroying its nature. We say that the domain R graded by any G is **divisorially graded** if $R = \oplus_{\sigma\in G} R_\sigma$ with R_σ divisorial and $(R_\sigma R_\tau)^{**} = R_{\sigma\tau}$ for all $\sigma, \tau \in G$. As in the strongly graded case it is not hard to see that a divisorial gradation corresponds to a monoid morphism $\Phi : G \to Cl(R_e)$, where $Im\Phi$ is a group of course, together with a factor set obtained from the isomorphisms $R_\sigma \perp_{R_e} R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$, where $\perp_{R_e}$ is the modified tensor product $(- \otimes_{R_e} -)^{**}$. Obviously a strongly graded ring is also divisorially graded.

I.1.15. Proposition. Let the commutative domain R be divisorially graded by a torsion free abelian group G. If R_e is integrally closed in its field of fractions Q_e then R is integrally closed in its field of fractions Q.

Proof. By Proposition I.1.1. (1), Q^g is completely integrally closed and in order to show that R is integrally closed it will suffice to establish that R is almost normal. Let $q \in Q^g$ satisfy the equation :

(*) $X^m + a_1 x^{m-1} + \ldots + a_m = 0$, with $a_i \in R$, $i = 1, \ldots, m$. From $(R_\tau R_{\tau^{-1}})^{**} = R_e$ for all $\tau \in G$, it follows that every element of $h(R)$ will be inverted in the ring obtained by inverting the nonzero elements of R_e. So we may select a $c \in R_e$ such that $cq_{\sigma_i} \in R$ for all q_{σ_i} appearing in the decomposition $q = q_{\sigma_1} + \ldots + q_{\sigma_n}$. Let $\mathcal{E}$ be the set of homogeneous elements obtained as homogeneous components in R of c, $cq, a_1, \ldots, a_m$. Let H be the subgroup of G generated by the degrees of the elements in C and let S be the graded ring generated by $\mathcal{E}$ over the prime ring ($\mathbb{Z}$ or $\mathbb{Z}_p$) of R. Obviously q is integral over S and the latter is an H-graded Noetherian domain such that its field of fractions contains q. Since H is a free abelian group it is totally ordered. Combinations of (*) and the fact that $c^m q^j \in S$ for $j = 0, \ldots, m$, yields that $c^m S[q] \subset S$. Suppose $\sigma_1 < \ldots < \sigma_n$ then $c^m q^j_{\sigma_n} \in S$ for all $j \in \mathbb{N}$, hence $S[q_{\sigma_n}] \subset Sc^{-m}$ and the latter is a finite S-module. Consequently, $S[q_{\sigma_n}]$ is a finite S-module and q_{σ_n} is integral over S. We reduced the problem to $q' = q_{\sigma_{n-1}} + \ldots + q_{\sigma_1}$, so by induction, we arrive at the situation where q_{σ_i} is integral over S for $i = 1, \ldots, n$, hence q_{σ_i} is integral over R for $i = 1, \ldots, n$. If one of the q_{σ_i} is not in R we fix it and denote it by y_σ from now on. If $R_{\sigma^{-1}} y_\sigma \subset R_e$ then $R_\sigma R_{\sigma^{-1}} y_\sigma \subset R_\sigma \subset R$ implies $(R_\sigma R_{\sigma^{-1}})^{**} y_\sigma \subset R_\sigma$ since R_σ is divisorial, hence $y_\sigma \in R_\sigma$, a contradiction. Therefore there is an $x \in (\overline{R})_e - R_e$. If x satisfies an equation $x^t + d_1 x^{t-1} + \ldots + d_t = 0$ with $d_j \in R$ then x also satisfies the equation :

(**) $x^t + (d_1)_e x^{t-1} + \ldots + (d_t)_e = 0$. Since $y_\sigma \in Q^g$ it follows that $x \in R_{\sigma^{-1}} y_\sigma \in Q_e$ and it is integral over R_e because of **. The assumptions entail that $x \in R_e$, a contradiction again. Consequently $q_{\sigma_1}, \ldots, q_{\sigma_n} \in R$ and $q \in R$, or R is integrally closed. □

I.1.16. Proposition. Let R be a Krull domain which is divisorially graded by a torsion free abelian group then R_e is a Krull domain and we have an exact sequence of abelian groups :

$$1 \to Im(\Phi) \to Cl(R_e) \xrightarrow{\pi} Cl(R) \to 1$$

where $\Phi : G \to Cl(R_e)$ is the group morphism associated to the divisorial gradation of R.

Proof. Since $R_e = Q_e \cap R$ in A^g it is obviously a Krull domain. Since Q^g is a Krull domain it is factorial and hence $Cl(R) = Cl^g(R)$; therefore π is surjective. If $[I] \in Im\Phi$, say $[I] = [R_\sigma]$ then $(RI)^{**} = (RR_{\sigma^{-1}} I)^{**} = J$ is such that $[J] = 1$ because $[(R_{\sigma^{-1}} I)^{**}] = 1$ in $Cl[R_e]$. On the other hand, if $[I] \in Cl(R_e)$ maps to 1 in $Cl(R)$ then $(RI)^{**} = Ra$ for some homogeneous element a (since $(RI)^{**}$ is graded !), say $\deg(a) = \gamma$. Taking parts of degree e yields : $I = R_{\gamma^{-1}} a$ hence $[I] = [R_{\gamma^{-1}}] \in Im\varphi$. □

I.1.17. Proposition. Suppose that the commutative domain R is divisorially graded by a torsion free abelian group G satisfying the ascending chain condition on cyclic subgroups. If R_e is a Krull domain then R is a Krull domain too.

Proof. First let us show that R is completely integrally closed. Since this property is preserved under direct limits it will be sufficient to show that $R_{(H)}$ is completely integrally closed for each finitely generated subgroup H of G which is then a free abelian group. Clearly $R_{(H)} = R \cap Q^g_{(H)}$. Now Q^g and $Q^g_{(H)}$ are factorial Krull domains (see also Proposition I.1.1.(4) and Corollary I.1.2.) and as H is free abelian : $Q^g_{(H)} \simeq Q(R_e)[H]$ is completely integrally closed. The fact that R is divisorially graded entails that Q^g is obtained by inverting the nonzero elements of R_e. If $x = a^{-1}y$ with $a \in R_e$, $y \in R_{(H)}$ is almost integral over $R_{(H)}$, say $x \in (I : I)$ then all powers of x have a common denominator b in R_e such that $ba^{-1}y^i \subset R_{(H)}$ for all i. If a is not a unit in R_e then in some essential valuation for R_e one may calculate that for some i, $a^{-i}(by^i)_\sigma \notin I_\sigma$ for some σ. Therefore $a^{-1} \in R_e$ and $x \in R_{(H)}$ follows. From this fact we also retain that a maximal proper divisorial ideal of R is necessarily a minimal prime ideal (but not conversely, a priori). If $P \in X^{(1)}(R)$ then either $P_g = 0$ or $P = P_g$. In case $P_g = 0$, R_P is also a localisation of the Krull domain Q^g at the minimal prime ideal PQ^g; therefore R_P is a principal valuation ring. In case $P = P_g$ we note that every homogeneous element of $R - P$ may be multiplied into $R_e - P_e$ and it is evident that the graded ring of fractions R^g_P equals R_p where $p = P \cap R_e$. For any $p \in X^{(i)}(R_e)$ we have that $(Rp)^{**}$ is a graded prime ideal; indeed if $IJ \subset (Rp)^{**}$ for some graded ideals I, J of R then $I_e \subset p$ say, but since R is divisorially graded $I^{**} = (RI_e)^{**}$ hence $I^{**} \subset (Rp)^{**}$ and $I \subset (Rp)^{**}$ follows. Consequently p_e is in $X^{(1)}(R_e)$ and $(R_e)_p = (R_p)_e$ is a principal valuation ring. From this fact we derive that $R_p = (R_p)_e G^t$ (divisorial R_p ideals are free), and this is a Krull domain because G satisfies the ascending chain condition on cyclic subgroups . It follows that $R_P = (R_p)_P$ is again a principal valuation ring. The finite characteristic property for R can easily be checked because if $x \in P$, $P \in X^{(1)}(R)$ then one considers $x \in PQ^g$ if $P_g = 0$ or $x \in (Rp)^{**}$ if $P = P_g$ and using the finite character properties for Q^g and R_e one easily derives the property for R. It remains to verify : $R = \cap\{R_P, P \in X^{(1)}(R)\}$. The latter intersection equals $S = Q^g \cap [\{R_P, P = P_g, P \in X^{(1)}(R)\}$. Consider a homogeneous $z \in Q^g$ and suppose that $Iz \subset R$ for some graded ideal I in R not contained in any of the $P \in X^{(1)}(R)$ i.e. suppose z is in the intersection $Q^g \cap [\cap\{R^g_P, P = P_g \text{ with } P \in X^{(1)}(R)\}] \overset{(!)}{=} S$. Since R is completely integrally closed and since I^{**} cannot be contained in a prime ideal of height one it follows that $I^{**} = R$. However $Iz \subset R$ yields $I^{**}z \subset R$ by definition of I^{**} in Q, hence $z \in R$. Thus $R = S$ follows.

Note that $P = P_g$ with $P \in X^{(1)}(R)$ entails $p = P_e \in X^{(1)}(R_e)$ but for an arbitrary $p \in X^{(1)}(R_e)$ it might a priori be true that $(Rp)^{**} = R$, (cf. also Proposition I.1.16.). We have established that R is a Krull domain.

I.1.18. Remark. 1. The condition on G in the theorem may be replaced by the condition : Q^g is a Krull domain; the proof is unaffected.

2. The equality marked (!) follows from $Q^g \cap R_P = R^g_P$ for all graded $P \in X^{(1)}(R)$. Indeed if $z \in Q^g \cap R_P$ then $I = [R : z]$ has $I_e = I \cap R_e \neq 0$ and $I \not\subset P$. From $I_e z \subset R$ we derive $I_e z_\sigma \subset R$ for every homogeneous component z_σ of z in Q^g. Consequently $RI_e z_\sigma \subset R$ and $(RI_e)^{**} z_\sigma \subset R$ or $Iz_\sigma \subset R$ (using $I^{**} = (RI_e)^{**}$, because R is divisorially graded and I is graded by Theorem I.1.4.(I), Theorem I.1.5.(2) and the fact that R is completely integrally closed).

Hence $z_\sigma \in R^g_P$ and $z \in R^g_P$ follows.

I.2. Valuations and the structure of Gr-Dedekind Rings.

In this section we consider $\mathbb{Z}$-graded rings only; we leave to the reader the verification of the claim that all concepts may be defined in case the grading group is torsionfree abelian. Although some results generalize to the torsion free abelian case, serious problems may arise and modifications are necessary, e.g. in the structure theory for gr-Dedekind domains.

We consider a gr-field K^g; since $G = \mathbb{Z}$ we have $K^g \simeq k[T, T^{-1}]$ where k is a field. A graded subring 0_v of K^g is said to be a **gr-valuation ring** of K^g if for every homogeneous $x \in K^g$ we have that either x or x^{-1} is in K^g. One easily verifies the following properties of a gr-valuation ring.

I.2.1. Proposition. Let O_v be a gr-valuation ring of K^g, then

1. The graded ideals of O_v are linearly ordered by inclusion.

2. For any given $x_1, \ldots, x_n \in h(K^g) - O_v$ there is an $i \in \{1, \ldots, n\}$ such that $x_i^{-1} x_j \in O_v$ for all $j \in \{1, \ldots, n\}$.

3. A gr-valuation ring is a gr-local ring; M_v will denote its maximal ideal.

We include the following theorem in order to establish that gr-valuation rings correspond to usual valuations on $K = Q(K^g)$.

I.2.2. Theorem. Let O_v be a gr-valuation ring of K^g. To O_v we may

correspond a valuation function $v : (K^g)^* \to \Gamma$ for some ordered groups Γ. This function v extends to a valuation $v : K^* \to \Gamma$. If $x \in K^g$ then $x \in O_v$ if and only if $v(x) \geq 0$.

Proof. On $h((K^g)^*)$ we define an equivalence relation $a \underset{v}{\simeq} b$ if there exist $x_1, x_2 \in O_v$ such that $a = x_1 b$ and $b = x_2 a$. Let Γ be the ordered group obtained from the equivalence classes with respect to the relation. As in the ungraded case one defines a valuation function $v^g : h((K^g)^*) \to \Gamma$. If $x \in (K^g)^*$, say $x = x_1 + \ldots + x_n$ with $\deg x_i = d_i$, $d_1 < \ldots < d_n$, then we put $v(x) = \min\{v^g(x_1), \ldots, v^g(x_n)\}$. The reader may verify by a rather straightforward computation that V is a valuation function $(K^g)^* \to \Gamma$; actually only the condition $v(xy) = v(x) + v(y)$ takes some work, cf. [37] Theorem I.3.13. p. 167. This definition also makes it perfectly clear that $O_v = \{x \in (K^g)^*, v(x) \geq 0\} \cup \{0\}$. It is not hard to extend v to a valuation $\widetilde{v} : K^* \to \Gamma$. Let us show directly that $(O_v)_{M_v}$ is a valuation ring of K. Pick $z \in K$; say $z = (x_1 + \ldots + x_n)(y_1 + \ldots + y_v)^{-1}$ for $x_1, \ldots, x_n, y_1, \ldots, y_s \in O_v$. By Proposition I.2.1. (2) there is a $\xi \in h(K^g)$ such that $\xi x_j, \xi y_i \in h(O_v) - \{0\}$ and some ξx_j or ξy_i is equal to 1.

Looking at $z = (\xi x_1 + \ldots + \xi x_n)(\xi y_1 + \ldots + \xi y_s)^{-1}$ it is clear that either its nominator or its denominator cannot be in M_v (since M_v is graded, $1 \in M_v$ would follow if both nominator and denominator were contained in it) hence z or $z^{-1} \in (O_v)_{M_v}$. This proves that the latter ring is a valuation ring associated to a valuation $\widetilde{v} : K^* \to \Gamma$ which extends v. $\square$

If v is a gr-valuation with $\Gamma = \mathbb{Z}$ then we say that v is a **discrete** gr-valuation or a **principal** gr-valuation. For $\mathbb{Z}$-graded rings R it is well-known that R is gr-Noetherian i.e. R satisfies the ascending chain condition on graded ideals, if and only if R is Noetherian, cf. [37] Theorem II.3.5. p. 88. This allows to state and prove :

I.2.3. Proposition. Let O_v be a gr-valuation ring of K^g with associated valuation v. Then v is discrete if and only if O_v is Noetherian (gr-Noetherian).

Proof. If O_v is Noetherian then so is $(O_v)_{M_v}$ hence $\widetilde{v}$ is discrete, $\Gamma = \mathbb{Z}$ and v is discrete. For the converse it suffices to check that $I \underset{\neq}{\subset} J \subset M_v$ are graded ideals then $(O_v)_{M_v} I \underset{\neq}{\subset} (O_v)_{M_v} J$. But if the latter inclusion were to be an equality then any $x \in h(J)$ has the property that $cx \in I$ for some $c \in O_v - M_v$. Write $c = c_1 + \ldots + c_m$, $\deg c_i = d_i, d_1 < \ldots < d_n$. From $cx \in I$, $c_i x \in I$ follows for $i = 1, \ldots, m$. Since some $c_\nu \notin M_v$ it then follows that $x \in I$. So that $I = J$ would follow too. The assumption that v is discrete leads to the fact that $\widetilde{v}$ is discrete, hence $(O_v)_{M_v}$ is Noetherian and the above argument entails that O_v is gr-Noetherian as well. □

Combining our knowledge of gr-valuation rings obtained so far we may conclude that a discrete gr-valuation ring will be a Noetherian integrally closed domain, hence a graded Krull domain. The graded Krull dimension of O_v is clearly equal to one, therefore O_v has $\mathrm{Kdim} O_v \leq 2$. A prime ideal $P \in X^{(1)}(O_v)$ is either equal to M or else $P_g = 0$. Consequently if $P \in X^{(1)}(O_v)$ is such that $P \neq M$ then $(O_v)_P$ is a localization of K^g at PK^g and therefore it is a discrete valuation ring of K (but the associated valuation on K does not come from a graded valuation on K^g !). Perhaps the easiest examples of discrete gr-valuation rings of $k[t, T^{-1}]$ are $k[T]$ and $k[T^{-1}]$; actually there are essentially the only discrete gr-valuation rings of $k[T, T^{-1}]$ where $\mathrm{Kdim} = 1$, and not two. This is an easy consequence of the consequent structure theorem.

I.2.4. Theorem. Let $K^g = k[T, T^{-1}]$ be a gr-field with $\deg T = 1$. A discrete gr-valuation ring R is necessarily one of the following types :

a. $R = k[T]$

b. $R = k[T^{-1}]$

c. $R = \sum_{n \in \mathbb{Z}} M_0^{-n\alpha} T^n$ where $\alpha \in \mathbb{Q}$ and M_0 is the maximal ideal of a discrete valuation ring R_0 of k.

Proof. Let $v : h(K^g) \to \mathbb{Z}$ be the graded valuation corresponding to R in K^g. First, if $v(k) = 0$ then $R \supset k$, either $R \supset k[T]$ or $R \supset k[T^{-1}]$. Since both rings mentioned are gr-valuation rings and discrete they are also maximal graded subrings of $k[T, T^{-1}]$ hence we arrive at the possibilities mentioned in a. en b.

Secondly, $v(k) = n\mathbb{Z}$. Then R_0 is a discrete valuation ring of k with corresponding valuation v/k. It is harmless to normalize v such that $v(k) = \mathbb{Z}$ (actually, in the sequel we assume that all valuations will be normalized in this way). Then we put $v(T) = \alpha$. By definition of v, $h(R) = \{x \in h((K^g)^*), v(x) \geq 0\} \cup \{0\}$ and so we find $R_i = \{x \in (K^g)_i - \{0\}, v(x) \geq 0\} \cup \{0\} = \{yT^i, y \in k^*, v(yT^i) \geq 0\} \cup \{0\} = \{yT^i, y \in M_0^{-i\alpha}\} = M^{-i\alpha} T^i$. □

Up to replacing T by $\pi^m T$ where π is the uniformising element of R_0 we may always assume that $0 \leq \alpha < 1$. We define the **type** of R, denoted by $t(R)$, to be the number α mod 1. It is obvious that R_0 and $t(R)$ determine the discrete gr-valuation ring R.

I.2.5. Proposition. Let R be a discrete gr-valuation ring of $k[T, T^{-1}]$ of $t(R) = p/e$ where $0 \leq p < e$ and $(p, e) = 1$ (if $t(R) = 0$ put $p = 0$ and $e = 1$). Let μ denote the maximal ideal of R_0 and M the gr-maximal ideal of R. The following properties hold :

1. $M^e = R\mu$; e may be called the ramification index of R_0 in R.
2. The units of R are homogeneous of degree he with $h \in \mathbb{Z}$.
3. Uniformizing elements of R have degree p' with $pp' = 1 \mathrm{mod}(e)$.
4. We have : $R/M \simeq R_0/\mu[T^e, T^{-e}]$ with $\deg T = 1$.

Proof. We may exclude the trivial cases and assume that $R \simeq \sum_{n \in \mathbb{Z}} \mu^{-n\alpha} X^n$. It is clear that $\mu^{-n\alpha} X^n$ contains a unit of R if and only if $\mu^{-n\alpha} \mu^{n\alpha} = R_0$, if and only if $\mu^{i(-n\alpha) + i(n\alpha)} = R_0$, (where $i(-)$ denotes the upper integral part of the number considered), if and only if $i(-n\alpha) + i(n\alpha) = 0$ or $n\alpha \in \mathbb{Z}$ i.e. $e|n$ and 2 follows. It is clear that the minimal value that may be attained by the valuation of an element in $\mu^{-n\alpha} T^n$ is $i(-n\alpha) + n\alpha$. So $\mu^{-n\alpha} T^n$ can only contain a uniformizing element if $i(-n\alpha) + n\alpha$ is as small as possible i.e. equal to $1/e$. For this to happen it is necessary that e divides $1 - np$ or equivalently that $np = 1$ mod e. Conversely if $np \equiv 1$ mod (e) then a direct calculation shows that $i(-n\alpha) + n\alpha = 1$.e. and 3. follows, as well as 1.
Since R/M is a gr-field, $R/M = R_0/\mu[T^f, T^{-f}]$ and f is the smallest number such that $R_f \not\subset M_f$. We have observed that $M_i = \{x \in (K^g)_i, v(x) \geq \frac{1}{e}\} \cup \{0\}$ hence $M_i \neq R_i$ if and only if there exists an

$x \in R_i$ with $v(x) \neq 0$, if and only if R contains an homogeneous unit of degree i. The first observation in this proof learned that the latter happens if and only if $i = he$, so the smallest possible i is indeed equal to f and this proves statement 4. □

I.2.6. Remark. The valuation of a homogeneous element of K^g is not completely independent of its degree (that is as far as gr-valuations are concerned). Indeed, one easily establishes the following relation for any $a \in h(K^g)$:

$$v(a) \equiv \alpha \deg(a) \bmod 1$$

As an easy consequence of this one retains that not every element of negative valuation has degree zero (Corollary I.3.15. of [37]).
We now turn to the structure theory of gr-Dedekind domains, the material we present here is a combination of results of M. Van den Bergh [51] and F. Van Oystaeyen [53].

A $\mathbb{Z}$-graded commutative domain is called a **gr-principal ideal domain** if every graded ideal is principal. A $\mathbb{Z}$-graded domain is said to be a **gr-Dedekind** ring if every graded ideal is projective. The following two results may be proved in a way much similar to the ungraded equivalents, for a little more detail we refer to [37], p. 179.
I.2.7. Theorem For a $\mathbb{Z}$-graded domain R the following statements are equivalent.
1. R is a gr-Dedekind ring.
2. R is a Krull domain and nonzero graded prime ideals are maximal graded ideals.
3. R is Noetherian and integrally closed in its field of fractions K, and nonzero graded prime ideals are gr-maximal ideals.
4. Every graded ideal of R is invertible.
5. Every graded ideal of R is in a unique way a product of graded prime ideals.
6. R is Noetherian and homogeneously integrally closed in K^g and every nonzero graded prime ideal of R is a gr-maximal ideal.
7. The graded fractional ideals of R form a multiplicative group.
8. R is Noetherian and each R_N at a gr-maximal ideal M of R is a principal ideal domain.

9. R is Noetherian and each graded localization R^g_P at a graded prime ideal P of R is a gr-principal ideal domain.
10. R is Noetherian and each graded localization R^g_M at a gr-maximal ideal M of R is a gr-principal ideal ring.
11. All graded R-modules which are gr-divisable are injective in R-gr.

I.2.8. Proposition. Let O_v be a gr-valuation of the gr-field K^g. The following statements are equivalent

1. O_v is a discrete gr-valuation ring.
2. O_v is a gr-Dedekind domain.
3. O_v is a gr-principal ideal domain.
4. M_v is generated by one homogeneous element of O_v.
5. O_v is factorial.
6. O_v satisfies the ascending conditions for principal ideals
7. O_v is a Krull domain and $ht(M_v) = 1$.

I.2.9. Lemma. Let R be a Dedekind domain.

1. R_0 is a Dedekind domain.
2. There is an $e \in \mathbb{N}$ such that $R = \oplus_{n \in \mathbb{Z}} R_{en}$ with $R_e \neq 0$.
3. Every fractional graded ideal of R can be generated by two homogeneous elements, one chosen arbitrary in the ideal.

Proof. 1. If I is an ideal of R_0 then RI is projective hence a direct summand of a gr-free R-module L which we may assume to be freely generated by elements $v_1, \ldots, v_n$ of degree zero. It is obvious that L_0 is a free R_0-module and $(RI)_0 = I$ is a direct summand of L_0. Consequently, ideals of R_0 are projective and R_0 is a Dedekind domain.

2. Choose $d, e > 0$ minimal such that $R_{-d} \neq 0$, $R_e \neq 0$. Note that if the gradation is left or right limited then $R \simeq k[Y]$ and we may neglect these trivial cases i.e. we may assume that both e and d as above exist. Write $RR_e = P_1^{\nu_1} \ldots . P_n^{\nu_m}$ where the P_i are graded prime ideals of R. If $d > e$ then $0 = (RR_e)_0 \supset (P_1^{\nu_1})_0 \ldots . (P_n^{\nu_n})_0$ hence $(P_i)_0 = 0$ for some $i \in \{1, \ldots, n\}$. Therefore $P_i \cap R_{md} = 0$ for all $m \in \mathbb{Z}$ and hence every element of $h(P_i)$ has to be nilpotent (of order d) i.e. $P_i = 0$. The case $e > d$ may be dealt with in a similar way. So we arrive at

$e = d$. Consider R_{ne+m} with $0 < m < e$. Then $R_{-ne}R_{ne+m} = 0$ with $R_{-ne} \neq 0$ yields $R_{ne+m} = 0$.

3. If I is a graded fractional ideal of R then an argument similar to the ungraded case yields that I may be generated by a and b with $a \in h(I)$ but not necessarily homogeneous. Write $b = b_1 + \ldots + b_n$, deg $b_i = t_i, t_1 < \ldots < t_n$. Since $b_i \in I$ we have relations :

(*) $b_i = x_i a + y_i b_1 + \ldots + y_i b_i + \ldots + y_i b_n$, with $x_i, y_i \in h(R)$. If $b_i = x_i a$ then we replace b by $b - b_i$ and repeat the argument. If $b \neq b_i \neq x_i a$ then comparing degrees in (*) yields either $y_i = 0$ which would contradict $b_i \neq x_i a$, or $y_i = 1$ and $x_i a + b_1 + \ldots + b_i + \ldots + b_n = 0$. In the latter case $b - b_i \in R$ and we may replace b by b_i. Finally this leads to the fact that I is generated by a and some b_i. □

Consider an extension of commutative rings $R \subset T$. An ideal I of R is said to be **invertible in** T if there is an R-(bi)module J in T such that $IJ = R$. The **generalized Rees ring** $\check{R}(I)$ with respect such an ideal I is defined to be the graded subring of $T[X, X^{-1}]$ given as $\check{R}(I) = \bigoplus_{n \in \mathbb{Z}} I^n X^n$. Note that the pospive part of $\check{R}(I)$ is nothing but the usual Rees ring $R(I)$. If T is the field of fractions of R then we do not refer to it i.e. "invertible in T" becomes "invertible". It is easy to prove that $\check{R}(I)$ and $R(I)$ are Noetherian if R is Noetherian. We now describe the structure of strongly graded gr-Dedekind rings; R is strongly graded exactly then when $RR_1 = R$. Since a gr-Dedekind domain is a graded Krull domain we may invoke Lemma I.1.14. bearing in mind that $\text{Pic}(R) = \text{Pic}^g(R)$, $Cl(R) = Cl^g(R)$ and $\text{Pic}(R) = Cl^g(R)$ because all graded fractional ideals of R are projective. Hence $\text{Pic}(R) = Cl(R)$ although R may (and will usually) have Kdim$R = 2$. Since $G = \mathbb{Z}$ the map $\Phi : \mathbb{Z} \to \text{Pic}(R_0)$ is determined by $\Phi(1) = [I]$. Lemma I.1.14. and the generalities preceding it yields :

I.2.10. Theorem. If R is a strongly graded gr-Dedekind domain then there is an invertible R_0-ideal I such that $R = \check{R}_0(I)$. There is a canonical epimorphism $\pi : Cl(R_0) \to Cl(R)$ and Ker(π) is the subgroup generated by the class [I] of I. Moreover, π is an isomorphism if and only if I is principal and in this case $R \cong R_0[X, X^{-1}]$ with deg$X = 1$.

Conversely each $\check{R}_0(I)$ for an invertible R_0-ideal I is a strongly graded gr-Dedekind domain. Furthermore $\check{R}_0(I) = \check{R}_0[J]$ if and only if $[I] = [J]$ in $Cl(R_0)$.

Proof. All statements except perhaps the last one follow at once from the preceding remarks. For the last statement we may assume without loss of generality that I and J are integral R_0-ideals. Assume first that $R = \check{R}_0(I) \cong \check{R}_0(J)$. Then $RI = RX^{-1}$ yields $I = R_1 X^{-1} \cong JYX^{-1}$ if we write $\check{R}_0(J) = \bigoplus_{n \in \mathbb{Z}} J^n Y^n$. Consequently $IJ^{-1}$is principal. Conversely if we assume $I = Jz$ for some $z \in K_0$ then $\check{R}_0(I) = \bigoplus_{n \in \mathbb{Z}} (Jz)^n X^n = \bigoplus_{n \in \mathbb{Z}} J^n (zX)^n \simeq \check{R}_0(J)$. □

Starting from the structure result in the strongly graded case we may proceed to unraffle the general case. Recall that for any $\mathbb{Z}$-graded ring A we define for each $e \in \mathbb{Z}$, $A^{(e)} = \bigoplus_{m \in \mathbb{Z}} A_{me}$ with gradation $(A^{(e)})_m = A_{em}$.

I.2.11. Lemma. If R is a gr-Dedekind domain then for all $e \in \mathbb{Z}_+$, $R^{(e)}$ is a gr-Dedekind domain.

Proof. It is clear that $R^{(e)}$ is Noetherian. Considering $R^{(e)}$ in $(K^g)^{(e)} = Q^g(R^{(e)})$ one verifies immediately that $R^{(e)}$ is integrally closed. The correspondence $P \mapsto P^{(e)}$ defines a bijective correspondence between $\text{Spec}_g(R)$ and $\text{Spec}_g(R^{(e)})$, the inverse correspondence being given by $Q \mapsto \text{rad}(R(\sum_{m \in \mathbb{Z}} Q_{me}))$. Hence graded prime ideals of $R^{(e)}$ are also gr-maximal. It now suffices to evoke Theorem I.2.7., $6 \Leftrightarrow 1$, to conclude the proof. □

Even if a gr-Dedekind domain is not strongly graded still the graded rings of fractions at graded prime ideals are determined by what happens in degree zero ! This is established by the following elementary but basic lemma.

I.2.12. Lemma. Let R be a gr-Dedekind domain and let P be a graded prime ideal of R. The graded ring of fractions at P, R^g_P say, is obtained by localizing at $R_0 - P_0$.

Proof. We know by Theorem I.2.7.(9) that R^g_P is a discrete gr-valuation ring. Now R/P is a gr-field. If $\overline{y} \in (R/P)_n$ then there is a $\overline{y}^{-1} \in (R/P)_{-n}$ and if y represents $\overline{y}$, y' represents $\overline{y}_{-1}$ then it suffices to invert $yy' \in P_0$ in order to invert $y \notin P$. So we only have to worry when R/P is trivially graded i.e. P contains all R_n with $n \neq 0$. The final observation in Remark I.2.6. may be applied to R^g_P in order to exclude this situation.

Proof. I.2.13. Theorem. Let R be a gr-Dedekind domain then there exists an $e \in I\!N$ and a fractional ideal of R_0 such that $R^{(e)} = \check{R}_0(I)$.

If $RR_1 = R$ then we may take $e = 1$ by Theorem I.2.10. If $RR_1 \neq R$, write $RR_1 = P_1^{\nu_1}\cdot\ldots\cdot P^{\nu_n}$. Each R/P_i is a gr-field thus each graded prime P_i contains $\underset{n\in \mathbb{Z}}{\oplus} R_{me_i+r}$ for some $e \in I\!N$, some r, $0 < r < e$. The argument at the end of Remark I.2.6. yields that for each P_i containing R_1 we find an $e_i \underset{\neq}{>} 0$ in $I\!N$ such that $P_i^{(e_i)}$ does not contain $(R^{(e_i)})_1 = R_{e_i}$. Let $e = \text{s.c.m.}\ (e_1,\ldots,e_n)$. If $P \not\subset R$, then $P \not\subset R_m$ for any $m > 0$ hence $R^{(e)}(R^{(e)})_1 = R^{(e)}$ because $(R^{(e)})_1$ cannot be contained in any graded prime ideal of $R^{(e)}$. Apply Theorem I.2.10. to $R^{(e)}$ and the statement follows.

Graded integrally closed domains R such that some $R^{(e)}$ is a generalized Rees rings have a well defined structure let us present this structural result now after having introduced some necessary terminology.

Let R_0 be integrally closed and $I \in \text{Inv}(R_0)$. For $p \in \mathbb{Z}$ we define $I^{p^{-1}}$ as the sum of all ideals J of R_0 satisfying $J^p \subset I$. In order to check $(I^{p^{-1}})^p \subset I$ it suffices to check that $J_1^p \subset I$, $J_2^p \subset I$ entails $(J_1+J_2)^p \subset I$ and the problem comes down to checking the validity of $J_1^iJ_2^{p-i} \subset I$ for $i = 1,\ldots,p$. But $(J_1^iJ_2^{p-i}I^{-1})^p \subset (J_1^p)^i(J_2^p)^{p-i}I^{-p} \subset R_0$ and that R_0 is integrally closed yield $J_1^iJ_2^{p-i} \subset I$. For $\alpha = p/q$, $q \in I\!N$ we define I^α as $(I^p)^{q^{-1}}$. This definitions make sense because $(I^p)^{q^{-1}} = (I^{np})^{(nq)^{-1}}$ for all $n \in I\!N$, as one easily checks. If R_0 is a Dedekind domain then the definitions obtain their classical meaning in terms of invertible ideals.

I.2.14. Proposition. Let I be an invertible ideal in an integrally closed domain R_0, then the following properties hold :

1. For all $\alpha,\beta \in \mathbb{Q}$, $I^\alpha I^\beta \subset I^{\alpha+\beta}$
2. For all $\alpha \in \mathbb{Q}$, $I^\alpha J^\alpha \subset (IJ)^\alpha$
3. For all $\alpha,\beta,\gamma \in \mathbb{Q}$ such that $\alpha < \beta < \gamma$, $I^\alpha \cap I^\gamma \subset I^\beta$.

Proof. Direct verification. □

I.2.15. Theorem. Let R be a $\mathbb{Z}$-graded integrally closed ring such that $R^{(e)}$ is a generalized Rees ring for some $e \in I\!N$, then $R = \sum_{n\in\mathbb{Z}} I^{n/e}X^n$ where $I \in \text{Inv}(R_0)$ is given by $R^{(e)} = \check{R}_0(I)$. Conversely, if R_0 is integrally closed then every graded ring of the type of R is also integrally closed.

Proof. Put $K^g = k[X,X^{-1}]$ and write $R = \sum_i I_iX^i$ where I_i is an R_0-module in k. Since $R^{(e)} = \check{R}_0(I)$ we have $Q(R_0) = k$ and each I_i is a fractional R_0-ideal. Since R is integrally closed, $x \in h(R)$ if and only if $x^e \in h(R^{(e)})$.
Therefore $I_i = \{x \in R_0, x^e \in I_{ei} = I^i\} = I^{i/e}$ and $R = \sum_n I^{n/e}X^n$ follows. Conversely if R is such that $R^{(e)} = \check{R}_0(I)$ then the integral closure R' of R has the property that $(R')^{(e)} = R^{(e)}$ hence by the foregoing $R' = \sum I^{n/e}X^n$ and $R' = R$. □
We now return to gr-Dedekind domains.

I.2.16. Proposition. Let K^g be a fixed gr-fixed and let $\{R_i, i \in \mathcal{I}\}$ be a family of discrete gr-valuation rings of K^g. Suppose that $\cap(R_i)_o$ is a Dedekind domain such that $(R_i)_0 \neq (R_j)_0$ for $i \neq j$ and $t(R_i) = 0$ for allmost all $i \in \mathcal{I}$, then $R = \cap\{R_i, i \in \mathcal{I}\}$ is a gr-Dedekind domain.

Proof. (For definition of $t(R_i)$ and e_i see before Proposition I.2.5.). That R is graded and integrally closed is clear. Let e be l.c.m. $\{e_i, e_i$ the ramification index of $R_i\}$; this makes sense since $t(R_i) = 0$ for almost all $i \in \mathcal{J}$. Since $(R_i)_0 \neq (R_j)_0$ if $i \neq j$ it follows that $R^{(e)}$ is strongly graded (as in Theorem I.2.13.), hence $R^{(e)} = \check{R}_0(I)$ for some invertible R_0-ideal I. The structure of $R^{(e)}$ and the assumption on R_0 entail that $R^{(e)}$ is a Noetherian domain of Krull dimension at most two. Since R is the integral closure of $R^{(e)}$ the classical result of Nagata yields that R is Noetherian. Every nonzero graded prime ideal of $R^{(e)}$

is gr-maximal, hence by integrally of R over $R^{(e)}$, the same is true in R and then we nay apply Theorem I.2.7.(6) and include that R is a gr-Dedekind domain. □

I.2.17. Note. The condition $(R_i)_0 \neq (R_j)_0$ is necessary because if one takes $R_1 = \sum_{n\in\mathbb{Z}} M^n X^n$, $R_2 = \sum_{n\in\mathbb{Z}} M^{-n}X^n$ where M is the maximal ideal of a discrete valuation ring then $R_1 \cap R_2$ is not gr-Dedekind because it has graded Kdim equal to two and Kdim equal to three.

If R is a gr-Dedekind domain then there is a bijective correspondence between maximal ideals of R_0 and gr-maximal ideals of M (the latter set is denoted by $\Omega_g(R)$).
If $M \in \Omega_g(R)$ and $m = M \cap R_0$ then $(R^g_M)_0 = (R_0)_m$ and the corresponding gr-valuation and ramification index will be denoted by v_M, e_M respectively. We say that R satisfies the **graded approximation property**, G.A.P., if for any finite subset S of $\Omega_q(R)$, together with any given set of integers $\{n_M, M \in S\}$, there exists an $x \in h(K^g)$ such that $e_M v_M(x) = n_M$ for all $M \in S$ and $v_M(x) \geq 0$ for $M \notin S$.

I.2.18. Theorem. A gr-Dedekind domain R satisfies G.A.P. if and only if for every $P \neq Q$ in $\Omega_g(R)$ we have that $(e_P, e_Q) = 1$.

Proof. First suppose R satisfies G.A.P. For $Q \neq Q$ in $\Omega_g(K)$ there exists a $z \in h(R)$ such that $v_P(z) = 0$ and $v_Q(z) = e_Q - 1$. By Remark I.2.6. : deg$z \equiv 0$ modulo e_P and deg$z \equiv 1$ modulo e_Q, this yields $(e_P, e_Q) = 1$. Conversely, just as in the ungraded case G.A.P. will follow if we show that $h(P_1) \not\subset h(P_1^2) \cup h(P_2) \cup \ldots \cup h(P_n)$ for $P_1, \ldots, P_n \in \Omega_g(R)$. Let π_1 be a uniformizing element for R_{P_1}, $\alpha_1 =$ degπ_1. Pick l such that $l \equiv \alpha_1$ modulo e_{P_1} and $l \equiv 0$ module e_{P_i}, $i = 2, \ldots, n$. In this case : $(P_i)_l = (P_i)_0^{a_i} R_l$ with $a_i > 0$ for $i = z, \ldots, n$, $(P_1^2)_l = (P_1)_0^{a_1} R_l$ with $a_1' > a_1$. The ungraded approximation theorem yields $(P_1)_l \not\subset (P_1^2)_l \cup \ldots \cup (P_m)_l$. □

We include some results concerning the category R-gr for a gr-Dedekind ring R. The proof of the following propospion is an easy technical modification of the corresponding ungraded statements, so we omitted it here.

I.2.19. Proposition. Let R be a gr-Dedekind domain.
1. Any finitely generated graded R-module M decomposes as $M \cong N \oplus T$ where $N \in R$-gr is torsion free, $T \in R$-gr is torsion.
2. If $T \in R$-gr is a finitely generated torsion module then $T \cong \bigoplus_{i=1}^{n} R/P_i^{n_i}$ with $P_i \in \Omega_g(R)$.
3. If M is a graded R-lattice (i.e. a graded torsion free finitely generated R-module) then $M = I_1 \oplus \ldots \oplus I_n$, where $I_1, \ldots, I_n$ are graded fractional R-ideals. □

Let R be any $\mathbb{Z}$-graded ring. For any $n \in \mathbb{Z}$ we define the shift functor $T_n : R-\text{gr} \to R-gr$ by associating to a graded R-module M the graded R-module $M(n)$ obtained by considering the ungraded R-module $\underline{M}$ and equip it with the new gradation defined by : $M(n)_m = M_{n+m}$, for all $m \in \mathbb{Z}$.

I.2.20. Proposition Let R be a discrete gr-valuation ring. Then $R(n_1) \oplus \ldots \oplus R(n_k) \simeq R(m_1) \oplus \ldots \oplus R(m_k)$ with $n_i, m_i \in \mathbb{Z}$ if and only if there is a permutation σ of the set $\{1, \ldots, k\}$ such that $n_{\sigma(i)} \equiv m_i$ modulo e, where e is the ramification index of R.

Proof. If M is the gr-maximal ideal of R then by definition of e we have :

$$R(\underline{n})/MR(\underline{n}) = k[X^e, X^{-e}](\underline{n})$$
$$R(\underline{m})/MR(\underline{m}) = k[X^e, X^{-e}](\underline{m})$$

where for any graded R-module N, $N(\underline{m}) = \bigoplus_i N(m_i)$.
Comparing $\dim_k((R(\underline{n})/MR(\underline{n}))_i)$ and $\dim_k((R(\underline{m})/MR(\underline{m}))_i)$ for every i, yields the result. □

If M, N are graded R-lattices over the $\mathbb{Z}$-graded gr-Dedekind domain R then we say that M and N have the same **genus**, and we write $M \sim N$, if $M_P \cong N_P$ for all $P \in \Omega_g(R)$. If M is a graded R-lattice of rank n then we let det$M$ denote the graded module $\Lambda^n M$ which has rank one. Consequently, if we take M to be the graded lattice $I_i \oplus \ldots \oplus I_n$ then det$M \cong I_1 \ldots . I_n$.

I.2.21 Lemma. Let $I_i, J_i, i = 1, \ldots n$, be graded fractional ideals of the gr-Dedekind domain R such that $I_1 \oplus \ldots \oplus I_n \sim J_1 \oplus \ldots \oplus J_n$. The following statements are equivalent, for $t \in R_0$:

1. $I_1 \cdot \ldots \cdot I_n J_1^{-1} \cdot \ldots \cdot J_n^{-1} = tR$

2. $(I_1)_0 (I_2)_0 \cdot \ldots \cdot (I_n)_0 (J_1)_0^{-1} \cdot \ldots \cdot (J_n)_0^{-1} = tR_0$.

Proof. The assumptions make it clear that we may reduce the problem to the gr-local case i.e. we may assume that R is a discrete gr-valuation ring. Write $t(R) = p/e$ with $(p,e) = 1$, $0 \leq p < e$, and let p' be the degree of a uniformizing element π of R. Say, $I_i = \pi^{r_i} R, J_i = \pi^{s_i} R$. By Proposition I.2.20. we can write $r_i - s_i = l_i e$ and thus $I_1 \cdot \ldots \cdot I_n\ J_1^{-1}, \ldots, J_n^{-1} = R\pi^{e\Sigma l_i}$. We also have that $\pi^e = \pi^0 u$ where π^0 is a uniformizing element of R_0 and u is a homogeneous unit, hence $R\pi^{e\Sigma l_i} = R(\pi^0)^{\Sigma l_i}$. We calculate : $(I_i)_0 = \pi^{r_i} R_{-r_i p'}$; $(J_i)_0 = \pi^{s_i} R_{-s_i p'}$,

$$\begin{aligned}(I_i)_0 (J_i)_0^{-1} &= \pi^{l_i e} (\pi^0)^{i\left(\frac{pp' r_i}{e}\right)} X^{-r_i p'} (\pi^0)^{-i\left(\frac{pp' s_i}{e}\right)} X^{s_i p'} R_0 \\ &= (\pi^0)^{l_i(1+pp')} \left((\pi^0)^{-pp'} X^{p'e}\right)^{l_i} X^{-l_i e p'} R_0 \\ &= (\pi^0)^{l_i} R_0\end{aligned}$$

Consequently $(I_1)_0 \cdot \ldots \cdot (I_n)_0 (J_1)_0^{-1} \cdot \ldots \cdot (J_n)_0^{-1} = (\pi^0)^{\Sigma l_i} R_0$ and the statement is evident. □

I.2.22. Corollaries. Let R be a gr-Dedekind domain.

1. If M and N are graded R-lattices with $M \sim N$ then an $f \in \mathrm{Hom}_{R-gr}(M,N)$ is an isomorphism if and only if $f_0 = f|M_0$ is an isomorphism.

2. For M, N as in 1., we have $\det M \cong \det N$ if and only if $M_0 \cong N_0$.

3. If I and J are graded fractional R-ideals such that $I \sim J$ then $J = HI$ for some fractional R_0-ideal H.

Proof. 1. We may assume $M \cong I_i \oplus \ldots \oplus I_n$, $N \cong J_1 \oplus \ldots \oplus J_n$ for some graded fractional ideals I_i, J_i. Then f may be viewed as an element of degree zero of $\oplus_i I_i^{-1} J_{j_i}$. Now f_0 will be an isomorphism exactly when $R_0 \det f$ equals $(I_1)_0^{-1} \cdot \ldots \cdot (I_n)_0^{-1} (J_1)_0 \ldots (J_n)_0$. By the lemma this is equivalent to $R \det f = I_1^{-1} \cdot \ldots \cdot I_n^{-1} . J_1 \cdot \ldots \cdot J_n$, so f is an isomorphism.

2. With notation as in 1., $\det M \cong I_1 \ldots I_n$ and $\det N \cong J_1 \cdot \ldots \cdot J_n$. The Lemma and Steinitz theorem prove the statement.

3. From $I \sim J$ we derive that $IJ^{-1} \sim R$. Since $(IJ^{-1})_0 R \subset IJ^{-1}$ it follows from 1 that $IJ^{-1} = (IJ^{-1})_0 R$. □

I.2.23. Lemma. Let M and N be graded lattices over the gr-Dedekind domain R such that $M \sim N$ and $M \cong I_1 \oplus \ldots \oplus I_n$ for some graded fractional ideals $I_1, \ldots, I_n$ of R. Then there exist $J_1, \ldots, J_n$, graded fractional ideals of R such that $N \cong J_1 \oplus \ldots \oplus J_n$ and $I_i \sim J_i$ for all $i = 1, \ldots, n$.

Proof. We may write $N = H_1 \oplus \ldots \oplus H_n$ for certain graded fractional R-ideals $H_1, \ldots, H_n$. By the approximation property for R_0 we may select $t_1, \ldots, t_n \in R_0$ satisfying the following conditions :

(a) $t_1 = 1$.

(b) Let $P \in \Omega_g(R)$ be such that $(I_1)_P \neq R_P$, or $(H_i)_P \neq R_P$ for some i. Put : $v_P(t_i) + v_P(H_i) = q_i$. Then we demand : $q_i \leq \mathrm{Min}\{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n\}$. Note that only finitely many P satisfy the condition.

(c) For $P \in \Omega_g(R)$ not included in (b) we demand : $0 \leq v_P(t_r), \ldots, 0 \leq v_P(t_P(t_n)$.

Let then J_1 be equal to $t_1 H_1 \oplus \ldots \oplus t_n H_n$. It is clear that $J_1 \sim I_1$. Let N' be the kernel of $\underline{t} : N \to J_1$ given by componentwise multiplication by t_i. We obtain $N \cong N' \oplus J_1$ with $N' \sim I_2 \oplus \ldots \oplus I_n$. Repeating the argument will yield a proof of the statement. □

I.2.24. Lemma. Let $I_{11}, I_{12}, I_{21}, I_{22}$ be fractional ideals of a Dedekind domain D. If $I_{11}I_{22} + I_{12}I_{21} = D$ then there exist elements $a_{11}, a_{12}, a_{21}, a_{22} \in Q(D) = K$, such that $a_{11}a_{22} - a_{12}a_{21} = 1$.

Proof. Put $\mathcal{A} = \{P \in \Omega(D), v_p(I_{ij}) \neq 0$ for some $I_{ij}\}$. Choose a'_{11}, a'_{22} such that the following conditions hold : if $P \in \mathcal{A}$ then $v_P(a'_{11}) = v_P(I_{11}), v_P(a'_{22}) = v_P(I_{22})$; if $P \notin \mathcal{A}$ then $v_P(a'_{11}) \geq 0, v_P(a'_{12}) \geq 0$. Choose a'_{21} and a'_{12} such that : if $P \in \mathcal{A}$, $v_P(a'_{21}) = v_P(I_{21})$ and $v_P(a'_{12}) = v_P(I_{12})$. If $P \in \mathcal{A}$ then we want $v_P(a'_{12}) = v_P(a'_{21}) = 0$ if $v_P(a'_{11}) \neq 0$ or $v_P(a'_{22}) \neq 0$, but $v_P(a'_{12}) \geq 0$ and $v_P(a'_{21}) \geq 0$ in case $v_P(a'_{11}) = v_P(a'_{22}) = 0$.

The choice of the elements a'_{ij} yields : $R = Ra'_{11}a'_{22} + Ra'_{12}a'_{21}$. The lemma is thus proved. $\square$

I.2.25. Lemma. If I and J are graded fractional ideals of the gr-Dedekind domain R and if H and K are fractional ideals of R_0 then :

$$HI \oplus KJ \cong I \oplus KHJ$$

Proof. Put $M = HI \oplus KJ$, $N = I \oplus KHJ$, and write $\mathrm{Hom}(M, N)$ as the ring

$$\begin{pmatrix} H^{-1}R & K^{-1}J^{-1}I \\ KI^{-1}J & HR \end{pmatrix}$$

Since $(H^{-1}R)_0(HR)_0 + (K^{-1}J^{-1}I)_0(KI^{-1}J)_0 = R_0$ it follows from the foregoing lemma, combined with Lemma I.2.21., that $M \cong N$ as graded R-modules. $\square$

I.2.26. Theorem. Let M and N be graded R-lattices over the gr-Dedekind domain R. The following statements are equivalent :

1. $M \cong N$ as graded R-modules.
2. $M \sim N$ and $M_0 \cong N_0$.
3. $M \sim N$ and $\det M \cong \det N$
4. $M \sim N$ and $\det M_0 \cong \det N_0$

Proof. 3. $\Leftrightarrow$ 4. is Steinitz' theorem because R_0 is a Dedekind domain.
2. $\Leftrightarrow$ 3. follows from Corollaries I.2.22.
We prove 1. $\Leftrightarrow$ 3. We may assume that $M = I_1 \oplus \ldots \oplus I_n$. From I.2.24. and I.2.25. we retain that $N \cong H_1I_1 \oplus \ldots \oplus H_nI_n$ for some fractional R_0-ideals H_i. Because $M \cong (H_1 \ldots H_nI_1) \oplus I_2 \oplus \ldots \oplus I_n$ and $\det M \cong \det N$ we obtain $H_1 \ldots . H_n = R_0$, proving 3. $\Rightarrow$ 1. The converse implication is trivial of course. $\square$
As a consequence of this we also have the cancelation theorem for gr-Dedekind domains.

I.2.27. Theorem. If M, N, P are graded lattices over the gr-Dedekind domain R then $M \oplus T \cong N \oplus T$ implies $M \simeq N$.

The fact that any gr-Dedekind domain contains a generalized Rees ring of a well-described type allows to do much better than Lemma I.1.14.

when it comes to the determination of the class groups, even if R is not necessarily strongly graded. In addition to the graded class group $Cl^g(R)$ introduced in Section I we also consider the following groups.
1. $Cl_g(R)$: the group of graded isomorphism (i.e. in degree zero) classes of graded divisorial R-ideals.
2. $g(R)$: the genus group of R, is given as $g(R) = \sum_{P \in \Omega_g(R)} \mathbb{Z}/e_P\mathbb{Z}$.
3. $g_0(R) = g(R)/ < (1, \ldots, 1, \ldots, 1) >$.

I.2.28. Proposition. Let R be a gr-Dedekind domain, suppose that $R^{(e)}$ is given as $\check{R}_0(I)$ for some fractional R_0-ideal I. The following commutative diagram is exact :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & <[I]> \arrow[r] \arrow[d] & <R(1)> \arrow[r] \arrow[d] & <(1,\ldots,1)> \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Cl(R_0) \arrow[r] \arrow[d] & Cl_g(R) \arrow[r] \arrow[d] & g(R) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Cl(R_0)/<[I]> \arrow[r] \arrow[d] & Cl^g(R) \arrow[r] \arrow[d] & g_0(R) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. Exactness of the columns and the first row follows by definition of the groups involved. Exactness of the middle row is consequence of Corollaries I.2.22.. Exactness of the bottom row is a technical consequence of the snake lemma.

I.2.29. Corollary. Let R be a gr-Dedekind domain.

1. If R satisfies G.A.P. then $Cl^g(R) \cong Cl(R_0)/ < [I] >$, this is a generalization of Lemma I.1.14. in this case.
2. If R is gr-semilocal then R is a graded principal ideal domain if and only if R satisfies G.A.P. (it had been noted before that gr-semilocal rings are not as nice as their ungraded equivalents e.g. Pic^g need not vanish, the corollary shows that it is the failure in G.A.P. that causes

these problems of purely graded nature).

Proof. **1.** If R satisfies G.A.P. then $(e_P, e_Q) = 1$ for all $P, Q \in \Omega_g(R)$ by Theorem I.2.18. Therefore the subgroup of $g(R)$ generated by $(1, \ldots, 1)$ is the whole of $g(R)$, hence $g_0(R) = 0$.

2. If R is gr-semilocal then R_0 is semilocal; again combining Theorem I.2.18. and Proposition I.2.28., the statements follow easily. □

II. Graded Rings and Orders.

II.1. Graded Rings over Goldie Rings.

The rings considered in consequent chapters will allways be P.I. rings, nevertheless in the present section the restriction to the P.I. case is really superfluous and it would serve no aim at all since none of the proofs given in this section would simplify if one restricts attention to P.I. rings. Hence, the fact that we prefer to include the following results in a generality exceding that of the main body of this book is a choice inspired by an esthetic evaluation rather than by pragmatic arguments.

A ring R is said to be a **Graded Goldie Ring** if it satisfies the ascending chain condition on graded left annihilators and if it has finite Goldie dimension viewed as an object of the category, R-gr, of graded left R-modules. It has been pointed out in [37] that a graded Goldie ring does not necessarily have a ring of homogeneous fractions which is gr-simple gr-Artinian (concepts defined as in the ungraded case but now intrinsically in R-gr). Also a graded Goldie ring is not necessarily a Goldie ring that happens to be graded, but our first result will remedy this at least in the cases we will be considering throughout this book.

II.1.1. Lemma. A graded ring R of type G, where G is an ordered group, satisfying the ascending chain condition for graded left annihilators has nilpotent left singular radical , $t_s(R)$.

Proof. First one easily verifies for any graded left ideal L of R that R is an essential extension of L in R-gr if and only if it is an essential extension in R-mod. If I is any essential left ideal in R then the left ideal $I^{\sim}$ generated by the components of highest degree appearing in the decompositions of elements of I is essential as a graded left ideal of R. Therefore $I^{\sim}$ is then also essential as a left ideal. If $r = r_{i_1} + \ldots + r_{i_n} \in R$, say $i_1 > \ldots > i_n$ where $i_j = \deg r_{i_j}$, then $Ir = 0$ entails $I^{\sim} r_{i_1} = 0$ and then it follows from the foregoing remarks that $r_{i_1} \in t_s(R)$. Repeating this argument finally leads to the conclusion that $t_s(R)$ is a graded (left) ideal. Write $J = t_s(R)$. From the ascending chain of left annihilators : $l(J) \subset l(J^r) \subset \ldots$, we derive that $l(J^n) = l(J^{n+1}) = \ldots$, for some $n \in I\!N$. If $J^{n+1} \neq 0$ then there is an homogeneous a in R such that a $J^n \neq 0$ and we may assume that a is chosen such that $l(a)$ is maximal with respect to the forementioned property. Let b be an homogeneous element of J, then $l(b) \cap Ra \neq 0$ since $l(b)$ is an essential left ideal of R. Consequently we may select an homogeneous c in R such that $ca \neq 0$ but $cab = 0$. Since $l(a) \underset{\neq}{\subset} l(ab)$, the maximality hypothesis entails that $abJ^n = 0$. Since J is generated by its homogeneous elements it follows that $aJ^{n+1} = 0$, i.e. $a \in l(J^{n+1}) = l(J^n)$ a contradiction. Therefore $J^{n+1} = 0$ follows. □

II.1.2. Theorem. Let R be a semiprime ring graded by a finitely generated ordered group then the following statements are equivalent :

1. R is a graded Goldie ring.
2. R is a Goldie ring.

Proof. The implication $2 \Rightarrow 1$ is obvious.

$1. \Rightarrow 2.$ The lemma entails that $t_s(R) = 0$. If we can show that R has finite Goldie dimension then the injective hull $E(R)$ of R is semisimple, hence R will then also satisfy the ascending chain condition for left annihilators and the proof will be finished. By 1. we know that R has finite Goldie dimension in R-gr, the fact that G is finitely generated allows us to derive that R has finite Goldie dimension in R-mod. □

An obvious draw-back of the result mentioned in Theorem II.1.2. is

that rings graded by finite groups are a priori excluded. It is therefore natural to look for mild conditions on the gradation ensuring the existence of rings of homogeneous fractions even when we consider arbitrary grading groups. The condition (E) we come up with is mild enough so as to include all the particular examples we will study in this and consequent chapters. In the sequel of this section we consider an arbitrary group G with unit element e and a graded ring $R = \oplus_{\sigma \in G} R_\sigma$ of type G. We say that R has property (E) if each nonzero graded left ideal of R intersects R_e in a nontrivial way, or equivalently, if for every nonzero $r_\sigma \in R_\sigma$ we have that $R_{\sigma^{-1}} r_\sigma \neq 0$. In case R_e is semiprime, this condition is left-right symmetric i.e. for $r_\sigma \neq 0$ in R_σ we have $R_{\sigma^{-1}} r_\sigma \neq 0$ if and only if $r_\sigma R_{\sigma^{-1}} \neq 0$. It is easily verified that a graded Goldie ring of type G satisfying (E) has a Goldie ring R_e as its part of degree e. Up to assuming that property (E) holds; we therefore have reduced the problems concerning graded Goldie rings to problems concerning graded rings over Goldie rings.

II.1.3. Lemma. Let R be a graded ring of type G such that property (E); holds then the following properties hold too :

1. R_e has finite (left) Goldie dimension if and only if R has finite (left) graded Goldie dimension.
2. A graded left ideal L of R is (graded) essential in R if and only if $L \cap R_e$ is essential as a left ideal of R_e.

Proof. An easy consequence of property (E). □

II.1.4. Proposition. Let R be a graded ring of type G such that R satisfies (E) and R_e is a semiprime (left) Goldie ring, then the following properties hold :

1. The set $S = \{s \in R, s \text{ is regular and homogeneous } R\}$ is a multiplicative (left) Ore set of R.
2. The multiplicative set $S_e = \{s \in R_e, s \text{ is regular in } R_e\}$ is a regular (left) Ore set of R and the (left) ring of fractions $S^{-1}R$ is also a (left), ring of fractions of R with respect to S_e.
3. The ring $S^{-1}R$ is graded of type G and it satisfies (E). Moreover, $S^{-1}R$ is a (left) gr-semisimple ring such that $(S^{-1}R)_e = S_e^{-1} R_e$.

Proof 1. Pick $s \in S$ and suppose that L is a nonzero graded left ideal of R such that $Rs \cap L = 0$. Then we obtain an infinite direct sum of nonzero graded left ideals of R : $L \oplus Ls \oplus \ldots \oplus Ls^n \oplus \ldots, n \in \mathbb{N}$. By the lemma this yields a contradiction, i.e. Rs is essential as a graded left ideal of R. For $r \in R$ homogeneous one easily verifies that $(Rs : r) = \{x \in R, xr \in Rs\}$ is essential as a graded left ideal of R. Again by the lemma $(Rs : r) \cap R_e$ is an essential left ideal of R_e and since R_e is a semiprime Goldie ring there exists a regular element of R_e in $(Rs : r) \cap R_e$. For an arbitrary $y \in R$, say $y = y_{\sigma_1} + \ldots + y_{\sigma_n}$, $\sigma_i \in G$, $i = 1, \ldots, n$, there exist : $t_1 \in S_e$ and $x_1 \in R$ such that $t_1 y_{\sigma_1} = x_1 s$, $t_2 \in S_e$ and $x_2 \in R$ such that $t_2(t_1 y_{\sigma_2}) = x_2 s$, and so on,... . Finally, there exist $t \in S_e$ and $x \in R$ such that $ty = xs$. Furthermore if r_σ is nonzero in R_σ then $R_{\sigma^{-1}} r_\sigma \neq 0$ and $r_\sigma R_{\sigma^{-1}} \neq 0$, i.e. if an element $t \in R_e$ is regular in R_e then it is also regular in R.

2. From the above proof it follows that S_e is a regular (left) Ore set of R. If $s \in S$ then there exist $t \in S_e$ and $x \in R$ such that $t1 = xs$, consequently, $S^{-1}R$ is a leftring of fractions with respect to S_e.

3. That $S^{-1}R$ satisfies (E) is readily verified and it is equaly obvious that $S^{-1}R$ does not contain a proper essential graded left ideal. As in the ungraded situation one now derives in a straightforward way that $S^{-1}R$ is (left) gr-semisimple i.e. $S^{-1}R$ is a finite direct sum of minimal graded left ideals. □

Note that the hypothesis of Proposition II.1.4. does imply that R is a left graded Goldie ring because the fact that $S^{-1}R$ is left gr-Noetherian (cf [37]) implies that R satisfies the ascending chain condition on graded left annihilators. An easy elaboration of the foregoing results leads to :

II.1.5. Proposition. Let R be a graded ring of type G such that property (E) holds and suppose that R_e is a prime left Goldie ring, then the following properties hold :

1. $S^{-1}R$ has no nontrivial graded ideals.

2. If $R_\tau r_\sigma = 0$ with $r_\sigma \in R_\sigma$ then either $R_\tau = 0$ or $r_\sigma = 0$

3. $G' = \{\sigma \in G, R_\sigma \neq 0\}$ is a subgroup of G. Since $(S^{-1}R)_e$ is simple Artinian we obtain that $(S^{-1}R)_{\sigma^{-1}}(S^{-1}R)_\sigma = (S^{-1}R)_e$ if $R_\sigma \neq 0$,

hence $S^{-1}R$ is strongly graded by G'. In this situation $S^{-1}R$ is even a crossed product (cf. [35]).

Recall that R is said to be **strongly graded** by G if $R_\sigma T_\tau = R_{\sigma\tau}$ holds for all $\sigma, \tau \in G$.
In the sequel of this section we consider the following situation : R is a graded ring of type G such that S is a left Ore set of R and $S^{-1}R = Q^g$ is a gr-simple gr-Artinian ring (in particular : R is a gr-prime left graded Goldie ring). This assumption has the advantage that we do not only cover the case of graded rings with property (E) over prime left Goldie rings but also all positively $\mathbb{Z}$-graded prime left graded Goldie rings are included . The structure of Q^g is given by Theorem I.5.8. in [37] i.e. there exists a gr-division ring D of type G (i.e. all homogeneopus elements are invertible) together with an element $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in G^n$ such that $Q^g \cong M_n(D)(\underline{\lambda})$ as graded rings, the gradation of $M_n(D)(\underline{\lambda})$ is defined by $(a_{ij}) \in (M_n(D)(\underline{\lambda}))_\sigma$ if and only if $a_{ij} \in D_{\lambda_i \sigma \lambda_j^{-1}}$. Since D_e is a division algebra (skewfield), D is a crossed product of D_e and $G^1 = \{\sigma \in G, D_\sigma \neq 0\}$.

II.1.6. Proposition. Let D be a gr-division ring of type G.

1. If $G = \mathbb{Z}$ then D is a left and right principal ideal domain.

2. If G is poly-infinite-cyclic then D is a Noetherian domain.

3. If G is torsion-free abelian then D is an Ore domain.

4. If G is finite then D is an Artinian ring.

5. If G is polycyclic-by-finite then D is Noetherian and it has an Artinian classical ring of fractions.

Proof. We may assume that D is nontrivially graded.
1. If $G = \mathbb{Z}$ then $D \cong D_0[X, X^{-1}, \varphi]$ where X is an indeterminate, $\varphi \in \mathrm{Aut}(D_0)$ and $Xa = \varphi(a)X$ for all $a \in D_0$, cf. [37]
2. Obviously G^1 is again poly-infinite-cyclic, hence there exists a finite series $\{e\} = G^1_0 \subset G^1_1 \subset \ldots \subset G^1_n = G^1$ of subgroups such that each G^1_{i-1} is normal in G^1_i and G^1_i/G^1_{i-1} is infinite cyclic for each i. If $n = 1$ then the statement holds, so we proceed by induction on n. Put $H = G^1_{n-1}$; then D is strongly graded by $G^1/H \cong \mathbb{Z}$ over the subring $D^{(H)} = \bigoplus_{\sigma \in H} D_\sigma$. Now, by induction, $D^{(H)}$ is a Noetherian domain.

From [37] p. 88, Lemma II.3.7., it follows that D is Noetherian. Since G^1/H is ordered, D is also a domain.

3. Since G is an ordered group the fact that D does not contain homogeneous zero-divisors implies that it is a domain. Pick nonzero a and b in D, $a = a_{\sigma_1} + \ldots + a_{\sigma_n}$ and $b = b_{\tau_1} + \ldots + b_{\tau_m}$ say. The subgroup H of G generated by $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m$ is torsion-free abelian, hence $H \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$. The subring $D^{(H)}$ of D is again a gr-division ring, hence it follows (see 2.) that $D^{(H)}$ is a Noetherian domain and consequently $D^{(H)}$ is an Ore domain. Thus a and b satisfy the Ore condition in $D^{(H)}$ hence in D.

4. Corollary II.3.3. in [37].

5. Obviously G^1 is again polycyclic-by-finite. Therefore there exists a poly-infinite-cyclic normal subgroup H of G^1 such that $[G^1 : H]$ is finite. By 2 the subring $D^{(H)} = \bigoplus_{\sigma \in H} D_\sigma$ of D is a Noetherian domain and as such it has a classical ring of fractions Σ which is a division ring. Now D is strongly graded by G^1/H over the subring $D^{(H)}$. According to Proposition II.1.4. the set $T = D^{(H)} - \{o\}$ is a regular Ore set of D and $T^{-1}D \cong DT^{-1}$ is strongly graded by G^1/H over the division ring $K = T^{-1}D^{(H)}$. Since G^1/H is a finite group it follows that $T^{-1}D$ is an Artinian ring (cf. 4.) hence it is a classical ring of fractions of D. Moreover, D is a Noetherian ring, cf. [37] p. 86. □

II.1.7. Corollary. Let R be a graded ring of type G possessing a graded left ring of fractions Q^g which is a gr-simple gr-Artinian ring (e.g. R satisfies (E) and R_e is a prime left Goldie ring) then the following properties hold :

1. If G is poly-infinite-cyclic or torsion-free abelian then R is a prime left Goldie ring.

2. If G is poly-cyclic-by-finite then R has an Artinian classical left ring of fractions, hence R is a left Goldie ring.

Proof. We have that $Q^g \cong M_n(D)$ for some gr-division ring D (the isomorphism is only considered here as an isomorphism of ungraded rings). Clearly $M_n(Q_{cl}(D))$ is an Artinian classical (left) ring of fractions for Q^g thus it is a (left) ring of fractions of R. □

Even if a prime graded (left) Goldie ring is also a (left) Goldie ring

in the ungraded sense it need not have a graded ring of homogeneous fractions (in the absence of property (E)). If we are contented to obtain a nice graded localization that is not necessary a ring of (homogeneous) fractions in the classical sense then general graded Goldie rings do indeed allow such a localization. We write $E^g(R)$ for the injective hull of R (up to graded isomorphism) in R-gr.

II.1.8. Proposition. Let R be a prime graded Goldie ring then $E^g(R)$ is a gr-simple gr-Artinian ring such that the canonical ring morphism $R \to E^g(R)$ is a left flat ring epimorphism.

Proof. In view of Theorem II.1.2., the injective hull $E(R)$ of R in R-mod is a simple Artinian ring and it is clear that this entails that $E^g(R)$ is a Goldie ring. The gr-injectivity of $E^g(R)$ entails that it is a graded Von Neumann regular ring in the sense of [55]. As in the ungraded case one may deduce from this that $E^g(R)$ is indeed gr-simple gr-Artinian. Since $E^g(R)$ is the localization of R at a graded torsion theory (associated to the filter of essential graded left ideals of R) it follows that this localization is perfect (since essential graded left ideals extend to $E^g(R)$ by localizing). Then the final statement is a consequence of the properties of perfect localizations. □

II.2. Divisorially Graded Rings and Generalized Rees Rings.

The concepts of divisorially graded rings and generalized Rees rings have been introduced by F. Van Oystaeyen first in the commutative situation, then extended to the case of rings satisfying polynomial identities, cf. L. Le Bruyn, F. Van Oystaeyen [30], and then to more general noncommutative rings cf. H. Marubayashi[31], C. Năstăsescu, E. Nauwelaerts, F. Van Oystaeyen [35]. We start here from the most general definition of a divisorially graded ring given by H. Marubayashi in [31].

Let R be a graded ring of type G such that R_e is a prime Goldie ring with classical ring of fractions $Q_{cl}(R_e) = Q_e$ and let $E_e = E(Q_e/R_e)$ be the injective hull of the left R_e-module Q_e/R_e. Consider the idempotent kernel functor κ on R_e-mod with Gabriel filter $\mathcal{L}(\kappa) = \{H$ left ideal of $R_e, \mathrm{Hom}_{R_e}(R_e/H, E_e) = 0\}$; in other words, $H \in \mathcal{L}(\kappa)$ if and only if for $r \in R_e$, $q \in Q_e$ such that $(H:r)q \subset R_e$ we have that $q \in R_e$. For an $M \in R_e$-mod we write $Q_\kappa(M)$ for the module of quotients of M with respect to κ, and we let $j_\kappa : M \to Q_\kappa(M)$ be the canonical localization morphism. In a left-right symmetric way one obtains the idempotent kernel functor κ' on the right R_e-module category mod-R_e which is then associated to the injective hull of Q_e/R_e as a right R_e-module etc ... . Since R_e is a prime Goldie ring, $\kappa(R_e) = 0$ and $Q_\kappa(R_e) = R_e$ and similar statements hold for κ'. For a left ideal L of R_e we define the κ-closure of L as $cl(L) = \{x \in R_e, Hx \subset L$ for some $H \in \mathcal{L}(\kappa)\}$ and we say that L is κ-closed if and only if $L = cl(L)$. Actually, since $Q_\kappa(R_e) = R_e$ it is not difficult to verify that $cl(L) \cong Q_\kappa(L)$ as a left

R_e-module.

We say that R is divisorially graded if the following properties hold :

1. R is κ- and κ'-torsion free,

2. For all $\sigma, \tau \in G$, $Q_\kappa(R_\sigma R_\tau) = R_{\sigma\tau} = Q_{\kappa'}(R_\sigma R_\tau)$. Note that the latter condition implies $Q_\kappa(R_\sigma) = Q_{\kappa'}(R_\kappa)$ for all $\sigma \in G$; moreover, for each $\sigma \in G$ we have that $Q_\kappa(R_\sigma R_{\sigma^{-1}}) = Q_\sigma(R_{\sigma^{-1}} R_\sigma) = R_e$, hence $R_\sigma R_{\sigma^{-1}}$ and $R_{\sigma^{-1}} R_\sigma$ are both in $\mathcal{L}(\kappa)$ (and in $\mathcal{L}(\kappa')$).

It is clear that a divisorially graded ring R of type G satisfies (E), so by the results of Section II.1., $S_e = \{s \in R_e, s$ is regular in $R_e\}$ is a regular left and right Ore set of R. The left ring of fractions with respect to S_e is isomorphic to the right ring of fractions and it will be denoted by Q^g. Since no R_σ is zero it follows that Q^g is strongly graded by G and $(Q^g)_e = Q_{cl}(R_e) = Q_e$. If $H \in \mathcal{L}(\kappa)$ then H is an essential left ideal of R_e and hence $H \cap S_e \neq$ because R_e is a prime Goldie ring. Consequently, Q^g is κ-torsion free and $Q_\kappa(Q^g) = Q^g$. Similarly, we derive that $Q_{\kappa'}(Q^g) = Q^g$. From the foregoing remarks one easily deduces that :

$$Q_\kappa(R_\sigma) = \{q \in (Q^g)_\sigma, Hq \subset R_\sigma \text{ for some } H \in \mathcal{L}(\kappa)\}$$
$$Q_\kappa(R) = \{q \in Q^g, Hq \subset \text{ for some } H \in \mathcal{L}(\kappa)\}$$

and from $Q_\kappa(R_\sigma) = R_\sigma$ it then follows that $Q_\kappa(R) = R$. Similar properties hold with respect to κ'.

II.2.1. Example. If R is a strongly graded ring of type G then R is automatically divisorially graded if R_e is a prime Goldie ring. To prove this claim it will be sufficient to show that $Q_\kappa(R_\sigma) = R_\sigma = Q_{\kappa'}(R_\sigma)$ for all $\sigma \in G$. Since R is strongly graded property (E) holds and thus we may apply the preceding remarks. If $q \in Q_\kappa(R_\sigma)$ then $Hq \subset R_\sigma$ for some $H \in \mathcal{L}(\kappa)$ and thus $HqR_{\sigma^{-1}} \subset R_e$. But then $qR_{\sigma^{-1}} \subset (Q^g)_e = Q_e$ and $H(qR_{\sigma^{-1}}) \subset R_e$ imply that $qR_{\sigma^{-1}} \subset R_e$, hence $qR_{\sigma^{-1}}R_\sigma \subset R_\sigma$ or $q \in R_\sigma$.

Let us now investigate somewhat closer the notion of a strongly graded ring; we restrict to the bare necessities and refer to C. Năstăsescu, F. Van Oystaeyen [36], [37] for full detail on the basic theory.

Let R be strongly graded by a group G. From $R_\sigma R_{\sigma^{-1}} = R_e$ it is easily deduced that R_σ is a finitely generated left (and right) projective R_e-module and then $R_e = R_\sigma R_{\sigma^{-1}} \cong R_\sigma \otimes_{R_e} R_{\sigma^{-1}}$ for all $\sigma \in G$. It follows that R_σ for all $\sigma \in G$, is an invertible R_e-bimodule and hence its isomorphism class (of R_e-bimodules), $[R_\sigma]$ say, determines an element of the Picard group $\mathrm{Pic}(R_e)$. The Picard group of a noncommutative ring consists of the isomorphism classes of invertible bimodules equipped with the group operation induced by the tensorproduct. The fundamental theory of the Picard group of a noncommutative ring, in particular of an order, is well established by A. Fröhlich in [21]. Giving a strongly graded ring R over R_e of type G comes down to giving a group morphism $\Phi : G \to \mathrm{Pic}(R_e), \sigma \mapsto [R_\sigma]$ and to define multiplication in $\oplus_{\sigma\in G} R_\sigma$ by a set of R_e-bimodule isomorphisms $f_{\sigma,\tau} : R_\sigma \underset{R_e}{\otimes} R_\tau \to R_{\sigma\tau}$ for all $\sigma, \tau \in G$ satisfying the associativity conditions expressed by the commutativity of the following diagram for $\sigma, \tau, \gamma \in G$:

$$\begin{tikzcd}
R_\sigma \underset{R_e}{\otimes} R_\tau \underset{R_e}{\otimes} R_\gamma \arrow[r, "f_{\sigma,\tau}\otimes 1_\gamma"'] \arrow[d, "I_\sigma \otimes f_{\tau,\gamma}"'] & R_{\sigma\tau} \otimes_{R_e} R_\gamma \arrow[d, "f_{\sigma\tau,\gamma}"] \\
R_\sigma \underset{R_e}{\otimes} R_{\tau\gamma} \arrow[r, "f_{\sigma,\tau\gamma}"'] & R_{\sigma\tau\gamma}
\end{tikzcd}$$

The family $\{f_{\sigma,\tau}, \sigma, \tau \in G\}$ is called a **factor system** describing the gradation of R.

II.2.2. Proposition. Let A be any ring with centre $Z(A) = C$.

1. There is a canonical group morphism, $\varphi : \mathrm{Pic}(A) \to \mathrm{Aut}(Z(A))$.

2. The following sequence of groups is exact :

$$0 \to \mathrm{Inn}(A) \to \mathrm{Aut}(A) \to \mathrm{Pic}(A).$$

Proof. Well-known, cf. [21] or [37]. Let us just indicate how the maps are defined. If $[P] \in \mathrm{Pic}(A)$ then we have that $\mathrm{End}_A(P_A) \cong A$, $\mathrm{End}_A({}_AP) \cong A^\circ$ and $\mathrm{End}_{A-A}(P) = Z(A) = C$. So, if $c \in C$ then there is a uniquely determined element of C, $\varphi_P(c)$ say, such that $Pc = \varphi_P(c)P$ holds elementwise. The map φ is then defined by $\varphi([P]) = \varphi_P$. If $\alpha \in \mathrm{Aut}(A)$ then we write ${}_1A_\alpha$ for the A-bimodule obtained

by considering the left A-module structure of A induced by the ring structure but for the right A-module structure we use $x.a = x\alpha(a)$ i.e. we "twist" by the automorphism α on the right. Clearly ${}_1A_\alpha \cong {}_1A_1$ as A-bimodules if and only if α is an inner automorphism. □

Applying the foregoing result to the situation where R is a strongly graded ring of type G we obtain group morphism :

$$G \underset{\Phi}{\longrightarrow} \mathrm{Pic}(R_e) \underset{\varphi}{\longrightarrow} \mathrm{Aut}(Z(R_e)),$$

and the composition $\varphi \circ \Phi$ defines an action of G on $Z(R_e)$ (this action need not be extendable to the whole of R_e in general !). Since property (E) holds we may apply Proposition II.1.5. to conclude that $Q^g = S_e^{-1}R$ is strongly graded of type G over $(Q^g)_e = S_e^{-1}R_e = Q_{cl}(R_e) = Q_e$, when R_e is a prime Goldie ring. Since $(Q^g)_e$ is a simple Artinian ring Q^g is a crossed product $Q_e * G = \underset{\sigma\in G}{\oplus} Q_e.u_\sigma$, with $u_\sigma q = q^{\varphi_\sigma}u_\sigma$, $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma\tau}$ for all $\sigma,\tau \in G$ (where φ_σ is determined by $[(Q^g)_\sigma] = [{}_1(Q_e)_{\varphi_\sigma}]$ for some $\varphi_\sigma \in \mathrm{Aut}(Q_e)$ and $c : G \times G \to U(Z(Q_e))$ is some 2-cocycle). The subring R of Q^g may thus be written as $R = \oplus_{\sigma\in G} I_\sigma u_\sigma$ where I_σ is an invertible R_e-bimodule in Q_e for each σ, satisfying $I_\sigma I_\tau = I_{\sigma\tau}$. With these notations we have the following :

II.2.3. Definition. A strongly graded ring of type G such that R_e is a prime Goldie ring is said to be a **generalized Rees ring** if $\varphi_\sigma(R_e) \subset R_e$ for all $\sigma \in G$ i.e. $\varphi_\sigma|R_e \in \mathrm{Aut}(R_e)$ for all $\sigma \in G$. Since the properties of Q^g mentioned above remain true even if R is only divisorially graded, **we will also use the term generalized Rees ring in the divisorially graded case.** If R_e is a semiprime (not necessarily prime) Goldie ring then Q_e will be a semisimple algebra and the Q^g need not be a crossed product of the form $Q_e * G$ (cf. the counter example given by C. Năstăsescu, F. Van Oystaeyen in [37]) but then we define (the most general) a generalized Rees ring to be a strongly or divisorially graded subring R of type G in $Q_e * G$ such that $Q^g(R) = Q_e * G$ and $\varphi_\sigma|R_e \in \mathrm{Aut}(R_e)$ for each σ, $(\varphi_\sigma \in \mathrm{Aut}(Q_e))$.
Let us round off this section by giving some specific results for strongly graded rings of type $\mathbb{Z}$ over prime Goldie rings.

II.2.4. Proposition. Let R be strongly graded of type $\mathbb{Z}$.

1. If P is a (semi) prime ideal of R_0 such that R then RP is an ideal of RP is a (semi prime ideal.

2. Suppose that R_0 satisfies the ascending chain condition on ideals. If P is a semiprime ideal of R then $P \cap R_0$ is a semiprime ideal of R_0. If P is a prime ideal of R then $P \cap R_0$ is of the form $M \cap R_1MR_{-1} \cap \ldots \cap RkMR_{-k}, k \in \mathbb{N}$, for some prime ideal M of R_0 minimal amongst prime ideals containing $M \cap R_0$.

Proof. 1. Easy.

2. There is a bijective correspondence between prime ideals of R_0 containing I and prime ideals containing R_nIR_{-n} given by $M \mapsto R_nMR_{-n}$. Therefore if $\mathrm{rad}(I) = \bigcap_i M_i$ than $\mathrm{rad}(R_nIR_{-n})$ equals $\bigcap_i R_nM_iR_{-n}$. Now, $R_{-n}(\bigcap_i R_nM_iR_{-n})R_n \subset \bigcap_i M_i$ yields $\bigcap_i R_nM_iR_{-n} \subset R_n(\bigcap_i M_i)R_{-n}$, hence $\bigcap_i R_nM_iR_{-n} = R_n(\bigcap_i M_i)R_{-n}$. Consequently $\mathrm{rad}(R_nIR_{-n}) = R_n\mathrm{rad}(I)R_{-n}$ for all n. For all n we also have : $R_n(P \cap R_0) = P \cap R_n = (P \cap R_0)R_n$. Hence, $R_n\mathrm{rad}(P \cap R_0)R_{-n} = \mathrm{rad}(P \cap R_0)$ and $R\mathrm{rad}(P \cap R_0) = \mathrm{rad}(P \cap R_0)R$. The ascending chain condition on ideals of R_0 entails that $(\mathrm{rad}(P \cap R_0))^m \subset P \cap R_0$ for some $m \in \mathbb{N}$. Hence, we obtain that $(R\mathrm{rad}(P \cap R_0))^m \subset R(P \cap R_0) \subset P$ or $R\mathrm{rad}(P \cap R_0) \subset P$ and also $\mathrm{rad}(P \cap R_0) = P \cap R_0$. By the remark above we then obtain : $P \cap R_0 = \bigcap_i(\bigcap_n R_nM_iR_{-n})$, where the M_i are minimal amongst the prime ideals of R_0 containing $P \cap R_0$, and the intersection over the i may be taken to be finite. Now, for each i we see that $R(\bigcap_n R_nM_iR_{-n}) = (\bigcap_n R_nM_iR_{-n})R$. So if P is a prime ideal of R then it follows that $R(\bigcap_n R_nM_iR_{-n}) \subset P$ (because $\bigcap_i R(\bigcap_u R_uM_iR_{-n}) \subset P$!) and hence $P \cap R_0 = \bigcap_n R_nM_1R_{-n}$ and the intersection is finite. □

II.2.5. Remarks. 1. It is clear that the restriction to $G = \mathbb{Z}$ is not really necessary in the foregoing proposition.

2. If P is a (semi) prime ideal of R such that $P \cap R_0 = 0$ then R is a (semi) prime ring (here "G is ordered" is a necessary assumption).

Indeed, if P_g is the graded ideal generated by the homogeneous elements of P then P_g is a (semi) prime ideal of R since G is ordered; from $P_g = R(P_g \cap R_0) \subset R(P \cap R_0)$ it follows that $P_g = 0$; hence R is (semi) prime.

II.2.6. Proposition. Let R be strongly graded of type $\mathbb{Z}$ over a prime left Goldie ring R_0.

1. If I is an ideal of R such that R/I satisfies the ascending chain condition on left annihilators then Q^gI is an ideal of Q^g.

2. If R_0 is a Goldie ring (i.e. left and right) then Q^gI is an ideal of Q^g for each ideal I of R.

Proof. It is clear that $Q^gI \neq Q^g$ if and only if I does not contain a regular element of degree zero.

1. Let S_0 be the set of regular elements in R_0, then its image $\overline{S}_0$ in R/I is a left Ore set and therefore $\overline{S}_0$ is left reversible in R/I. For $r \in R$, $s \in S_0$ such that $rs \in I$ we obtain $s'r \in I$ for some $s' \in S_0$. If $a \in I$, $s \in S_0$ then there exist $r \in R, t \in S_0$ such that $ta = rs$; but now $rs \in I$ implies $s'r \in I$ for some $s' \in S_0$, hence $s'ta = s'rs$ with $s't \in S_0$ and $s'r \in I$. Consequently $IQ^g \subset Q^gI$ follows.

2. By Proposition II.1.6., Q^g is a prime left and right principal ideal ring and thus $Q^gIQ^g = Q^ga = aQ^g$. We claim that a be chosen in I. Indeed, since R_0 is a left and right Goldie ring there exist $s, t \in S_0$ such that $ast \in I$. Now, $Q^gsat = Q^gat = A^gaQ^g = Q^ga$ and similarly $satQ^g = aQ^g$. Obviously $Q^gI = IQ^g$ follows. □

II.2.7. Proposition. Let R be strongly graded of type $\mathbb{Z}$ such that R_0 is a prime left Goldie ring.

1. If P is a prime ideal of R such that $P \cap R_0 = 0$ and R/P is a left Goldie ring then $Q^gP \cap R = P$.

2. Suppose for each prime ideal P of R with $P \cap R_0 = 0$ that R/P is a left Goldie ring. Then there is a bijective correspondence between prime ideals of R lying over zero in R_0 and proper prime ideals of Q^g. Moreover each such prime ideal P has $ht(P) \leq 1$.

Proof. 1. The ideal $I = Q^gP \cap R$ contains P. If $I/P \neq 0$ then it has to contain a regular element $\overline{a}$ of the prime left Goldie ring R/P. Choose a representative $a \in I$ for $\overline{a}$; for this a there exists an $s \in S_0$ such that $sa \in P$, but then $s \in P$ leads to a contradiction. Consequently $I/P = 0$ or $I = P$.

2. If P is a prime ideal of R such that $P \cap R_0 = 0$ then Q^gP is a proper prime ideal of Q^g because $Q^gP \cap R = P$ (see 1.). On the other hand, if M is a proper prime ideal of Q^g and $M \neq 0$, then $M \cap R \subset P$ for some ideal P of R which is maximal with respect to the property of lying over zero in R_0. It is clear that P is a prime ideal of R. Now, $M = Q^g(M \cap R) \subset Q^g P$. Since Q^g is a left and right principal ideal ring, $M = Q^g P$ follows and $M \cap R = Q^g P \cap R = P$. For the final statement consider a prime ideal $P' \neq 0$ of $R, P' \subset P$. From $P' \cap R_0 = 0$ it follows that $Q^g P' \subset Q^g P$ are prime ideals of the prime left and right principal ideal ring Q^g, hence $Q^g P' = Q^g P$ and $P' = P$ follows. □

II.2.8. Proposition. Let R be strongly graded of type $\mathbb{Z}$ over the prime left Goldie ring R_0. Assume that for all prime ideals P of R such that $P \cap R_0 = 0$ the ring R/P is a left Goldie ring. Then for each prime ideal P of R lying over zero in R_0 we have that $C(P)$ (the multiplicative set associated to P) is a regular left Ore set of R containing S_0. Moreover, $R \subset Q_P(R) = Q_M(Q^g)$ where $M = Q^g P$ and $Q_P(R)$ is the ring of fractions of R at $C(P)$, the latter ring is a bounded prime left and right principal ideal ring with unique maximal ideal $Q_P(R)P$.

Proof. First, if $rs \in P$ for some $s \in S_0$, $r \in R$, then there is an $s' \in S_0$ such that $s'r \in P$, hence $r \in P$ and thus $S_0 \subset C(P)$. We now divide the proof in three steps.

1. $C(P) \subset C(Q^g P)$ Take $c \in C(P)$ and let $q \in Q^g$ be such that $qc \in Q^g P$. Write $q = s^{-1}r$ for certain $s \in S_0$, $r \in R$. Then $qc \in Q^g P$ yields $rc \in Q^g P \cap R = P$, hence $r \in P$ and $q \in Q^g P$ follows. This states that $c \in C(Q^g P)$.

2. If $s^{-1}c \in C(Q^g P)$ with $s \in S_0$, $c \in R$, then $c \in C(P)$.
Indeed, if $rc \in P$ for some $r \in R$ then $rss^{-1}c \in Q^g P$ yields $rs \in Q^g P$ and $r \in Q^g P \cap R = P$.

3. $C(M)$ is a regular left and right Ore set of Q^g.
Obvious because Q^g is a prime left and right principal ideal ring.

4. $Q_M(Q^g)$ is the left ring of fractions of R with respect to $C(P)$.
Clearly, $R \subset Q_M(Q^g)$. If $c \in C(P)$ then $c \in C(M)$ and hence c is invertible in $Q_M(Q^g)$. On the other hand for any $q \in Q_M(Q^g)$ we write $q = u^{-1}v$ for some $u \in C(M)$, $v \in Q^g$ and $u = s^{-1}c$, $v = s^{-1}r$ for some $s \in S_0, c, rinR$. From $u \in C(M)$ it follows that $c \in C(P)$ (see 2.) and thus c is invertible in $Q_M(Q^g)$. This yields $q = (s^{-1}c)^{-1}s^{-1}r = c^{-1}r$

and thus it follows that $Q_M(Q^g) = Q_P(R)$ and $C(P)$ is a regular left Ore set of R. □

In concluding this section let us point out that the assumption : "R/P is a left Goldie ring for each prime ideal P of R such that $P \cap R_0 = 0$" is trivially satisfied in the case where R is a left Noetherian ring or a ring, satisfying polynomial identities, hence from our point of view this condition is a very mild one.

II.3. Graded Rings Satisfying Polynomial Identities.

For the general theory of rings satisfying polynomial identities (termed P.I. rings in this work) we refer to the books by C. Procesi and L. Rowen resp [42] and [48]. The study of a gradation of type $\mathbb{Z}$, or more general groups like polycyclic-by-finite groups, is a very natural idea. Indeed, the ring of generic $n \times n$ matrices as well as its trace ring is a positively graded ring. The consideration of P.I. rings graded by a group G presents in some sense a reversion (of a generalization) of the problem of determining which group rings satisfy polynomial identities, cf. D. Passman [41]. Both the gradations of type $\mathbb{Z}$ and the more general ones of type G will play an effective part in further chapters of this book. We say that G is a pi-group if kG satisfies some polynomial identities for some commutative ring k.

II.3.1. Proposition. (D. Passman) Let K be a field and suppose that KG satisfies a proper polynomial identity of degree n. If $\Delta(G)$ denotes the "finite conjugation" subgroup of G, i.e. $g \in \Delta(G)$ whenever the centralizer $C_G(g)$ of g in G has finite index in G, then $[G : \Delta(G)] \leq \frac{n}{2}$ and $|\Delta(G)'| < \infty$.

Proof. cf. [41] Theorem 2.14. □

If in the foregoing proposition KG is a prime ring then $\Delta(G)$ is a torsion-free abelian group. In general for any finitely generated subgroup H of $\Delta(G)$ in the pi-group G we have : $[H : Z(H)] < \infty, |H'| < \infty, [G : C_G(H)] < \infty$ and H/H_{tors} is a free abelian group. So if G itself is a finitely generated pi-group then it contains a normal abelian subgroup of finite index; the same is true for a general pi-group if $\text{char}(K) = 0$. It is the latter result we aim to extend to the graded situation. First note that we may take K to be a commutative ring in Proposition II.3.1.; actually, using A. Regev's result on the tensor product of P.I. rings being again a PI ring, it is possible to prove that over a semiprime P.I. ring K, KG satisfies polynomial identities only if $[G : \Delta(G)] < \infty$ and $|\Delta(G)|' < \infty$.

II.3.2. Theorem (F. Van Oystaeyen). Let R be a G-graded ring over the semiprime Goldie ring R_e of characteristic zero such that property (E) holds. Assume that there is a subgroup $G(R)$ of finite index in G and consisting of $\sigma \in G$ with $R_\sigma \neq 0$. If R is a P.I. ring then G contains a normal abelian subgroup of finite index; in particular if G is finitely generated then G is polycyclic-by-finite and $|G'| < \infty$.

Proof. Up to reducing to $R^{(G(R))} = \oplus_{\sigma \in G(R)} R_\sigma$ if necessary, we may assume that $G = G(R)$. At this point let us point out that some conditions like $R_\sigma \neq 0$ for almost all $\sigma \in G$ are unavoidable if one aims to relate properties of G to the structure of R and vice-versa.
Without a condition like this we may consider R as an $H \times G$-graded ring by putting $R_h = 0$ for all $h \in H$. The fact that H is completely arbitrary makes it completely clear that it would be impossible to obtain any decent result about the structure of G in the absence of the imposed condition (which most people would agree to call : not restrictive at all). From the foregoing section we know that $Q^g = S^{-1}R$ is a gr-simple gr-Artinian P.I. ring with $Q_e = S_e^{-1}R_e$ being Artinian and with $Q^g_\sigma \neq 0$ for all $\sigma \in G$. Decompose Q_e as $L_1 \oplus \ldots \oplus L_r$ where each L_i is a minimal left ideal of Q_e. From condition (E) it follows that $Q^g L_i$ is a minimal graded left ideal of Q^g and also that $Q^g L_i \cap Q^g L_j = 0$ if $i \neq j$, i.e. $Q^g = Q^g L_1 \oplus \ldots \oplus Q^g L_r$. Let Q_i be the gr-simple gr-Artinian components of Q^g (i.e. the minimal graded ideals obtained by grouping together the suitable $Q^g L_j$) and we write $Q^g = Q_1 \oplus \ldots \oplus Q_t$. Because each $Q_i, i = 1, \ldots, t$, is a graded ideal of Q^g we obtain for each $\sigma \in G$ that $Q^g_\sigma = (Q_1)_\sigma \oplus \ldots \oplus (Q_t)_\sigma$, hence for every $\sigma \in G$ there is a $i(\sigma) \in \{1, \ldots, t\}$ such that $(Q_{i(\sigma)}) \neq 0$. The graded version of the Artin-

Wedderburn theorem as stated in [37] yields that $Q_i \cong M_{n_i}(D_i)(\underline{\sigma}^i)$ for all $i = 1, \ldots, t$, where D_i is a gr-division ring and $\underline{\sigma} \in G^t$ determines the gradation of Q_i (as explained just before Proposition II.1.6.). For each D_i we let $G_i = G(D_i)$ be the subgroup of G consisting of all $\sigma \in G$ such that $(D_i)_\sigma \neq 0$ (this is indeed a subgroup !). If $\delta \in G$ is such that $(Q_i)_\delta \neq 0$ then some entry in $\left(M_{n_i}(D_i)(\underline{\sigma}^i)\right)_\delta$ is nonzero, i.e. $\sigma^i_\lambda \delta (\sigma^i_\mu)^{-1} \in G_i$ for certain λ, μ.

Consequently $(Q_i)_\delta \neq 0$ implies that $\delta \in (\sigma^i_\lambda)^{-1} G_i \sigma^i_\mu$. Now we look at the finite set of subgroups of G, $\{G_i, (\sigma^i_\lambda)^{-1} G_i \sigma^i_\lambda, i = 1, \ldots, t, \lambda = 1, \ldots n_i\}$. Since for every $\delta \in G$ some $(Q_i)_\delta \neq 0$ it follows that G is a finite union of right cosets of the forementioned groups, i.e.

$$G = \underset{i}{U} \left\{ G_i \underset{\lambda,\mu}{U} \{ (\sigma^i_\lambda)^{-1} G_i \sigma^i_\lambda . ((\sigma^i_\lambda)^{-1} \sigma^i_\mu) \} \right\}.$$

A result of B. Neumann, cf. [39], yields that at least one of the groups considered must have finite index in G. But if some $(\sigma^i_\lambda)^{-1} G_i \sigma^i_\lambda$ has finite index in G then so has G_i so we may assume that G_i has finite index in G, and we have reduced the proof of the theorem to the case of a gr-division ring strongly graded by G, i.e. $G = G_i$, $R = D_i$ and $R = R_e * G$ is a crossed product, where G acts on $Z(R_e)$ via the group morphism $\varphi : G \to \mathrm{Aut}(Z(R_e))$ deriving from the strong gradation on R. We claim that $\left[Z(R_e) : Z(R_e)^G\right] < \infty$. Suppose this were not true. We first show that $Z(R_e)$ and $Z(R)$ are linearly disjoint over $Z(R_e)^G$, i.e. $Z(R_e)Z(R) \cong Z(R_e) \underset{Z(R_e)^G}{\otimes} Z(R)$. Consider $Z(R_e)^G$-independent $x_1, \ldots, x_m \in Z(R_e)$ and suppose that they are linearly dependent over $Z(R)$, say :

$$x_1 y_1 + \ldots + x_m y_m = 0 \tag{$*$}$$

with $y_i \in Z(R)$. Write $y_i = \sum_{\sigma \in G} (y_i)_\sigma$, $i = 1, \ldots, m$. Since the x_j are of degree e it follows that :

$$x_1 (y_1)_\sigma + \ldots + x_m (y_m)_\sigma = 0 \tag{$**$}$$

for each σ. Since $y_i \in Z(R)$, $z_\tau y_i = y_i z_\tau$ for all $z_\tau \in R_\tau, \tau \in G$, $z_\tau (y_i)_\sigma = (y_i)_\gamma z_\tau$ for some γ. It is clear that $\gamma = \tau\sigma\tau^{-1}$. Then if $(y_i)_\sigma \neq 0$ then $z_\tau (y_i)_\sigma \neq 0$ and thus $(y_i)_{\tau\sigma\tau^{-1}} \neq 0$. It follows that we may assume that in the relation ($*$) the y_i are such that the degrees

appearing in their homogeneous decomposition are **exactly** the finitely many (!) conjugates $\{\tau\sigma\tau^{-1}, \tau \in G\}$. From ($**$) we then derive for all $\sigma \in G : x_1 (y_1)_\sigma (y_m)_\sigma^{-1} + \ldots + x_m = 0$, and after summation over the conjugation class $C(\sigma)$ of a fixed σ we obtain the relation :

$$x_1 \sum_{\gamma \in C(\sigma)} (y_1)_\gamma (y_m)_\gamma^{-1} + x_2 \sum_{\gamma \in C(\sigma)} (y_2)_\gamma (y_m)_\gamma^{-1} + \ldots + d x_m = 0 \quad (^{\ *}_{**})$$

where d is the number of conjugates of σ, i.e. $d = [G : C_G(\sigma)]$. Since $d \neq 0$ and because $\mathrm{char} R_e = 0$, the foregoing relation is not trivial.

For any $\tau \in G, z_\tau \in R_\tau$ we calculate :

$z_\tau (y_i)_\sigma (y_m)_\sigma^{-1} = (u_i)_{\tau\sigma\tau^{-1}} z_\tau (y_m)_\sigma^{-1}$

$z_\tau (y_m)_\sigma = (y_m)_{\tau\sigma\tau^{-1}}, z_\tau$ or $(y_m)_{\tau\sigma\tau^{-1}} z_\tau = z_\tau = z_\tau (y_m)_\sigma^{-1}$

Consequently we obtain the following relations.

$$z_\tau (y_i)_\sigma (y_m)_\sigma^{-1} = (y_i)_{\tau\sigma\tau^{-1}} (y_m)^{-1}_{\tau\sigma\tau^{-1}} z_\tau$$

$$\begin{aligned} z_\tau \left(\sum_{\gamma \in C(\sigma)} (y_i)_\gamma (y_m)_\gamma^{-1} \right) &= \left(\sum_{\gamma \in C(\sigma)} (y_i)_{\tau\gamma\tau^{-1}} (y_m)^{-1}_{\tau\gamma\tau^{-1}} \right) z_\tau \\ &= \left(\sum_{\gamma \in C(\sigma)} (y_i)_\gamma (y_m)_\gamma^{-1} \right) z_\tau \end{aligned}$$

The coefficients in $(^{\ *}_{**})$ are therefore in $Z(R) \cap R_e = Z(R_e)^G$, contradicting the assumption on $x_1, \ldots, x_m$.

So $Z(R_e)Z(R)$ is a free $Z(R)$-module of infinite rank. Since this leads to the existence of a free $Q(Z(R))$-module in the central simple algebra $Q(R) = Q(Z(R)) \otimes_{Z(R)} R$ it yields a contradiction. Consequently $Z(R_e)$ is a Galois extension of $Z(R_e)^G$ with Galois group $Im\varphi$ which is a finite group. Crossed products of $Z(R_e)$ and G with G-action on $Z(R_e)$ given by φ and two-cocycle $c : G \times G \to U(Z(R_e))$ will be denoted by $(Z(R_e), G, \varphi, c)$. Since $R = R_e * G$ is a P.I. ring $Z(R_e) * G$ is a P.I. ring too. Now if $A = (Z(R_e), G, \varphi, c)$ and $B = (Z(R_e), G, \varphi, c')$ then we claim that $A \otimes_{Z(R_e)^G} B$ contains the $Z(R_e)^G$-algebra $(Z(R_e), G, \varphi, cc')$ as a subring. Actually, the proof of this claim is the same as the proof of the product theorem for 2-cocycles in Galois cohomology cf. Pierce [] p. 258, up to verifying that the finiteness of G assumed in loc. cit. is not neccessary as long as Im φ, i.e. $Z(R_e)$ over $Z(R_e)^G$, is finite. Applying this general argument to $(Z(R_e), G, \varphi, c) \otimes_{Z(R_e)^G} (Z(R_e), G, \varphi, c)^0 = S$ we see that S is a P.I. ring by A. Regev's result and S contains $(Z(R_e), G, \varphi, 1)$ as a subring. It follows that the latter ring as well as the subring $Z(R_e)^G G$ is a P.I. ring. Since $\mathrm{char}(Z(R_e)^G) = 0$, D. Passman's result entails that G contains a normal abelian subgroup of finite index. If G is finitely generated then so is the normal abelian subgroup contained in it, hence G will be polycyclic-by-finite, also $|G'| < \infty$ follows from $|\Delta(G)' < \infty$ where $\Delta(G)$ has finite index in G. □

II.3.3. Remark. If $\mathrm{char}(R) \neq 0$ in the foregoing theorem then the proof breaks down at the point where one has to establish that for a gr-division ring D which satisfies polynomial identities and which is strongly graded by G, i.e. $D = D_e * G$ the extension of fields $Z(D_e)/Z(D_e)^G$ is finite. Now it is easy to see that $\mathrm{Im}\varphi$ is a torsion group ($\varphi : G \to \mathrm{Aut}(Z(De))$) and its exponent is bounded by the pi-degree of D. So if G is finitely generated then the cases where the Burnside problem may be answered in the affirmative for torsion groups of bounded exponent (e.g. groups which may be embedded in some matrix ring) allow to establish Theorem II.3.2. even if $\mathrm{char}(R_e)$ is nonzero.

The existence of a multilinear central polynomial for (certain) P.I. rings provides very elementary proofs for the following results.

II.3.4. Proposition. 1. Let R be a P.I. ring graded by an abelian torsion-free group G and assume that R satisfies the identities of $n \times n$ matrices. A prime ideal P of R will have pi-degree n if and only if P_g has pi-degree n; consequently the radical of the Formanek centre of R is a graded ideal.

2. Let R be graded by an arbitrary abelian group such that R satisfies the identities of $n \times n$-matrices. Suppose that every central homogeneous element of R is invertible, then R is an Azumaya algebra.

Proof. In both cases considered in the statement of the proposition we know that R allows a multilinear central polynomial, f say, which is not an identity for R (since we assume that R does not satisfy the identities of $n-1 \times n-1$ matrices, i.e. the n in the statement is supposed to be minimal as such). Since f is not an identity for R it

cannot vanish at all the homogeneous substitutions for the variables appearing in f (note that the fact that G is abelian entails that homogeneous substitutions by elements of R in f lead to homogeneous values. If $c = f(\lambda_1, \ldots, \lambda_r) \neq 0$ for some homogeneous λ_i in R then the assumptions in e yield that c is invertible and hence the Formanek centre (generated by all evaluations of central polynomials) of R equals $Z(R)$ and then R is an Azumaya algebra, cf. [42], what proves 2. The ideal of $Z(R)$ generated by all evaluations of f is a graded ideal (using the fact that G is abelian). Since the hypothesis of 1 makes G into an ordered group it is clear that P_g is a prime ideal of R for any prime ideal P of R. Consequently P will contain all evaluations of f if and only if P_g contains all these evaluations. The statement of 1. follows immediately from this because a prime ideal of R can only have pi-degree n when it does not contain the evaluations of F. □

II.3.5. Proposition. If a prime P.I. ring R is graded by an abelian group G then every graded ideal I of R contains nonzero homogeneous elements in $I \cap Z(R)$.

Proof. $I \cap Z(R)$ is graded and $I \cap Z(R)$ is nonzero by a theorem of L. Rowen, cf. [42] or [48]. □

II.3.6. Corollary. A P.I. ring R graded by an abelian group G such that R is gr-simple is an Azumaya algebra.

Proof. If $c \in Z(R)$ is homogeneous then $Rc = R$ because R is gr-simple, i.e. c is invertible and we may apply Proposition II.3.4.(2) □

Some results in the same vein as the foregoing but for not necessarily abelian groups will be included in Section II.3. Let us conclude this section by showing that Proposition II.1.8. may be strengthened as follows :

II.3.7. Proposition : Let R be a prime P.I. ring graded by an abelian group then R has a graded ring of homogeneous fractions, Q^g say, and $Q^g \cong E^g(R)$ (as defined in Proposition II.1.8.).

Proof. Let L be an essential graded left ideal of R. Since L is essential as a left ideal and R is prime P.I. it follows that L contains a nontrivial ideal and thus $L \cap Z(R) \neq 0$. Since $L \cap Z(R)$ is a graded ideal it contains a nonzero homogeneous element which is regular since R is prime. Consequently, if we put $Q_h = S_c^{-1}R$ where S_c is the set of regular homogeneous central elements then $Q_h L = Q_h$ follows. For $x \in E^g(R)$ there exists an essential graded left ideal L of R such that $Lx \subset R$ hence $Q_h Lx \subset Q_h$ and $x \in Q_h$. It follows that $E^g(R) = Q_h$ is a ring of homogeneous fractions and we have even verified that it suffices to invert central homogeneous elements in order to obtain it. □

II.4. Orders and Graded Orders.

The theory of orders and maximal orders is expounded in I. Reiner's book [44]; here we include only some perhaps less accessible facts about tame orders, class groups and reflexive algebras. In this section we also introduce graded orders and (gr-) maximal orders and some applications of these concepts. The main results concerning the ungraded properties of graded orders will be in Chapter III and further applications will appear frequently in Chapter IV. As a basic reference for maximal orders over Krull domains the reader may use R.M. Fossum's paper [20] and for details on tame orders we may refer to L. Silver [49].

In this section R will be a Krull domain with field of fractions K and A will be a central simple algebra over K. A ring $R \subset \Lambda \subset A$ such that $K\Lambda = A$ and each element of Λ is integral over R will be said to be an **R-order of A**. The reduced trace $Tr : A \to K$ induces $t : A \to \mathrm{Hom}_K(A, K)$, $a \mapsto t(a)$ $t(a)(b) = Tr(ab)$ for $b \in A$, and t is a K-vector space isomorphism. If $a \in A$ is integral over R then $Tr(a) \in R$. For a K-basis $\{a_1, \ldots, a_n\}$ of A there are $a_1^*, \ldots, a_n^* \in A$ such that $Tr(a_i^* a_j) = \delta_{ij}$ and we put $F = Ra_1 + \ldots + Ra_n, F^0 = Ra_1^* + \ldots + Ra_n^*$ then T restricts to an isomorphism $t : F^0 \to F^* = [R : F]$.

II.4.1. Proposition. 1. Consider R-orders $\Lambda \subset \Gamma$ of A and suppose that Λ contains a free R-module F, then $F^0 \supset \Gamma$.

2. The R-orders of A are R-lattices in A.

Proof. 1. It is clear that $\Gamma F \subset \Gamma$. Since $Tr(\Gamma) \subset R$ it follows that $Tr(\Gamma F) \subset R$ hence $\Gamma \subset F^0$.
2. By definition, each R-order contains a free R-module, so by 1. Λ is contained in a free R-lattice in A. □

Recall some facts about R-lattices. Consider a finite dimensional vector space V over K. An R-submodule of V is said to be an **R-lattice in V** if it contains a K-basis for V and if it is contained in a finitely generated R-submodule of V. Let us recall some of the operations on lattices that are used most frequently (cf. R. Fossum [20], N. Bourbaki [9]).

Proposition. Let $W, U, V.V_1, \ldots, V_m$ be finite dimensional vector spaces over K.
1. If M and N are R-lattices in V then so are $M + N$ and $M \cap N$.
2. If $U \subset V$ and M is an R-lattice in V then $M \cap U$ is an R-lattice in U.
3. Consider R-lattices $M_1, \ldots, M_m$ in $V_1, \ldots, V_m$ resp. and let $\mu : V_1 \times \ldots \times V_m \to U$ be a multilinear form, then the R-module generated by $\mu(M_1 \times \ldots M_m)$ is an R-lattice in the subspace of U spanned by $\mu(V_1 \times \ldots \times V_m)$.
4. Let M be an R-lattice in V, N an R-lattice in W, and define $(N : M) = \{\alpha \in \mathrm{Hom}_K(V, W), \alpha(M) \subset N\}$, then $(N : M)$ is an R-lattice in $\mathrm{Hom}_K(V, W)$ and $(N : M) = \mathrm{Hom}_R(M, N)$.
5. If $S \supset R$ is a domain with field of fractions L and M is an R-lattice in V then the image of $S \underset{R}{\otimes} M$ in $L \underset{K}{\otimes} V$ is an S-lattice.

If we identify V and $\mathrm{Hom}_K(\mathrm{Hom}_K(V, K), K)$ then M may be viewed as an R-submodule of $(R : (R : M))$. Clearly, if M is a free R-lattice then $M = (R : (R : M))$. In general $(R : M) = (R : (R : (R : M)))$. We say that an R-lattice M is **divisorial** if $M = (R : (R : M))$.
For an arbitrary torsion-free R-module M we say that M is **divisorially** if in $K \underset{R}{\otimes} M$ we have that. $M = \cap\{M_p = R_p \underset{R}{\otimes} M, p \in X^1(R)\}$. The **rank** of M is the K-dimension of $K \underset{R}{\otimes} M$. We define $(R : M) = \{f \in \mathrm{Hom}_R(K \underset{R}{\otimes} M, K), f(M) \subset R\}$, and we write M^* for $\mathrm{Hom}_R(M, R)$. The canonical isomorphism $K \otimes_R M \cong (K \underset{R}{\otimes} M)^{**}$ allows to view $(R : (R : M))$ as an R-submodule of $K \underset{R}{\otimes} M$. If M is

an R-lattice then $(R : M)$ is isomorphic to M^* and therefore M is divisorial if and only if $M = M^{**}$.
An R-order Λ is said to be a **maximal R-order** if it is not contained properly in another R-order. Every R-order of A is contained in a maximal R-order of A, cf. [20] Theorem 1.4., p. 323.

II.4.3. Proposition. An R-order Λ of A is a maximal R-order of A if and only if the following properties hold :
1. Λ is a divisorial R-lattice in A.
2. Λ is a maximal R_p-order of A for each $p \in X^1(R)$.

The so-called tame orders may be viewed as globalizations of hereditary orders over discrete valuation rings. In L. Silver's treatment of these rings, cf. [49], the orders considered are by assumption finite modules over their centres whereas in R.M. Fossum's approach, cf. [20], this is not necessarily the case. The following result of A. Braun, cf. [10], relates both approaches.

II.4.4. Proposition. If Λ is an R-order such that Λ is an affine P.I. algebra then Λ is a finitely generated R-module.

An R-order Λ is said to be a **tame R-order** if it is a divisorial R-lattice such that Λ_p is a hereditary R_p-order for each $p \in X^1(R)$. If Λ' is a maximal R-order containing a tame order Λ then $\Lambda'_p = \Lambda_p$ for almost all $p \in X^1(R)$.

II.4.5. Lemma. 1. Let E and F be finitely generated Λ-modules, where Λ is an R-algebra and F is reflexive as an R-module, then $\mathrm{Hom}_\Lambda(E, F)$ is reflexive as an R-module.
2. If Λ is a reflexive R-algebra then a finitely generated reflexive Λ-module is also reflexive as an R-module.

Proof. 1. $\mathrm{Hom}_\Lambda(E, F) = \mathrm{Hom}_\Lambda(E, F^{**}) \cong \mathrm{Hom}_R(F^* \underset{\Lambda}{\otimes} E, R)$ by Cartan-Eilenberg [13], II.5.2.
2. Put $F = \Lambda$ in 1. □

II.4.6. Proposition. 1. If Λ is a tame order over the Krull domain

R then every divisorial R-order containing Λ is again tame.
2. An ideal I of Λ is quasi-idempotent if $I = (I^2)^{**}$. There is a canonical one-to-one correspondence between divisorial R-orders containing Λ and quasi-idempotent ideals of Λ.

Proof. Along the lines of L. Silver's Theorem 1.8. in [49]. □

II.4.7. Corollary. If Λ is a tame R-order then the number of divisorial R-orders containing Λ is finite. For each $p \in X^1(R)$ let $e(p)$ be the number of hereditary R_p-orders containing Λ_p, then there are exactly $e = \sum_{p \in X^1(R)} e(p)$ tame orders containing Λ.

An R-lattice M in A which is a two-sided ideal Λ is called a divisor of Λ if M is a divisorial R-lattice and M_p is an invertible Λ_p-module for each $p \in X^1(R)$. If E is a right Γ-module, where Γ is some arbitrary ring, then we define the (right) **trace ideal** $t_\Gamma(E) = t_\Gamma^r(E)$ to be the image of t, $t : E \underset{\gamma}{\otimes} \mathrm{Hom}_\Gamma(E, \Gamma_\Gamma) \to \Gamma$, $e \otimes f \mapsto f(e)$. The left trace ideal $t_\Gamma^l(E)$ of a left Γ-module is defined in a similar way. It is clear that each of these trace ideals is an ideal (two-sided !).

II.4.8. Proposition. Let Λ be a tame R-order in A and let M be a two-sided Λ-module in A which is a divisorial R-lattice, then the following statements are equivalent :

1. M is a Λ-divisor.
2. $\mathrm{End}_\Lambda(M_\Lambda) = \Lambda$
3. $t_\Lambda^l(M)^{**} = \Lambda$
4. $t_\Lambda^r(M)^{**} = \Lambda$
5. $\mathrm{End}({}_\Lambda M) = \Lambda$

Proof. It suffices to prove these equivalences locally at each $p \in X^1(R)$ i.e. for hereditary orders. Therefore the equivalences $2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$ follow from Proposition 2.1. in M. Harada's paper [22].
Since M_p is an invertible Λ_p-module for each $p \in X^1(R)$, the equivalence of 1. and 2. (or 5.) is evident. □

Let $D(\Lambda)$ be the set of Λ-divisors, we derive :

II.4.9. Theorem. 1. $D(\Lambda)$ is an abelian group. In fact, $D(\Lambda)$ is the free abelian group generated by the ideals $J(\Lambda_p) \cap \Lambda$, for all $p \in X^1(R)$, where $J(\Lambda_p)$ is the Jacobson radical of Λ_p.

2. The Λ-divisor M is invertible if and only if $MM^{-1} = \Lambda = M^{-1}M$.

3. If M and N are invertible divisors, then M^{-1} and $M * N = (MN)^{**}$ are invertible. The invertible divisors form a subgroup of $D(\Lambda)$.

Proof. 1. The product in $D(\Lambda)$ is defined by $M * N = (MN)^{**}$ for $M, N \in D(\Lambda)$. The proof is now completely similar to the proof of Theorem 2.3. of L. Silver, [49].

2. cf. loc. cit. Proposition 3.1.
3. cf. loc. cit Proposition 3.2. □

II.4.10. Proposition. Let R be a Krull domain with field of fractions K, let A be a central simple algebra over K and let Λ_1, Λ_2 be maximal R-orders in A. The conductor $(\Lambda_1 : \Lambda_2)$ induces an isomorphism $d(\Lambda_2, \Lambda_1) : D(\Lambda_1) \to D(\Lambda_2)$. If Λ_3 is another maximal R-order in A then we have : $d(\Lambda_3, \Lambda_2)d(\Lambda_2, \Lambda_1) = d(\Lambda_3, \Lambda_1)$.

Proof. The conductor $(\Lambda_1 : \Lambda_2)$ is a divisorial R-lattice and it is also a $\Lambda_2 - \Lambda_1$-bimodule. It is easily checked that $((\Lambda_2 : \Lambda_3)(\Lambda_1 : \Lambda_2))^{**} = (\Lambda_1 : \Lambda_3)$. Define $d(\Lambda_2, \Lambda_1)(M)$ to be equal to $((\Lambda_1 : \Lambda_2)M(\Lambda_2 : \Lambda_1))^{**}$ for any $M \in D(\Lambda_1)$. The statements of the proposition are easily checked. □

II.4.11. Proposition. Let Λ be a tame R-order in A and let $Y \subset X^1(R)$ be some subset. If $S = \bigcap_{p \in y} R_p$ then $\Gamma = \bigcap_{p \in y} \Lambda_p$ is a tame S-order in A. Moreover if Λ is a maximal R-order then Γ is a maximal S-order.

Proof. S is a Krull domain and Γ is a divisorial S-order. By localisation at each $p \in Y$ the result follows. □

II.4.12. Corollary Let S be a multiplicatively closed subset of $R, 0 \neq S$.
If Λ is a tame (maximal) R-order then $S^{-1}\Lambda$ is a tame (maximal) S^{-1}-

order.

In general a maximal R-order Λ exists in A only when R is a completely integrally closed domain and R will be a Krull domain when Λ satisfies the ascending chain condition on reflexive ideal of Λ. In the literature there exist several different types of non-commutative Krull rings, cf. M. Chamarie [14], H. Marubayashi [31],..., but all of these notions coincide if one considers Krull rings satisfying polynomial identities. In the latter case all the different types of Krul rings reduce to the notion of a maximal order over a Krull domain in a central simple algebra. In general a Noetherian maximal order Λ need not have a Noetherian centre and it certainly needs not be a finitely generated module over its centre. On the positive side we have :

II.4.13. Proposition. Let Λ be a Noetherian maximal order in a central simple algebra A of pi-degree n. If n is invertible in A then $Z(\Lambda) = R$ and Λ is a finitely generated R-module.

Proof. Since Λ is integral over R and R is completely integrally closed, $\text{Tr}(\Lambda) \subset R = Z(\Lambda)$ follows. If I is an ideal of $Z(\Lambda)$ and $z \in \Lambda I \cap Z(\Lambda)$ then $\text{Tr}(z) \in I$ and $z = n^{-1}\text{Tr}(z) \in I$ follows, thus $I = \Lambda I \cap Z(\Lambda)$. It is now very easy to verify that $Z(\Lambda) = R$ is a Noetherian ring. A well-known result of G. Cauchon then entails that Λ is a finitely generated R-module. □

II.4.14. Lemma. Let Λ be a maximal order over the Krull domain R in A. Let $p \subset q$ be prime ideals of R and assume that there exists a prime ideal Q of Λ such that $Q \cap R = q$, then there is a prime ideal P of Λ such that $P \subset Q$ and $P \cap R = p$.

Proof. Let S be the multiplicatively closed subset of Λ generated by $R-p$ and $C(Q)$. As in the commutative case one shows that $Ap \cap S = \emptyset$, Theorem 6 on p. 262 in [62]. Let P be an ideal of Λ containing Ap and maximal such that it has no intersection with S. It is easy enough to verify that P is a prime ideal of Λ such that $P \cap R = p$. □

II.4.15. Lemma. Let Λ be a maximal R-order and let P be a prime

ideal of Λ.Then P is reflexive if and only if P has height one, or equivalently if $p = P \cap R \in X^1(R)$. If this holds then P is the unique prime ideal of Λ lying over p.

Proof. The first assertion is clear because here the terms "reflexive" and "divisorial" have the same meaning (on R-lattices !). By the going-down result in the foregoing lemma it follows that the extension $R \to \Lambda$ satisfies P.D.E., i.e. $ht(p) = 1$. Since R_p is a discrete valuation ring and $\Lambda_p = R_p \underset{R}{\otimes} \Lambda$ has a unique maximal ideal $R_p \underset{R}{\otimes} P$, the final assertion also follows immediately. □

II.4.16. Proposition. Let Λ be a maximal R-order in A and let P be a prime ideal of Λ, $p = P \cap R$; then : **1.** $Q_P(\Lambda) = R_p \underset{R}{\otimes} \Lambda$ where $Q_P(\Lambda)$ is the localization of Λ at the m-set $\Lambda - P$. Note that $Q_P(M)$ may well be different of $R_p \underset{R}{\otimes} M$ for left Λ-modules $M \neq \Lambda$.
2. The left and right Ore conditions with respect to $C(P)$ hold if and only if P is the unique prime ideal over p.

Proof. 1. That $\Lambda_p \subset Q_P(\Lambda)$ is clear. Conversely, let $x \in Q_P(\Lambda)$;then $\Lambda x \Lambda$ is a (fractional) Λ-ideal and so there exists a left Λ-ideal I of finite type such that $I \subset \Lambda x \Lambda$ and $I^{-1} = (\Lambda x \Lambda)^{-1}$. Therefore $Iz \subset \Lambda$ for some $z \in C(P)$ and hence there is an ideal J of Λ such that $J \not\subset P$ and $xJ \subset \Lambda$. In $D(\Lambda)$ we write $J^{**} = \prod_{i=1}^{n} P_i^{(m_i)}$ where the P_i are reflexive rime ideals and (m_i) denotes the m_i-fold product in $D(\Lambda)$. If $J^{**} \cap R \subset p$ then $P_i \cap R \subset p$ for some $i \in \{1, \ldots, n\}$. By the going down property and the foregoing lemma : $P_i \subset P$. Consequently $J \subset P$ but this is a contradiction. But $J^{**} \cap R \not\subset p$ entails that $x \in \Lambda_p$ hence $\Lambda_p = R_p \underset{R}{\otimes} \Lambda = Q_P(\Lambda)$.
2. If Λ satisfies the left Ore condition with respect to $C(P)$ then $C(P)$ consists of regular elements of Λ. It follows that $Q_P(P)$ is the unique maximal ideal of $Q_P(\Lambda)$; the first part entails that P is the unique prime ideal of Λ lying over p. Conversely, if P is the unique prime ideal of Λ lying over p then P_p is the unique maximal ideal of Λ_p and thus $Q_P(P)$ is the unique maximal ideal of $Q_P(\Lambda)$. Since Λ_p is a P.I. ring it follows that $Q_p(\Lambda)$ has Jacobson radical $Q_P(P)$ and by a result of A. Heinicke [21] it follows that Λ satisfies the (left) Ore conditions with

respect to $C(P)$. $\square$

A maximal order Λ over a Krull domain R is said to be a **reflexive Azumaya algebra** if for every $p \in X^1(R)$, Λ_p is an Azumaya algebra over R_p. Two reflexive Azumaya algebras are said to be equivalent if $(A \underset{R}{\otimes} \mathrm{End}_R(Q))^{**} \cong (B \underset{R}{\otimes} \mathrm{End}_R(P))^{**}$ or equivalently if $(A \underset{R}{\otimes} B^\circ)^{**} \cong \mathrm{End}_R(P)$, for some divisorial R-lattices P and Q. Note that the property $(A \underset{R}{\otimes} A^\circ)^{**} \cong \mathrm{End}_R(P)$ where P is a divisorial R-lattice, may be used to characterize reflexive Azumaya algebras. The set of classes of reflexive Azumaya algebras is a group with respect to $(- \underset{R}{\otimes} -)^{**}$ and it is called **the reflexive Brauer group**, denoted by $\beta(R)$. More details about this group can be found in [61] or [58] where some geometric interpretations of this group are highlighted. It is not hard to verify that a reflexive Azumaya algebra Λ is an Azumaya algebra if and only if Λ is a flat R-module.

II.4.17 Proposition. If Λ is a reflexive Azumaya algebra then $D(\Lambda) = D(R)$ (where $D(R)$ is as defined in Chapter I, after Corollary I.1.2.).

Proof. A consequence of the fact that Λ-bimodules over R which are divisorial Λ-lattices correspond bijectively to divisorial R-lattices under the correspondence $M \to M^{(\Lambda)} = \{m \in M, \lambda m = m\lambda \text{ for all } \lambda \in \Lambda\}$ where M is such a Λ-bimodule centralizing the action of R, and $N \mapsto (\Lambda \underset{R}{\otimes} N)^{**}$ for a divisorial R-lattice N. $\square$

We have introduced the reflexive Azumaya algebras in order to provide a nice class of maximal orders over Krull domains. In [28], the study of maximal orders over Krull domains is reduced to the study of reflexive Azumaya algebras plus the investigation of certain properties of divisorially graded generalized Rees rings. We briefly recall these techniques here in order to provide some fundaments for Chapter IV.

Let Λ be a maximal order over the Krull domain R. The $D(\Lambda)$ is a free abelian group generated by the prime ideals of height one of Λ. We consider the subgroup $\mathcal{P}^c(\Lambda)$ of $D(\Lambda)$ generated by the principal ideals generated by a central element, and $I(\Lambda)$ will be the subgroup of invertible ideals in $D(\Lambda)$. The **central class group** of Λ is defined

to be $CCl(\Lambda) = D(\Lambda)/\mathcal{P}^c(\Lambda)$. It is not hard to check that, in the terminology of A. Frölich, [21], $\mathrm{Pic}_R(\Lambda) = \mathrm{Picent}(\Lambda) = I(\Lambda)/\mathcal{P}^c(\Lambda)$. Now consider an extension of Krull domains $R \hookrightarrow S$ and an "extension" of maximal orders $\Lambda \hookrightarrow \Gamma$ resp. over R, S (i.e. we assume $\Gamma = \Lambda C_\Gamma(\Lambda), C_\Gamma(\Lambda) = \{\gamma \in \Gamma, \gamma\lambda = \lambda\gamma \text{ for all } \lambda \in \Lambda\}$). Define a map of sets, $\Psi : D(\Lambda) \to D(\Gamma), I \mapsto (\Gamma I)^{**}$. This map need not be a group morphism even if Λ, Γ are commutative but $\Psi | I(\Lambda)$ defines a morphism and it induces a group morphism $\widetilde{\Psi}$: Picent $(\Lambda) \to$ Picent (Γ). On the other hand we may define $\Phi : D(\Lambda) \to D(\Gamma)$ from the divisor point of view as follows. If $P \in X^1(\Lambda), Q \in X^1(\Gamma)$ then $\overline{P}$, resp. $\overline{Q}$ denotes the unique maximal ideal of $Q_P(\Lambda)$ resp. $Q_Q(\Gamma)$. If $Q \cap \Lambda = P$ then $Q_Q(\Gamma)P = (\overline{Q})^{e_{P,Q}}$ for some $e_{P,Q} \in \mathbb{N}$, called the **ramification index of Q over P**. If $Q_1, \ldots, Q_n$ are the prime ideals of $X^1(\Gamma)$ lying over P then we put $\Phi(P) = \prod_{i=1}^{n} Q_i^{e_i}$ where $e_i = e_{P,Q_i}$. The map $\Phi : D(\Lambda) \to D(\Gamma)$ is a group morphism by definition but it need not induce a map on $CCl(\Lambda)$. As for commutatative rings we say that $\Lambda \hookrightarrow \Gamma$ satisfies $\underline{PDE}$ if for all $Q \in X^1(\Gamma)$ we have $\mathrm{ht}(\Lambda \cap Q) \leq 1$. The following is now just an easy exercise :

II.4.18. Proposition.
The following statements are equivalent :
1. $\Phi = \Psi$.
2. $\Phi(\Lambda x) = \Gamma x$ for all nonzero x in $Z(\Lambda)$.
3. $\Lambda \hookrightarrow \Gamma$ satisfies PDE.

II.4.19. Corollary. 1. If $\Lambda \hookrightarrow \Gamma$ satisfies PDE then Ψ induces a morphism $\widetilde{\Psi} : CCl(\Lambda) \to CCl(\Gamma)$ which restricts to the morphism $\mathrm{Picent}(\Lambda) \to \mathrm{Picent}(\Gamma)$ introduced before. If we apply this to the extension $R \hookrightarrow \Lambda$ there it follows that $Cl(R) \subset CCl(\Lambda)$.
2. $\Lambda \hookrightarrow \Gamma$ satisfies PDE if and only if $Z(\Lambda) \hookrightarrow Z(\Gamma)$ satisfies PDE.

II.4.20. Proposition. For any maximal order Λ over a Krull domain R, the following sequence is exact :

$$1 \to Cl(R) \to CCl(\Lambda) \to G = \bigoplus_P \mathbb{Z}/e_P\mathbb{Z} \to 1$$

where the e_P are the ramification indices of the essential valutions of

K which ramify in A. The exponent of G is bounded by the P.I. degree of A.

Proof. For $p \in X^1(R)$, $P = \Psi(p)$ satisfies $(\Lambda p)^{**} = (P^{e_P})^{**}$ for some $e_P \in \mathbb{N}$. That e_P equals the ramification index of the discrete valuation of K associated to R_p in the central simple algebra A is clear. Since $\Psi(P(R)) = \mathcal{P}^c(\Lambda)$, the sequence $1 \to D(R) \to D(\Lambda) \to \oplus_P \mathbb{Z}/e_P\mathbb{Z}$ is evident and the proposition follows if we show that the number of ramifying primes is finite. To verify this consider $c \neq 0$ in the Formanek centre of Λ and note that $V = \{p \in X^1(R), c \in p\}$ is finite, while the Λ_p with $p \in X^1(R) - V$ are Azumaya algebras. Consequently the $p \in X^1(K) - V$ correspond to $P \in X^1(\Lambda)$ such that $e_P = 1$. The claim concerning the exponent of G follows from a result of G. Bergman, L. Small or as follows : for $P \in X^1(\Lambda)$, the reduced norm $\mathrm{nr}(P)$ satisfies $(\mathrm{nr}(P))^{**} = (p^f)^{**}$ for $f \in \mathbb{N}$, so $(\Lambda p^f)^{**} = (P^{e_P f})^{**}$ yields $(p^{f_n})^{**} = (p^{(f^2 e_p)})^{**}$ and since $D(R)$ is free on the $p \in X^1(R)$ it follows that $n = fe_P$. □

II.4.21. Corollary. Let Λ and Γ be maximal R-orders in A, then $CCl(\Lambda) = CCl(\Gamma)$.

The foregoing corollary may be a desillusion for those who hoped to use the properties of the central class group for the determination of specific structural features of maximal orders. On the positive side it follows from Proposition II.4.20. that the condition $CCl(\Lambda) = Cl(R)$ may be translated into a condition on the ramification of the essential valuations of R in A i.e. all of these have to be unramified.

II.4.22. Lemma. Let Λ be a maximal order over a discrete valuation ring R. If $CCl(\Lambda) = 1$ (i.e. A is unramified) then either A is an Azumaya algebra over C, or else the center of A/mA is a purely inseparable extension of $R/mR = \overline{R}$, where m is the maximal ideal of R.

Proof. If $Z(\Lambda/m\Lambda) = R/m$ then pi-deg(Λm) = pi-degΛ, so by the Artin-Procesi theorem it follows that Λ is an Azumaya algebra; the converse of this is clear. The existence of nontrivial separable elements in $Z(\Lambda/m\Lambda)$ over R/m would provide nonzero irreducible polynomials

p', q' in $Z(\Lambda/m\Lambda)[T]$ lying over a separable p in $(R/mR)[T]$ and hence this leads to prime ideals P, Q of $\Lambda[T]$ such that $P \cap \Lambda = Q \cap \Lambda = \Lambda m$, and P and Q lie over the same prime ideal of $K[T]$. Since $\Lambda[T]$ is a maximal order over the Krull domain $R[T]$, we derive that $Q_P(\Lambda[T])P$ is the unique maximal ideal of $Q_P(\Lambda[T])$ hence it is the Jacobson radical too, by applying proposition II.4.26. Moreover : $Q_P(\Lambda[T])/Q_P(P) = \Lambda[T]/P = (\Lambda/m\Lambda)[T]/P'$ for some P' generated by an irreducible polynomial $p(T) \in Z(\Lambda|m\Lambda)[T]$. Thus, the elements of $C(P)$ are invertible and since $\Lambda[T]$ is Noetherian we may apply the results of A. Heinicke, [], to derive that $\Lambda[T]$ satisfies the Ore conditions with respect to $C(P)$. Again, by Proposition II.4.16. it follows that $P = Q$, or $p' = q'$ or $Z(\Lambda/m\Lambda)$ is purely inseparable over R/mR. □

II.4.23. Lemma. Let Λ be a maximal order over a discrete valuation ring R. The following statements are equivalent :

1. Λ is an Azumaya algebra.

2. $CCl(\Lambda) = 1$ and there exists a separable splitting field L of A such that $\Lambda \underset{R}{\otimes} D$ is hereditary, where D is the integral closure of R in L.

3. $CCl(\Lambda) = 1$ and there exist a separable splitting field L of A such that $\Lambda \underset{R}{\otimes} D$ is a tame D-order in $A \underset{K}{\otimes} L$.

4. There exists a faithfully flat extension S of R such that S/mS is separable over R/mR and S splits A, i.e. the field of fractions of S splits A.

Proof. 1 ⇒ 2,3,4 are obvious.

2. ⇒ 1. Suppose $Z(\Lambda/m\Lambda)$ is purely inseparable over R/mR. In that case $\mathrm{Br}(R/mR) \to \mathrm{Br}(Z(\Lambda/m\Lambda))$ is surjective, so up to changing A into a suitable matrix ring over A we may assume that $\Lambda/m\Lambda$ contains an R/mR-central simple algebra B such that $\Lambda/m\Lambda = B \underset{R/mR}{\otimes} Z(\Lambda/m\Lambda)$. The pre image Λ_1 of B in Λ is an R-order contained in Λ and $m\Lambda$ is a common ideal for Λ and Λ_1. Obviously, $m\Lambda$ is the unique nonzero prime ideal of Λ_1 and $Z(\Lambda_1/m\Lambda) = R/mR$. By definition D is a semilocal Dedekind domain and its nonzero prime ideals $m_1, \ldots, m_q$ ly over mD. Let D_j be the discrete valuation ring $D_{m_j}, j = 1 \ldots q$, and write m'_j for its maximal ideal. If we localize the inclusions $(\Lambda \underset{R}{\otimes} D)m \subset \Lambda_1 \underset{R}{\otimes} D \subset$

$\Lambda \underset{R}{\otimes} D$ at m_j, then we obtain $(\Lambda \underset{R}{\otimes} D_j)m'_j \subset \Lambda_1 \underset{R}{\otimes} D_j \subset \Lambda \underset{R}{\otimes} D_j$.

Since $(\Lambda_1 \underset{R}{\otimes} D_j)/(\Lambda \underset{R}{\otimes} D_j)m'_j = B \underset{R/mR}{\otimes} (D/m_jD)$ is central simple it follows that $(\Lambda \underset{R}{\otimes} D_j)m'_j$ is the unique nonzero prime ideal of $\Lambda_1 \underset{R}{\otimes} D_j$. Therefore, ideals of $\Lambda \underset{R}{\otimes} D_j$ intersect $\Lambda_1 \underset{R}{\otimes} D_j$ in subsets of $(\Lambda \underset{R}{\otimes} D_j)m'_j$. Since we assume that $Z(\Lambda/m\Lambda)$ is purely inseparable over C/mC, the unique prime ideal of $Z(\Lambda/m\Lambda) \underset{R/mR}{\otimes} (D/m_jD)$ is the nilradical (note : D/m_jD is algebraic over R/mR). Consequently, there is only one prime of $\Lambda \underset{R}{\otimes} D_j$ lying over $(\Lambda \underset{R}{\otimes} D_j)m'_j$. M. Harada's determination of the structure of hereditary orders in matrix rings (cf. also M. Artin []) yields that the number of prime ideals of $\Lambda \underset{R}{\otimes} D_j$ lying over m'_j is at least as big as the number of diagonal blocks of type $M_{n_j}(D_j)$ in the structure of $\Lambda \underset{R}{\otimes} D_j$ (which is hereditary since $\Lambda \underset{C}{\otimes} D$ is.) It follows that $\Lambda \underset{R}{\otimes} D_j = M_n(D_j)$ where $n =$ pi-deg(Λ) and therefore $\Lambda \underset{R}{\otimes} D$ is a reflexive Azumaya algebra over the Dedekind domain D i.e. it is an Azumaya algebra. But then Λ is an Azumaya algebra over R and $Z(\Lambda/m\Lambda) = R/mR$. So condition 2. excludes the alternative for Λ being an Azumaya algebra given in Lemma II.4.22.
3,4 $\Rightarrow$ 1. Exactly as above, modifying the argument at the end (or using a "local" version of it in case 3 $\Rightarrow$ 1 is considered) in order to derive that Λ is Azumaya over R (e.g. in 4 $\Rightarrow$ 1 use faithfully flat descend). □

For a maximal order Λ over a Krull domain R we may also consider a separable splitting field of A. We say that Λ is **L-tame** if $\Lambda \underset{R}{\otimes} D$ is a tame D-order in $A \underset{K}{\otimes} L$, where D is the integral closure of R in L. We say that Λ is **Zariski tamifiable** if for every $p \in X^1(R)$ there exists a separable splitting field $L(p)$ of A such that $\Lambda \underset{R}{\otimes} D(p)$ is a tame $D(p)$-order, where $D(p)$ is the integral closure of R_p in L.

II.4.24. Theorem. The following properties of a maximal order Λ over a Krull domain R are equivalent :
1. Λ is a reflexive Azumaya algebra over R.
2. $CCl(\Lambda) = Cl(R)$ and Λ is Zariski tamifiable.
3. $CCl(\Lambda) = Cl(R)$ and Λ is L-tame for some separable splitting field L of A.

Proof. 1 $\Rightarrow$ 3 $\Rightarrow$ 2 obvious.
2 $\Rightarrow$ 1. For $p \in X^1(R)$, Λ_p is a maximal R_p-order with $CCl(\Lambda_p) = 1$. By the lemma's, Λ_p is an Azumaya algebra over R_p. By definition it follows that Λ is a reflexive Azumaya algebra over R. □

In the sequel of this section we consider graded orders i.e. Λ is now an order over a Krull domain R which we assume to be graded by a torsion free abelian group G. We let K^g be the gr-field of fractions of R and $A^g = Q^g(\Lambda)$ the graded ring of fractions of Λ which is an Azumaya algebra over K^g (cf. Proposition II.3.4., or Corollary II.3.6.). For $p \in X^1_g(R)$, the set of graded minimal prime ideals, we let R^g_p be the graded ring of fractions at the multiplicative set $h(R-p)$, cf. C. Năstăsescu, F. Van Oystaeyen [37]. Since R is a Krull domain we have that $R = \cap_{p \in X^1(R)} R_p$ and we also have (as is easily seen) :

$$R = K^g \bigcap \left(\bigcap_{p \in X^1(R)} R_p \right) = \bigcap_{p \in X^1_g(R)} R^g_p.$$

Since G is torsion free abelian it is an orderded group and hence $p \in X^1(R)$ yields $p_g = p$ or $p_g = 0$. If $p_g = 0$ then $R^g_p = K^g$ is clear. For every $p \in X^1_g(R)$ we have that R^g_p is a discrete gr-valuation ring; then we also have that R_p is a discrete valuation ring of K and the associated valuation v is "graded" in the sense that for any $x \in R$ with homogeneous decomposition $x = x_{\sigma_1} + \ldots + x_{\sigma_n}$ we have that $v(x) = \min\{v(x_i)\}$ (see Section I.2. up to, as indicated in the beginning of that section, checking that the result given there for $G = \mathbb{Z}$ generalizes in the obvious way to the case where G is torison free abelian). We know that A^g is a maximal order in A (note that A^g need not be a Krull order since K^g is only a Krull domain when we assume that G satisfies the ascending chain condition on cyclic subgroups). We say that Λ is a ***gr*-maximal *R*-order** if it is not properly contained in another graded R-order in A. In general there need not exist graded orders in a given central simple algebra over K; indeed, an obvious necessary condition on A is that it should represent an element of $Br^g(K^g)$, cf. [12], and it is straightforward to verify that this condition is also sufficient.
A graded R-order Λ is ***gr*-hereditary** if the graded left ideals of Λ are projective. A graded R-order is ***gr*-tame** if Λ^g_p is gr-hereditary for each $p \in X^1_g(R)$.

II.4.25. Lemma. A gr-maximal R-order Λ of A is a divisorial R-lattice.

Proof. That Λ is an R lattice is obvious. It is also very easy to verify that Λ^{**} is an R-order, $\Lambda \subset \Lambda^{**}$ (along the lines of Proposition 1.3. in M. Auslander, O. Goldman [7]). Since $A^g \subset K^g.\Lambda^{**}$ and A^g being a maximal K^g-order, it follows that $\Lambda^{**} \subset A^g$. Let Γ be the ring (it is an R-order !) generated by the homogeneous components in A^g of elements in Λ^{**}. Then Γ is a G-graded R-order, hence the gr-maximality of Λ entails that $\Lambda = \Gamma$ and in particular $\Lambda = \Lambda^{**}$. □

II.4.26. Remark. One may verify that a G-graded R-order Λ over a G-graded Krull domain R is a gr-maximal order if and only if : $\Lambda = \Lambda^{**} = \cap_{p \in X_g^1(R)} \Lambda_p^g$ and each Λ_p^g is a gr-maximal R_p^g-order.

Consider a discrete gr-valuation ring S in K^g and let Γ be a gr-maximal S-order in A. Modifying a classical result of M. Deuring [19], p. 74 and p. 108, we easily verify that Γ is a gr-local ring in the sense that it has a unique gr-maximal ideal, M say, and Γ/M is a gr-c.s.a.

II.4.27. Lemma. Let S and Γ be as above. If E is a finitely generated left Γ-module then $hd_\Gamma^g E = hd_S^g E$, where hd_Γ^g, resp. hd_S^g denotes the homological dimension in Γ-gr, resp. S-gr.

Proof. Dimensions in the category of graded modules have been studied in [37]. An easy graded version of Theorem 2.2., M. Auslander, O. Goldman [7], proves the claim (note that S is Noetherian here !). □

II.4.28. Corollary. Γ is gr-hereditary.

II.4.29. Proposition. A graded S-order Γ is gr-maximal if and only if the graded Jacobson radical $J^g(\Gamma)$ is a gr-maximal ideal of Γ. We have that Γ is gr-hereditary if $J^g(\Gamma)$ is also a projective left (or right) Γ-module.

Proof. Following Theorem 2.3. of M. Auslander, O. Goldman [7]. □

II.4.30. Remark. Using obvious properties of the trace map it is

possible to show that the different of a gr-maximal order Λ is a graded ideal. So if a prime ideal P divides the different (i.e. if P is of the first kind this means that P^2 divides $\Lambda(P \cap Z(\Lambda))$) then P_g divides the different.

II.4.31. Theorem. If Λ is a gr-tame order over the graded Krull domain R then Λ is a tame R-order in A.

Proof. If $p \in X^1(R)$ is such that $p_g = 0$ then Λ_p is a localization of A^g hence it is an Azumaya algebra over its center R_p which is a discrete valuation ring, and therefore Λ_p is a fortiori hereditary (even a maximal order). If $p \in X_g^1(R)$ then $\Lambda' = \Lambda_p^g$ is by assumption a gr-hereditary R'-order where $R' = R_p^g$ is a discrete gr-valuation ring. Putting $p' = pR_p^g$, we have to check whether $\Lambda'_{p'}$ is a hereditary $R'_{p'}$-order. One checks that the graded Jacobson radical $J^g(\Lambda')$ localizes to the Jacobson radical $J(\Lambda_p)$ (using the "unique lying over" properties observed earlier over Krull domains applied here to Λ'). Since $J^g(\Lambda')$ is a projective Λ'-module it follows that $J(\Lambda_p)$ is projective as a Λ_p-module. So Λ_p is left (and right) hereditary. Since for $p \in X^1(R)$ either $p_g = 0$ or $p \in X_g^1(R)$ the cases considered cover all possibilities for Λ_p, i.e. Λ is a tame R-order in A. □

If I is a graded fractional ideal of Λ then I^{**} is graded too :

$$\begin{aligned} I^{**} &= \bigcap_{p \in X^1(R)} I_p = \Big(\bigcap_{p_g = 0} I_p\Big) \cap \Big(\bigcap_{p \in X_g^1(R)} I_p\Big) \\ &= A^g \cap \Big(\bigcap_{p \in X_g^1(R)} I_p\Big) = \bigcap_{p \in X_g^1(R)} (A^g \cap I_p) = \bigcap_{p \in X_g^1(R)} I_p^g \end{aligned}$$

The group $D^g(\Lambda)$ is defined to be the subgroup of $D(\Lambda)$ consisting of the graded elements in $D(\Lambda)$. As for the ungraded case one establishes that $D^g(\Lambda)$ is the free group generated by $X_g^1(\Lambda)$.
Let us mention some consequences of results of M. Chamarie in particular Theorem 4.2.3. in [14].

II.4.32. Lemma. Let Λ be a maximal order over a Krull domain R and consider an intermediate ring $\Lambda \subset \Gamma \subset A$.

The following statements are equivalent :
1. Γ is a central localization of Λ.
2. Γ is a subintersection of Λ, i.e. $\Gamma = \cap\{\Lambda_p, p \in Y \subset X^1(\Lambda)\}$.
If these conditions hold then we have an exact sequence :

$$1 \to H \to CCl(\Lambda) \to CCl(\Gamma) \to 1$$

where H is generated by the $p \in X^1(\Lambda) - Y$.

In Section II.1. it will be shown that gr-maximal orders (for torsion free abelian grading groups) are in fact maximal orders; it is necessary to point this out in order to make it clear that the following theorem holds for gr-maximal orders in the sense of the foregoing results.

II.4.33. Theorem. Let Λ be a maximal order over a Krull domain R and suppose that Λ is graded by an abelian group. The following sequence is exact :

$$1 \to H \to CCl(\Lambda) \to Cl(K^g) \to 1$$

Proof. Since A^g is an Azumaya algebra over K^g we have $CCl(A^g) = Cl(K^g)$ and then it suffices to apply Lemma II.4.32 to $\Lambda \subset A^g \subset A$. Note that H is also the subgroup of $CCl(\Lambda)$ generated by the classes of $P \in X^1(\Lambda)$ such that $p = P \cap R$ contains nonzero elements of $h(R)$.
□

Note that, if $x \in K^g$ then $\Lambda c \cap K^g = Rc$, hence for every gr-maximal order we have an injection $Cl^g(R) \hookrightarrow CCl^g(\Lambda)$ where $CCl^g(\Lambda)$ is $D^g(\Lambda)$ being the subgroup of graded ideals of Λ generated by a central homogeneous element. By Corollary II.3.6. it follows that H in Theorem II.4.33. is in fact equal to $CCl^g(\Lambda)$, so we obtain :

II.4.34. Corollary.
Let Λ be a maximal order over the Krull domain R and suppose that Λ is graded by the torsion free abelian group G, then the following sequence is exact :

$$1 \to CCl^g(\Lambda) \to CCl(\Lambda) \to Cl(K^g) \to 1$$

But K^g is a Krull domain, hence it follows from Proposition I.1.1.(3) that K^g is factorial, hence $CCl^g(\Lambda) = CCl(\Lambda)$ follows.

In a straightforward way one may prove that $\underline{\underline{\mathrm{P}}}\mathrm{ic}(\Lambda) = \underline{\underline{\mathrm{P}}}\mathrm{ic}^g(\Lambda)$, $\mathrm{Picent}(\Lambda) = \mathrm{Picent}^g(\Lambda)$ where in each case the graded version of the group considered is defined by taking only the classes represented by graded elements. Note that $\underline{\underline{\mathrm{P}}}\mathrm{ic}(\Lambda)$ (a subgroup of $CCl(\Lambda)$ determined by the invertible ideals) may be different from $\mathrm{Picent}(\Lambda)$ defined (following A. Fröhlich [21]) by using invertible bimodules in general (in particular non-Noetherian situations).
Finally let us mention some results over strongly graded Krull domains, a situation that may arise if one applies the construction of generalized Rees rings to an order Λ in a "central way" i.e. by applying the construction to the center.

II.4.35. Lemma. Let R be a Krull domain strongly graded by a torsion free abelian group G and let Λ be a gr-maximal (i.e. graded and maximal, in view of results to come, III.1.) R-order in A. Then we have the following properties :
1. Λ is a central extension of Λ_e.
2. Λ_e is a maximal order over the Krull domain R_e in the $K_e = Q(R_e)$-central simple algebra $(A^g)_e$.
3. Λ is a left and right flat Λ_e-module.
4. The (central) extension of orders $\Lambda_e \hookrightarrow \Lambda$ satisfies P.D.E.
5. The map $\varphi : D(\Lambda_e) \to D(\Lambda), I \mapsto (\Lambda I)^{**}$ induces a group morphism $CCl(\Lambda_e) \to CCl(\Lambda)$ and also a group morphism $\mathrm{Pic}(\Lambda_e) \to \mathrm{Pic}(\Lambda)$. Actually $(\Lambda I)^{**} = \Lambda I$ for every $I \in D(\Lambda_e)$.

Proof. 1. If $x \in \Lambda_\sigma, \sigma \in G$, then $x \in R_\sigma R_{\sigma^{-1}} \Lambda_\sigma \subset R\Lambda_e$.

2. If I is an ideal of Λ_e then ΛI is a proper graded ideal (by 1.) of Λ and thus $(\Lambda I \underset{\Lambda}{:} \Lambda I) = \Lambda$ follows from the fact that Λ is gr-maximal. Consequently $(I \underset{\Lambda_e}{:} I) \subset \Lambda_e$. That A^g_e is simple Artinian is a direct consequence of the fact A^g is gr-simple gr-Artinian. If $x \in Z(A^g_e)$ then, since $R\Lambda_e = \Lambda$ yields $RA^g_e = A^g$, it follows that $x \in Z(A^g) = K^g$ and thus $x \in K^g \cap A^g_e = K^g_e = Q(R_e)$ or $Z(A^g_e) = Q(R_e)$. So it is clear that Λ_e is a maximal R_e-order in A^g_e.

3. Since $\Lambda = \bigoplus_{\sigma\in G} \Lambda_\sigma$ and each Λ_σ is a projective left and right Λ_e-module.

4. PDE for Λ over Λ_e follows from PDE for R over R_e and the latter is a consequence of flatness of R over R_e.

5. One easily checks for any graded ideal I of $\Lambda, I = \Lambda I_e$, that $T^{**} = I_e^{**} \underset{\Lambda_e}{\otimes} \Lambda = \Lambda(I_e)^{**}$ (using the graded localizations *at* $p \in X^1(-)$ both for Λ and Λ_e and the correspondence between these which does exist in view of the strong gradation on R and the equivalence of the categories R-gr and R_e). All the claims are easy consequences of this fact. □

II.4.36. Theorem. With assumptions as in the lemma we obtain the following exact and commutative diagram :

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & Cl(R) \arrow[u] \arrow[r, "i"] & CCl(\Lambda) \arrow[u] \arrow[r, "\gamma"] & \bigoplus_{P\in X^1_g(\Lambda)} \mathbb{Z}/e_P\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & Cl(R_e) \arrow[u] \arrow[r, "i_e"] & CCl(\Lambda_e) \arrow[u] \arrow[r, "\gamma_e"] & \bigoplus_{P_e\in X^1(\Lambda_e)} \mathbb{Z}/e_{P_e}\mathbb{Z} \arrow[r] & 0 \\
 & <Im\varphi> \arrow[u] & <Im\widetilde{\varphi}> \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $\varphi : G \to Cl(R_e), \sigma \mapsto [R_\sigma], \widetilde{\varphi} : G \to CCl(\Lambda_e), \sigma \mapsto [\Lambda_\sigma]$ and the $e_P, e_{P_e} \in \mathbb{N}$ are the ramification indexes for P, resp. P_e.

Proof. Define γ by $P_1^{\nu_1} * \ldots * P_r^{\nu_2} \mapsto (\nu_1 \mathrm{mode}_{P_1}, \ldots, \nu_r \mathrm{mode}_{P_r})$; this is well defined because I and Ic with $c \in R$ have the same image under γ (we used $CCl(\Lambda) = CCl^g(\Lambda)$ here !). If $\gamma(P_1^{\nu_1} * \ldots * P_r^{\nu_r}) = 0$

then $e_i = e_{P_i}$ divides ν_i and therefore $P_1^{\nu_1} * \ldots * P_r^{\nu_r}$ then has to be an extension of $p_1^{\nu_1/e_1} * \ldots * p_r^{\nu_r/e_r}$ in $Cl(R)$. Exactness of the first row follows from this, exactness of the second row follows in the same way. Exactness of the first column follows from the fact that R is a generalized Rees ring (cf. Proportion I.1.16.). Consider the diagram of extensions :

$$\begin{tikzcd}
R \arrow[r] & \Lambda \\
R_e \arrow[u] \arrow[r] & \Lambda_e \arrow[u]
\end{tikzcd} \qquad (*)$$

and note that (by flatness) each extension satisfies P.D.E. It is clear that $e_P = e_{P_e}$ because graded ideals of R, Λ are generated by their parts of degree e in R_e, Λ_e resp. The diagram $CCl(*) = CCl^g(*)$ is commutative. The class of $P_{e,1}^{t_1} * \ldots * P_{e,q}^{t_q}$ in $CCl(\Lambda_e)$ maps to the class of $P_1^{t_1} * \ldots * P_q^{t_q}$ in $CCl(\Lambda)$, where $P_i = \Lambda P_{e,i}, i = 1, \ldots, q$. The latter image will be trivial in $CCl(\Lambda)$ when $e_i | t_i$, $i = 1, \ldots, q$, and then the class of $p_1^{t_1/e_1} * \ldots * p_q^{t_1/e_q}$ maps to 1 in $Cl(R)$ where $p_i = P_i \cap R$ $i = 1, \ldots, q$. Putting $p_{e,i} = p_i \cap R_e$ the foregoing leads to : $p_{e,1}^{t_1/e_1} * \ldots * p_{e,q}^{t_q/e_q} = c(I_e^m)$ for some $c \in K^g$, $m \in \mathbb{N}$ and I_e being $(P_1^{\nu_1} * \ldots * P_r^{\nu_q})_e$. Since the ramification of $P_{e,i}$ over the $p_{e,i}$ equals e_i for $i = 1, \ldots, q$ it follows that : $P_{e,1}^{t_1} * \ldots * P_{e,q}^{t_1} = (\Lambda_e I_e))^m c$ and exactness of the second column follows from the fact that $\Lambda_e I_e$ is divisorial (here even invertible). □

II.4.37. Remark. The theorem holds for divisorially graded orders too.

Proof. As above, adding $()^{**}$ where necessary and noting that Proposition I.1.16. holds for divisorially graded rings as stated. □

In the final paragraph of this section we deduce the main result concerning class groups of generalized Rees rings following F. Van Oystaeyen, L. Le Bruyn [30] and L. Le Bruyn [28]. This result reduces the study of maximal orders over Krull domains to the study of reflexive graded Azumaya algebras over certain generalized Rees rings plus the investigation of the extension $R \to R_1$ where R_1 is the generalized Rees ring mentioned before.

Let Λ be a maximal order over a Krull domain (we leave it to the reader to verify that the results can be extended to the case of tame orders). Let $P_1, \ldots, P_n$ be the finite number of prime ideals of Λ which are not centrally generated and let $e_1, \ldots, e_n$ be their ramification indexes. We will sometimes say that $\Sigma_i e_i P_i = \mathcal{P}$ is the **ramification divisor** of Λ and $\Sigma_i e_i p_i = \mathcal{P}^c$ is the **central ramification divisor** where $p_i = P \cap R$. In general, to every divisor D for Λ we associate the $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$-graded subring of $A[X_1 . X_1^{-1}, \ldots, X_n, X_n^{-1}]$ (where n is the number of nonzero elements appearing in D defined by :

$$\Lambda[D](m_1, \ldots, m_n) = (P_1^{m_1} * \ldots * P_n^{m_n}) X_1^{m_1} \ldots X_n^{m_n}$$

This divisorially graded ring is called the **generalized Rees ring for** D. We call $\Lambda[\mathcal{P}]$ **the ramification Rees ring** of Λ.

II.4.38. Theorem. Let $\mathcal{P}$ be the ramification divisor of a maximal order Λ over the Krull domain R.
1. $\Lambda[\mathcal{P}]$ is a P.I. ring and a maximal order over its center $R[\mathcal{P}^c, \underline{e}]$ (where $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$) which is a Krull domain.
2. $CCl(\Lambda[\mathcal{P}]) = Cl(R[\mathcal{P}^c, \underline{e}])$.

Proof. 1. See Section III.1. The ring $R[\mathcal{P}^c, c]$ is the so-called **scaled Rees ring** with step $(e_1, \ldots, e_n)$, i.e. $R[\mathcal{P}^c, \underline{e}](m_1, \ldots, m_n) = (p_1^{\alpha_1} * \ldots * p_n^{\alpha_m}) X_1^{m_1} \ldots . X_n^{m_n}$, where α_i is the integral part of $\frac{m_i}{e_i}$, $i = 1, \ldots, n$. Note that such rings also appeared in Chapter I. e.g. Proposition I.2.5. Theorem I.2.15.

2. Since $A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is an Azumaya algebra over a factorial domain we may apply Corollary II.4.34., hence $CCl^g(\Lambda[\mathcal{P}]) = CCl(\Lambda[\mathcal{P}])$. Then it is easy to verify exactness of the following sequence :

$$0 \to < [P_1], \ldots, [P_n] > \to CCl(\Lambda) \to CCl^g(\Lambda[\mathcal{P}]) \to 0$$

In a similar way : $Cl^g(R[\mathcal{P}^c, \underline{e}]) \cong Cl(R[\mathcal{P}^c, \underline{e}])$ and the following sequence is also exact :

$$0 \to < [p_1], \ldots, [p_n] > \to Cl(R) \to Cl^g(R[\mathcal{P}^c, \underline{e}]) \to 0$$

We obtain an exact commutative diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & < [p_1], \ldots, [p_2] > \arrow[r] \arrow[d] & Cl(R) \arrow[r] \arrow[d] & Cl(R[\mathcal{P}^c, \underline{e}]) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & < [P_1], \ldots, [P_n] > \arrow[r] \arrow[d] & CCl(\Lambda) \arrow[r] \arrow[d] & CCl(\Lambda[\mathcal{P}]) \arrow[r] & 0 \\
 & \bigoplus_{J=1}^{n} \mathbb{Z}/e_i\mathbb{Z} & \bigoplus_{J=1}^{n} \mathbb{Z}/e_i\mathbb{Z} & &
\end{tikzcd}$$

hence $Cl(R[\mathcal{P}^c, \underline{e}]) = CCl(\Lambda[\mathcal{P}])$ follows. □

If we are in a situation where Theorem II.4.24. may be applied to conclude that $\Lambda[\mathcal{P}]$ is a reflexive Azumaya algebra then we have reached the goal set out in the beginning of this paragraph. Note that this is the case for applications over rings appearing in algebraic geometry if the characteristic of the ground field is zero. As an extra example of a situation where this technique works we mention the following :

II.4.39. Proposition. If Λ is Zariski-tamifiable then so is $\Lambda[\mathcal{P}]$.

Proof. Let L be a separable splitting field for A, then $L(X_1, \ldots, X_n)$ splits $A(X_1, \ldots, X_n)$. Let $S[\mathcal{P}^c]$ be the integral closure of $R[\mathcal{P}^c, \underline{e}]$ in $L(X_1, \ldots, X_n)$. Since $R[\mathcal{P}^c, \underline{e}]$ is a graded Krull domain the same is true for $S[\mathcal{P}^c]$. Let $P \in X^1(S[\mathcal{P}^c])$, then either $P_g = 0$ or $P = P_g$. If $P_g = 0$ then the localization of $\Lambda[\mathcal{P}] \underset{R[\mathcal{P}^c, \underline{e}]}{\otimes} S[\mathcal{P}^c]$ at P is a localization of $A \underset{K^g}{\otimes} L[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ and therefore it will be an Azumaya algebra over the Krull domain $S[\mathcal{P}^c]_P$, hence it is a maximal order and obviously tame as well. If $P = P_g$, look at $p = P \cap R$. If p is not appearing in P^c then the localization of $\Lambda[\mathcal{P}] \underset{R[\mathcal{P}^c, \underline{e}]}{\otimes} S[\mathcal{P}^c]$ at p is a localization of $(\Lambda_p \underset{R_p}{\otimes} S_p)[X_1, X_1^{-1}, , X_n, X_n^{-1}]$; therefore it is a tame order because the class of tame orders is closed under polynomial

extensions and central localizations. If p appears in $\mathcal{P}^c$, say $p = P_1 \cap R$ then $(\Lambda[\mathcal{P}] \underset{R[\mathcal{P},\underline{e}]}{\otimes} S[\mathcal{P}^c])_P = (\Lambda[\Theta] \otimes S[\Theta])_q[X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$ where $\Lambda[\Theta]$ is defined by $\Lambda[\Theta]_n = P_1^n X_1^n$ $S[\Theta]$ is the integral closure of $R[\Theta] = Z(\Lambda[\Theta])$ in $L(X_1)$, $q = P \cap S[\Theta]$. Clearly, $\Lambda[\Theta] \underset{R[\Theta]}{\otimes} S[\Theta]$ is an overring of $(\Lambda \underset{R}{\otimes} S)[\mathcal{P}]$ in $(A \underset{R}{\otimes} S)[\mathcal{P}]$ in $(A \underset{K}{\otimes} L)(X_1)$.
Furthermore, $(\Lambda \underset{R}{\otimes} S)(\mathcal{P})$ will be tame (see section III.1., or derive it from Theorem II.4.31. in a rather straightforward way) and therefore the overring $\Lambda[\Theta] \underset{R[\Theta]}{\otimes} S[\Theta]$ will be tame too. □

II.4.40. Remark. The extension of Theorem II.4.38. and Proposition II.4.39 to the case where Λ is a tame order over the Krull domain R presents no problems (as is evident in the proofs given).

II.5. Some General Techniques.

Our aim in this section is to give some techniques that are useful in the study of arbitrary graded rings. In Section II.5.1. we introduce the notion of a separable functor. This is an attempt to give a unified treatment of the various versions of Maschke's theorem that occur in the literature. We give a set of examples in an attempt to convince the reader that separable functors occur very naturally. In Section 5.2. we introduce group rings over graded rings. This is a technique that is very often useful to prove a theorem for arbitrary graded rings once the corresponding result for group rings is known.
A typical example of this technique is an easy proof of the Bergman conjecture [16]. As another example we give an upper bound for the global dimension of an arbitrary graded ring that is only slightly weaker than Rosset's bound [47] on the global dimension of a crossed product, but has a conceptually much simpler proof.

II.5.1. Separable Functors.

II.5.1.1. Definition.
A functor $F : C \to D$ between two arbitrary categories is called separable if for all objects M, N in C there are maps :

$$\varphi^F_{M,N} : \mathrm{Hom}_D(FM, FN) \to \mathrm{Hom}_C(M, N)$$

satisfying the following compatibility conditions.

(1) If $\alpha \in \mathrm{Hom}_C(M,N)$ then $\varphi^F_{M,N}(\alpha) = \alpha$.

(2) If there are $M', N' \in C$ and $\alpha \in \mathrm{Hom}_C(M,N), \beta \in \mathrm{Hom}_C(M',N')$, $f \in \mathrm{Hom}_D(FM, FN), g \in \mathrm{Hom}_D(FM', FN')$ such that the diagram

$$\begin{array}{ccc} FM & \xrightarrow{f} & FN \\ F\alpha \downarrow & & \downarrow F\beta \\ FM' & \xrightarrow{g} & FN' \end{array}$$

is commutative, then the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi^F_{M,N}(f)} & N \\ \alpha \downarrow & & \downarrow \beta \\ M' & \xrightarrow{\varphi^F_{M',N'}(g)} & N' \end{array}$$

is commutative.
If there is no confusion possible we will usually denote $\varphi^F_{M,N}(?)$ by $\widetilde{?}$. The following Proposition gives us some elementary properties of separable functors.

II.5.1.2. Proposition. - If $F : C \to D, G : D \to E$ are separable functors then $G \circ F$ is separable.
- If $G \circ F$ is separable then F is separable.

Proof. Just checking. One deduces that

$$\varphi^{G \circ F}_{M,M'}(f) = \varphi^G_{M,M'}(\varphi^F_{M,M'}(f))$$

and

$$\varphi^G_{M,M'}(f) = F(\varphi^{G\circ F}_{M,M'}(f)) \qquad \square$$

If a functor $F : C \to D$ is separable and if M is an object in C then it is often possible to deduce properties of M from the corresponding properties of FM. Proposition 5.1.3. contains some examples of this phenomenon.

II.5.1.3. Proposition. Suppose that $F : C \to D$ is a separable functor and let M, N be objects in C.

(a) If $f : M \to N$ is a map in C such that Ff is split then f is split. For the following properties we assume that C, D are abelian categories.

(b) If FM is semisimple (every submodule splits off) then M is semisimple.

(c) If FM is projective then M is projective.

Proof. An easy exercise using the defining property of a separable functor. $\square$

Let $\alpha : R \to S$ be a ring morphism. Then there are two classical functors associated to α.

(1) $_\alpha(?)$: S-mod $\to$ R-mod which makes an S module M into an R-module $_\alpha M$ by defining the multiplication as $r.M = \alpha(r)m$. This is called the change of rings functor.

(2) $S \underset{R}{\otimes} ? : R\text{-mod} \to S\text{-mod}$ of which the definition is obvious.

We will examine what separability means for these two functors.

II.5.1.4. Proposition. (a) $_\alpha(?)$ is separable if and only if the map $\Psi : S \underset{R}{\otimes} S \to S : s \otimes s' \to ss'$ splits as a map of S-S-bimodules.

(b) $S \otimes_R ?$ is separable if and only if α splits as an R-bimodule map.

Proof. (a) Suppose that $_\alpha(?)$ is separable. Ψ splits as an R-S bimodule map by sending $s \to 1 \otimes s$. Then by the separability we can change this splitting to an S-S-bimodule splitting.

Conversely assume that Ψ is split by some S-S-bimodule map ϑ.

Let M, N be S-modules and let f be an R linear map $_\alpha M \to_\alpha N$. We then define $\tilde{f}$ by the following commutative diagram of S modules.

$$\begin{tikzcd}
S \underset{R}{\otimes} M \arrow[r, "1\otimes f"] & S \underset{R}{\otimes} N \arrow[d, "\cong"] \\
S \underset{R}{\otimes} S \underset{S}{\otimes} M \arrow[u, "\cong"] & S \underset{R}{\otimes} S \underset{S}{\otimes} M \arrow[d, "\Psi \otimes 1"] \\
S \underset{S}{\otimes} M \arrow[u, "\vartheta \otimes 1"] & S \underset{S}{\otimes} N \arrow[d, "\cong"] \\
M \arrow[u, "\cong"] \arrow[r, "\tilde{f}"] & N
\end{tikzcd}$$

From this diagram one easily verifies that $\tilde{f} = f$ if f is S-linear. Let u_M resp. v_M denote the composition of the vertical maps on the left resp. on the right.

If M', N' are different S-modules and $\alpha \in \mathrm{Hom}_S(M, M')$, $\beta \in \mathrm{Hom}_S(N, N')$ $g \in \mathrm{Hom}_R(M', N')$ then one has the diagram

$$\begin{tikzcd}
S \underset{R}{\otimes} M \arrow[rrr, "1\otimes f"] \arrow[dr, shift left, "v_M"'] \arrow[ddd, "1\otimes\alpha"'] & & & S \underset{R}{\otimes} N \arrow[dl, shift left, "v_M"'] \arrow[ddd, "1\otimes\beta"] \\
& M \arrow[ul, shift left, "u_M"'] \arrow[r, "\tilde{f}"] \arrow[d, "\alpha"] & N \arrow[ur, shift left, "u_N"'] \arrow[d, "\beta"] & \\
& M' \arrow[r, "\tilde{g}"] \arrow[dl, shift left, "u_{M'}"'] & N' \arrow[dr, shift left, "u_{N'}"'] & \\
S \underset{R}{\otimes} M' \arrow[rrr, "1\otimes g"'] \arrow[ur, shift left, "v_{M'}"'] & & & S \underset{R}{\otimes} N' \arrow[ul, shift left, "v_{N'}"']
\end{tikzcd}$$

Here the diagonal maps are splittings. One deduces that if the outer square is commutative then the inner square is it too.

(b) Suppose that $S \otimes_R ?$ is separable. The composite of the maps $i \otimes \alpha : S \otimes_R R \to S \otimes_R S$ and $v : S \otimes_R S \to S \otimes_R R : s \otimes s' \to ss' \otimes 1$ is the identity. Hence $\tilde{v}$ provides a splitting for α.

Suppose now that α is split by a map β. If $f : S \otimes_R M \to S \otimes_R N$ is S-linear then we define $\tilde{f}$ by the commutative diagram.

$$\begin{tikzcd}
S \otimes_R M \arrow[r, "f"] & S \otimes_R N \arrow[d, "\otimes 1"] \\
R \otimes_R M \arrow[u, "\alpha \otimes 1"] & R \otimes_R N \arrow[d] \\
M \arrow[u, "\cong"] \arrow[r, "\tilde{f}"] & N
\end{tikzcd}$$

A routine verification as is (a) then shows that $S \otimes_R ?$ is separable. □

If $_\alpha(?)$ is separable we say that S/R is separable. Some times we use a more informal terminology by saying that S/R satisfies a version of Maschke's theorem. Our notion of separability is an extension of the classical notion [17] which is defined when R is commutative.
Suppose now that R is strongly graded by a group G. It is well known [56] that in this situation there is a unique G action on $Z(R_e)$ satisfying for $a \in R_\sigma$, $b \in Z(R_e) : ab = \sigma(b)a$.

II.5.1.5. Proposition. (Maschke's theorem) R is separable over R_e if and only if the trace map $t : Z(R_e) \to Z(R_e) : a \to \Sigma\sigma(a)$ is surjective.

Proof. Suppose that R/R_e is separable i.e. there is an R-bimodule splitting of the product map $\Psi : R \otimes_{R_e} R \to R$. But $R \otimes_{R_e} R = \Sigma_{\sigma,\tau} R_{\sigma\tau}$ for a strongly graded ring.
Hence we have to find a splitting of the sum map $\Psi' : \sum\limits_{\sigma,\tau} R_{\sigma\tau} \to R$. Such a splitting is determined by an R-centralizing element s in $\sum\limits_{\sigma,\tau} R_{\sigma\tau}$ mapping to 1. It is easily seen that s can be chosen in $\sigma \in G\, R_e$ i.e. $s = (s_{\sigma,\sigma^{-1}})_\sigma$ where $s_{\sigma,\sigma^{-1}} \in R_e$. From the fact that s is R_e-centralizing we deduce that $s_{\sigma,\sigma^{-1}} \in Z(R_e)$. Let $t \in R_\tau$. From the fact that $st = ts$ we deduce that

$$(s_{\tau\sigma,\sigma^{-1}\tau^{-1}} - \tau(s_{\sigma,\sigma^{-1}}))t = 0$$

and hence

$$(s_{\tau\sigma,\sigma^{-1}\tau^{-1}} - \tau(s_{\sigma,\sigma^{-1}}))R_\tau R_{\tau^{-1}} = 0$$

Therefore, by the fact that R is strongly graded : $s_{\tau,\tau^{-1}} - \tau(s_{\sigma,\sigma^{-1}}) = 0$ in particular $s_{\tau,\tau^{-1}} = (s_{e,e})$. So $1 = \Psi'(s) = \sum_{\sigma,\sigma^{-1}} = t(s_{e,e})$
Conversely if there is an element u in $Z(R_e)$ of trace 1 we may produce a splitting of Ψ' by sending 1 to $(\sigma(u))_{\sigma\in G}$.
It is possible is this situation to describe $\widetilde{?}$ explicitly i.e. if $f \in \mathrm{Hom}_{R_e}(M,N)$ then $\widetilde{f}(m) = \Sigma_\tau\Sigma_i u_i^{(\tau)} f(uv_i^{(\tau^{-1})}m)$ where the $u_i^{(\tau)} \in R_\tau, v_i^{(\tau^{-1})} \in R_{\tau^{-1}}, 1 = \Sigma_i u_i^{(\tau)} v^{(\tau^{-1})}$ and u is the element of trace one. □

II.5.1.6. Remark : If R is an arbitrary graded ring by a finite group G we do not know whether separability of R over R_e implies that R is strongly graded. This seems very likely however. The following proposition is true in general.

II.5.1.7. Proposition. If R is an arbitrary graded ring then the forgetful functor R-mod $\to$ R-gr is separable.

Proof. Let M, N be graded R-modules and $f \in \mathrm{Hom}_R(M,N)$. If $m \in M$ with homogeneous decomposition $m = m_{\sigma 1} + \ldots + m_{\sigma_k}$ then one defines $\widetilde{f}(m) = \Sigma_i f(m_{\sigma_i})_{\sigma_i}$. It is clear that $\widetilde{f} \in \mathrm{Hom}_{R-gr}(M,N)$. □

A particularly useful special case of a separable extension of rings is a Galois extension.

II.5.1.8 Definition : Assume $R \subset S$ and that there is a finite group acting on S leaving R invariant. Then we say that S/R is G-Galois if S is a right R progenerator and the canonical map

$$S \otimes_R S \to \oplus_{\sigma\in G}({}_\sigma S_1) : a \otimes b \mapsto \oplus_{\sigma\in G}\sigma(a)b$$

is an isomorphism of S - S bimodules.

II.5.1.9 Lemma : If S/R is G-Galois then S/R is separable.

Proof : Immediate.

II.5.1.10 Proposition : Assume that S/R is G-Galois. Then the following are true :
(1) The canonical map

$$V : S * G \to End_R(S_R), V(s * \sigma)(a) = s\sigma(a),$$

is an isomorphism.
(2) $S^G = R$
(3) The trace map $Tr : S \to R : a \to \sum \sigma(a)$ is surjective. (Hence R is a direct summand of S).

Proof :
(1) We construct the following diagram :

$$\begin{array}{ccc} S * G & \xrightarrow{V} & End_R(S) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ Homs_S(\oplus_\sigma S_1, S) & \xrightarrow{V^*} & Homs_S(S \otimes_R S, S) \end{array}$$

where the maps are defined as follows (all Hom's are for right modules) : the map V is as above, furthermore $\beta(\phi)(a \otimes b) = \phi(a)b$, and $\alpha(s * \sigma)({}_\tau a) = \delta_{\sigma\tau} s_\tau a$ where ${}_\tau a \in_\sigma S_1$.

One quickly verifies that this diagram is commutative. Furthermore α and β are obviously isomorphisms. V^* is also an isomorphism since V is by definition an isomorphism. (Note that we did not use the fact that S is a R progenerator.)
(2) Since S is a right R progenerator S is faithfully flat and hence

$$0 \to R \xrightarrow{\alpha} S \xrightarrow{\beta} S \otimes_R S$$

with $\alpha(a) = a$ and $\beta(a) = a \otimes 1 - 1 \otimes a$ is exact. By using the isomorphism of II.5.1.8., we see that Ker$\alpha = S^G$. Hence $R = S^G$.
(3) Again using the fact that S is a right progenerator it suffices to show that $S \otimes_R S \to S : a \otimes b \to \sum a\sigma^{-1}(b)$ is surjective. Using the isomorphism of II.5.1.8. this becomes obvious.

II.5.1.11 Remark : The condition that S is a progenerator is necessary, otherwise any finite ring epimorphism would qualify as a Galois extension for the trivial action. Clearly (2) and (3) are not true in this case.

II.5.1.12 Proposition : Assume $R \subset S \subset T$ and that a finite group G acts on T leaving R invariant. Assume furthermore that T/S and S/R are Galois, resp. for a normal subgroup H of G and for G/H. Then S/R is G-Galois.

Proof : An easy verification after observing that $T \otimes_R T \cong T \otimes_S S \otimes_R S \otimes_S T$. □

Clearly if S and R are commutative then our notion of Galois extension coincides with the classical notion as defined for example in [17]. Starting from commutative Galois extensions one can construct new ones by base extension. The following is easily verified.

II.5.1.13 Lemma : Suppose that $R \subset S$ are commutative rings with S/R Galois for some finite group G. If T is an arbitrary R-algebra then $S \otimes_R T/T$ is G Galois.

To apply the previous proposition, it is useful to have a method for constructing the group G if it is not already given. This is accomplished by the following proposition :

II.5.1.14 Proposition : Assume that $R \subset S \subset T$ and that T/S and S/R are resp. H and H' Galois for finite groups H and H'. For each $h' \in H'$ let there be given an element $h'^* \in Aut_R(T)$ lifting the action of h' on S and let $H'^* = \{h'^* | h' \in H\}$ (this might not be a group). Assume that the following conditions hold :

(a) H acts faithfully on S.
(b) $(h_1'^* H_2'^*)(h_1' h_2')^{*-1} \in H$ for $h_1', h_2' \in H'$,
(c) $h'^{*-1} h h'^* \in H$ for $h, h' \in H$.

II.5.1.15 Lemma : Let G be the group generated by H and H'^* in $Aut_R(T)$. Then $G/H \cong H'$.

Proof : One easily deduces that $G = H'H$ and $H' \cap H = \{e\}$. This proves the claim. □

A different way to construct Galois extensions is through strongly graded rings.

II.5.1.16 Proposition : Assume that R is strongly graded for some commutative finite group G. Assume furthermore that R contains a field containing a primitive p^{th} root of unity for every prime divisor p of $|G|$. Then G^* acts on R through multiplication by characters $(\chi a_\sigma = \chi(\sigma) a_\sigma)$ and R/R_e is G^* Galois for this action.

Proof : Verification of the defining conditions for being a Galois extension. □

II.5.1.17 Remark : The proposition as stated above looks somewhat unnatural due to the condition on the field contained in R. The reason for this is that it would be more natural to look at the action of the dual of the Hopf algebra kG : $(kG)^*$ (k the prime ring). Then if G is an arbitrary finite group and R/R_e is strongly G graded, R/R_e will be $(kG)^*$ Galois. The conditions we put on the field in R_e guarantee that

$(kG)^* \cong k(G^*)$ as Hopf algebras. The reason that we didn't use $(kG)^*$ is that it would involve going into the theory of Galois actions of Hopf algebras [52] which is more difficult. Proposition II.5.1.16 as stated is enough for our purposes (see Chapter V).

We can extend the above material to reflexive modules. Let us adopt temporarily (i.e. till the end of II.5.1) the following generalized notion of order. If R is a Krull ring then an R-order will be simply an R-algebra refelexive as an R-module. By Λ-ref we denote the full subcategory of Λ-mod consisting of Λ-modules reflexive over R. Modules in Λ-ref are called reflexive Λ-modules. It is clear that Λ-ref does not depend on the particular Krull domain over which Λ is an order. Furthermore if M is a Λ-module then M^{**} is defined as $\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ but this definition is also independent of R. Suppose that $\Lambda \subset \Gamma$ are R-orders then one can define the change of rings functor ${}_\Lambda(?) : \Gamma\text{-ref} \to \Lambda\text{-ref}$ and the functor $\Gamma \perp_\Lambda ?\; : \Lambda\text{-ref} \to \Gamma\text{-ref}$ which sends a reflexive Λ-module M to $(\Gamma \otimes_\Lambda M)^{**}$. Then the following analogon of Proposition 5.1.5. is true.

II.5.1.18. Proposition :

(a) ${}_\Lambda(?)$ is separable if and only if $\Psi : \Gamma \perp_\Lambda \Gamma \to \Gamma$ splits as $\Gamma - \Gamma$ bimodules where Ψ is induced from the product map $\Gamma \otimes_\Lambda \Gamma \to \Gamma$: $s \otimes s' \to ss'$

(b) $\Gamma \perp_\Lambda ?$ is separable if and only if Λ splits off as a Λ bimodule.

Proof. The proof of this lemma is essentially the same as the proof of Proposition 5.1.5. □

If ${}_\Lambda(?)$ is separable then we say that Γ is **reflexive separable** over Λ. It is a natural question to ask whether reflexive separability is a local notion, i.e. is it true that Γ/Λ is reflexive separable if Γ_p/Λ_p is separable for all $p \in X^1(R)$. We give an answer to this problem in the case that Λ is commutative.

II.5.1.19. Proposition : Suppose that $\Lambda \subset \Gamma$ are R-orders where Λ is commutative. Suppose that for all $p \in X^1(R)$ Γ_p/Λ_p is separable.

Then the following is true :

(a) If Γ is commutative then Γ/Λ is reflexive separable.

(b) If $Z(\Gamma) = \Lambda$ then Γ/Λ is reflexive separable if and only if Λ is a Λ-direct summand of Γ.

(c) In general, Γ is reflexive separable over Λ if and only if $Z(\Gamma)$ is a $Z(\Gamma)$-direct summand of Γ.

Proof. (c) is just a combination of (a) and (b).

(a) Consider the exact sequence of $\Gamma \perp_\Lambda \Gamma$-modules :

$$0 \longrightarrow J \longrightarrow \Gamma \perp_\Lambda \Gamma \longrightarrow \Gamma \longrightarrow 0$$

It is clear that in this situation $(\Gamma \perp_\Lambda \Gamma)^\Gamma$ is a reflexive $\Gamma \otimes^1_\Lambda \Gamma$ ideal. We claim that $J \cap (\Gamma \perp_\Lambda \Gamma)^\Gamma = 0$. It suffices to check this at each $p \in X^1(R)$. Since Γ_p/Λ_p is separable there is a separability idempotent e in $\Gamma_p \otimes_{\Lambda_p} \Gamma_p$ mapping to 1 in Γ_p. Hence $J_p + (\Gamma_p \otimes_{\Lambda_p} \Gamma_p)^{\Gamma_p} = \Gamma_p$ (*). Furthermore $J_p(\Gamma_p \otimes_{\Lambda_p} \Gamma_p)^{\Gamma_p} = 0$ since J_p is generated by elements of the form $a \otimes 1 - 1 \otimes a$. Hence $J_p \cap (\Gamma_p \otimes_{\Lambda_p} \Gamma_p)^{\Gamma_p} = 0$. Therefore $J + (\Gamma \perp_\Lambda \Gamma)^\Gamma$ is a reflexive submodule of $\Gamma \perp_\Lambda \Gamma$. Then we deduce from * that $\Gamma \perp_\Lambda \Gamma = (\Gamma \perp_\Lambda \Gamma)^\Gamma \oplus J$. Hence $\Gamma \perp_\Gamma \Gamma \to \Gamma$ splits.

(b) There is a canonical map $\varphi : \Gamma \perp_\Lambda \Gamma^\circ \longrightarrow \mathrm{Hom}_\Lambda(\Gamma,\Gamma)$ obtained from the map $\Gamma \underset{\Lambda}{\otimes} \Gamma^0 \longrightarrow \mathrm{Hom}_\Lambda(\Gamma,\Gamma)$ which sends $x \otimes y$ to the map $a \mapsto xay$ in $\mathrm{Hom}_\Lambda(\Gamma,\Gamma)$. The map φ is an isomorphism since it is an isomorphism for any $p \in X^1(R)$, [58].
As in the classical case reflexive separability of Γ/Λ is equivalent with the existence of an $e \in \Gamma \perp_\Lambda \Gamma$ such that : $(\mathrm{Ker}(\Gamma \perp_\Lambda \Gamma \longrightarrow \Gamma))e = 0$ and $\Psi(e) = 1$. Furthermore such e's correspond to projection maps : $\Gamma \to \Lambda$ under φ. This proves (b). □
For completeness we give an example of an R-order Γ separable over its center $Z(\Gamma)$ in each height one prime but not containing $Z(\Gamma)$ as a direct summand.

II.5.1.20. Example : Let R be a positively graded Krull domain with $R_0 = k$, an algebraically closed field of characteris-

tic $p > 0$ and let M be an indecomposable reflexive R module of rank p. An easy example of this situation is $p = 2, R = k[X,Y,Z], M = \mathrm{Ker}(k[X,Y,Z]^3 \xrightarrow{f} k[X,Y,Z])$, where f is given by the column (X,Y,Z).
Let $\Gamma = \mathrm{End}_R(M)$. Then $k \cong \Gamma/J_g(\Gamma)$ where J_g denotes the graded Jacobson radical, and by the fact that the reduced trace $\Gamma \to R$ is a graded map one sees that there is no element of reduced trace 1 in Γ. Since every R-map $\Gamma \to R$ is of the form $a \mapsto tr(xa)$ this implies that Γ does not contain R as a direct summand.
The natural generalisation of a Galois extension becomes a reflexive Galois extension. If Λ is an R-order and M is a left reflexive Λ-module then we say that M is a (left) reflexive generator if M_p is a Λ_p progenerator for all $p \in X^{(1)}(R)$.

II.5.1.21 Definition : If $\Lambda \subset \Gamma$ are R orders and G is some finite group acting on Γ leaving Λ invariant then we say that Γ/Λ is reflexive G-Galois if Γ is a right Λ reflexive generator and the natural map $\Gamma \otimes'_\Lambda \Gamma \to \oplus_{\sigma \in G} \sigma \Gamma_1 : a \otimes b \to \oplus_{\sigma \in G} \sigma(a) b$ is an isomorphism as Λ-Λ bimodules.

It is quickly verified that being reflexive Galois is a local notion. I.e. Γ/Λ is Galois if and only if Γ_p/Λ_p is Galois for all $p \in X^{(1)}(R)$. Furthermore one verifies easily that Propositions II.5.1.10, II.5.1.12, II.5.1.13 etc... have an equivalent in this setting. For example Prop. II.5.1.16 becomes true for divisorially graded rings. We will not state explicitly all the generalisations since they are clear in each case.

Finally we note that proposition 5.1.5. has an extension to divisorially graded orders. Let Γ be an R-order graded by a finite group such that $(\Gamma_\sigma \Gamma_\tau)^{**} = \Gamma_{\sigma\tau}$. Then there is a G-action on $Z(\Gamma e)$ satisfying for $a \in Z(\Gamma_e) : ab = \sigma(b)a$ cf. o.a. [37].

II.5.1.22. Proposition. (Maschke's theorem for divisorially graded rings).
Γ is reflexive separable over Γ_e if the trace map $Z(\Gamma e) \to Z(\Gamma e)^G : a \to \Sigma\sigma(a)$ is surjective.

Proof. Exactly as in Proposition 5.1.4. □

II.5.2. Group rings over graded rings.

If R is a G-graded ring, then the forgetful functor R-gr $\to$ R-mod has a right adjoint R-mod $\to$ R-gr that sends an R-module N to the graded R-module $\underset{\sigma \in G}{\oplus} N_\sigma$ where $N_\sigma = N$ and the R multiplication is defined by $R_\sigma \times N_\tau \to N_{\sigma\tau} : (r_\sigma, n_\tau) \to r_\sigma n_\tau$. It turns out however that there is a more convenient way to visualize this adjoint. Let RG be the groupring over R graded in the usual way, i.e. $\deg(R\sigma) = \sigma$. RG has a graded subring $S = \sum_{\sigma \in G} R_\sigma \sigma$ isomorphic to R as a graded ring. Note however that the standard copy of R in RG,(R_e), is not graded. If N is an R-module then NG is a graded RG module. Then ${}_S(NG)$ defines a graded-S module and the reader can verify that after identifying S with R this is the graded module associated to N defined above.

II.5.2.1. Lemma : If G is a finite group then the change of rings functor R-gr $\to$ S-gr is separable if and only if $|G|^{-1} \in R$.

Proof. As before one has to construct a graded splitting of the map : $RG \otimes_S SG \longrightarrow RG$. It is easy to see that such a splitting exists if and only if $|G|^{-1} \in R$. □

This lemma gives an explanation for the fact that for rings graded by a finite group graded properties and the corresponding ungraded properties usually are related. In fact one can develop the theory in the general setting of categories with separable adjoint functors. Even the case where $|G|^{-1} \notin R$ can be incorporated in this formalism if one works in the localized category R-gr$/\mathcal{C}$ where a $\mathcal{C}$ is the Serre-subcategory of R-gr consisting of graded modules that are annihilated by a power of $|G|$. We think however that such a generality would be too much a burden on the reader who is only interested in concrete results. As an illustration of the concepts described above we give a proof of the Bergman conjecture. This conjecture was first proved in [16] using a method inspired by the theory of Hopf algebras.

Let $J(?), J_g(?)$ denote resp. the Jacobson radical and the graded Jacobson radical of a graded ring.

II.5.2.2. Proposition. Let R be graded by a finite group, then $J_g(R) \subset J(R)$. If $|G|^{-1} \in R$ then $J_g(R) = J(R)$

Proof. Let V be an irreducible R-module. Then $W = VG = \sum_{\sigma\in G} V\sigma$ is a graded irreducible RG-module. The graded version of Nakayama's lemma now implies that $J_g(S)W \neq W$. Since $J_g(S)W$ is a graded submodule of $W, J_g(S)W = 0$. But $J_g(S)W = (J_g(R)V)G$ and hence $J_g(R)V = 0$. So $J_g(R \subset J(R)$. Assume now $|G|^{-1} \in R$. If M is a graded S-module then $RG \otimes_S M = \sum_{\tau\in G} M(\tau)$ where $M(\tau)$ is isomorphic to M as an ungraded module but has a shifted grading : $(M(\tau))_\sigma = M_{\tau\sigma}$. Since $M \to M(\tau)$ defines an auto-equivalence on R-gr it is clear that if V is a graded irreducible S-module then $RG \otimes_S V$ is graded completely reducible as an S-module. By Proposiotion II.5.2.1., $RG \otimes_S V$ is graded completely reducible as an RG-module. Hence $J_g(RG)(RG \otimes_S V) = 0$. Hence $J(R)(RG \otimes_S V) = 0$ and $J(R)V = 0$ since V is a submodule of $RG \otimes_S V$. So we have proved that $J(R) \subset J_g(R)$ is not restricted to finite groups as Proposition II.5.2.3. shows If R is a graded ring denote by gl.dim$R$ resp. gr.gl.dim R the global dimension and the graded global dimension of R. If M is an R module then $pd_R M$ is the projective dimension of M. It is superfluous to introduce a similar graded notion since the projective dimension of a module is equal to its graded projective dimension.

□

II.5.2.3. Proposition. Suppose that R is graded by a group G and let k be a commutative subring of R_e central in R. Then gl.dim $R \leq$

gr.gl.dim $R + cd_k(G)$ where $\mathrm{cd}_k(G) = \mathrm{pd}_{kG}(k)$ where k has trivial G action.

Proof. Let $h = \mathrm{cd}_k(G)$. Since gl.dim$R$ =gr.gl.dimRG it is sufficient to prove that for a graded RG-module M, which is projective as an S-module : $\mathrm{pd}_{RG}(M) \leq cd_k(G)$. There is an exact sequence of graded kG-bimodules

$$\ldots \longrightarrow kG \otimes_k kG \otimes_k kG \xrightarrow{\vartheta_2} kG \otimes_k kG \xrightarrow{\vartheta} kG \longrightarrow 0 \qquad (*)$$

Tensoring (*) on the left by S/k one obtains an exact sequence of graded RG-bimodules

$$\longrightarrow RG \otimes_S \ldots \otimes_S RG \longrightarrow \ldots \longrightarrow RG \otimes_S RG \longrightarrow RG \longrightarrow 0$$

This sequence is split as a sequence of **right** RG-modules. Therefore tensoring on the right by M is exact. We then obtain an exact sequence

$$RG \otimes_S RG \otimes_S M \longrightarrow \ldots \longrightarrow RG \otimes_S M \longrightarrow M \longrightarrow 0 \qquad (**)$$

yielding a projective resolution of M as a left RG-module. If we can show that $\mathrm{pd}_{kG\otimes kG^o}(kG) = h$ then we know that (*) and hence (**), splits at step h. So $\mathrm{pd}_{RG}(M) \leq h$. □

The fact $\mathrm{pd}_{kG\otimes kG^o}(kG) = \mathrm{pd}_{kG}(k)$ is well known. We include a proof that has a fundamentally graded nature.

If $k \subset R$ are rings, k commutative and central in R then we denote by R-bimod$_k$ the category of R-R-bimodules that are k-centralizing. If R_e is graded, $k \subset Re$ then R-gr-bimod$_k$ is the corresponding graded notion.

Lemma II.5.2.4. The category kG-gr-bimod$_k$ is naturally equivalent to k-mod.

Proof. Let M be a kG-bimodule, then G acts on M_e by conjugation and hence M_e is a kG-module. Conversely if N is a kG-module then we define $\tilde{N} = N \otimes_k kG$ where kG acts diagonally at the left and on kG at the right. It is clear that $(\tilde{N})_e \cong N$, $(\widetilde{M})_e \cong M$. It is clear that $(kG \otimes_k kG)_e = kG$ and $(kG)_e = k$. Hence $(?)_e$ transforms (*) into a projective resolution of k. Since if (*) splits at a certain stage there always is a graded splitting (as in II.5.1.7.) we have proved that $\mathrm{pd}_{kG\otimes_k kG}(kG) = \mathrm{pd}_{kG}(k) = \mathrm{cd}_k(G)$. It is interesting to note that the projective resolution of k obtained is equivalent to the usual one. □

The inequality in II.5.2.3. may be strict.

Consider $R = k[X, X^{-1}]\ [Y, Y^{-1}]/\ c(YX = (XY)$ where k is a field of characteristic zero and c is not a root of unity. R is graded by $\mathbb{Z} \times \mathbb{Z}$ $(\deg X^k Y^l = (k, l))$ and $\mathrm{gl.dim} R = 1$, $\mathrm{gr.gl.dim} R = 0$ but $\mathrm{cd}_k(\mathbb{Z} \times \mathbb{Z}) = 2$. This immediately provides a counter example against a conjecture in [47].

II.5.3 Some Morita theory.

In this section we review some classical facts about Morita theory and we also show how these notions extend to the reflexive case. Our main aim will be to prove II.5.3.7. Apart from the fact that this proposition (or rather its reflexive equivalent) will be useful in Chapter V, it shows that the theory for strongly graded rings for finite groups is essentially equivalent to the theory of skew group rings.

II.5.3.1 Definition : Assume that R, S are arbitrary rings. We say that R and S are **Morita equivalent** if there exists bimodules ${}_R M_S$ and ${}_S N_R$ together with bimodule isomorphisms

$$i : {}_R M_S \otimes_S {}_S N_R \to R \text{ and } j : {}_S N_R \otimes_R {}_R M_S \to S$$

satisfying the following associativity conditions : for all $a, a' \in M$, $b, b' \in N$: $i(a \otimes b)a' = aj(b \otimes a')$ and $a'i(b \otimes a) = j(a' \otimes b)a$ We say that (M, N, i, j) define a **Morita equivalence** between R and S.

II.5.3.2 Remark : The associativity condition is less important since, if one has isomorphisms i and j, one can always choose j in such a way that the associativity conditions are satisfied.

II.5.3.3 Proposition : Assume R, S, M, N, i, j as above. Then the

following are true.

(1) $M \otimes_S ?$ and $N \otimes_R ?$ induce inverse equivalences between $S - mod$ and $R - mod$.
(2) $? \otimes_R M$ and $? \otimes_S N$ induce inverse equivalences between $mod - R$ and $mod - S$.
(3) $M \otimes_S ? \otimes_S N$ and $N \otimes_R ? \otimes_R M$ induce inverse equivalences between $R - bimod$ and $S - bimod$.
(4) The equivalences introduced in (1) (2) and (3) are compatible with the various possible tensor products, e.g. $mod{-}R \times R{-}mod \to Ab$, $R - bimod \times R - mod \to R - mod$, etc...
(5) $S \cong End_R(N_R)$, $R \cong End_S(M_S)$ as rings.
(6) ${}_R N_S \cong Hom_R({}_S M_R, {}_R R_R)$
(7) N is a finitely generated right R progenerator.
(8) $Z(R) \cong Z(S)$ canonically.

Proof : Easy and classical. (For (8) one uses the fact that $Z(R) = Hom_{R-bimod}(R)$, $Z(S) = Hom_{S-bimod}(S)$.) □

Properties (5),(6) and (7) give us a clue how a Morita equivalence looks in general. This is strengthened in the following Theorem :

II.5.3.4 Theorem :

(1) Let R be a ring and N a finitely generated right R progenerator. If $S = End_R(N_R)$ then N is a $S{-}R$ bimodule. Let $N^* = Hom_R(N_R, R_R)$. This is canonically an R-S bimodule. Let $j : N \otimes_R N^* \to S : j(n \otimes \phi)(m) = n\phi(m)$ and $i : N^* \otimes_S N \to R : i(\phi \otimes n) = \phi(n)$ be the canonical maps. Then (N^*, N, i, j) defines a Morita equivalence between R and S.

(2) For any equivalence $F : R - mod \to S - mod$ there are (M, N, i, j) as in Def. II.5.3.1 such that F is naturally isomorphic to $M \otimes ?$. □

Proof : Classical.

If I is an invertible $S - S$ bimodule then we have seen that I induces an automorphism σ_I on $Z(S)$ (II 2.2.1). This automorphism has the property that $ic = \sigma_I(c)i$ for all $c \in Z(S)$, $i \in I$. We will need the fact that σ_I is invariant under Morita equivalence in Chapter V.

II.5.3.5 Lemma : Let R and S be Morita equivalent and assume that I is an invertible $S-S$ bimodule. Let I' be the corresponding $R-R$ bimodule. Then the automorphisms σ_I, $\sigma_{I'}$ induced on $Z(S)$ and $Z(R)$ are the same under the canonical isomorphism between $Z(R)$ and $Z(S)$.

Proof : Easy. □

Now we turn to automorphisms.
Let M, N, i, j, R, S be as in Definition II.5.3.1.

II.5.3.6 Proposition : Assume that $\alpha : S \to S$ is a ring automorphism. Then there is a couple (α', I_α) unique up to unique isomorphism where I_α is an invertible R-R bimodule and α' is a right R-module isomorphism $N \to N \otimes_R I_\alpha$ such that α^{-1} is given by the composition

$$S \cong End_R(N_R) \xrightarrow{?\otimes id} End_R(N_R \otimes T_\alpha) \xrightarrow{\alpha'^{-1} \circ ? \circ \alpha'} End_R(N_R) \cong S \qquad (*)$$

Proof : ${}_1S_\alpha$ is an invertible S-S bimodule. Hence under the equivalence defined in Prop. II.5.3.3 it will correspond to an invertible R-R bimodule, say I_α. Furthermore the isomorphism of right S modules $S_1 \xrightarrow{\alpha} S_\alpha \cong S \otimes_S {}_1S_\alpha$ corresponds to an isomorphism of right R-modules $N \xrightarrow{\alpha'} N \otimes_R I_\alpha$.
One easily verifies that the composition

$$S \cong End_S(S_S) \xrightarrow{?\otimes id} End_S(S \otimes_S {}_1S_\alpha) \longrightarrow \alpha^{-1} \circ ? \circ \alpha End_S(S_S) \cong S$$

is α^{-1}. Using Morita equivalence this composition translates into (*).

If $\alpha, \beta : S \to S$ are two ring automorphisms and if (α', I_α), (β', I_β) are the corresponding pairs then there is a unique isomorphism $f_{\alpha,\beta} : I_\alpha \otimes I_\beta \to I_{\alpha\beta}$ such that the following diagram is commutative.

$$\begin{array}{ccc} N & \xrightarrow{\beta'} & N \otimes_R I_\beta \\ \downarrow{\scriptstyle (\alpha\beta)'} & & \downarrow{\scriptstyle \alpha' \otimes 1} \\ N \otimes_R I_{\alpha\beta} & \xleftarrow{1 \otimes f_{\alpha,\beta}} & N \otimes_R I_\alpha \otimes_R I_\beta \end{array}$$

The maps $f_{\alpha,\beta}$ must necessarily satisfy the cocycle condition given by the commutativity of the following diagram :

$$\begin{array}{ccc} I_\alpha \otimes I_\beta \otimes I_\gamma & \xrightarrow{f_{\alpha\beta}\otimes 1} & I_{\alpha\beta} \otimes I_\gamma \\ \downarrow{\scriptstyle 1\otimes f_{\beta,\gamma}\otimes 1} & & \downarrow{\scriptstyle f_{\alpha\beta,\gamma}} \\ I_\alpha \otimes I_{\beta\gamma} & \xrightarrow{f_{\alpha,\beta\gamma}} & I_{\alpha\beta\gamma} \end{array}$$

□

Morita theory has of course a graded equivalent. Let R, S be rings graded by a not necessarily finite group G. Then a graded Morita equivalence is a quadruple $({}_RM_S, {}_SN_R, i, j)$ as before but this time we require M, N to be graded bimodules and i, j should be graded maps. One can easily verify that Prop. II.5.3.3 and Theorem II.5.3.4 have graded equivalents. In particular the equivalences defined in Prop. II.5.3.3 respect the shift functor. Furthermore any category equivalence between $R-gr$ and $S-gr$ respecting the shift functor is obtained from a graded Morita equivalence.

Now we come to our main result :

II.5.3.7 Proposition :
(1) Let R be an arbitrary ring and let N be some finitely generated right R progenerator such that a finite group G acts on $S = End_R(N)$. Let $S' = S * G$ be G-graded in the usual way. Then S' is graded Morita equivalent to a strongly graded ring R' with $R_e = R$.
(2) If R' is strongly graded with $R'_e = R$ then there is a finitely generated right R-progenerator N and an action of G on $S = End_R(N_R)$ such that R' is graded Morita equivalent to $S * G$.

Proof of (1) :
As above, elements σ of G give rise to pairs (σ', I_σ). Define $R' = \oplus_{\sigma \in G} I_\sigma$. Then the maps $f_{\sigma,\tau}$ define a graded ring structure on R. Clearly R' is strongly graded. Now let $N' = N \otimes_R R'$. N' is a left S-module. Let $\sigma \in G$. Then we define a σ-action $\tilde{\sigma} : N \otimes I_\tau \to N \otimes I_{(\sigma\tau)}$ as the composition $N \otimes I_\tau \longrightarrow \sigma' \otimes 1 N \otimes I_\sigma \otimes I_\tau \longrightarrow f_{\sigma\tau} N \otimes I_{(\sigma\tau)}$
The reader may verify the following facts :

(a) G defines a left action on N' i.e. $\tilde{\sigma}(\tilde{\tau}(m)) = \widetilde{\sigma\tau}(m)$

(b) The G-action is compatible with the action of S in the sense that the following diagram is commutative for $f \in S = End_R(N_R)$.

$$\begin{array}{ccc} N \otimes I_{\tau^{-1}} & \xrightarrow{\tilde{\sigma}} & N \otimes I_{(\sigma\tau)} \\ \Big\downarrow{\scriptstyle f\otimes 1} & & \Big\downarrow{\scriptstyle \sigma^{-1}(f)\otimes 1} \\ N \otimes I_{\tau} & \xrightarrow{\tilde{\sigma}} & N \otimes I_{(\sigma\tau)} \end{array}$$

Hence N' is a left $S' = S * G$ module. From the category equivalence of $mod{-}R_e$ and $gr{-}R'$ it follows that N' is a right R' graded progenerator. Left multiplication induces a map $i : S' \to End_{R'}(N_{R'})$. This map is an isomorphism in degree zero and since it is for example a map as left S' module (S' strongly graded) we see that i must be an isomorphism.

Proof of (2)

Let the $R - R$ bimodule I_σ be defined as R'_σ and let the maps $f_{\sigma\tau}$: $I_\sigma \otimes I_\tau \to I_{\sigma\tau}$ be defined by the multiplication in R'. If we try to invert the proof of (1) we see that we need a finitely generated right R-progenerator N_R and suitable maps

$$N \longrightarrow \sigma' N \otimes I_\sigma$$

such that

$$\begin{array}{ccc} N & \xrightarrow{\tau'} & N \otimes I_{\tau} \\ \Big\downarrow{\scriptstyle (\sigma\tau)'} & & \Big\downarrow{\scriptstyle \sigma'\otimes 1} \\ N \otimes I_{\sigma\tau} & \xleftarrow{1\otimes f_{\sigma\tau}} & N \otimes I_{\sigma} \otimes I_{\tau} \end{array}$$

is commutative.

Now it is easily verified that it suffices to take $N = R'$ and $\sigma' = \oplus_{\tau\in G} f_{\sigma\tau}^{-1}$. □

Now suppose that $i : R \to S$ is an arbitrary extension of rings and assume that S is a right R progenerator. Let $R_1 = End_R(S_R)$ and let $j : S \to R_1$ be the canonical morphism by viewing S as a left S-module. From the above we know that R_1 is Morita equivalent with R. Now let M be an R-module, then one can extend it to $M_1 = S \otimes_R M$. On the other hand, under the Morita equivalence $R - mod \to R_1 - mod$, M corresponds to $M_2 = S \otimes_R M$. It is easy to see that M_2 viewed as an

S-module via j is isomorphic to M_1. Hence extension of scalars via i corresponds to restriction of scalars via j.

Similarly if N is a left S-module then we can view it as an R-module via i. Via the Morita equivalence $R - mod \to R_1 - mod$ this corresponds to the R_1-module $N_1 = S \otimes_R N$. On the other hand let $N_2 = R_1 \otimes_S N$. It would be nice if N_1 and N_2 were functorially isomorphic because this would mean that restriction of scalars for i would correspond to extension of scalars for j.

To construct a canonical isomorphism between N_1 and N_2 it is clearly sufficient to give an $R_1 - S$-bimodule isomorphism between $S \otimes_R S$ and R_1 since $N_1 = S \otimes_R S \otimes_S N$

Of course such an isomorphism does not exist in general but it does exist for example in the case that S/R is Galois for a finite group G. It is easily verified that in that case the map $S \otimes_R S \to End_R(S) : a \otimes b \to aTr(b-)$ has the required properties.

Let us summarize what we have shown:

II.5.3.8 Proposition : With notation as above we obtain :

(1) The composition of the Morita equivalence $R - mod \to R_1 - mod$ and the restriction functor $R_1 - mod \to S - mod$ is naturally isomorphic to the extension of scalars functor $R - mod \to S - mod$.

(2) Suppose that S/R is Galois for a finite group G. The restriction of scalars functor $S - mod \to R - mod$ is naturally isomorphic with the composition of the extension of scalars functor $S - mod \to R_1 - mod$ and the Morita equivalence $R_1 - mod \to R - mod$.

As usual all the above definitions and theorems have reflexive interpretations. These will be mainly useful in Chapter V. We will just state the definitions and leave the task of making the other obvious generalisations to the reader. We revert to the situation introduced just after II.5.1.7.

II.5.3.9 Definition : Assume that Γ and Λ are R-orders. We say that Γ and Λ are **reflexive Morita equivalent** if there exists R-commuting reflexive bimodules ${}_\Lambda M_\Gamma$ and ${}_\Gamma N_\Lambda$ together with bimodule isomorphisms $i : {}_\Lambda M_\Gamma \otimes'_\Gamma {}_\Gamma N_\Lambda \to \Gamma$ and $j : {}_\Gamma N_\Lambda \otimes'_\Lambda {}_\Lambda M_\Gamma \to \Gamma$ satisfying the same associativity conditions as in II.5.3.1. We say that the quadruple (M, N, i, j) defines a **reflexive Morita equivalence.**

The role of finitely generated progenerators in classical Morita theory is played by reflexive generators in reflexive Morita theory (See II.5.1).

III. Artihmetically Graded Rings over Orders

III.1. : Orders Graded by a Torsion Free Abelian Group.

Throughout this section, Λ will be a prime p.i. ring, graded by a torsion free Abelian group G. We want to prove under mild conditions on the grading (in particular, Λ has to be divisorially graded) that arithmetical properties of Λ_e, e being the neutral element of G, extend to Λ. Further, we include counterexamples in order to show that the converse implications do not hold.

Let us first consider the case that Λ_e is a maximal order, i.e. for every two-sided Λ_e-ideal I we have $(I :_l I) = (I :_r I) = \Lambda_e$. Similarly, we may define a gr-maximal order Γ to be the a G-graded ring with graded ring of quotients $Q^g(\Gamma)$ such that for every homogeneous ideal I of Γ we have :

$$\begin{cases} (I :_l I)^g = \{x \in Q^g(\Gamma) : xI \subset I\} = \Gamma \\ (I :_r I)^g = \{x \in Q^g(\Gamma) : Ix \subset I\} = \Gamma \end{cases}$$

In II.4. we have seen that gr-tame implies tame. We will now show that a similar result holds for gr-maximal orders.

III.1.1. Lemma. If Λ is a G-graded prime p.i. ring, where G is a torsion free Abelian group, then Λ is a gr-maximal order if and only if Λ is a maximal order.

Proof. Let Σ be the full ring of quotients of Λ and Σ^g the G-graded ring of quotients. In III.3. we will see that Σ^g is an Azumaya algebra over K^g where K^g is the graded field of fractions of the center of Λ.

Further, the results of I.1. entail that K^g is completely integrally closed. Combining all these facts it is easy to show that Σ^g is a maximal order in Σ.

Clearly, $\Lambda \subset \Sigma^g$ is a central extension whence two-sided ideals of Λ extend to two-sided ideals of Σ^g. So, let $x \in \Sigma$ and I an ideal of Λ such that $xI \subset I$, then $xI\Sigma^g \subset I\Sigma^g$ whence $x \subset \Sigma^g$ by maximality of Σ^g.

For each $\sigma \in G$, we denote by $C_\sigma(I)$ the set of elements from Λ_σ which appear as a leading coefficient of an element from I (note that this definition makes sense since G is ordered !). Now, decompose x into homogeneous components $x = x_{\sigma_1} + \ldots + x_{\sigma_k}$ where $\sigma_1 < \ldots < \sigma_k$, the $xI \subset I$ entails that $x_{\sigma_k}(\oplus_\sigma C_\sigma(I)) \subset (\oplus_\sigma C_\sigma(I))$. Since $\oplus_\sigma C_\sigma(I)$ is homogeneous and a twosided ideal of Λ (and Λ is gr-maximal) we obtain $x_{\sigma_k} \in \Lambda$. Replacing x by $x - x_{\sigma_k}$ and arguing as before we finally obtain that $x \in \Lambda$, finishing the proof.

The other implication is, of course, trivial. □

Using this result, we will prove :

III.1.2. Theorem. Let Λ be a divisorially G-graded prime p.i. ring, where G is a torsion-free Abelian group. If Λ_e is a maximal order in its ring of quotients $Q(\Lambda_e)$, then Λ is a maximal order.

Proof. In view of the foregoing lemma, it suffices to show that Λ is a gr-maximal order. So, let $I = \oplus_\sigma I_\sigma$ be an homogeneous twosided ideal of Λ and $x \in \Sigma_\tau$ such that $xI \subset I$. This inclusion entails $xI^{**} \subset I^{**}$, so we may assume that I is divisorial, hence that it is "generated" by I_e. Similarly, $(I :_e I)$ is divisorial and homogeneous, whence "generated" by $(I :_e I)_e = (I_e :_e I_e) = \Lambda_e$ finishing the proof. □

We know that a maximal order Λ is a maximal order over a Krull domain provided Λ satisfies the ascending chain condition on divisorially (twosided) ideals. To prove this fact, we need to impose another condition on our grading group G. This is already clear from the fact that a G-graded field is a Krull domain if and only if G is torsion free Abelian satisfying the ascending chain condition on cyclic subgroups. We will now show that this condition is sufficient :

III.1.3. Theorem. Let Λ be a divisorially G-graded prime p.i.-ring, where G is a torsion Abelian group satisfying the ascending chain condition on cyclic subgroups. If Λ_e is a maximal order over a Krull domain, then Λ is a maximal order over a Krull domain.

Proof. The fact that Λ is a maximal order follows from Theorem III.1.2. We have to verify the ascending chain condition on divisorial ideals. Therefore, let $I_1 \subset I_2 c \ldots$ be an ascending sequence of such ideals. Since Σ^g is an Azumaya algebra over a unique factorization domain it follows that $(\Sigma^g I_1)^{**} \subset (\Sigma^g I_2)^{**} c, \ldots$ becomes constant, i.e. $(\Sigma^g I_N)^{**} = (\Sigma^g I_m)^{**}$ for all $m \geq N$. Further, one has an ascending sequence of divisorial Λ_e- ideals : $C_e(I_1) \subset C_e(I_2) C \ldots$ which terminates by our assumption, i.e. $C_e(I_M) = C_e(I_n)$ for all $n \geq M$. Let $\alpha = \sup\{M, N\}$. We now claim that $I_\alpha = I_n$ for all $n \geq \alpha$. For take $J = (I_n : I_\alpha)$ and $j \in J$, then $j.I_n j \subset I_\alpha$ entails $(\Sigma^g I_n)^{**} j \subset (\Sigma^g I_\alpha)^{**} = (\Sigma^g I_n)^{**}$ whence $j \in \Sigma^g$. So, decompose j into homogeneous components $j = j_{\sigma_1} + \ldots + j_{\sigma_k}$ where $\sigma_1 < \ldots < \sigma_k$. Then, $C_e(I_n).j_{\sigma_k} \subset C_{\sigma_k}(I_\alpha)$. Further, using the fact that Λ is divisorially graded, it is easy to verify that : $C_\tau(I_j) = (\Lambda_\tau C_e(I_j))^{**} = (C_e(I_j)\Lambda_\tau^{**}$ This entails that $C_e(I_n).j_{\sigma_k} \subset (\Lambda_{\sigma_k}.C_e(I_\alpha))^{**} = (\Lambda_{\sigma_k}.C_e(I_n))^{**}$ whence $C_e(I_n)^{-1} * C_e(I_n).j_{\sigma_k} \subset \Lambda_{\sigma_k}.C_e(I_n) * C_e(I_n)^{-1}$ or, $\Lambda_e.j_{\sigma_k} \subset \Lambda_{\sigma_k}$ So, $j_{\sigma_k} \in \Lambda_{\sigma_k}$ and replacing j by $j - j_{\sigma_k}$ one can continue in this way and obtain finally, $j \in \Lambda$. Therefore: $\Lambda = (I_n : I_\alpha) = I_n * I_\alpha^{-1}$ whence $I_n = I_\alpha$, finishing the proof. □

However, the inverse implication is far from being true. For example, let R be a discrete valuation ring with uniformizing parameter π and consider

$$\Lambda_e = \begin{pmatrix} R & (\pi) \\ R & R \end{pmatrix}$$

Then, Λ is clearly hereditary but not maximal. Its Jacobson radical is clearly invertible

$$J(\Lambda_e) = \begin{pmatrix} (\pi) & (\pi) \\ R & (\pi) \end{pmatrix} = \Lambda_e. \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} .\Lambda_e = x\Lambda_e$$

Then form the strongly graded ring $\Lambda_e[J(\Lambda_e)]$. This ring is isomorphic to $\Lambda_e[Y, Y^{-1}, \varphi]$ where φ is the automorphism given by conjugation in

Λ_e with $x = \binom{0\pi}{10}$. In the next theorem we will see that this ring is a tame order. It is easy to see that every localization at an height one prime different from $p = Z(\Lambda_e[Y, Y^{-1}, \varphi])\pi$ is an Azumaya algebra and that the localization at p gives an hereditary algebra with a unique maximal ideal generated by Y^{-1} and hence is a maximal order. In conclusion $\Lambda_e[Y.Y^{-1}, \varphi]$ is a strongly graded maximal order whose part of degree e, Λ_e is not maximal.

Theoren III.1.3. has an immediate consequence for gradation by a polycyclic infinite (but not necessarily abelian) group.

III.1.4. Corollary. Let Λ be strongly graded by a poly-infinite-cyclic group G. If Λ_e is a maximal order (over a Krull domain) then Λ is a maximal order (over a Krull domain).

Proof. There is a finite series $\{e\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$ of subgroups of G such that each G_{i-1} is normal in G_i and $G_i/G_{i-1} \cong \mathbb{Z}$. If $n = 0$ then the statement holds and we can apply induction on n. Put $H = G_{n-1}$, then Λ is strongly graded by $G/H \cong \mathbb{Z}$ over $\Lambda^{(H)} = \bigoplus_{\sigma \in H} A_\sigma$. By induction, $\Lambda^{(H)}$ is a maximal order, whence by Theorem III.1.3., Λ is a maximal order, too (the same for maximal orders over Krull domains). □

Now, we will consider the similar problem for tame orders.

III.1.5. Theorem. Let Λ be divisorially G-graded, where G is a torsion free Abelian group satisfying the ascending chain condition on cyclic subgroups. If Λ is prime p.i. and if Λ_e is a tame order, then Λ is a tame order.

Proof. First, we claim that $Z(\Lambda_e)$ is an integral extension of $Z(\Lambda)_e$. For, take the graded localization Σ^g of Λ, then $Z(\Sigma^g_e)$ is the field of fractions of $Z(\Lambda_e)$ and $Z(\Sigma^g)_e$ that of $Z(\Lambda)_e$. We can use the proof of Theorem II.3.2. to show that $Z(\Sigma^g_e)$ is finite dimensional over $Z(\Sigma^g)_e$. Moreover, this is a Galois extension and $Z(\Lambda)_e = Z(\Lambda_e) \cap Z(\Sigma^g)_e$, so this finishes the proof of our claim.

In view of Theorem II.3.21. we merely have to verify that Λ is gr-tame. Let q be an height one prime of $Z(\Lambda)_e$, then by integrality $Z(\Lambda_e)_q$

has only height one primes and is therefore a Dedekind domain. So, $(\Lambda_e)_q$ is a hereditary order and Λ_q is strongly graded. But then Λ_q is gr-hereditary whence tame and $Z(\Lambda_q)$ is gr-Dedekind whence Krull in view of the results of I.1. Therefore,

$$(*) \qquad \Lambda_q = \cap\{\Lambda_p; p \in X^{(1)}(Z(\Lambda)) : p \cap Z(\Lambda)_e = q\}$$

We will first show that Λ is a reflexive $Z(\Lambda)$-module.
Since :

$$\Lambda \subset \cap\{\Lambda_q; q \in X^{(1)}(Z(\Lambda)_e\},$$

and both rings are divisorially graded with the same part of degree e, so they are equal. The reflexivity property now follows from (*). But then it also follows that $Z(\Lambda)$ is a Krull domain since $Z(\Lambda) = \cap\{Z(\Lambda_q); q \in X^{(1)}(Z(\Lambda)_e)\}$ and all $Z(\Lambda_q)$ are Krull domains.
We are left to prove that graded localizations at graded height one prime ideals of $Z(\Lambda)$ are gr-hereditary. Take such a prime p and consider $p_e \cap Z(\Lambda)_e = q$. But Λ_{p_e} is gr-hereditary by the argument above and Λ_p is a graded localization of Λ_{p_e}, so, Λ_p is gr-hereditary too, finishing the proof. □

III.2. Orders Graded by a Finite Group.

In this section we aim to prove similar results as in III.1. when the grading group is supposed to be finite. Let us first fix notations. Λ will be a prime p.i.-ring, divisorially graded by a finite group G and suppose that Λ_e is also prime. Let A be the center of Λ_e and L its fields of fractions. The classical ring of fractions Σ_e of Λ_e is a central simple L-algebra. If we invert all central non-zero elements of Λ_e we obtain a strongly G-graded ring Σ^g s.t. $\Sigma^g_e = \Sigma_e$. Since Σ^g_e is a finitely generated Σ_e-module, it is Artinian, hence Σ^g is the classical ring of quotients of Λ.

As explained before, we have an action of G on A and this action extends to an action on L defined by the grouphomomorphism $G \to \mathrm{Pic}(\Sigma_e) \to \mathrm{Aut}(L)$. Clearly, L is a finite Galois extension of L^G and so A is an integral extension of A^G.

III.2.1. Theorem. Let Λ be divisorially graded by a finite group G such that $\mathrm{tr}(a_0) = 1$ for some $a_0 \in A$. Assume that Λ is a prime p.i. ring whose part of degree e is a tame A-order. Then $C = Z(\Lambda)$ is a Krull domain and Λ is a tame C-order.

Proof. Since Λ_e is an A-order, Λ_e is integral over A which is integral over A^σ, so Λ_e is integral over A^G. If $r \in \Lambda_\sigma$ then $r^n \in \Lambda_e$ for $n = |G|$ and so r is integral over A^G. Now, let $0 \neq r = r_{\sigma_1} + \ldots + r_{\sigma_k}$ be the homogeneous decomposition of r in Λ. Let $T = A^G\{r_{\sigma_1}, \ldots, r_{\sigma_k}\}$ be the A^G-subalgebra of Λ generated by $\{r_{\sigma_1}, \ldots, r_{\sigma_k}\}$. From [42] p. 152

it follows that T is a finitely generated A^G-module, whence integral. This entails that r is integral over A^G.

The extension $A^G \subset A$ satisfies the PDE condition and for every $q \in X^{(1)}(A^G)$ there is a $p \in X^{(1)}(A)$ lying over q. Now, let $q \in X^{(1)}(A^G)$, then A_q is the integral closure of $(A^G)_q$ in L. Hence A_q is a Dedekind domain and even a principal ideal domain (only a finite number of maximal ideals). Clearly $(\Lambda_e)_q = \cap\{(\Lambda_e)_p; p \in X^{(1)}(A)$ lying over $q\}$ and since Λ_e is a divisorial A-module we have $\Lambda_e = \cap\{(\Lambda_e)_q; q \in X^{(1)}(A^G)\}$.

Further, for each $q \in X^{(1)}(A^G)$, Λ_q is strongly graded and $(\Lambda_q)_e = (\Lambda_e)_q$ is hereditary, so by the results from II.5., Λ_q is hereditary, too. (Here, we use $tr(ao) = 1$!).

Clearly, $\Lambda = \cap\{\Lambda_q; q \in X^{(1)}(A^G)$ since both sides are G- divisorially graded with the same part of degree e. It remains to show that Λ is a tame C-order. Since $A^G \subset C$ is integral, it satisfies lying over and the PDE-condition. Since Λ is a reflexive A^G-module, it is reflexive as a C-module. Finally, since Λ_q is hereditary for $q \in X^{(1)}(A^G)$ and Λ_p for $p \in X^{(1)}(C)$ is a localization of some Λ_q, Λ_p is hereditary, too. Finally, C is a Krull domain since $C = \cap\{C_q; q \in X^{(1)}(A^G)\}$ which follows from $\Lambda = \cap\{\Lambda_q : q \in X^{(1)}(A^G)\}$. Each C_q is a Dedekind domain, being the integral closure in a finite field extension of the discrete valuation ring. A_1. □

The corresponding situation for maximal orders is more complicated. This is clear from the Auslander-Goldman-Rim result (which we will generalize in the next theorem) which states that the skew group ring $S \circ G$ is a maximal R-order (R and S Dedekind domains) if and only if S/R is unramified, that is if and only if the discriminant ideal $d(S/R)$ is equal to R.

III.2.2. Theorem. Let Λ be divisorially graded by a finite group G such that $\mathrm{tr}(a_0) = 1$ for some element $a_0 \in A$. Assume that Λ is a prime p.i. ring whose part of degree e is a maximal order over the Krull domain A. Suppose further that for every height one prime ideal q of A^G we have that $(q\Lambda_e)^{**} = (P_1 \ldots P_t)^{**}$ where the P_i are distinct height one prime ideals of Λ_e, then Λ is a maximal order over the Krull domain C.

Proof. In Theorem III.2.1. we have proved that C is a Krull domain and that Λ is a tame order. Therefore, it suffices that the localizations $\Lambda_q, q \in X^{(1)}(A^G)$ are maximal orders. So, we aim to prove that every twosided $\Gamma = \Lambda_q$-ideal is invertible. So, consider a maximal Γ-ideal M, then $M \cap (A^G)_q = q(A^G)_q$ and $q(A^G)_q.\Gamma_e = P_1 \ldots P_e = P_1 \cap \ldots \cap P_t$ where the P_i are distinct maximal ideals of Γ_e. Now, $\Gamma/(M \cap A_q^G)$. Γ is a strongly graded ring, hence by a result of Cohen and Montgomery [CM], it follows that $(M \cap A_q^G)\Gamma$ is a semiprime ideal of Γ, but then $(M \cap A_q^G).\Gamma = M \cap J$ where J is an ideal of Γ not contained in M and it is easily seen that $(M \cap A_q^G).\Gamma = M.J = J.M$. The inverse of $(M \cap A_q^G).\Gamma_e$ in Σ_e^g is then used to construct an inverse for $(M \cap A_q^G).\Gamma$ in Σ^g. Thus, $JM = MJ$ has an inverse in Σ^g and this entails that M is invertible in Σ^g. It is well known that an hereditary order such that every maximal ideal is invertible is maximal, finishing the proof. □

III.3. Generalized Crossed Product Azumaya Algebras.

In Section II.4. we have seen that it is possible to construct the ramification Rees ring over a maximal order (which is not an Azumaya algebra) to obtain a reflexive Azumaya algebra or sometimes even an Azumaya algebra. In the case of group rings, such phenomena do not exist, in fact one may prove the following result :

III.3.1. Proposition. Let R be a ring, G a group. The groupring RG is an Azumaya algebra if and only if :

1. R is an Azumaya algebra.
2. $|G'|$ is finite and $|G'|^{-1} \in R$.
3. $[G : Z(G)] < \infty$.

Proof. cf. F. De Meyer, G. Janusz [18], Theorem 1. □

Taking into account the phenomena described above it is clear that the determination by strongly graded Azumaya algebras of properties of the group and of the part of degree e cannot be as nice as in the case of group rings. In fact the natural question to ask is the following : if A is strongly graded by G over A_e such that A_e is an Azumaya algebra, what conditions on G make A into an Azumaya algebra ? We will deal with this problem in the quasi-inner case. A gradation of type G on the A is said to be **quasi-inner** if $Z(A_e) \subset Z(A)$ i.e. if A is an A_e-bimodule over $Z(A_e)$. If A is morever strongly graded then the gradation is quasi-inner exactly when the canonical morphism

$$G \underset{\Phi}{\longrightarrow} \mathrm{Pic}(A_e) \longrightarrow \mathrm{Aut}(Z(A_e))$$

is trivial.

III.3.2. Theorem. Let A be a strongly graded ring of type G such that A_e is an Azumaya algebra. Assume that $|G|^{-1} \in A_e$ (i.e. G is finite) and that the gradation is quasi-inner. Then A is an Azumaya algebra.

Proof. Since A is an A_e-bimodule centralizing the $Z(A_e)$-action it follows that $A = A_e \underset{Z(A_e)}{\otimes} C_A(A_e)$, where $C_A(A_e) = \{a \in A, ab = ba$ for all $b \in A_e\}$. It is obvious that $C_A(A_e)$ is a graded ring of type G and from $A_\sigma \underset{A_e}{\otimes} A_{\sigma^{-1}} \cong A_e$ it follows that $A_\sigma^{(A_e)} \underset{Z(A_e)}{\otimes} A_{\sigma^{-1}}^{(A_e)} \cong Z(A_e)$, where the functor $(-)^{(A_e)} : A_e\text{-}A_e\text{-mod} \to Z(A_e)\text{-mod}$ associates to an A_e-bimodule M over $Z(A_e)$ the $Z(A_e)$-module $M^{(A_e)} = \{m \in M, am = ma$ for all $a \in A_e\}$, because the functor $(-)^{(A_e)}$ defines an equivalence of categories. Consequently, $C_A(A_e)$ is strongly graded by G over $Z(A_e)$, $C_A(A_e) = \oplus_{\sigma \in G} A_\sigma^{(A_e)}$. If $C_A(A_e)$ is separable over $Z(A_e)$ then it will also follow that A is separable over $Z(A_e)$ hence over $Z(A)$ and then A will be an Azumaya algebra.
Consider a maximal ideal m in $Z(A_e)$ and look at the strongly graded ring $B = C_A(A_e)/mC_A(A_e)$ over $Z(A_e)/m$. The field $Z(A_e)/m$ is central in B, so $B \cong (Z(A_e)/m)G^t$ is a twisted group ring with respect to some $t \in H^2(G, Z(A_e)/m)$. Because $|G|^{-1} \in Z(A_e)/m$ we may conclude that B is separable over $Z(A_e)/m$. The local-global property for separability, cf. F. De Meyer, E. Ingraham [17], entails that $C_A(A_e)$ is separable over $Z(A_e)$. □
The next theorem extends Corollary II.3.6.

II.3.3. Theorem. Let A be a P.I. ring strongly graded by G such that A_e is semiprime. Assume that G' is finite and that each torsion element of G has an exponent not divisible by char (A_e). If A is gr-semisimple gr-Artinian then A is an Azumaya algebra.

Proof. Let S_e be the set of regular elements in $Z(A_e)$. Since A_e is semiprime and a P.I. ring it is a semiprime Goldie ring. By Proposition

II.1.4., S_e is a left Ore set in A and as such it must be invertible in A. Obviously an $s \in S_e$ has inverse s^{-1} in A_e, hence $S_e^{-1}A^{(G')} = A^{(G')}$ is a gr-semisimple gr-Artinian ring over the semisimple Artinian A_e, where $A^{(G')} = \oplus_{\sigma \in G'} A_\sigma$. Since $|G'|^{-1} \in A_e$ it follows that $A^{(G')}$ is a semisimple Artinian ring. Indeed, that $A^{(G')}$ is Artinian follows from the fact that each $A_\sigma, \sigma \in G$ is a finitely generated projective A_e-module (left and right), hence $A^{(G')}$ is finite dimensional over $Z(A_e)$; from $J(A^{(G')}) = J^g(A^{(G')})$ it follows that $J(A^{(G')}) = 0$ because the Jacobson radical is generated by $J^g(A^{(G')})_e = J(A_e) = 0$. Now, A is strongly graded by G/G' over $A^{(G')} = A_{\overline{\sigma}}, \overline{\sigma} \in G/G'$ being the zero element. Let $S_{\overline{\sigma}}$ be the set of regular elements in $A_{\overline{\sigma}}$, then $S_{\overline{\sigma}}^{-1}A = A$ because $S_{\overline{\sigma}}A = A$ because $S_{\overline{\sigma}}$ is invertible in $A_{\overline{\sigma}}$. Again from Proposition II.1.4. it follows that A is gr-semisimple gr-Artinian as a G/G'-graded ring too ! So we may write $A = A_1 \oplus \ldots \oplus A_r$ where each A_i is gr-simple gr-Artinian in the G/G'-gradation. The fact that G/G' is abelian entails that each $Z(A_i)$ is a G/G'-graded field.
Since G' is finite we have that $G' \subset G_t$ where G_t is the torsion subgroup of G. It is clear that $Z(A_i)^{(G_t)}$ is semiprime and then so is $Z(A_i)$ because G/G_t is torsionfree abelian. Therefore A_i is semiprime and so we may pick a nontrivial central polynomial and proceed as in Corollary II.3.6. in order to deduce that each A_i is an Azumaya algebra. □

III.3.4. Corollary. The foregoing applies to twisted group rings and group rings. If G is a group such that $[G : Z(G)] < \infty$, $|G'|^{-1} \in k$ and kG^t is the twisted group ring with respect to the twocycle $t : G \times G \to k^*$ then, if kG^t satisfies polynomial identities it follows that $[Z(G) : Z(G)_{\mathrm{reg}}] < \infty$ where $Z(G)_{\mathrm{reg}}$ is the central subgroup of t-regular elements α in $Z(G)$, where α is t-regular if $t(\alpha, \beta) = t(\beta, \alpha)$ for all $\beta \in Z(G)$. Therefore $kZ(G)^t_{\mathrm{reg}}$ is in the centre of kG^t and kG^t is finitely generated over it.

In the sequel we will just write G_c for G_{reg} with respect to the cocycle c. When considering a quasi-inner graded ring, there are two ways to make the cocycle describing the graded structure visible. One is to look at the generalized crossed product structure i. e. consider the A_e-bimodule isomorphisms, $f_{\sigma,\tau} : A_\sigma \otimes_{A_e} A_\tau \to A_{\sigma\tau}$, determining the factor set $\{f_{\sigma,\tau}, \sigma, \tau \in G\}$ which in turn determines an element of

$H^2(G, U(Z(A_e)))$. The other is to look at the graded ring of fractions $Q^g(A)$ and the subring kG^c where $k = Z(Q_{Cl}(A_e))$ (this is a subring in the quasi-inner situation because the $c(\sigma,\tau)$ will be G-invariant). So, if we refer to "the" cocycle of a quasi-inner graded ring we mean the cocycle (up to equivalence) obtained in the natural way indicated above. With these conventions it is unambiguous to refer to the group G_c in G associated to a quasi-inner gradation of type G (G_c will be the group generated by the c-regular elements in G).

In the sequel of this section we consider quasi-inner G-strongly graded rings A over an Azumaya algebra A_e. The quasi-inner property entails that ideals of A_e extend to ideals of A in this case. In particular $I = A\ rad\ (A_e)$ is an ideal (and a nil ideal contained in rad (A)) and A/I is strongly graded by G over $A_e/\text{rad}\ (\overline{A}_e) = \overline{A}_e$. Since $Z(\overline{A}_e) = \overline{Z(A_e)}$ it is clear that A/I is quasi-inner graded by G too.
If we know already that A is a finite module over its center then A will be an Azumaya algebra if and only if A/I is an Azumaya algebra over $\text{Im}(Z(A)) = Z(A)/I \cap Z(A)$ because for every m maximal in $Z(A)$ we have that $m \supset \text{rad}(Z(A_e))$ hence A/mA is an epimorphic image of A/I and consequently A/mA is an Azumaya algebra with centre $ImZ(A/I)$ (note : $\text{Br}(Z(A)) = \text{Br}(Z(A)/I \cap Z(A))$).
In order to check that A is an Azumaya algebra whenever A_e is one we may reduce the problem to the situation where A_e is commutative because $A = A_e \underset{Z(A_e)}{\otimes} C_A(A_e)$ follows from the quasi-inner property and, just like in the proof of Theorem III.3.2. we may replace A by $C_A(A_e)$ which is then quasi-inner strongly graded $C_A(A_e)_\sigma = (A_\sigma)^{(A_e)}$. However for generality's sake we present the following theorem in its more natural form. □

III.3.5. Proposition. Let A be a semiprime P.I. ring quasi-inner strongly graded by G over A_e. Let c be the two-cocycle describing the gradation of A. The set of c-regular elements in G has finite index and they generate a normal subgroup G_c of G such that $[G : G_c] < \infty$.

Proof. First note that A_e is semiprime to. Indeed, if rad (A_e) were nonzero then $E = \text{rad}(A_e) \cap Z(A_e) \neq 0$ hence from the quasi-inner property it follows that for $x \in E$, Ax is a nil-ideal of A and $Ax \neq 0$ if $x \neq 0$.

Since A is a semiprime P.I. ring there are no such nil-ideals, hence rad $(A_e) = 0$. Put $Q^g = Q^g(A)$, $C = Z(Q^g)$; since there is no action on $Z(A_e)$ we have that $Q^g = KG^c$ where $K = Q(Z(A_e))$ (note : $Q_{cl}(A_e)$ is obtained by inverting the regular elements of $Z(A_e)$). As a K-module, C is freely generated by the ray class sums $\rho_i = \sum_{\overline{g} \in G/C_G(\alpha_i)} u_g u_{\alpha_i} u_g^{-1}$ where α_i is a c-regular element of G. Recall that α_i is i-regular if $c(\alpha_i, \beta) = c(\beta, \alpha_i)$ for all $\beta \in G$ that commute with α_i and that every conjugate of a c-regular element is again c-regular. Note that we assume that c is **normalized**, i.e. $c(\beta, \beta^{-1}) = 1$ for all $\beta \in G$, in writing this formula, indeed if $g' = gz$ with $z \in C_G(\alpha)$ then $u_{g'} u_{\alpha_i} u_{g'}^{-1} = c(z,\alpha)c(z\alpha)c(z\alpha, z^{-1}) u_g u_{\alpha_i} u_g^{-1} = c(z, z^{-1}) u_g u_{\alpha_i} u_g^{-1}$ so $c(z, z^{-1}) = 1$ is necessary to make φ_i well defined. However the normalized condition can be avoided either by allowing a free extension of K as in [54] or by using some more complicated calculations and formula. $Q(A)$ may be obtained by inverting central elements in A and $Q(A)$ is finitely generated over $Q(C)$, say by $x_1, \ldots, x_t$, which may be taken in KG^t or even in A. Write : $x_i = \sum k_{ij} g_{ij}$ with $k_{i_j} \in Q(C)$, $g_{i_j} \in G$.

Take $\lambda \in C$ such that $\lambda x_i = \Sigma_j \lambda_{ij} g_{ij}$ with $\lambda, \lambda_{ij} \in C$ and λ regular in C. For $g \in G$ we may write $g = \Sigma a_i x_i$ with $a_i \in Q(C)$, hence $ag = \Sigma_i a'_i x_i$ with $a, a'_i \in C$ and a regular in C. So, $\lambda a g = \Sigma_i \Sigma_j a'_i \lambda_{ij} a_{ij}$ with $\lambda a \neq 0$ in C and $a'_i \lambda_{ij} \in C$. Using the fact that λa and $a'_i \lambda_{ij}$ are K-linear combinations of ray class sums we obtain that $\alpha_i g = \alpha_{ij} a_{ij}$ for certain c- regular elements α_i and α_{ij}. This establishes that the set of c-regular element $\mathcal{R}_c$ has finite index in G (i.e. G is a finite union of cosets of $\mathcal{R}_c$). It is clear that $x \in G_c$ entails $x^{-1} \in \mathcal{R}_c$ and $gxg^{-1} \in \mathcal{R}_c$ for all $g \in G$, hence $\mathcal{R}_c^* = \mathcal{R}_c \cup \{1\} \cup \mathcal{R}_c^{-1} = \mathcal{R}_c$. Lemma 2.3. of [41] p. 182 yields that $R_c^{\prime k}$ is a subgroup of G (where k is the number of g_{ij} in a decomposition $G = \bigcup_{i,j} \mathcal{R}_c g_{ij}$). Since a conjugate of conjugates in $\mathcal{R}_c$ in the obvious way, it is clear that $G_c = \mathcal{R}_c^{4^k}$ is a normal subgroup of finite index. □

III.3.6. Corollary. If c is normalized then we have that $\mathcal{R}_c \cap Z(G)$ equals $G_c \cap Z(G)$. We refer to [54] for the proof.

III.3.7. Theorem. Let A be a P.I. ring quasi-inner strongly graded

over the semiprime ring A_e which is supposed to be a $\mathbb{Q}$-algebra. Then A is an Azumaya algebra if the following conditions hold.
1. A_e is an Azumaya algebra.
2. $|G'| < \infty$.

Proof. Since $|G'| < \infty$ we may consider $G_t = \{g \in G, g^n = e$ for some $n \in \mathbb{N}\}$ as a normal subgroup. Since A is a $\mathbb{Q}$-algebra $A^{(G_t)}$ is easily seen to be semiprime; since G/G_t is torsion-free abelian, rad (A) is G/G_t-graded hence rad $(A) \cap A^{(G_t)}$ is nonzero whenever rad (A) is nonzero. Consequently A is semiprime. Suppose that A satisfies the $n \times n$-identities but not the $n-1 \times n-1$-identities and assume that for every finitely generated subgroup H of G, $A^{(H)}$ does satisfy these identities. If f is an $n-1 \times n-1$ identity not satisfied by A then $f(a_1, \ldots, a_r) \neq 0$ for certain $a_i \in A$. The degrees of the homogeneous components of those a_i generate a finitely subgroup H_f of G such that $A^{(H_f)}$ does **not** satisfy the identities of $n-1 \times n-1$-matrices. We claim that A is an Azumaya algebra if $A^{(H)}$ (write $H_f = H$ again) is an Azumaya algebra. Indeed, let g be a multilinear central polynomial for A then it is also a multilinear central polynomial for $A^{(H)}$ and it does not vanish on the (semiprime !) ring $A^{(H)}$ since the latter does not satisfy the identities of $n-1 \times n-1$-matrices. If $A^{(H)}$ is an Azumaya algebra then the Formanek centre of $A^{(H)}$ equals $Z(A^{(H)})$. By multilinearity it follows that $g(A^{(H)})$ generates the Formanek centre of $A^{(H)}$ additively. Since $g(A) \supset g(A^{(H)})$ it is clear that 1 is in the additive group generated by $g(A)$ in $Z(A)$, hence the Formanek centre of A equals $Z(A)$ or A is an Azumaya algebra. So we reduce the problem to the situation where A_e is commutative and G is finitely generated. Then $G = \Delta(G)$ and hence $Z(G)$ has finite index in G in view of Theorem II.3.2. Put $Z = Z(G)_c = Z(G) \cap G_c$. Then $A^{(Z)}$ is commutative and central in A (we have denoted the algebra obtained after the reductions we described above by A again). Since G/Z is finite and $|G/Z|^{-1} \in A_e$ it follows from Theorem III.3.2. that A is an Azumaya algebra. □

III.3.8. Remark. If k is a field and kG is semiprime then kG is an Azumaya algebra if and only if $[G : Z(G)] < \infty$. Indeed, then kG is a P.I. ring and from $[G : Z(G)] < \infty$ then $|G'| < \infty$ follows. Moreover since kG is semiprime either char $k = 0$ and $k \supset Q$, or G has no finite

normal subgroups H with char $k \| H|$, hence $|G'|^{-1} \in k$ and the theorem may be applied, so the above statement results. □

In the following A is a $\mathbb{Q}$-algebra.

III.3.9. Theorem. Let A be a prime P.I. ring ring which is strongly graded by a group G such that the action of G in $Z(A_e)$ (via the morphism $G \to \mathrm{Pic}(A_e) \to \mathrm{Aut}(Z(A_e))$ makes $Z(A_e)$ into a finitely generated separable extension of $Z(A_e)^G$. If A_e is an Azumaya algebra and $[G : Z(G)] < \infty$, $|G'|^{-1} \in A$, then A is an Azumaya algebra too.

Proof (Sketch). Since $Z(A)$ is a domain, $Z(A)^G$ is a domain and $Z(A_e)$ is a Galois extension of $Z(A_e)^G$ with finite Galois group $G_1 = \mathrm{Im}\varphi$, $\varphi : G \to \mathrm{Aut}(Z(A_e))$ the canonical morphism. Put $H = \mathrm{Ker}\varphi$, $G_1 = G/H$. The ring $C_A(A_e)$ is strongly graded by G over $Z(A_e)$ but in $C_A(A_e)^{(H)}$ the ring $Z(A_e)$ is central. Since $[H : Z(H)] < \infty$, $|H'| < \infty$ and $|H'|^{-1} \in A$ follow from the conditions on G (note $[G : H] < \infty$) then it follows from Theorem III.3.5. that $C_A(A_e)^{(H)}$ is an Azumaya algebra. From $C_{A^{(H)}}(A_e) = (C_A(A_e))^{(H)}$ it is easily derived that $A^{(H)}$ is an Azumaya algebra. Up to another "commutator ring" reduction we reduce the problem to a G_1-strongly graded ring over a commutative ring C such that C is Galois over C^{G_1}. Hence one can pass to quotients modulo maximal ideals of C^{G_1} and then one arrives at the problem of proving that a crossed product $k * G_1$, where k is a Galois ring over the field k^{G_1} is an Azumaya algebra with centre k^{G_1}, but the latter is a classical fact. □

III.4. Extensions of Tame Orders and Related Sequences.

In this section B will be a tame order over a Noetherian integrally closed domain R and A is a ring containing B. It is thus understood that whenever we mention divisorial B-(bi)modules these will be divisorial R-lattices ! A two-sided B-submodule P of A is said to be **divisorial in** A if $P = P^{**}$ where $(-)^* = \mathrm{Hom}_B(-,_B B)$ and there exists a two-sided B-submodule Q of A such that $Q = Q^{**}$ and $(PQ)^{**} = (QP)^{**} = B$. By the first remark such a B-bimodule P will be reflexive as a B-module (left) if and only if it is reflexive as an R-module and then it is finitely generated (and presented) as a B-module and an R-module. The two-sided B -submodules of A which are divisorial in A form a group $D_B(A)$ with respect to the multiplication $P.Q = (PQ)^{**}$.

We write $\mathrm{Aut}_B(A)$ for the group of all B-automorphisms of A. Clearly, $\mathrm{Aut}_B(A)$ acts on $D_B(A)$ by $P \mapsto P^\sigma$ if $\sigma \in \mathrm{Aut}_B(A)$. Taking isomorphism classes of B-bimodules defined a group morphism $D_B(A) \to \mathrm{IIic}(B)$, $P \mapsto [P]$, where $\mathrm{IIic}(B)$ is the **reflexive Picard group** of the tame order B, defined to be the set of isomorphism classes of reflexive B-bimodules M such that there exists a reflexive B-bimodule N such that $(M \underset{B}{\otimes} N)^{**} \cong (N \underset{B}{\otimes} M)^{**} \cong B$ as B-bimodules equipped with the operation induced by $(- \underset{B}{\otimes} -)^{**}$. If A is divisorially graded by G over B, i.e. with $B = A_e$, then each $A_\sigma, \sigma \in G$, is in $D_B(A)$ and the maps $G \to D_B(A)$ and $D_B(A) \to \mathrm{IIic}(B)$ are group morphisms giving rise to the morphism $G \to \mathrm{IIic}(B), \sigma \mapsto [A_\sigma]$. If we

consider the classes of $P \in \mathrm{IIic}(B)$ with the property that the action of R is centralized, i.e. $xp = px$ for all $x \in R$, $p \in P$ then we write $[P] \in \mathrm{IIic}_R(B)$. With the assumptions on B set forth above it is easily verified that $Cl(B) = \mathrm{IIic}_R(B)$. However, since we do not assume that $R \subset Z(A)$, we cannot map $D_B(A)$ to $CCl(B)$. In general we may extend Y. Miyashita's work, cf. [33], to the case where A is a divisorial R-module cf. [50], but here we restrict at once to the **arithmetical situation** i.e. we assume that the A.S. conditions hold :

1. R is integral over $R \cap Z(A) = C$

2. The extension $C \hookrightarrow Z(A)$ satisfies P.D.E.

3. A is a tame order over $Z(A)$, in paricular $Z(A)$ is a Krull domain (which is not necessarily Noetherian in general).

Put $C_B(A) = \{x \in A, xb = bx \text{ for all } b \in B\}$. Take $P \in D_B(A)$. Since $p \in X^1(R)$ is central in B, A_p is a B_p-bimodule and P_p is an invertible B_p-bimodule. From a decomposition of 1 in $(P_p)(P_p)^{-1} = B_p$, say $1 = \sum_i u_i v_i$ with $u_i \in P_p$, $P_p, v_i \in (P_p)^{-1}$ we may derive an explicit form of the automorphsim $\sigma_{P_p} \in \mathrm{Aut}_{Z(A_p)}(C_{B_p}(A_p))$ defined by $P_p y = \sigma_{P_p}(y)P_p$ (elementwise) for all $y \in C_{B_p}(A_p)$, i.e. $\sigma_{P_p}(y) = \sum_i u_i y v_i$. Since $A \subset A_p$ for all $p \in X^1(R)$ it follows that $C_B(A) \subset C_{B_p}(A_p)$ (note that A_p need not be a ring a priori). Consequently, for all $a \in C_B(A)$, $x \in P$ we have $xa = \sigma_{P_p}(a)x$ because $P \subset P_p$ as $P \in D_B(A)$. From $C_B(A) \subset C_{B_p}(A_p)$ it follows that $C_{B_p}(A_p) \cong (C_B(A))_p$ hence, for all $x \in P$ we obtain $xa = \sigma_{P_p}(a)x = \sigma_{P_q}(a)x$ for all $q \in X^1(R)$. Consequently $(\sigma_{P_q}(a) - \sigma_{P_p}(a))P = 0$, but then, $(\sigma_{P_p}(a) - \sigma_{P_p}(a))PQ = 0$ for some Q such that $(PQ)^{**} = B$ yields that $\sigma_{P_q}(a) = \sigma_{P_p}(a)$ for all $q \in X^1(R)$ (for clarity's sake all localizations appearing here are to be interpreted in $K \underset{R}{\otimes} A$ where $K = Q(R)$). We established that $\sigma_{P_p}(a) \in \cap\{(C_B(A))_q, q \in X^1(R)\} \subset A$ and thus $\sigma_{P_p}(a) \in C_B(A)$. In this way we obtained a uniquely determined $\sigma(P) \in \mathrm{Aut}_{Z(A)}(C_B(A))$ corresponding to P such that $xa = \sigma(P)(a)x$ holds for all $a \in C_B(A), x \in P$. It is easily verified that we may replace A by any A-bimodule M which is R-compatible. So we obtain :

III.4.1. Proposition. Let $P \in D_B(A)$ and let M be an A-bimodule which is R-compatible (i.e. $rm = mr$ for all $r \in R$, $m \in M$), and divisorial as an R-module. To P we associate $\sigma(P)$ in $\mathrm{Aut}_{Z(A)}(C_B(A))$ which may be induced by choosing any decomposition $1 = \sum_i u_i v_i$ in $P_p Q_p = B_p$ for some $p \in X^1(R)$, in the sense that $\sigma(P)(a) = \sum_i u_i a v_i$ and $Pa = \sigma(P)(a)P$ elementwise, for all $a \in C_B(A)$.

In a similar way one obatins a map $\sigma_M(P) : C_B(M) \to C_B(M)$ which is a semilinear (left and right) $C_B(A)$-automorphism.

Proof. An easy elaboration of the preceding remarks; note that $\sigma(P)$ as defined in the statement does not depend on the choice of the decomposition $1 = \sum_i u_i v_i$. □

III.4.2. Corollary. Let there be given a group morphism $\Phi : G \to D_B(A), g \mapsto P_g$, for some group G. Then there is a canonical action of G on $C_B(A)$ given by $\Psi_A : G \to D_B(A) \to \mathrm{Aut}(C_B(A)), g \mapsto P_g \mapsto \sigma(P_g) = \sigma_g$. This action of G is compatible with the canonical action of G on $C_B(M)$ defined by the composed group morphism : $\Psi_M : G \to D_B(A) \to \mathrm{Aut}_{\Psi_A}(C_B(M))$, where the latter group is the group of left and right Ψ_A-semilinear $C_B(A)$-bimodule automorphisms.

In [50], $M \in D_B(A)$ is defined to be **centrally controlled** if for every $P \in X^1(R)$ localization at $p = P \cap Z(A)$ yields $M_p \in \mathrm{Pic}(B_p)$. In the A.S. situation the condition that R is integral over C entails that each $M \in D_B(A)$ is in fact centrally controlled so we do not consider this notion further (cf [50] for some more general results in the weak AS case). A further consequence of the A.S. condition is that the double dual of an R-module or a B-bimodule coincides with the double dual of it as a C-module ! The importance of this being that C is central in A and so for $P \in D_B(A)$ left and right does not matter when calculating P^{**}.

III.4.3. Proposition. For $P \in D_B(A)$ we have $A = (P \underset{B}{\otimes} A)^{**} = (A \underset{B}{\otimes} P^{-1})^{**}$.

Proof. Let us establish the first equality, the other may be proved in a

very similar way. The B-bimodule map $A \underset{B}{\otimes} P^{-1} \to AP^{-1} \subset A$ yields a B-bimodule map $\gamma : (A \underset{B}{\otimes} P^{-1})^{**} \to (AP^{-1})^{**} = A$ (because $P \subset A$). Put $K = \mathrm{Ker}\gamma$. Localizing at $q = Q \cap Z(A)$ for some $Q \in X^1(R)$ yields $P_q^{-1} \in \mathrm{Pic}(B_q)$, hence $K_q = 0$ for every $Q \in X^1(R)$. It follows that $K_q = 0$ and thus γ is injective. On the other hand $\mathrm{Im}\gamma \supset AP^{-1}$ and $\mathrm{Im}\gamma$ is reflexive, hence γ is surjective too. □

II.4.4. Examples.

A. Let B be a tame order over R and let A be a prime ring divisorially graded by a finite group G such that $|G|^{-1} \in A$ and $B = A_e$. Results of Section III.2. yield that A is a tame order over $Z(A)$ and both R and $Z(A)$ are integral over $R \cap Z(A) = R^G$, hence we are in the A.S.-situation.

B. Let B be a maximal R-order and suppose that A is divisorially graded by a torsionfree abelian group G over $B = A_e$. By results of section III.1., A is a maximal order in $Q_{cl}(A)$. In this case $Z(A)$ is G-graded over $Z(A)_0 = R \cap Z(A) = C$ and one easily checks that $Z(A)$ is integral over some subring D (assuming that A is a P.I. ring) which is divisorially graded over $Z(A)_0$. Localization at $q \in X^1(Z(A)_0)$ makes A_q strongly graded and $Z(A_q)$ will be a scaled Rees ring. Consequently R and $Z(A)$ will satisfy the P.D.E. condition over $Z(A)_0 = R^G$. We are in the weak arithmetical situation considered in [50]. If R is integral over R^G then we are in the A.S. situation considered above.

C. Let A and B be tame orders over Noetherian integrally closed domains $Z(A), Z(B)$ resp. assume that A is divisorially graded by an arbitrary group G. We will show that we are again in the a.s. situation by proving the following :

III.4.5. Proposition. Let A and B be tame orders over Noetherian integrally closed domains $Z(A), Z(B)$ respectively, and assume that A is divisorially graded by an arbitrary group G over B, i.e. $B = A_e$. Then B is finitely generated over $Z(A) \cap Z(B) = C$.

Proof. If the statement is false we may consider a strictly ascending chain of C-modules in B, say

$$(*) \qquad Cm_1 \subset Cm_1 + Cm_2 \subset \ldots \subset Cm_1 + \ldots + Cm_n \subset \ldots$$

This chain extends to a chain of $Z(A)$-modules in A,
(**)

$$Z(A)m_1 \subset Z(A)m_1 + Z(A)m_2 \subset \ldots \subset Z(A)m_1 + \ldots + Z(A)m_n \subset \ldots$$

We claim that this chain is strictly ascencing. Indeed, $Z(A) \cap B = Z(A)_e = \{x_e, x \in Z(A)\} \subset Z(A)$ is a direct factor of $Z(A)$ because if $x = x_{\sigma_1} + \ldots + x_{\sigma_n} \in Z(A)$ then we may decompose x as $x = \sum_{\tau \in G} x_{\tau\sigma_1\tau^{-1}} + \ldots + \sum_{\tau \in G} x_{\tau\sigma_r\tau^{-1}}$ for some $\sigma_1, \ldots, \sigma_r$ appearing in the homogeneous decomposition of x. Clearly each part $\sum_{\tau \in G} x_{\tau\sigma_1\tau^{-1}} \in Z(A)$ as one easily checks from the relation $z_\gamma x = x z_\gamma$ for any $z_\gamma \in A_\gamma, \gamma \in G$. Therefore, $(Z(A)m_1 + \ldots + Z(A)m_n) \cap B = Z(A)_e m_1 + \ldots + Z(A)_e m_n = Cm_1 + \ldots + Cm_n$ and the claim follows. Since A is a Noetherian $Z(A)$-module, the sequence (**) terminates hence (*) terminates, a contradiction. □

That we are in the A.S. situation in Example C above is now clear except that we have to verify that $C \hookrightarrow Z(A)$ satisfies the P.D.E. condition but this follows from the fact that A is divisorially graded over B.
Next we consider B-bimodules P, P' and A-bimodules Q, Q' together with left and right B-linear maps $\phi : P \to Q, \phi' : P' \to Q'$. An isomorphism between ϕ and ϕ' is given by a commutative diagram :

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ P' & \xrightarrow[\phi']{} & Q' \end{array}$$

where f is a B-bimodule isomorphism, g an A-bimodule isomorphism and all bimodules and isomorphisms are compatible with the C-structure. We now extend some results of Y. Miyashita to the divisorially graded case but restricting, as before, attention to the A.S. situation. Let $R_B(A)$ be the set of isomorphism classes $[\phi]$ of $\phi : P \to Q$

as introduced above, where $P \in \Pi ic_R(B)$ and Q is such that the morphism $(A \underset{B}{\otimes} P)^{**} \to Q$ induced by $a \otimes p \mapsto a\phi(p)$ is an isomorphism. We may introduce an operation in $R_B(A)$ denoted by $\perp$ (or by $(- \otimes -)^{**}$ by "abus de symboles") where $[\phi].[\phi'] = [\phi \perp \phi']$ is defined by the composition :

$$(P \underset{B}{\otimes} P')^{**} \to (Q \underset{A}{\otimes} Q')^{**B} \to (Q \underset{A}{\otimes} Q')^{**}.$$

Since $(A \underset{B}{\otimes} (P \underset{B}{\otimes} P')^{**})^{**} = (A \underset{B}{\otimes} P \underset{B}{\otimes} P')^{**} = ((A \underset{B}{\otimes} P) \underset{A}{\otimes} (A \underset{B}{\otimes} P'))^{**} = (Q \underset{A}{\otimes} Q)^{**}$, where $**$ at the end of a formula refers to $**$ in A-mod, it follows that $[\phi \perp \phi']$ is in $R_B(A)$. The inclusion $B \hookrightarrow A$ is the identity element for this operation and if $\phi : P \to Q$ represents $[\phi] \in R_B(A)$ then one easily verifies that $[\phi^*]$ with $\phi^* : P^* \to Q^*$, $P^* = \mathrm{Hom}_B({}_BP, {}_BB)$ and $Q^* = \mathrm{Hom}_A({}_AQ, {}_AA)$ and $\phi^*(p^*)$ for $p^* \in P^*$ is defined by sending $a\phi(p)$ to $ap^*(p)$ for all $a \in A$, $p \in P$, yields an inverse for $[\phi]$ in $P_B(A)$.

III.4.6. Theorem. Consider $A \supset B$ as before and assume that we are in the arithmetical situation, then we obtain the following exact sequences.

(a) $1 \to U(Z(A)) \to U(C_A(B)) \xrightarrow{\alpha} D_B(A) \to \Pi ic_C(B)$
(b) $1 \to U(Z(A)) \to U(C_A(B)) \xrightarrow{\beta} \mathrm{Aut}_B(A) \to \Pi ic_C(A)$
(c) $1 \to U(C) \to U(Z(A)) \xrightarrow{\alpha} D_B(A) \xrightarrow{\epsilon} R_B(A) \xrightarrow{\pi_A} \Pi ic_C(A)$
(d) $1 \to U(C) \to U(R) \xrightarrow{\beta} \mathrm{Aut}_B(A) \xrightarrow{\eta} R_B(A) \xrightarrow{\pi_B} \Pi rc_C(B)$.

Proof. Once that all maps are defined the verification of the exactness of the sequences is straightforward and boring, so we leave the details to the reader and we just point out how the non-obvious maps are defined.

(a) For $d \in U(C_A(B))$ define $\alpha(d) = Bd$.

(b) For $d \in U(C_A(B))$ define $\beta(d)(a) = dad^{-1}$ and note that the inner automorphism of A given by d maps to the trivial element in $\Pi ic_C(A)$.

(c) If $P \in D_B(A)$ then $\epsilon(P) = [i]$ where i is the inclusion $i : P \to A$. If $\phi : P \to Q$ represents $[\phi] \in R_B(A)$ then $\pi_A([\phi]) = [Q]$.

(d) To $f \in \mathrm{Aut}_B(A)$ we associate $[\psi], \psi : B \to Au_f, b \mapsto bu_f$, where $Au_f =_1 A_f$ i.e. $Au_f.a = Af(a)u_f$ for all $a \in A$.

If $[\phi] \in R_B(A)$ then $\pi_B(\phi) = [P]$ where $\phi : P \to Q$ represents $[\phi]$. $\square$

We consider the following situation throughout the sequel of this section. Let B be a tame order over the Noetherian integrally closed domain R and let A be divisorially graded by a group G over $A_e = B$ such that $B \hookrightarrow A$ verifies the A.S. condition. For B-bimodules N and M we say that $N|M$ if N is a direct summand of a finite number of copies of M. We say that N is **similar** to M, written $N \sim M$, if $N|M$ and $M|N$. We say that $N||M$ if there exists a B-linear surjection $M \oplus \ldots \oplus M \to N$ which splits locally at every $p = P \cap R^G$ for each $P \in X^1(R)$. We say that M is **divisorially similar** to N, written $N \approx M$ if $N||M$ and $M||N$.

III.4.7. Lemma. Let M be divisorially similar to N over B, then

1. If $[N] \in \Pi ic(B)$ then $[M] \in \Pi ic(B)$.
2. If $[N] \in \Pi ic_{R^G}(B)$ then $[M] \in \Pi ic_{R^G}(B)$.

Proof. In the situations described in 1. or 2. we have that N_P is finitely presented and then so is M_P, for all $P \in X^1(R)$ and $N_P \sim M_P$ holds too because localisation at P is a further localization at $p = P \cap R^G$. Hence $M_P \in Pic(B_P)$ for all $P \in X^1(R)$ in case 1 and $M_P \in \mathrm{Pic}_{(R^G}(B_P)$ in case 2. The local-global characterization of finitely presented divisorial modules entails the statements in the lemma. $\square$

We define $\Gamma_B(A)$ to be the set of graded isomorphism classes of divisorially graded rings of type G, $\underset{\sigma \in G}{\oplus} H_\sigma$ over $B = H_e$, such that for all $\sigma \in G$ we have $H_\sigma \approx A_\sigma$. The multiplication of $\oplus_{\sigma \in G} H_\sigma$ is given by a factor system $\{h_{\sigma,\tau} : (H_\sigma \otimes H_\tau)^{**} \to H_{\sigma\tau}, \sigma, \tau \in G\}$ consisting of B-bimodule isomorphisms. We will write (H, h) for this divisorially graded ring and $[H, h]$ for its graded isomorphism class, e.g. $A = [J, j]$ will be used (for symmetry in notation) with $J_\sigma = A_\sigma$ for all $\sigma \in G$ and j given by the B-bimodule isomorphisms $j_{\sigma,\tau} : (A_\sigma \otimes A_\tau)^{**} \to A_{\sigma\tau}$. Consider $[V, v]$ and $[W, w]$ in $\Gamma_B(A)$ and define their product $[U, u]$ by putting $U_\sigma = (V_\sigma \underset{B}{\otimes} J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma)^{**}$, $u_{\sigma,\tau} = (v_{\sigma,\tau} \otimes j_{\sigma^{-1},\tau^{-1}} \otimes \omega_{\sigma,\tau})^{**}$.

III.4.8. Lemma 1. 1. If M is a divisorial R-module such that $M||R$ then $(\mathrm{End}_B(B \underset{R}{\otimes} M))^{**} = (B \underset{R}{\otimes} \mathrm{End}_R(M))^{**}$ and $(B \underset{R}{\otimes} M)^{**}||B$. Moreover $C_B(B \underset{B}{\otimes} M) = M$ and if $[M] \in Cl(R)$ then $B \underset{R}{\otimes} M \in CCl(B)$.

2. If $M||B$ then $M = (B.C_B(M))^{**} \cong (B \underset{R}{\otimes} C_B(M))^{**}$ and $C_B(M)||R$. Moreover, $\mathrm{Rnd}_R(C_B(M)) \cong \mathrm{End}_B({}_BM_B)$, $\mathrm{End}_B({}_BM) \cong (B \underset{R}{\otimes} \mathrm{End}_B(M))^{**}$. If $M||B$ and $M'||B$ then M and M' are isomorphic B-bimodules if and only if $C_B(M) \cong C_B(M')$ are R-isomorphic.

3. If $M||B$ and $M'||B$ then $(C_B(M \underset{B}{\otimes} M'))^{**} \cong ((C_B(M)) \underset{R}{\otimes} C_B(M'))^{**}$ and then there exists an isomorphism t, $t : (M \underset{B}{\otimes} M')^{**} \to (M' \underset{B}{\otimes} M)^{**}$, $m \otimes m' \mapsto m' \otimes m, m \in M, m' \in C_B(M')$ (extended by right linearity to the whole $(M \underset{B}{\otimes} M')^{**}$).

Proof. If $M||B$ then the existence of a B-linear morphism $B \oplus \ldots \oplus B \to M$ proves that M is finitely generated as a left (and right) B-module and M is generated as a left (and right) B-module by $C_B(M)$, i.e. $M = BC_B(M)$. All statements in the lemma are now derived from Lemma 2.3., Lemma 2.4. and Corollaries 1, 2, 3, in [33] (modifying by $**$ where necessary). $\square$

For the definitions of (U, u) we note :

$$\begin{aligned}((V_\sigma \underset{B}{\otimes} J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma)^{**} &\underset{B}{\otimes} (V_\tau \underset{B}{\otimes} J_{\tau^{-1}} \underset{B}{\otimes} W_\tau)^{**})^{**} \\ &= (V_\sigma \underset{B}{\otimes} J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma \underset{B}{\otimes} V_\tau \underset{B}{\otimes} J_{\tau^{-1}} \underset{B}{\otimes} W_\tau)^{**}\end{aligned}$$

Furthermore $(V_\tau \underset{B}{\otimes} J_{\tau^{-1}})^{**}||B$ and $(J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma)^{**}||B$, and

$$(J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma)^{**} \underset{B}{\otimes} (V_\tau \underset{B}{\otimes} J_{\tau^{-1}})^{**} \xrightarrow[\cong]{t} (V_\tau \underset{B}{\otimes} J_{\tau^{-1}})^{**} \underset{B}{\otimes} (J_{\sigma^{-1}} \otimes W_\sigma)^{**}$$

By localizing $v_{\sigma,\tau} \otimes j_{\tau^{-1},\sigma^{-1}} \otimes \omega_{\sigma,\tau}$ (note $(-)^{**} = \underset{P \in X^1(R)}{\cap} (-)_P$) we obtain a B-bilinear map, $(v_{\sigma,\tau} \underset{B}{\otimes} j_{\tau^{-1},\sigma^{-1}} \otimes \omega_{\sigma,\tau})^{**}$:

$$(V_\sigma \underset{B}{\otimes} V_\tau \underset{B}{\otimes} J_{\tau^{-1}} \underset{B}{\otimes} J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma \underset{B}{\otimes} W_\tau) \xrightarrow{**} (V_{\sigma\tau} \underset{B}{\otimes} J_{(\sigma\tau)^{-1}} \underset{B}{\otimes} W_{\sigma\tau})^{**}$$

which defines the map, $u_{\sigma,\tau}$:

$$((V_\sigma \underset{B}{\otimes} J_{\sigma^{-1}} \underset{B}{\otimes} W_\sigma)^{**} \underset{B}{\otimes} (V_\tau \underset{B}{\otimes} J_{\tau^{-1}} \underset{B}{\otimes} W_\tau)^{**}) \xrightarrow{**} (V_{\sigma\tau} \underset{B}{\otimes} J_{(\sigma\tau)^{-1}} \underset{B}{\otimes} W_{\sigma\tau})^{**}$$

It is easily checked that $\{u_{\sigma,\tau}, \sigma, \tau \in G\}$ is a factor system and (using the transposition map t frequently) one may check that we have defined an associative multiplication on $\Gamma_B(A)$ such that $[A] = [J, j]$ is the unit element. The inverse of $[V, v]$ is defined by considering $W_\sigma = (J_\sigma \otimes V_\sigma^* \otimes J_{\sigma^{-1}})^{**}$ where $V_\sigma^* = \mathrm{Hom}_B({}_BV_\sigma, {}_BB) \in \Pi ic(B)$ and the map $v^*_{\sigma,\tau} : (J_\sigma \underset{B}{\otimes} V_\sigma^* \underset{B}{\otimes} J_\sigma \underset{B}{\otimes} J_\tau \underset{B}{\otimes} V_\tau^* \underset{B}{\otimes} J_\tau)^{**} \to (J_{\sigma\tau} \underset{B}{\otimes} V^*_{\sigma\tau} \otimes J_{\sigma\tau})^{**}$ is obtained via the transposition map because the first term is isomorphic to $(J_\sigma \underset{B}{\otimes} J_\tau \underset{B}{\otimes} V_\tau^* \underset{B}{\otimes} V_\sigma^* \underset{B}{\otimes} J_\sigma \underset{B}{\otimes} J_\tau)^{**}$ as a B-bimodule. Note that $[V^*]$ is indeed the inverse of $[V]$ in $\Pi ic(B)$; this can easily be checked locally at $P \in X^1(R)$. Using the transposition again one easily proves that $\Gamma_B(A)$ is an abelian group. In the foregoing argumentation we freely use (implicitely) that the "local properties" at $P \in X^1(R)$ also hold at $p = P \cap R^G$ and that we may replace $\Pi ic(B)$ by $\Pi ic_{R^G}(B)$ where appropriate. We write $\Gamma_B^g(A)$ for the subgroup of $\Gamma_B(A)$ consisting of the divisorially graded rings over B which are isomorphic to A as a graded ring i.e. those $[V, v]$ such that $V_\sigma \cong J_\sigma = A_\sigma$ as B-bimodules for all $\sigma \in G$ with v equivalent to j.

We define an action of G on $\Pi ic(B)$ by ${}^\sigma[P] = (J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**}$ (writing unadorned tensor products for those over B). We put : $\Pi ic_C(B)^G = \{[P] \in \Pi ic_C(B), {}^\sigma[P] = [P] \text{ for all } \sigma \in G\}$. $\Pi ic_C(B)^{(G)} = \{[P] \in \Pi ic_C(B), (P \otimes J_\sigma \otimes^* P)^{**} \approx J_\sigma \text{ for all } \sigma \in G\}$, where ${}^*P$ is the right dual of P, ${}^*P = \mathrm{Hom}_B(P_B, B_B), C = R^G$. We have $(P \otimes^* P)^{**} = B$. If $[P] \in \Pi ic_C(B)^G$ then $(J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**} = P$ for all $\sigma \in G$, hence $(J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P)^{**} = B$. Thus we obtain : $J_\sigma \otimes (J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P)^{**} = J_\sigma$, or $(P \otimes J_\sigma \otimes^* P)^{**} = J_\sigma$ and therefore $[P] \in \Pi ic_C(B)^{(G)}$. We define a group morphism $\Pi ic_C(B) \to \Gamma_B(A), [P] \mapsto [\oplus_{\sigma\in G}(P \otimes J_\sigma \otimes^* P)^{**}, j^P_{\sigma,\tau}]$, where $j^P_{\sigma,\tau} : (P \otimes J_\sigma \otimes^* P \otimes P \otimes J_\tau \otimes^* P)^{**} \to (P \otimes J_\sigma \otimes J_\tau \otimes^* P)^{**} \to (P \otimes J_{\sigma\tau} \otimes^* P)^{**}$, defines the multiplication of the divisorially graded ring $\underset{\sigma in G}{\oplus}(P \otimes J_\sigma \otimes^* P)^{**}$.
So we proved :

III.4.9. Proposition. With conventions as before, there is a commu-

tative diagram of abelian groups :

$$\begin{array}{ccc} \Pi ic_C(B)^G & \longrightarrow & \Gamma_B^g(A) \\ \downarrow & & \downarrow \\ \Pi ic_C(B)^{(G)} & \longrightarrow & \Gamma_B(A) \end{array}$$

In $R_B(A)$ we define $R_B(A)^{(G)} = \{[\phi] \in R_B(A), \phi : P \to M, (J_\sigma\phi(P))^{**} = (\phi(P)J_\sigma^{**}) \text{ for all } \sigma \in H\}$. Note that the divisoriality of A over B entails that $P \hookrightarrow A \underset{B}{\otimes} P$, hence $(A \underset{B}{\otimes} P)^{**} = M$, i.e. we may identify P and $\phi(P)$ in M. Obviously $[\phi] \in R_B(A)^{(G)}$ exactly then when there exists a B-bimodule isomorphism $f_\sigma : P \to (J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**}$ making the following diagram commutative :

$$(D) \qquad \begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "f_\sigma"'] & M \\ (J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**} \arrow[ur, "\sigma_\phi^{**}"'] & \end{tikzcd}$$

where ${}^\sigma\phi(x_\sigma \otimes p \otimes x'_{\sigma^{-1}}) = x_\sigma\phi(p)x'_{\sigma^{-1}}$ for all $x_\sigma \in J_\sigma, p \in P, x'_{\sigma^{-1}} \in J_{\sigma^{-1}}$. Indeed, if for all $\sigma \in G$ we have : $(J_\sigma\phi(P))^{**} = (\phi(P)J_\sigma)^{**}$ then from $(J_\sigma \otimes P)^{**} \to (A \otimes P)^{**}$ we obtain that $(J_\sigma \otimes P)^{**} = (J_\sigma\phi(P))^{**}$. On the other hand if K is the kernel of $\phi(P) \otimes J_\sigma \to M, p \otimes x \mapsto px$, then by localization at $p = P \cap R^G$ with $P \in X^1(R)$ it follows that $(\phi(P) \otimes J_\sigma)^{**} \cong (\phi(P)J_\sigma)^{**}$, i.e. we obtain $(\phi(P) \otimes J_\sigma)^{**} = (\phi(P)J_\sigma)^{**} = (J_\sigma\phi(P))^{**} = (J_\sigma \otimes \phi(P))^{**}$. Identifying P and $\phi(P)$ in M we obtain : $P = (P \otimes J_\sigma \otimes J_{\sigma^{-1}})^{**} = ((P \otimes J_\sigma)^{**} \otimes J_{\sigma^{-1}})^{**} = (PJ_\sigma)^{**} \otimes J_{\sigma^{-1}})^{**} = (J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**}$, what defines the B-bimodule isomorphism f_σ used in the diagram above. Again by localizing at all $p = P \cap C$ for all $P \in X^1(R)$ it follows that the diagram leads to

$(PJ_\sigma)^{**} = (J_\sigma P)^{**}$ for all $\sigma \in G$. Considering $((D) \otimes J_\sigma)^{**}$ we obtain :

$$\begin{tikzcd}
(P\otimes J_\sigma)^{**} \arrow[r, "(\phi\otimes J_\sigma)^{**}"'] \arrow[d] & (M\otimes J_\sigma)^{**} \arrow[r, "\mu"] & M^{**} \\
(J_\sigma\otimes P\otimes J_{\sigma^{-1}}\otimes J_\sigma)^{**} \arrow[ur, "({}^\sigma\phi\otimes J_\sigma)^{**}"] \arrow[d] & & \\
(J_\sigma\otimes P)^{**} \arrow[uurr, "v"'] & &
\end{tikzcd}$$

where μ is obtained from $m \otimes x_\sigma \mapsto mx_\sigma$, $m \in M, x \in J_\sigma$, and v is the composition $\mu({}^\sigma\phi \otimes J_\sigma)^{**}$. We have $Im\mu = Imv$. Clearly $(Im\mu)^{**} = (PJ_\sigma)^{**}$. On the other hand, the definition of ϕ is such that $(Imv)^{**} = (J_\sigma P)^{**}$ is evidently true, so $Im\mu = Imv$ entails $(J_\sigma P)^{**} = (PJ_\sigma)^{**}$. (Note that these results also follow from Miyashita's results in [ ] by the usual "local" argumentation). The foregoing argument also applies to $^*P$, representing the inverse of $[P]$ and it is not hard to check that $\Phi^* : P^* \to M^*$, where $M^* = \mathrm{Hom}_A({}_AM, {}_AA)$, is again representing an element of $R_B(A)^{(G)}$. Therefore $R_B(A)^{(G)}$ is closed under taking inverses and it is a subgroup of $R_B(A)$ (this can also be checked "locally" in the usual way). Let us write $\mathrm{Aut}_B(A)^{(G)}$ for $\{\alpha \in \mathrm{Aut}_B(A), \alpha(J_\sigma) = J_\sigma \text{ for all } \sigma \in G\}$. We have now established :

III.4.10. Proposition. With conventions and notation as before, we obtain an exact sequence :

$$0 \to U(C) \to U(R) \xrightarrow{\beta} \mathrm{Aut}_B(A) \xrightarrow{\eta} R_B(A)^{(G)} \xrightarrow{\pi_B} \Pi ic_C(B)^{(G)}$$

Proof. This sequence is a subsequence of the one in Theorem III.4.6., d and it suffices to establish that an $f \in \mathrm{Aut}_B(A)$ maps to $R_B(A)^{(G)}$. Now $\eta(f) : B \to_1 A_f$ is in $R_B(A)^{(G)}$ if and only if $f(J_\sigma) = J_\sigma$ for all $\sigma \in G$. □

II.4.11. Lemma. The maps in Proposition III.4.9. yield an exact sequence : $R_B(A)^{(G)} \longrightarrow \Pi ic_C(B)^{(G)} \xrightarrow{\epsilon} \Gamma_B^g(A)$.

Proof. Let $\Psi : P \to M$, represent some $[\psi] \in R_B^g(A)^{(G)}$, then $[P] \in \Pi ic_C(B)$ and it maps to $[(P \otimes J_\sigma \otimes^* P)^{**}, P_j]$. Now $(P \otimes J_\sigma \otimes^* P)^{**} = ((J_\sigma \otimes P \otimes J_{\sigma^{-1}})^{**} \otimes J_\sigma \otimes^* P)^{**}$, since $[\Psi] \in R_B(A)^{(G)}$, and the latter is further isomorphic to : $(J_\sigma \otimes P \otimes J_{\sigma^{-1}} \otimes J_\sigma \otimes^* P)^{**} = (J_\sigma \otimes P \otimes (J_{\sigma^{-1}} \otimes J_\sigma)^{**} \otimes^* P)^{**} = (J_\sigma \otimes (P \otimes^* P)^{**})^{**} = J_\sigma$ (as B-bimodules). Let $h_\sigma : (P \otimes J_\sigma \otimes^* P) \longrightarrow J_\sigma$ be the isomorphism of B-bimodules just defined then we may fit h_σ, H_τ and $h_{\sigma\tau}$ in the following commutative diagram :

$$\begin{tikzcd}
((P\otimes J_\sigma\otimes^* P)^{**}\otimes(P\otimes J_\tau\otimes^* P)^{**}) \arrow[r, "**", "P_{j_{\sigma,\tau}}"'] \arrow[d] & (P\otimes J_{\sigma\tau}\otimes^* P)^{**} \arrow[d] \\
(J_\sigma\otimes J_\tau)^{**} \arrow[r, "j_{\sigma,\tau}"'] & J_{\sigma\tau}
\end{tikzcd}$$

If $[P] \in \Pi ic_C(B)$ is in the kernel of ϵ then, $A^1 = (\bigoplus_{\sigma \in G}(P \otimes J_\sigma \otimes^* P^{**}, P_j) = (\oplus J_\sigma, j) = A$. First we establish that $A^1 = \mathrm{End}_A((P \otimes A)_A^{**})$ and then it is evident from $A^1 = A$ that $(P \otimes A)^{**} \in \Pi ic_C(A)$, proving that the canonical $\phi : P \to (P \otimes A)^{**}$ determines an element $[\phi] \in R_B(A)^{(G)}$ that maps to $[P]$. For all $\tau \in G$, and for $p \otimes x \otimes p' \in P \otimes J_\sigma \otimes^* P$ we define :
$p \otimes x \otimes p' : P \otimes J_\tau \to P \otimes J_{\sigma\tau}, q \otimes y \mapsto p \otimes xp'(q)y$.Thus $(p \otimes x \otimes p')^{**} : (P \otimes J_\tau)^{**} \to (P \otimes J_{\sigma\tau})^{**}$ determines an element of degree σ in $\mathrm{HOM}_A((P \otimes A)^{**}, (P \otimes A)^{**})$and we have $\mathrm{Hom}_A((P \otimes A)^{**},(P \otimes A)^{**}) = \mathrm{HOM}_A((P \otimes A)^{**}, (P \otimes A)^{**})$ since $(P \otimes A)^{**}$ is finitely generated as an A- module. It is straightforward to check that we have actually defined a graded ring morphism (of degree zero) : $\Omega : \bigoplus_{\sigma \in G}(P \otimes J_\sigma \otimes^* P)^{**} \to \mathrm{HOM}_A((P \otimes A)_A^{**}, (P \otimes A)_A^{**})$, where both rings are divisorially graded by G over B. Let us first check that Ω is a monomorphism. Pick $x, y \in (P \otimes J_\sigma \otimes^* P)^{**}$ such that $\Omega(x) = \Omega(y)$. Then we obtain : $\Omega((P \otimes J_{\sigma^{-1}} \otimes^* P)^{**}x) = \Omega((P \otimes J_{\sigma^{-1}} \otimes^* P)^{**}y)$. Since the restriction $\Omega|B$ is an isomorphism it follows that : $(P \otimes J_{\sigma^{-1}} \otimes^* P)^{**}(x - y) = 0$ and further that $(P \otimes J_\sigma \otimes^* P \otimes P \otimes J_{\sigma^{-1}} \otimes^* P)^{**}(x - y) = 0$, i.e. $x - y = 0$. Now, divisorially graded rings over the same ring in degree zero such that one contains the other are necessarily equal, so Ω has to be an isomorphism. □

Let $\beta_B(A)$ be defined by the exactness of the sequence : $\Pi ic_C(B)^{(G)} \to \Gamma_B(A) \xrightarrow{\xi} \beta_B(A)$.

III.4.12. Proposition. With conventions as above, the following sequence is exact :

$$\Pi ic_C(B)^{(G)} \to \Gamma_B^G(A) \to \beta_B(A).$$

Proof. Semi-exactness is obvious. If $[\bigoplus_{\sigma\in G} H_\sigma, h_{\sigma,\tau}]$ is in the kernel of ξ then there exists a $[P]$ in $\Pi ic_C(B)^{(G)}$ such that $[P] \mapsto [\bigoplus_{\sigma\in G} H_\sigma, h_{\sigma,\tau}]$ under the map $\Pi ic_C(B)^{(G)} \to \Gamma_B(A)$. Therefore, $(P \otimes J_\sigma \otimes^* P)^{**} = J_\sigma$ and $(J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P)^{**} = B$ hence $(J_{\sigma^{-1}} \otimes P \otimes J_\sigma \otimes^* P)^{**} \otimes P = P$, and finally $(J_{\sigma^{-1}} \otimes P \otimes J_\sigma)^{**} = P$ or $P \in \Pi ic_C(B)^{(G)}$ follows. □

Put $\Pi ic_0(B) = \{[P] \in \Pi ic_C(B), P \approx B\}$. Since this group consists of B-bimodule classes $[M]$ where M is of the form $M = (B \otimes_R C_B(M))^{**} = BC_B(M)$ it is obvious that we have : $\Pi ic_0(B) = \Pi ic_C(R)$. It is now possible to define a group morphism $\Gamma_B(A) \to Z^1(B, \Pi ic_0(B))$, $[\bigoplus_{\sigma\in G} V_\sigma, v_\sigma] \mapsto (\sigma \to (V_\sigma \otimes J_{\sigma^{-1}})^{**})$, which obviously gives rise to the exact sequence :

$$0 \to \Gamma_B^g(A) \to \Gamma_B(A) \to Z^1(G, \Pi ic_0(B))$$

If we define the group $\overline{H}_1(G, \Pi ic_0(B))$ by the exactness of the following sequence :

$$\begin{tikzcd} \Pi ic_C(B) \arrow[r] \arrow[dr] & Z^1(G, \Pi ic_0(B)) \arrow[r] & \overline{H}_1(G, \Pi ic_0(B)) \\ & \Gamma_B(A) \arrow[u] & \end{tikzcd}$$

then we obtain an exact sequence :

$$\Gamma_B(A) \to \beta_B(A) \to \Pi^1(G, \Pi ic_0(B)) \to H^3(G, U(R))$$

Combining all of these sequences we obtain a long exact sequence generalizing the Chase-Harrison-Rosenberg sequence for the (reflexive) Brauer group of a Krull domain :

III.4.13. Theorem. Let B be a tame order over the Noetherian integrally closed domain R, let A be divisorially graded by some group G over $B = A_e$ such that the A. S. conditions hold. Then the following sequence is exact : $(C = R^G = R \cap Z(A))$ where we put :

$$0 \to U(C) \to U(R) \to \mathrm{Aut}_B(A)^G \to R_B(A)^{(G)} \to \Pi ic_0(B) \to$$
$$\to \Gamma_B^g(A) \to \beta_B(A) \to \overline{H}^1(G, \Pi ic_0(B)) \to H^3(G, U(R))$$

The groups in this sequence may deserve further attention e.g. in case B is commutative. Note that in the case where B is a Galois extension of C and a maximal commutative subring of a reflexive Azumaya algebra A then the sequence in the theorem (with G being a finite group) does indeed reduce to the reflexive version of the Chase-Harrison-Rosenberg sequence as explained in [12] and in [61].
In the particular case where $B \hookrightarrow A$ is an extension of tame orders i.e. $Z(B) \subset Z(A)$ or $R = C$ then the above reduces to a sequence including some more familiar groups.

III.4.14. Corollary. If in the situation of the theorem $B \hookrightarrow A$ is an extension then the exact sequence given reduces to :

$$0 \to \mathrm{Aut}_B(A)^G \to R_B(A)^{(G)} \to CCl(B) \to \Gamma_B^g(A) \to \beta_B(A) \to$$
$$H^1(G, CCl(B)) \to H^3(G, U(R))$$

IV. Regular Orders.

IV.1. Moderated Gorenstein and Regular Orders.

In this first section we will recall some basic results on orders having finite global dimension. Since we will freely use results from homological algebra, the reader is referred to [13] for more details.
Since regularity is a local condition, we will always assume that the order Λ is a finite module over a commutative base ring R, which is a local Noetherian domain with maximal ideal m and such that R is a subring of the center of Λ, which we will denote with C.

Let us briefly recall some standard definitions and results from commutative ring theory. Let M be a finitely generated R-module and let $\{x_1, \ldots, x_r\}$ be a sequence of elements from m.
If we denote

$$M_i = M/(x_1 M + \ldots x_i M)$$

Then we will say that the sequence $\{x_1, \ldots, x_r\}$ is M-regular if the short sequences

$$0 \longrightarrow M_i \xrightarrow{x_{i+1}} M_i$$

are exact for all $0 \leq i < r$, i.e. x_{i+1} is a non-zero divisor on M_i.

The depth of the R-module M, $\mathrm{depth}_R(M)$, is defined to be the supremum of all integers r such that there exists an M-regular sequence $\{x_1, \ldots, x_r\}$ of elements from m.

Recall that a prime ideal p of R is said to be associated to a finitely generated R-module M if there exists an element $x \in M$ such that p is the annihilator of x. With $\mathrm{Ass}(M)$ we will denote the set of all prime ideals of R associated to M.
The dimension of the R-module M, $\dim_R(M)$, is the infimum of the classical Krull dimensions of the quotients R/p where $p \in \mathrm{Ass}(M)$. A classical result from commutative ring theory, asserts that

$$\mathrm{depth}\,_R(M) \leq \dim_R(M)$$

A finitely generated R-module M is said to be Cohen-Macaulay, if $\mathrm{depth}_R(M) = \mathbf{dim}_R(M)$. The local Noetherian ring R is said to be a Cohen-Macaulay ring if it is a Cohen-Macaulay module over itself.
The order Λ is said to have selfinjective dimension $n < \infty$ if there is an exact sequence

$$0 \to \Lambda \to E_0 \to \ldots \to E_n \to 0$$

where all E_i are injective left Λ-modules and n is the least integer for which such an injective resolution exists.
Because Λ is a finite module over the local ring R, Λ is semi-local in the sense that it has only a finite number of maximal two sided prime ideals $\{P_1, \ldots, P_k\}$ all lying over m. If M is a finite generated left Λ-module, we define the Λ-Ext dimension to be the last integer p such that $\mathrm{Ext}^p_\Lambda(M, \Lambda) \neq 0$.

IV.1.1. Definition. The order Λ is called **a moderated Gorenstein algebra** if and only if :

1. Λ has finite selfinjective dimension n, where $n = \dim_R \Lambda$, which coincides of course with the Krull dimension of Λ.

2. For all $1 \leq i \leq k$ we have that $\Lambda - \mathrm{Ext}\ \dim(\Lambda/P_i) = n$

The first major result on moderated Gorenstein algebras is :

IV.1.2. Theorem. (d'après Vasconcelos). A moderated Gorenstein algebra over a local Noetherian ring R is a Cohen-Macaulay module over its center C.

Proof. The result follows if we prove that Λ is a Cohen-Macaulay R-module. So, assume that $x \in m$ is a Λ regular element, then it is well known, cf. [63], Th. 9.3.7 p. 248, that :
$\mathrm{depth}_R(\Lambda) = 1 + \mathrm{depth}_R(\Lambda/x\Lambda)$
$\dim_R(\Lambda) = 1 + \dim_R(\Lambda/x\Lambda)$
$\mathrm{id}_\Lambda(\Lambda) = 1 + \mathrm{id}_{\Lambda|x\Lambda}(\Lambda/x\Lambda)$.

So, by induction we can reduce by as many elements as there are in a regular Λ-sequence, i.e. we may assume that m is an associated prime of the R-module Λ. If we can prove that the injective dimension of Λ is zero, then Λ has to be Artinian, whence both $\mathrm{depth}_R(\Lambda)$ and $\dim_R(\Lambda)$ will be zero, yielding the result. Therefore, let us assume that $\mathrm{id}_\Lambda(\Lambda) = t > 0$ and let $0 \neq x \in \Lambda$ s.t. $mx = 0$. Then, it follows that the Λ-submodule Λx of Λ has finite length and so it will be contain a simple submodule S. From the short exact sequence

$$0 \to S \to \Lambda \to \Lambda/S \to 0$$

we derive the part of the long exact homology sequence

$$\mathrm{Ext}^t_\Lambda(\Lambda, \Lambda) \to \mathrm{Ext}^t_\Lambda(S, \Lambda) \to \mathrm{Ext}^{t+1}_\Lambda(\Lambda/S, \Lambda)$$

By the fact that $\mathrm{id}_\Lambda(\Lambda) = t$ and [63], Th. 9.8, p. 236, we get that $\mathrm{Ext}^{t+1}_\Lambda(\Lambda/S, \Lambda) = 0$. Since we have assumed that $t > 0$, also $\mathrm{Ext}^t_\Lambda(\Lambda, \Lambda) = 0$ follows, yielding that $\mathrm{Ext}^t_\Lambda(S, \Lambda) = 0$. Since $\mathrm{id}_\Lambda(\Lambda) = t$, there exists a finitely generated Λ-module M s.t. $\mathrm{Ext}^t_\Lambda(M, \Lambda) \neq 0$. If m is not associated to M, then we can find an element $a \in m$ such that

$$0 \to M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$$

is exact. This provides us with a long exact sequence :

$$\mathrm{Ext}^t_\Lambda(M, \Lambda) \xrightarrow{a.} \mathrm{Ext}^t_\Lambda(M, \Lambda) \longrightarrow \mathrm{Ext}^{t+1}_\Lambda(M/aM, \Lambda)$$

where the last term is zero since $\mathrm{id}_\Lambda(\Lambda) = t$. So, $m.\mathrm{Ext}^t_\Lambda(M, \Lambda) = \mathrm{Ext}^t_\Lambda(M, \Lambda)$ and since $\mathrm{Ext}^t_\Lambda(M, \Lambda)$ is finitely generated as an R-module, Nakayama's lemma entails that $\mathrm{Ext}^t_\Lambda(M, \Lambda) = 0$, a contradiction. Therefore, m is associated to M which therefore must contain simple submodules. Let N be the submodule consisting of all elements which are of finite length, i.e. which are annihilated by some power of m. Then, we obtain the exact sequence :

$$\mathrm{Ext}^t_\Lambda(M/N,\Lambda) \to \mathrm{Ext}^t_\Lambda(M,\Lambda) \to \mathrm{Ext}^t_\Lambda(N,\Lambda)$$

Here, $\mathrm{Ext}^t_\Lambda(M/N,\Lambda) = 0$ because M/N has no elements of finite length and then one can apply the Nakayama-argument as above. Further, N is an extension of simple modules S_i, and for all S_i we have $\mathrm{Ext}^t_\Lambda(S_i,\Lambda) = 0$ since $\mathrm{Ext}^t_\Lambda(M/N,\Lambda) = 0$ and condition (ii) of Definition 4.1.1. Therefore, $\mathrm{Ext}^t_\Lambda(N,\Lambda) = 0$ and we obtain a contradiction. Therefore, we have shown that $\dim_R(\Lambda) = \mathrm{id}_\Lambda(\Lambda) = \mathrm{depth}_R(\Lambda)$ i.e. Λ is a Cohen-Macaulay module over R and hence over C. □

Next, we want to investigate the behaviour of moderated Gorenstein algebras under localization.

IV.1.3. Theorem. Let Λ be a moderated Gorenstein algebra over the local Noetherian ring R and let p be a prime ideal of R, then Λ_p is a moderated Gorenstein algebra over R_p.

Proof. By induction we only have to prove the result for prime ideals of height $n-1$, where $n = \mathrm{Kdim}(R)$. Because Λ is a Cohen-Macaulay module over R, we can apply a result of J.P. Serre to obtain a regular Λ-sequence $\{a_1,\ldots,a_{n-1}\}$ of elements in p. By this it follows that $\mathrm{id}_{\Lambda_p}(\Lambda_p) \geq n-1$ and we have to verify equality. So, assume that there is a finitely generated left Λ_p-module M with $\mathrm{Ext}^n_{\Lambda_p}(M,\Lambda_p) \neq 0$. Let N_0 be a finitely generated left Λ-module such that $N_0 \underset{R}{\otimes} R_p \cong M$ and define N to be $N_0/\mathrm{Ker}\varphi$ where $\varphi : N_0 \to M$ is the natural morphism. Then, of course, $N \underset{R}{\otimes} R_p \cong M$ and m is not an associated prime ideal of N. By the Nakayama-argument of Theorem IV.1.2., we obtain that $\mathrm{Ext}^n_\Lambda(N,\Lambda) = 0$, implying that $\mathrm{Ext}^n_{\Lambda_p}(M,\Lambda_p) = 0$, a contradiction. We still have to verify condition (ii) of Definition IV.1.1. So, let Q be a maximal two sided ideal of Λ_p, then we have to check that $\mathrm{Ext}^{n-1}_{\Lambda_p}(\Lambda_p/Q,\Lambda_p) \neq 0$. Again, let $\{q_1,\ldots,q_{n-1}\}$ be a regular Λ-sequence of elements in p, then it is also a regular Λ_p-sequence of elements lying in Q. By an iterated version of the Rees' shifting lemma, cfr. [63]; Th. 9.37., p. 248, we get :

$$\mathrm{Ext}^{t-1}_{\Lambda_p}(\Lambda_p/Q,\Lambda_p) \cong \mathrm{Hom}_\Gamma(\Gamma/Q',\Gamma)$$

where $\Gamma = \Lambda_p/(q_1,\ldots,q_{t-1})\Lambda_p$ and Q' the image of Q in Γ. But, Γ is a self-injective Noetherian ring and hence $\mathrm{Hom}_\Gamma(\Gamma/Q',\Gamma) \neq 0$, what proves the claim. □

In Definition IV.1.1. we started with an order, i.e. in particular a prime ring, Λ. This condition can be removed if we replace semi-local by local. We now want to investigate the structure of the center of Λ, C, say if Λ is moderated regular.

IV.1.4. Definition : The order Λ is called a **moderated regular** algebra if and only if

(1) Λ is moderated Gorenstein.

(2) Λ is of finite global dimension $n = \mathrm{Kdim}\Lambda$.

IV.1.5. Proposition : If the the order Λ is moderated regular, then its center C is an integrally closed domain.

Proof. By assumption, C is a domain. To prove that it is integrally closed, we use the Serre normality criterion, see for example [9]. Let p be a height one prime ideal of C, then by Theorem IV.1.3., Λ_p is a moderated Gorenstein algebra which is of global dimension 1. But then, Λ_p is hereditary, so its center C_p must be a discrete valuation ring. Finally, let $x \in C$ then we have to show that Cx has no embedded primes. But since $C/Cx \hookrightarrow \Lambda/\Lambda x$ and Λ is a Cohen-Macauley module over C by Theorem IV.1.2., this follows. □

Provided that the p.i.-degree of Λ is a unit in R of C, a more strict result may be deduced.

IV.1.6. Proposition : If C is a C-direct summand of Λ, then C is a Cohen-Macaulay ring.

Proof. Since C is a C-direct summand of Λ we get $\mathrm{depth}_R\Lambda \leq \mathrm{depth}_R C$. By Theorem IV.1.2. we know that $\mathrm{depth}_R\Lambda = \mathrm{Kdim}C$ and one has always the inequality $\mathrm{depth}_R C \leq \mathrm{Kdim}C$, is always verified, thus finishing the proof.

If the condition on the p.i. degree is not satisfied, the conclusion fails too. Furthermore, it is well known that the center of a moderated regular (even $\mathbb{Q}$-) algebra, must not be regular. Skew formal power series are the easiest counterexamples. For example let $\Lambda = \mathbb{C}[[X, Y; \sigma]]$ where σ denotes the complex conjugation, then the center of Λ is equal to $\mathbb{R}[[X^2, Y^2, XY]]$ which is not regular. Ramras has shown that regularity of the center is equivalent to Λ being free over the center.

IV.1.7. Proposition : If Λ is a moderated regular order, then Λ is a tame order.

Proof. Since Λ is a Cohen-Macaulay module over its center, which is integraly closed, it follows that Λ is a reflexive C-module. Furthermore, as observed above, localizations of Λ at height one prime ideals of C are hereditary so the claim follows. □

It follows from a result of Vasconcelos [59], Th. 4.3., that one may consider maximal orders instead of tame orders at the cost of considering a local order Λ and not only a semi-local one.
Next, we want to investigate when a moderated Gorenstein algebra is moderated regular. For any order Λ we say that a left Λ-module is Cohen-Macaulay if it is Cohen-Macaulay over R. We include the following well-known result.

IV.1.8 Lemma. (Peskine - Szpiro acyclicity lemma)
Let R be a local Noetherian ring and let there be given a complex of R-modules of finite type :

$$0 \to L_s \to L_{s-1} \to \ldots \to L_1 \to L_0 \to 0 \qquad (L)$$

Assume that the following conditions hold :
(1) : $\operatorname{depth}(L_i) \geq i$
(2) : $\operatorname{depth}(H_i(L)) = 0$ or $H_i(L) = 0$
Then $H_i(L) = 0$ for all $i \geq 1$.

Using this lemma, it is possible to prove the next result due to H. Bass :

IV.1.9. Theorem : Let Λ be a tame order which is a Cohen-Macaulay module over R, then $\operatorname{gldim}(\Lambda) = \operatorname{Kdim}(\Lambda)$ if and only if every left Cohen-Macaulay Λ-module is projective.

Proof. As always, $n = \operatorname{Kdim}(\Lambda) = \operatorname{Kdim}(R)$. Passing to a subring of R if necessary, we may assume that R is a regular local ring. Over this ring, left Cohen-Macaulay Λ-modules are R-projective. Let M be any left finitely generated Λ-module, then since Λ is Cohen-Macaulay, the n-th *syzygy* module N in a resolution of M :

$$0 \to N \to \Lambda^{r_{n-1}} \to \ldots \Lambda^{r_0} \to M \to 0$$

is projective as an R-module, i.e. it is a Cohen-Macaulay Λ-module. Therefore, $\operatorname{gldim}(\Lambda) \leq n$ if every left Cohen-Macaulay Λ-module is projective and $\operatorname{gldim}(\Lambda) \geq \operatorname{Kdim}(\Lambda) = n$, is trivial. Conversely, assume that $\operatorname{gldim}(\Lambda) = n$ then, if we can realize every left Cohen-Macaulay Λ-module as an n-th *syzygy*, then this would prove its projectivity as an R-module. First, let us show that whenever M is torsion-free of $\operatorname{depth}_R(M) \geq 2$, then $M \cong M^{**}$. It suffices to prove this for the localization M_p at an height one prime p of R. Because Λ is tame, Λ_p is hereditary so M_p is projective, finishing the proof of the claim. Or, to realize M as an n-th *syzygy*, take a left resolution

$$0 \to P \to \Lambda^{r_{n-1}} \to \ldots \to \Lambda^{r_0} \to M^* \to 0$$

where P is necessary projective since $\operatorname{gldim}(\Lambda) = n$. Consider the dual sequence

$$0 \to M \to \Lambda^{r_0} \to \ldots \to \Lambda^{r_{n-1}} \to P^* \to 0 \qquad (*)$$

By induction on the Krull dimension of R, we may assume that the localization of M at any prime ideal of height less than n is projective. Therefore, the localization of M^* is projective, too. So, (*) is locally exact, whence the homology has finite length. We are now in a position to apply the acyclicity lemma of Peskine and Szpiro to obtain that (*) is actually exact, finishing the proof. □

In the next section, we will use generalized Rees constructions in order to give examples of moderated regular orders of Krull dimension two with a nasty ramification divisor.

IV.2. Orders of finite representation type.

First we will review Morita equivalence for reflexive modules. Let Λ be a tame order over a Noetherian local domain R in a central simple algebra Σ, and let M, N be a left and a right Λ-module. Since Λ is a tame order, M_p and N_p are left resp. right projective Λ_p-modules, whence $(N \underset{\Lambda}{\otimes} M)_p$ is torsion free. We denote by the modified tensor product $N \perp_\Lambda M$ the reflexive hull of $N \underset{\Lambda}{\otimes} M$, it is a reflexive R-module and the canonical morphism :

$$N \underset{\Lambda}{\otimes} M \to N \perp_\Lambda M$$

has kernel and cokernel of finite length. Given a nonzero right reflexive Λ-module N, then $\Gamma = \mathrm{End}_\Lambda(N)$ is a tame order and there is a Morita equivalence : left reflexive Λ-modules $\leftrightarrow$ Γ-modules

$$\begin{array}{ccc} M & \longmapsto & N^* \perp_\Lambda M \\ N \perp_\Gamma M & \longmapsto & M \end{array}$$

which follows immediately from reflexivity of the Λ-modules and the usual Morita equivalence after localizing at height one prime ideals of R. Note that the equivalence can also be defined by $M \longmapsto \mathrm{Hom}_\Lambda(N, M)$ since $\mathrm{Hom}_\Lambda(N, M) \cong N^* \perp'_\Lambda M$.

IV.2.1. Definition. An order Λ which is a Cohen-Macaulay module over R is said to be **of finite representation type** if the set of

isomorphism classes of finitely generated, indecomposable left Cohen-Macaulay Λ-modules is finite.
Of course, by Theorem IV.1.9. it is clear that having finite representation type is a necessary condition for an order to be moderated regular.
From now on we will assume that R is complete. Actually, it would be sufficient to assume R to be Henselian since we want a Krull-Remak-Schmidt-Azumaya decomposition for reflexive modules over R. In Krull dimension two, one has the following important result due to M. Auslander and I. Reiten.

IV.2.2. Theorem : Let R be a complete normal domain of Krull dimension two. Let Λ be a tame order over R, in a central simple algebra Σ, of finite representation type and let M be a left Cohen-Macaulay Λ-module containing every indecomposable left Cohen-Macaulay Λ-module as a direct summand, then $\Gamma = \mathrm{End}_\Lambda(M)$ is of global dimension two.

Proof. Let $M = \bigoplus_{i=1}^{n} M_i$ be a decomposition of a left Λ- module M into indecomposable modules which are Cohen-Macaulay R-modules, each isomorphism class occuring at least once. Then we can write

$$\Gamma = \mathrm{End}_\Lambda(M) = \sum_{i,j=1}^{n} \Gamma_{ij} = \sum_{i,j=1}^{n} e_i.\Gamma.e_j$$

where $\Gamma_{ij} = \mathrm{Hom}_\Lambda(M_i, M_j)$ and e_i is the projection of M onto M_i. By the Morita-equivalence for reflexive modules between Λ and Γ_{ii}, it follows that the i-th row $\Gamma_{i1}, \ldots, \Gamma_{in}$ lists all indecomposable left Γ_{ii}-modules which are Cohen-Macaulay over R. Since the situation is symmetric with respect to a permutation of the indices i, it suffices to show that $S_1 = \Gamma_{11}/J(\Gamma_{11})$ has projective dimension two.
Let $P_j = \Gamma.e_j$ be the j-th column of Γ, then P_j is clearly a projective left Γ-module and $\Gamma_{ij} = \mathrm{Hom}_\Lambda(M_1, M_j) = \mathrm{Hom}_\Gamma(P_i, P_j)$. Now, take a two-step resolution of S_1 as a left Γ-module :

$$0 \to N \to \bigoplus_k P_{j_k} \to P_1 \to S_1 \to 0$$

Considering its top row, we obtain

$$0 \to L \to \bigoplus_k M_{j_k} \to M_1 \to S_1 \to 0$$

as left Λ-modules. Therefore, L is a left Cohen-Macaulay Λ-module, whence $L \cong \bigoplus_l M_{i_e}$ as left Λ-modules. Because $P_j = \bigoplus_i \Gamma_{ij} = \bigoplus_i \mathrm{Hom}_\Lambda (M_i, M_j) = \bigoplus_i \mathrm{Hom}_\Lambda(M, M_j)$ as left Λ-modules and $\mathrm{Hom}_\Lambda(M, -)$ is left exact it follows from the exact sequence

$$0 \to \bigoplus_l M_{i_l} \to \bigoplus_k M_{j_k} \to M_1$$

that

$$0 \to \bigoplus_l P_{i_l} \to \bigoplus_k P_{i_k} \to P_1 \to S_1 \to 0$$

is the required resolution of S_1. The converse implication follows from Morita-equivalence and Theorem IV.1.9. □

That these rings are even moderated Gorenstein, follows from the next observation :

IV.2.3. Lemma : If Λ is a tame order of global dimension two, then Λ is moderated regular.

Proof. Let M be a maximal twosided ideal of Λ such that Λ-Ext dim $(\Lambda/M) < 2$. Then, the exact sequence :

$$0 \to M \to \Lambda \to \Lambda/M \to 0$$

gives rise (for every left Λ-module B) to a sequence :

$$\to \mathrm{Ext}^1_\Lambda(\Lambda, B) \to \mathrm{Ext}^1_\Lambda(M, B) \to \mathrm{Ext}^2_\Lambda(\Lambda/M, B) \to$$

Here, the last term vanishes since Λ-Ext dim $(\Lambda/M) < 2$ and the first term because Λ is projective. Hence $\mathrm{Ext}(M, B) = 0$ for all B, so M is projective. But then M has to be reflexive and hence of height one, a contradiction. □

The above lemma and Theorem IV.2.2. allow us to replace the problem of classifying moderated regular algebras of dimension two by the equivalent one of classifying tame orders of finite representation type and indecomposable (left) Cohen-Macaulay modules.
Before we can give our construction of tame algebras of finite representation type, we put forward some remarks on the invariance of finite representation type under change of ring.

IV.2.4. Lemma : Let $\Lambda \subset \Gamma$ be an extension of tame orders which are both Cohen-Macaulay modules over R and assume that Λ is a Λ-bimodule direct summand of Γ. If Γ has finite representation type, then so has Λ.

Proof. Assume that $\Gamma \cong \Lambda \oplus \Gamma_0$ as Λ-bimodules. Let M be an indecomposable left Cohen-Macaulay Λ-module, then :

$$\Gamma \perp_\Lambda M \cong M \oplus (\Gamma_0 \perp_\Lambda M)$$

By the Krull-Schmidt theorem, M is isomorphic to a Λ-direct summand of an indecomposable Γ-summand N of $\Gamma \perp_\Lambda M$. If Γ has finite representation type, there are finitely many possibilities for N and for M.

□

IV.2.5. Lemma : Let $\Lambda \subset \Gamma$ be an extension of tame orders both of which are Cohen-Macaulay R-modules. Assume that the natural map

$$\Gamma \perp_\Lambda \Gamma \to \Gamma$$

splits as a Γ-bimodule map. If Λ has finite representation type, then so does Γ.

Proof. Let M be an indecomposable left Cohen-Macaulay module and let $M = \oplus M_i$ be a decomposition of M in indecomposables over Λ. Because

$$\Gamma \perp_\Lambda M = \Gamma \perp_\Lambda \Gamma \perp_\Gamma M$$

and taking into account the assumption on the splitting, we may deduce that M is a Γ-direct summand of $\Gamma \perp_\Lambda M$. This entails that M is a Γ-direct summand of one of the $\Gamma \perp_\Lambda M_i$, so there can only be a finite number of possibilities for M. □

IV.2.6. Lemma : If Λ is a reflexive Azumaya algebra over a finite commutative extension S of R, then Λ has finite representation type if S has.

Proof. Let M be an indecomposable left Cohen-Macaulay Λ-module, then $M \perp_S \Lambda$ is a Cohen-Macaulay Λ-bimodule. Let $M \perp_S \Lambda = \oplus M_i$, be the decomposition of $M \perp_S \Lambda$ in indecomposable Cohen-Macaulay Λ-bimodules, then it is a direct summand of some M_i. Because there is an equivalence of categories between reflexive R-lattices and the reflexive Λ-bimodules, there are only a finite number of different M_i's, whence a finite number of possibilities for M. □

Let G be a finite group. If Λ is a G-graded ring we may form the groupring ΛG and we equip it with a G-gradation given by the formula $\deg(\Lambda_\sigma.\tau) = \tau$. Then, the ring $S = \sum_{\sigma \in G} \Lambda_\sigma.\sigma$ is a G-graded subring of ΛG which is graded isomorphic to Λ. From Chapter II we recall that there is a Maschke-type theorem between S and ΛG.

We say that a graded ring (by an arbitrary group G) is of **graded finite representation type**, if there are only a finite number of graded isomorphism classes of graded indecomposable, graded left Cohen-Macaulay modules. An immediate consequence is :

IV.2.7. Proposition : Let Λ be graded by a finite group G such that $|G| \in \Lambda^*$. Then if Λ is of graded finite representation type, Λ is of finite representation type.

Proof. If R is of graded finite representation type, then so is S because they are graded isomorphic. It is easy to verify that the natural morphism $\Lambda G \underset{S}{\otimes} \Lambda G \to \Lambda G$ is graded split. So, by a graded version of Lemma IV.2.5., ΛG is of graded finite representation type. In view of the natural category equivalence between ΛG-gr and Λ-mod, it follows that Λ is of finite representation type. □

Now, let R be a complete (or Henselian) local normal domain; $\mathcal{D} = \{p_1, \dots, p_n\} \subset X^{(1)}(R)$ are torsion elements of the class group $Cl(R)$ of R. If $g = \{g_1, \dots, g_n\}$ is a set of natural numbers, we recall that $R[\mathcal{D}, g]$ is the $\mathbb{Z}^{(n)}$-graded subring of $K[X_1, X_1^{-1}, \dots .X_n, X_n^{-1}]$ where $\deg(X_i) = (0, \dots, 1, \dots, 0)$, whose part of degree $(m_1, \dots, m_n)$ is given by

$$R[\mathcal{D}, g]_{(m_1,\dots,m_n)} = (p_1^{[\frac{m_1}{g_1}]} * \dots * p_n^{[\frac{m_n}{g_n}]})X_1^{m_1} \dots X_n^{m_n}$$

where $[\frac{a}{b}]$ is the least integer $\geq \frac{a}{b}$.

It is easy to verify that $R[\mathcal{D}, g]$ contains the $\mathbb{Z}^{(n)}$-graded Rees ring

$$T = \sum_{(j_1,\dots,j_n) \in \mathbb{Z}^n} (p_1^{j_1 l_1}) * \dots * p_1^{j_1 l_1} X_1^{j_1 l_1 g_1} \dots X_n^{j_n l_n g_n}$$

where we take l_i to be the order of $[p_i]$ in $Cl(R)$. So, $p_i^{l_i} = Ra_i$ for some element $a_i \in R$.

Then we define the **roll-up** of $\mathcal{R}[\mathcal{D}, g]$ to be the ring

$$R[\mathcal{D}, g]_! = R[\mathcal{D}, g]/(1 - a_1 X_1^{l_1, g_1}, \dots, 1 - a_n X^{l_n g_n})$$

which is clearly a $(\mathbb{Z}/l_1 g_1 \mathbb{Z}) \times \dots \times (\mathbb{Z}/l_n g_n \mathbb{Z})$ graded ring in the natural way.

Further, one verifies easily that there is an equivalence of categories between $R[\mathcal{D}, g]$-gr (as $\mathbb{Z}^{(n)}$-graded) and $R[\mathcal{D}, g]_!$-gr (as $\times \mathbb{Z}/e_i g_i \mathbb{Z}$-graded). We now aim to investigate the relation between (graded) finite representation type of $R, R[\mathcal{D}, g]$ and $R[\mathcal{D}, g]_!$. By Proposition IV.2.7. we know that graded finite representation type implies finite representation type for $R[\mathcal{D}, g]_!$ provided that the l.c.m.$(g_1 l_1, \dots, g_n l_n) \in R^*$.

Recall from the foregoing chapter, that whenever Λ is a tame order, there are only finitely many "prime"-divisors $P \in X^{(1)}(\Lambda)$ s.t. $P \neq \Lambda.(P \cap R)^{**}$. This finite set was denoted by $\mathcal{P}$ and called the ramification divisor of Λ. With P^c we denote $P \cap R$. Further, for every $P_i \in \mathcal{P}$ there is an $e_i \in \mathbb{N}$ s.t. $(P_i^{e_i})^{**} = \Lambda(P_i \cap R)^{**}$. The set $\underline{e} = \{e_i : P_i \in \mathcal{P}\}$ is said to be the set of ramification indices.

We can now state and prove the main result of this section.

IV.2.8. Theorem : If Λ is a tame order over a complete (or Henselian) local normal domain R satisfying (Et_1). Then Λ is of finite representation type if and only if $R[\mathcal{P}^c, e]_!$ is of finite representation type, provided *l.c.m.*$(\underline{e}) \in R^*$.

Proof. Suppose first that Λ is of finite representation type, then the usual generalized Rees ring $\Lambda[\mathcal{P}]$ is of $\mathbb{Z}^{(n)}$-graded finite representation type by the equivalence of categories between left reflexive graded graded $\Lambda[P]$-modules and reflexive left Λ-modules. By the fact that $\Lambda[\mathcal{P}]$ is a $\mathbb{Z}^{(n)}$-graded reflexive Azumaya algebra and using a graded version of Lemma IV.2.6., this entails that $R[\mathcal{P}^c, e]$ is of $\mathbb{Z}^{(n)}$-graded

finite representation type. By the equivalence of categories between $R[\mathcal{P}^c, e]$-gr and $R[\mathcal{P}^c, e]_!$-gr, the roll-up is of $(\times \mathbb{Z}/e_i\mathbb{Z})$ graded finite representation type, whence of finite representation type by Proposition IV.2.7. .
Conversely, if the roll-up $R[\mathcal{P}, e]_!$ is of finite representation type, then $R[\mathcal{P}, e]$ is of $\mathbb{Z}^{(n)}$-graded representation type, whence so is $\Lambda[\mathcal{P}]$ by a graded version of Lemma IV.2.5. . Finally, the equivalence of categories between Λ-ref and $\Lambda[\mathcal{P}]$-gref entails that Λ is of finite representation type, done what establishes the claims. □

We will now apply this result to construct tame orders of finite representation type having a nasty ramification divisor.

IV.2.9. Example : Let F be a field of characteristic different from 2 or 3 and consider $S = F[[x,y]]$. Consider the group

$$S_3 = < \sigma, \tau : \sigma^2 = 1, \tau^3 = 1, \sigma\tau\sigma = \tau^2 >$$

then S_3 acts in a natural way on S by the representation :

$$\sigma \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau \to \begin{pmatrix} \xi & 0 \\ 0 & \xi^2 \end{pmatrix}$$

where ξ is a primitive 3^{rd} root of unity. Then, the algebra of invariants

$$F[[x,y]]^{S_3} = R = F[[x^3 + y^3, xy]]$$

is a regular algebra
If we denote $u = x^3 + y^3$ and $v = xy$, then :

$$du = 3x^2dx + 3y^2dy$$

$$dv = ydx + xdy$$

whence the ramification divisor of the extension $R \subset S$ is given by

$$\det \begin{pmatrix} 3x^2 & 3y^2 \\ y & x \end{pmatrix} = 3x^3 - 3y^3 = 0$$

Therefore, $v^3 = x^6 = \frac{u^2}{4}$ and therefore the ramification divisor has a cusp.
Now, consider the skew group ring $\Lambda = S * S_3$ i.e. with commutation rule $s.\sigma = \sigma.\sigma(s)$, then Λ is an order in a matrixring and has the same ramification divisor as $R \subset S$. Furthermore, as Λ is strongly graded over the regular domain S, Λ has global dimension two and is tame whence moderated regular.

IV.3. Regularizable Domains.

In this section we aim to characterize those normal domains over R such that for some subset $\mathcal{D} \subset X^{(1)}(R)$ and for some set of natural numbers $g \in \mathbb{N}^{(n)}$ the scaled Rees ring $R[\mathcal{D}, g]$ is a graded regular domain in the sense that every finitely generated graded $R[\mathcal{D}, g]$-module has a finite resolution by graded projective modules.
If the Krull dimension of R is two we will give a complete description in terms of finite representation type.
Let us start by investigating the connection between regularity of R and graded regularity of $R[\mathcal{D}, g]$.

IV.3.1. Lemma : Let R be a Noetherian graded local domain with unique graded maximal ideal m. Then R is graded regular if and only if the (graded) dimension of m/m^2 over the graded field R/m is equal to the graded Krull dimension of R.

Proof. Clearly, it is sufficient to prove that R_m is a regular domain. In order to do this we have to prove that P is graded projective if P is a f. g. graded R-module such that P_m is R_m-projective. So, we have shown that $\mathrm{Hom}_R(P, -)$ is exact in R-gr. Let $f : M \to$ be an epimorphism of graded R-modules and let T be the cokernel of the induced map $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ then it follows from our assumption that $T_m = 0$. Now, let t be an homogeneous element of T, then there is an element $\mu \in R - m$ such that $\mu t = 0$. Decompose μ in its homogeneous components $\mu = \mu_{\sigma_1} + \ldots + \mu_{\sigma_n}$ then at least one of the μ_{σ_i} is not in

m. But this implies that $t = 0$. □

IV.3.2. Lemma. If R is a regular (local) domain, then the generalized Rees ring $R[\mathcal{D}]$ is graded regular for every set of divisors $\mathcal{D}$.

Proof. This follows trivially from the equivalence of categories between R-mod and $R[\mathcal{D}]$-gr which exists since every element of $\mathcal{D}$ is invertible (R is factorial !). □

We can now give a complete solution to the question when a scaled Rees ring $R[\mathcal{D}, g]$ is graded regular.

IV.3.3. Theorem : Let R be a regular local domain. Let $\mathcal{D} = \{p_1, \ldots, p_n\} \subset X^{(1)}(R)$ and $g = \{g_1, \ldots, g_n\}$ with all $g_i > 1$. Then the scaled Rees ring is regular if and only if the generators of the principal prime ideals p_i from part of a regular system of parameters of R.

Proof. Define $S_0 = R$ and by induction

$$S_{i+1} = \sum_{j \in \mathbb{Z}} p_{i+1}^{[\frac{j}{g_{i+1}}]}.S_i.X_{i+1}^j$$

Let M_{i+1} be the unique maximal graded ideal of S_{i+1}. The reader can easily verify that

$$M_{i+1} = \sum_{j \in \mathbb{Z}} I_j.p_{i+1}^{[\frac{j}{g_{i+1}}]}.S_i.X_{i+1}^j$$

where $I_j = M_i$ if $g_{i+1} + j$ and $I_j = S_i$ if $g_{i+1}|j$. Calculating the graded dimension of M_{i+1}/M_{i+1}^2 over S_{i+1}/M_{i+1} gives us

$$1 + \dim_{S_i/M_i}(M_i/M_i^2 + p_{i+1}S_i)$$

and therefore S_{i+1} is graded regular if and only if S_i is graded regular and $p_{i+1} \not\subset (M_i^2)_0$. Calculating $(M_{i+1}^2)_0$ yields $(M_i^2)_0 + p_{i+1}$ whence $(M_i^2)_0 = M_0^2 + \sum_j p_j$. Therefore, S_{i+1} is graded and $p_{i+1} \not\subset M_0^2 + p_1 + \ldots + p_i$, finishing the proof. □

As we will see below, there exist non-regular normal domains such that $R[\mathcal{D}, g]$ is graded regular for some suitable choice of $\mathcal{D}$ and g. The next theorem characterizes those normal domains.

IV.3.4. Theorem : Let R be a local normal domain containing an algebraically closed field of characteristic zero.
There exists a set of divisors $\mathcal{D} \subset X^{(1)}(R)$ representing torsion elements in the classgroup $Cl(R)$ and a set of natural numbers g such that the scaled Rees ring $R[\mathcal{D}, g]$ is graded regular if and only if there exists a regular overring S of R, finitely generated as R-module and a finite Abelian group G acting on S such that $S^G = R$.

Proof. Let S be the roll-up of $R[\mathcal{D}, g]$. Then S is graded regular because of the equivalence of categories of $R[\mathcal{D}, g]$-gr and S-gr contains a field of characteristic zero. It follows from the results of the foregoing chapter that S is regular.
Because R contains all roots of unity, the gradation of S may be changed into a group action having the required properties.
Conversely, because S is a regular ring, it is a direct sum of regular domains each containing R. Therefore, we can replace S by a domain and G by a subgroup of G. Since R contains an algebraically closed field, the group action of G may be turned into a gradation of S by G^*. Therefore, $S = \bigoplus_{\sigma \in G^*} I_\sigma$ with $I_e = R$. Since S is a reflexive R-lattice each of the I_σ is a reflexive R-lattice. Further, since S is a domain this entails that every I_σ has rank either zero or one. If rank $_R(S) = n$, then the class group of R is n-torsion. Let $\mathcal{D} = \{p_1, \ldots, p_k\}$ be the set of prime factors occuring in the decomposition of all $I_\sigma, \sigma \in G^*$ and let $\mathcal{D}' = \{(p_1S)^{**}, \ldots, (p_kS)^{**}\}$. Since each $(p_iS)^{**}$ is invertible, it is clear from the category-equivalence between S-mod and $S[\mathcal{D}]$-gr that $S[\mathcal{D}']$ is graded regular. Further, $S[\mathcal{D}']$ is a graded extension of $R[\mathcal{D}]$ entailing that $R[\mathcal{D}]$ is graded regular. □

If the Krull dimension of R is two, we can give an intrinsic characterization of regularizable domains.

IV.3.5. Theorem : Let R be a local normal domain of Krull dimensions two. There exists a set of divisors $\mathcal{D} = \{p_1, \ldots, p_n\} \subset X^{(1)}(R)$ representing torsion elements of $Cl(R)$ and a set of natural numbers $g = \{g_1, \ldots, g_n\}$ such that the scaled Rees ring $R[\mathcal{D}, g]$ is graded regular if and only if R has only a finite number of isomorphism classes of indecomposable reflexive modules, all of which are of rank one.

Proof. Since each one of the p_i is torsion in $Cl(R)$, the graded Krull dimension of $R[\mathcal{D}, g]$ is two, whence every graded reflexive $R[\mathcal{D}, g]$-module is graded free. Therefore, if M_0 is an indecomposable reflexive R-module, then $(M_0 R[\mathcal{D}, g])^{**}$ is graded free, yielding that $M_0 = (M_0 R[\mathcal{D}, g])^{**}_e$ is a direct sum of divisors which are products of the p_i's. Therefore, each indecomposable reflexive R-module has rank one. Since every p_i is assumed to be torsion, there are only a finite number of isomorphism classes.

Conversely, if R has only a finite number of isomorphism classes of indecomposable reflexive ideals, $Cl(R)$ is finite. Let $\mathcal{D} = \{p_1, \ldots, p_k\}$ be the prime factors of these indecomposable reflexives. Since there is an equivalence of categories between R-ref and $R(\mathcal{D})$-gref, every graded reflective $R[\mathcal{D}]$-module is gr-free, yielding that $R[\mathcal{D}]$ has graded global dimension two, finishing the proof. □

This provides us with a list of examples of non-regularizable domains.

IV.3.6. Example : Let R be the local normal domain $\mathbb{C}[[x, y, z]]$ $(XY - Z^2)$, then it is well-known that $Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$ and is generated by the ruling $p = (Y, Z)$. The Rees ring $R[\mathcal{D}]$ can be visualized as

$$\ldots \oplus (Y^{-1})X_1^{-2} \oplus p^{-1}X_1^{-1} \oplus \ldots \oplus R \oplus pX_1 \oplus (Y)X_1^2 \oplus p(Y)X_1^3 \oplus \ldots$$

and using Lemma IV.3.1. one easily shows that this ring is graded regular.

IV.4. Smooth Tame Orders.

We have seen in Section IV.2. that moderated regular orders can have a weird ramification divisor, even in dimension two. In this section we aim to study the Zariski and étale local stucture of moderated regular orders with an extremely nice ramification divisor : smooth orders.

IV.4.1. Definition : A tame order Λ over a normal local domain R is said to be smooth if there exists a set of divisors $\mathcal{D}$ of Λ such that the generalized Rees ring $\Lambda[\mathcal{D}]$ is an Azumaya algebra over a graded regular center and if every element of $\mathcal{D}$ is invertible.

Clearly, in view of the equivalence of categories between Λ-mod and $\Lambda[\mathcal{D}]$-gr, it follows that Λ has finite global dimension and is even moderated regular. Furthermore, it is trivial to verify that the class of smooth tame orders in closed under taking matrix rings and polynomial extensions. In most applications $\mathcal{D} = \mathcal{P}$ the ramification divisor of Λ.. However, there are smooth, tame orders Λ such that $\Lambda[\mathcal{P}]$ is not Azumaya.

IV.4.2. Example : Let R be the normal local domain $\mathbb{C}[[x, y, z]]/(xy - z^2)$ and let $p = (Y, Z)$ be the ruling which generates the classgroup. Let Λ be the reflexive Azumaya algebra :

$$\Lambda = \mathrm{End}_R(R \oplus p) = \begin{bmatrix} R & p \\ p^{-1} & R \end{bmatrix}$$

Now, let $\mathcal{D} = \{\Lambda\left(\begin{smallmatrix} 0 & y \\ 1 & 0 \end{smallmatrix}\right)\}$ which is clearly an invertible twosided Λ-ideal.

Then $\Lambda[\mathcal{D}]$ is seen to be the $\mathbb{Z}$-graded ring

$$\Lambda[\mathcal{D}] \cong \Lambda\left[\left(\begin{pmatrix}0 & y\\ 1 & 0\end{pmatrix}\right).X_1, \begin{pmatrix}0 & 1\\ y^{-1} & 0\end{pmatrix} X_1^{-1}\right]$$

which is easily checked to be an Azumaya algebra since its center is a graded regular domain of graded dimension two :

$$\ldots \oplus (Y^{-1})X_1^{-2} \oplus p^{-1}X_1^{-1} \oplus R \oplus pX_1 \oplus (Y)X_1^2 \oplus p(Y)X_1^3 \oplus \ldots$$

So, Λ is smooth, whereas $\Lambda[P] = \Lambda$ is not an Azumaya algebra.

For the rest of this section, we will impose the following extra assumptions : $\mathcal{D} = \mathcal{P}$ and Λ is an order in a p^2-dimensional division algebra Δ, p being a prime number. Furthermore, we assume that the ramified height one prime ideals of Λ are generated by a normalizing element. We believe that this condition is often (if not always) satisfied for smooth tame orders. The first consequence of the dimension assumption is that not too many primes can be ramified.

IV.4.3. Lemma : If Λ is a smooth tame order over a local normal domain R in a p^2-dimensional division algebra Δ, then $\#\mathcal{P} \leq 2$.

Proof. Let $n = \#\mathcal{P}$, then $\Lambda[\mathcal{P}]$ is a $\mathbb{Z}^{(n)}$-graded Azumaya algebra over its center $R[\mathcal{P}^c, e]$ which is a graded ring with unique maximal ideal

$$m[\mathcal{P}^c, e] = \sum_{\sigma \in H} m.R[\mathcal{P}, e]_\sigma \oplus \sum_{\sigma \in G \setminus H} R[\mathcal{P}^c, e]_\sigma$$

where $G = \mathbb{Z}^{(n)}$ and $H = p\mathbb{Z} \oplus \ldots \oplus p\mathbb{Z}$. But then we must have that $\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}].m[\mathcal{P}^c, e]$ is a $\mathbb{Z}^{(n)}$-graded central simple algebra of dimension p^2 over the $\mathbb{Z}^{(n)}$-graded field

$$R[\mathcal{P}^c, e]/m[\mathcal{P}^c, e] \cong R/m[X_1^p, X_1^{-p}; \ldots; X_n^p, X_n^{-p}]$$

Since we may assume that every prime ideal $P_i, 1 \leq i \leq n$, is generated by a normalizing element, an easy computation shows :

$$\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}]m[\mathcal{P}^c, e] = \bigoplus_{0 \leq i_j < p} \Lambda/(\Lambda m + P_1 + \ldots + P_n)X^{i_1} \ldots X_n^{i_n}$$

the isomorphism being one of graded $R/m[X_1^p, X_1^{-p}, \ldots, X_n^p, X_n^{-p}]$ modules. Calculating the dimensions on both sides yields :

$$p^2 = p_n.\dim_{R/m}(\Lambda/(\Lambda m + P_1 + \ldots + P_n))$$

which immediately implies that $n \leq 2$. $\square$

If $dim(\Delta) \neq p^2$ one can have a worse ramification divisor. For example, it is perfectly possible to have smooth maximal orders over regular local domains of dimension 4 in division algebras of dimension 16 with 4 central ramified height one primes, each having ramification index 2.

Let us recall from algebraic geometry the definition of a set of regular divisors with normal crossings. We say that a set of divisors $\mathcal{D} = \{D_i; i \in I\}$ has **strictly normal crossings** if for every prime ideal q of R lying in $\cup$ Supp(D_i) we have the next property : if $I_q = \{i : q \in \text{Supp}D_i\}$ then for $i \in I_q$ we have that $D_i = \sum_\lambda \text{div}(x_{i,\lambda})$ with $x_{i,\lambda} \in R_q$ and $\{x_{i,\lambda}\}_{i,\lambda}$ part of a regular system of parameters in R_q. We say that the set $\mathcal{D}$ **has normal crossings** if for every $q \in \cup\text{Supp}(D_i)$ there exists an étale neighbourhood of q in Spec(R), Spec(S) say, such that the family of inverse images of $\mathcal{D}$ on Spec(S) have strictly normal crossings. Finally, a divisor D of R is called **regular at** $q \in \text{Supp}(D)$ if the subscheme D of Spec(R) is regular at q. The divisor D is called **regular** if it is regular everywhere. For more details, the reader is referred to the monograph of Murre and Grothendieck [34].

Combining the foregoing result with the characterization of regularizable domains we get.

IV.4.4. Theorem : If Λ is a tame order over a local domain R in a p^2-dimensional division algebra Δ, then Λ is smooth if

(a) R has a regular ramification divisor with normal crossings in Σ.

(b) One of the following three cases occurs

case 0 : $\mathcal{P} = \emptyset$ i.e. Λ is an Azumaya algebra

case 1 : $\mathcal{P} = \{P\}$ and $\dim_{R/m}(\Lambda/\Lambda m + P) = p$

case 2 : $\mathcal{P} = \{P, Q\}$ and $\dim_{R/m}(\Lambda/\Lambda m + P + Q) = 1$

Let us give some explicit examples of smooth maximal orders in quaternion-algebras.

IV.4.5. Example : Let $\Lambda = \mathbb{C}[[X,-]]$ be the ring of skew formal power series over $\mathbb{C}$, where $-$ denotes the complex conjugation. It is clear that Λ is a maximal order with center $R = \mathbb{R}[t]]$ where $t = X^2$ and that $\mathcal{P} = \{(X)\}$. So, $\Lambda[\mathcal{P}]$ is the $\mathbb{Z}$-graded ring.

$$\ldots \oplus (X^{-2})X_1^{-2} \oplus (X^{-1})X_1^{-1} \oplus \Lambda \oplus (X)X_1 \oplus (X^2)X_1^2 \oplus \ldots$$

and $R[\mathcal{P}^c, e]$ is the $\mathbb{Z}$-graded ring

$$\ldots \oplus (t^{-1})X_1^{-2} \oplus RX_1^{-1} \oplus R \oplus (t)X_1 \oplus (t)X_1^2 \oplus (t^2)X_1^3 \oplus \ldots$$

Furthermore, we have

$$\dim_{\mathbb{R}[[t]]/(t)}(\mathbb{C}[[X,-]]/(X)) = \dim_{\mathbb{R}}\mathbb{C} = 2$$

so, Λ is smooth over R and in case 1.
Further, $R[\mathcal{P}^c, e]/m[\mathcal{P}^c, e] \cong \mathbb{R}[t_1, t_1^{-1}]$ where $t_1 = tX_1^2$ and $\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}]$ $m\ [\mathcal{P}^c, e]$ is the $\mathbb{Z}$-graded central simple algebra $\mathbb{C}[Y_1, Y_1^{-1}, \ldots]$ with $Y_1 = XX_1$ over $\mathbb{R}[t_1, t_1^{-1}]$.

IV.4.6. Example : Let R be a regular local domain of dimension two such that x and y generate the maximal ideal m. Let Δ be the quaternion-algebra $\left(\frac{x,y}{K}\right)$ and let

$$\Lambda = R.1 \oplus R.i \oplus R.j \oplus Rij$$

with the obvious relations, i.e. $i^2 = x, j^2 = y$ and $ij = -ji$. Then one can verify that Λ is a maximal R-order. Clearly, $\mathcal{P}^c = \{(x), (y)\}$ which is a set of regular ramification divisors with normal crossings. The generalized Rees ring $\Lambda[\mathcal{P}]$ is a $\mathbb{Z} \oplus \mathbb{Z}$-graded ring which can be visualized as (omitting powers of X_1 and X_2) :

$$\begin{array}{ccccccccccccc}
\ldots & \oplus & (i^{-2}j^2) & \oplus & (i^{-1}j^2) & \oplus & (jj^2) & \oplus & (ij^2) & \oplus & (i^2j^2) & \oplus & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\ldots & \oplus & (i^{-2}j) & \oplus & (i^{(-1)}j) & \oplus & (j) & \oplus & (ij) & \oplus & (i^2j) & \oplus & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\ldots & \oplus & (i^{-2}) & \oplus & (i^{-1}) & \oplus & \Lambda & \oplus & (i) & \oplus & (i^2) & \oplus & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\ldots & \oplus & (i^{-2}j^{-1}) & \oplus & (i^{-1}j^{-1}) & \oplus & (j^{-1}) & \oplus & (ij^{-1}) & \oplus & (i^2j^{-1}) & \oplus & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\ldots & \oplus & (i^{-2}j^{-2}) & \oplus & (i^{-1}j^{-2}) & \oplus & (j^{-2}) & \oplus & (ij^{-2}) & \oplus & (i^2j^{-2}) & \oplus & \ldots
\end{array}$$

and its center $R[\mathcal{P}, e]$ is the $\mathbb{Z} \oplus \mathbb{Z}$-graded ring which looks like :

$$\begin{array}{ccccccccccccccc}
 & \oplus & (X^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) & \oplus & \ldots & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
 & \oplus & (X^{-1}y) & \oplus & (y) & \oplus & (y) & \oplus & (xy) & \oplus & (xy) & \oplus & \ldots & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
\ldots & \oplus & (X^{-1}) & \oplus & R & \oplus & R & \oplus & (X) & \oplus & (X) & \oplus & (X^2) & \oplus & \ldots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
\ldots & \oplus & (X^{-1}) & \oplus & R & \oplus & R & \oplus & (X) & \oplus & (X) & \oplus & \ldots & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
\ldots & \oplus & (X^{-1}y^{-1}) & \oplus & (y^{-1}) & \oplus & (y^{-1}) & \oplus & (xy^{-1}) & \oplus & (xy^{-1}) & \oplus & \ldots & &
\end{array}$$

Further, $\dim_{R/m}(\Lambda/(\Lambda m + (i) + (j)) = \dim_{R/m}(R/m)$ i.e. Λ is a smooth maximal order of case 2. $R[\mathcal{P}^c, e]/m[\mathcal{P}^c, e] \cong R/m[Y_1^2, Y_1^{-2}, Y_2^2, Y_2^{-2}]$ where $Y_1 = iX_1$ and $Y_2 = jX_2$, whereas $\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}]m[\mathcal{P}^c, e]$ is the $\mathbb{Z} \oplus \mathbb{Z}$-graded central simple algebra $R/m[Y_1, Y_1^{-1}][Y_2, Y_2^{-1}, \sigma]$ where $\sigma(Y_1) = -Y_1$. Its homogeneous part of degree $(0,0)$ is equal to R/m corresponding to the fact that Λ is local with unique twosided ideal $M = (i,j)$.

IV.4.7. Example : Let F be any field with characteristic unequal to 2. Let R be $F[X,Y]_{(X,Y)}$ where X and Y are indeterminates. Let Δ be the quaternion algebra $\left(\frac{X\ \ 1+Y}{K}\right)$ and let

$$\Lambda = R.1 \oplus R.i \oplus .j \oplus R.ij$$

then Λ is a maximal order. Here $\mathcal{P} = \{(i)\}$ and the central ramification divisor is $\mathcal{P}_c = \{(X)\}$ and $(X) \not\subset m^2$. Since

$$\dim_{R/m}(\Lambda/\Lambda m + (i)) = \dim_F(F \oplus F) = 2$$

Λ is smooth of case 1. We have that $R[\mathcal{P}^c, e]/m[\mathcal{P}^c, e] \cong F[Y_1^2, Y_1^{-2}]$ where $Y_1 = iX_1$ and the quotient $\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}].m[\mathcal{P}^c, e]$ is the $\mathbb{Z}$-graded algebra

$$(F \oplus F\epsilon)[Y_1, Y_1^{-1}, \varphi] \xrightarrow[\cong]{\alpha} M_2(F[Y_1^2, Y_1^{-2}])$$

where $\varphi(a, b\epsilon) = (a, -b\epsilon)$ and α is given by

$$\alpha(1,0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \alpha(0,\epsilon) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\alpha(Y_1) = \begin{pmatrix} 0 & 1 \\ Y_1^2 & 0 \end{pmatrix}$$

Therefore, $\Lambda[\mathcal{P}]/\Lambda[\mathcal{P}^c, e]$ is a $\mathbb{Z}$-graded central simple algebra over $F[Y_1^2, Y_1^{-2}]$. However, its homogeneous part of degree zero is $F \oplus F\epsilon$ corresponding to the fact that Λ is not local. Each factor corresponds to one of the two maximal ideals of Λ lying over $m = (X, Y)$:

$$M_1 = \Lambda(i, j-1); M_2 = \Lambda(i, j+1)$$

In Theorem IV.4.4. we have characterized smooth tame orders with a regular local center. We will now study their Zariski local structure over such a domain. By this we mean the number of conjugacy classes of smooth orders. One of the basic ingredients in the proof will be the following result of A. Grothendieck [34, 2.5.8.].

IV.4.8. Theorem : Let R be a Noetherian (semi) local ring, Λ an R-algebra, which is a finite R-module and M_1, M_2 finitely generated left Λ-modules. Let $\varphi : R \to S$ be a faithfully flat morphism where S is also Noetherian. If $M_1 \otimes S \cong M_2 \otimes S$ as left $\Lambda \otimes S$-modules, then $M_1 \cong M_2$ as left Λ-modules.

In view of this result, it will be sufficient to calculate the conjugacy classes of the extended orders $\Lambda \underset{R}{\otimes} R^{sh}$ where Λ is a smooth maximal R-order and R^{sh} denotes the strict Henselization of R. Case 0 is easy. For, if one maximal order over R in Δ is Azumaya, then every smooth maximal order Λ is Azumaya, too. Since $\mathrm{Br}(R^{sh}) = 0$ we

have $\Lambda \underset{R}{\otimes}(R^{sh}) \cong M_n(R^{sh}) \cong \Gamma \underset{R}{\otimes} R^{sh}$ and by descent $\Lambda \cong \Gamma$ as R-algebras (apply Grothendieck's descent result to the conductor of Λ in Γ i.e. $\{x \in \Delta : \Lambda x \subset \Gamma\}$), so by the Skolem-Noether theorem they are conjugated.

Before we can look at the the other cases, we have to recall some results on graded Brauer groups, see [12]. Two $\mathbb{Z}^{(n)}$-graded Azumaya algebras over a $\mathbb{Z}^{(n)}$-graded ring T are said to be graded equivalent provided there are finitely generated $\mathbb{Z}^{(n)}$-graded projective T-modules P and Q such that there exists a gradation preserving isomorphism of T-algebras :

$$\Gamma \underset{T}{\otimes} \mathrm{END}_T(P) \cong \Omega \underset{T}{\otimes} \mathrm{END}_T(Q)$$

where the endomorphism rings and tensor products are equipped with the natural gradation, see loc. cit. for more details. The set of all graded equivalence classes of $\mathbb{Z}^{(n)}$-graded Azumaya algebras from a group with respect to the tensorproduct, $\mathrm{Br}^g(T)$, the so-called graded Brauer group of T.

If T is a $\mathbb{Z}^{(n)}$-graded normal domain, one can verify, cf. loc. cit., that the natural (i.e. gradation-forgetting) morphism $\mathrm{Br}^g(T) \to \mathrm{Br}(T)$ is monomorphic.

A graded local ring T is called graded Henselian if every finite graded commutative T-algebra is graded decomposed in the sense that it is a direct sum of graded local rings. It is easy to verify that a graded local ring is graded Henselian if and only if its homogeneous part of degree $(0, \ldots, 0)$ is Henselian. Further, if T is graded Henselian with unique graded maximal ideal m, then the natural morphism $Br^gT \to \mathrm{Br}^g(T/m)$, is monomorphic by a similar argument as in the ungraded case.
The following result describes the étale local structure of smooth maximal orders.

IV.4.9. Theorem : Let Λ be a smooth tame order over a local normal domain R in a p^2-dimensional divison algebra Δ such that char $(R/m) \neq p$

(1) If Λ is of case 1, R^{sh} splits Δ.

(2) If Λ is of case 2, R^{sh} does not split Δ.

Proof. The ring $\Lambda[\mathcal{P}] \underset{R}{\otimes} R^{sh}$ is a graded Azumaya algebra over $R[\mathcal{P}^c, e] \underset{R}{\otimes} R^{sh}$, both equipped with the natural gradation. $R[\mathcal{P}^c, e] \underset{R}{\otimes} R^{sh} \cong R^{sh}[\mathcal{P}]$ is graded Henselian because its part of degree $(0, \ldots, 0)$ is Henselian. Its unique maximal graded ideal will be denoted by $m^{sh}[\mathcal{P}]$.

(1) : In this case, $R^{sh}[\mathcal{P}]/m^{sh}[\mathcal{P}]$ is easily seen to be the graded field

$$R^{sh}/m^{sh}[Y_1^p, Y_p^{-1}]$$

where $Y_1^P = \pi X_1^P, (\pi)$ being the unique ramified central height one prime ideal. Now,

$$\overline{\Gamma} = (\Lambda[\mathcal{P}] \underset{R}{\otimes} R^{sh})/(\Lambda[\mathcal{P}] \underset{R}{\otimes} R^{sh}).m^{sh}[\mathcal{P}]$$

must be a $\mathbb{Z}$-graded central simple algebra of dimension p^2 over R^{sh}/m^{sh} $[Y_1^p, Y_1^{-p}]$. Using the formula at the end of Lemma IV.4.3., it turns out the homogeneous part of degree zero of $\overline{\Gamma}$ has to be an algebra of dimension p over R^{sh}/m^{sh}. Since R^{sh}/m^{sh} is separately closed and by assumption $\mathrm{char}(R^{sh}/m^{sh}) \neq p$, we have to conclude that

$$\overline{\Gamma}_0 \cong R^{sh}/m^{sh} \oplus \ldots \oplus R^{sh}/m^{sh} \qquad \text{(p copies)}$$

Therefore, $\overline{\Gamma}$ contains zero divisors whence

$$\overline{\Gamma} \cong M_p(R^{sh}/m^{sh}[Y_1^p, Y_1^{-p}])$$

with an appropriate gradation on the matrix ring.

Finally, using the injectivity of the natural morphsim

$$Br^g(R^{sh}[\mathcal{P}]) \to Br^g(R^{sh}[\mathcal{P}]/m^{sh}[\mathcal{P}])$$

we find that

$$\Lambda[\mathcal{P}] \underset{R}{\otimes} R^{sh} \cong \mathrm{END}_{R^{sh}[\mathcal{P}]}(P)$$

for some finitely generated graded projective (free) $R^{sh}[\mathcal{P}]$-module P. If we calculate the parts of degree zero on both sides we get a monomorphism

$$\Lambda \underset{R}{\otimes} R^{sh} \hookrightarrow M_p(K^{sh})$$

finishing the proof of (1).

(2) In this case, $R^{sh}[\mathcal{P}]/m^{sh}[\mathcal{P}]$ is the $\mathbb{Z} \oplus \mathbb{Z}$-graded field R^{sh}/m^{sh} $[Y_1^p, Y_1^{-1}, Y_2^p, Y_2^{-p}]$ where $Y_1^p = \pi X_1^p$, $Y_2^p = \pi' X_2^p$ with $\{(\pi), (\pi')\} = \mathcal{P}^c$ and $\deg Y_1^p = (p, 0)$ and $\deg Y_2^P = (0, p)$. Further,

$$\overline{\Gamma} = \Lambda[\mathcal{P}] \otimes R^{sh}/(\Lambda[\mathcal{P}] \otimes R^{sh}).m^{sh}[\mathcal{P}]$$

is a $\mathbb{Z} \oplus \mathbb{Z}$-graded central simple algebra of dimension p^2 over R^{sh}/m^{sh} $[Y_1^p, Y_1^{-p}, Y_2^p, Y_2^{-p}]$. Again, using the formula of the proof of Lemma 4.3. we obtain that all homogeneous parts should be one-dimensional. In particular,

$$\overline{\Gamma}_{(0,0)} = R^{sh}/m^{sh}$$

Let X be a generator of the $\overline{\Gamma}_{(0,0)}$-module $\overline{\Gamma}_{(1,0)}$ and Y a generator of the $\overline{\Gamma}_{0,0}$-module $\overline{\Gamma}_{(0,1)}$, then it turns out that $\overline{\Gamma}$ is a graded cyclic algebra determined by the relations

$$\begin{cases} X^p = aY_1^p, & a \in (R^{sh}/m^{sh})^* \\ Y^p = bY_1^p, & b \in (R^{sh}/m^{sh})^* \\ XY^i = \xi^i Y^i X & \text{for } 1 \leq i \leq p-1 \end{cases}$$

where ξ is any primitive p-th root of unity. Because R^{sh}/m^{sh} is separably closed and $\mathrm{char}(R^{sh}/m^{sh}) \neq p$, this algebra does not depend upon the choice of a or b, showing that $\overline{\Gamma}$ is graded isomorphic to the $\mathbb{Z} \oplus \mathbb{Z}$-graded cyclic algebra determined by

$$\begin{cases} X^p & = & Y_1^p \\ Y^p & = & Y_1^p \\ XY & = & \xi YX \end{cases}$$

and if we calculate its norm, it follows that this algebra is a domain. But then, clearly, $\Lambda[\mathcal{P}] \otimes R^{sh}$ represents a non-trivial element in $Br^g(R^{\mathrm{sh}}[\mathcal{P}])$ and so its part of degree $(0,0)$ cannot be an order in a matrix ring since $R[\mathcal{P}^c,e]\underset{R}{\otimes}R^{\mathrm{sh}}$ is graded regular and there is the monomorphism

$$Br^g(R^{\mathrm{sh}}[\mathcal{P}])\hookrightarrow Br^g(K^{\mathrm{sh}}[\mathcal{P}])$$

whilst $K^{\mathrm{sh}}[\mathcal{P}]$ is strongly graded. This finishes the proof. $\square$

IV.4.10. Theorem : All smooth tame orders over a local normal domain in a p^2-dimension division algebra are conjugated.

Proof.

Case 1 : Let Λ be a smooth maximal order in Δ. In the proof of the foregoing theorem we have seen that there exists an étale extension S of R such that :

$$\Lambda[\mathcal{P}]\underset{R}{\otimes}S\cong \mathrm{END}_{S[\mathcal{P}]}(P)$$

where P is a initely generated graded projective $S[\mathcal{P}]=R[\mathcal{P}^c,e]\underset{R}{\otimes}S$-module. Since $S[\mathcal{P}]$ is graded local, P is graded free i.e. of the form

$$P\cong S[\mathcal{P}](\sigma_1)\oplus\ldots\oplus S[\mathcal{P}](\sigma_p)$$

where $\sigma_i\in\mathbb{Z}$ and $S[\mathcal{P}](\sigma_i)$ is the $\mathbb{Z}$-graded $S[\mathcal{P}]$-module determined by taking for its homogeneous part of degree α : $S[\mathcal{P}](\sigma_i)_\alpha=S[\mathcal{P}]_{\sigma_i+\alpha}$. Therefore,

$$\Lambda[\mathcal{P}]\underset{R}{\otimes}S\cong M_p(S[\mathcal{P}])(\sigma_1,\ldots,\sigma_p)$$

where the homogeneous part of degree α of the ring on the right hand side is given by allowing as an entry on the (i,j)-place an element of $S[\mathcal{P}]_{\alpha+\sigma_i-\sigma_j}$.

An easy computation shows that up to conjugation in degree zero, all σ_i may be chosen to be elements of the set $\{0,1,\ldots,p-1\}$. Further, we may assume that $\sigma_1\leq\sigma_2\leq\ldots\leq\sigma_p$ for otherwise one simply has to conjugate by a permutation matrix. Because all isomorphisms occuring in the foregoing are gradation preserving we can look at the degree zero part : $\Lambda\otimes S\cong M_p(S)(\sigma_1,\ldots,\sigma_p)$ Only a few rings $M_p(S)(\sigma_1,\ldots,\sigma_p)$ can actually occur. For example, $\sigma_1=\sigma_2=\ldots=\sigma_p$, then $M_p(S)(\sigma,\ldots,\sigma_p)=M_p(S)$ which is an Azumaya algebra, contradicting the fact that Λ is ramified.

Further, if $\sigma_1\leq\sigma_2\leq\ldots\leq\sigma_p$ and if $\sigma_i=\sigma_{i+1}$ for some i, then $M_p(S)(\sigma_1,\ldots,\sigma_p)$ is no longer a tame order because it has the following matrix-form : the upper triangular entries are S, the lower triangular entries are either (π) of S and at least one of them, namely the entry at the position $(i+1,i)$ equals S. If we localize this ring at (π) and use the characterization of hereditary orders in [22], it follows that $(\Lambda\otimes S)_{(\pi)}$ is not hereditary, whence $\Lambda\otimes S$ is not tame. But this is impossible since S is étale over R.
The only remaining possibility is therefore $M_p(S)(0,1,2,\ldots,p-1)$ and we may now use Grothendieck descent to finish the proof.

Case 2 : From the foregoing theorem we retain that $\Lambda[\mathcal{P}]$ cannot be split by an étale extension of R. Nevertheless, mimicking the ungraded case, $\Lambda[\mathcal{P}]$ can be split by a graded étale extension of $R[\mathcal{P}^c,e]$ since this ring is graded local. Denote $R[X]/(X^P-\pi)$ by S, then $S(\Phi)$ will be defined to be the $\mathbb{Z}\oplus\mathbb{Z}$-graded ring whose $(\mathbb{Z},0)$-axis is the strongly graded ring $S[(X)]$ and $(0,\mathbb{Z})$-axis is the extended $(0,\mathbb{Z})$-axis of $R[\mathcal{P}^c,e]$. For example if $p=2$ we get the following pictural description of $-S(\Phi)$.

$$\begin{array}{ccccccccccccc}
\cdots & \oplus & (\pi'X^{-2}) & \oplus & (\pi'X^{-1}) & \oplus & (\pi') & \oplus & (\pi'X) & \oplus & (\pi'X^2) & \oplus & \cdots\\
 & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (\pi'X^{-2}) & \oplus & (\pi'X^{-1}) & \oplus & (\pi') & \oplus & (\pi'X) & \oplus & (\pi'X^2) & \oplus & \cdots\\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) & \oplus & \cdots\\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (X^{-2}) & \oplus & (X^{-1}) & \oplus & S & \oplus & (X) & \oplus & (X^2) & \oplus & \cdots\\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
\cdots & \oplus & (\pi'^{-1}X^{-2}) & \oplus & (\pi'^{-1}X^{-1}) & \oplus & (\pi'^{-1}) & \oplus & (\pi'^{-1}X) & \oplus & (\pi'^{-1}X^2) & \oplus & \cdots
\end{array}$$

It can be shown that $S(\Phi)$ is a graded étale splitting ring for $\Lambda[\mathcal{P}]$. Running along the lines of the argument given above for the first case we obtain :

$$\Lambda[\mathcal{P}]\underset{R[\mathcal{P}^c,e]}{\otimes}S(\Phi)\cong M_p(S(\Phi))(\sigma_1,\ldots,\sigma_p)$$

where the $\sigma_i\in\mathbb{Z}\oplus\mathbb{Z}$ may be chosen to stem from the set $\{(0,1),\ldots,(0,p-1)\}$. Furthermore, as in the first case the only possibility which yields a (graded) tame order is graded isomorphic to

$$M_p(S(\Phi))((0,0)(0,1),\ldots,(0,p-1))$$

Applying a graded version of the Grothendieck descent mentioned above, it follows that $\Lambda[\mathcal{P}] \cong \Gamma[\mathcal{P}]$ for any smooth maximal orders over R in Δ, the isomorphism being one of graded $R[\mathcal{P}^c, e]$-algebras.

As always, such an isomorphism is necessarily conjugation by a unit, α say, where $\alpha \in \Delta[X_1, X_1^{-1}, X_2, X_2^{-1}]$; because the latter ring is a $\mathbb{Z} \oplus \mathbb{Z}$-graded domain, α is homogeneous i.e. $\alpha = \delta X_1^{l_1} X_2^{l_2}$ with $\delta \in \Delta^*$. Finally, we calculate the homogeneous parts of degree $(0,0)$ in the generalized Rees rings and then we arrive at : $\Lambda = \delta^{-1}.\Gamma.\delta$, thus finishing the proof of the theorem. □

IV.5. Weak Brauer-Severi Schemes :

As an application of the graded techniques introduced before, we will construct here a big open subset of the Brauer-Severi scheme of a smooth maximal order. First we outline the general problem.

Let Λ be an order over a normal R in a central simple algebra Σ of dimension n^2, then one can define a functor from the category of all commutative R-algebras to the category of sets

$$F_\Lambda : \text{Comm Alg}_R \longrightarrow \text{Sets}$$

which associates to an R-algebra A the set of all left ideals of $\Lambda \underset{R}{\otimes} A$ which are split projective A-modules of rank n. The main problem is to determine whether this functor is representable. By this we mean the following : does there exist a scheme $\underline{\mathrm{BS}}_\Lambda$ over $\underline{\mathrm{Spec}}(R)$, called the **Brauer-Severi scheme of** Λ, such that for every commutative R-algebra A there is a natural one-to-one correspondence between elements L of $F_\Lambda(A)$ and scheme homomorphisms Ψ_L from $\underline{\mathrm{Spec}}(A)$ to $\underline{\mathrm{BS}}_\Lambda$ making the diagram below commutative :

$$\begin{tikzcd} & \underline{\mathrm{BS}}_\Lambda \arrow[d, "\varphi"] \\ \underline{\mathrm{Spec}}(A) \arrow[ur, "\Psi_L"] \arrow[r, "\varphi_A"'] & \underline{\mathrm{Spec}}(R) \end{tikzcd}$$

where φ and φ_A are the natural structural morphisms. Therefore, one may view the Brauer-Severi scheme as parametrizing the commutative R-algebras which split Σ. A first step in this study consists in determining the étale structure of the Brauer-Severi scheme, i.e. suppose that a point $x \in \mathrm{Spec}(R)$ has an étale neighbourhood S which splits Σ, then one calculates a representation of the functor

$$F_{\Lambda \otimes S} : \mathrm{Comm\ Alg}_S \to \mathrm{Sets}$$

This étale local structure has been derterminded in several cases. Grothendieck showed that the étale local structure of the Brauer-Severi scheme of an Azumaya algebra in $hbox P_S^{n-1}$. Artin and Mumford calculated the Brauer-Severi scheme of a maximal order over a smooth surface in a ramified quaternion algebra with a regular ramification-divisor. Recently, Artin calculated the étale local structure of the Brauer-Severi scheme for a maximal order over a Dedekind domain, see [2].
If one restricts attention in the first case to Azumaya algebras over regular domains, it turns out that all rings for which a description of the Brauer-Severi scheme exists in the literature, are smooth maximal orders. Therefore, one can ask whether the functor F_Λ is representable for every smooth maximal order. We will show in this section that the restriction of F_Λ to some nice subcategory $\underline{\mathcal{C}}$ of Comm Alg_R, which includes all étale and smooth extensions of R, is indeed representable. To be more precise, $\underline{\mathcal{C}}$ will be the full subcategory of Comm Alg_R consisting of all R-algebras A such that $A[\mathcal{P}] = R[\mathcal{P}^c, e] \underset{R}{\otimes} A$ is a regular domain.
Let us first reformulate the question in a graded context. Denote by F_Λ^g the functor

$$F_\Lambda^g : \underline{\mathcal{C}} \underset{R}{\otimes} R[\mathcal{P}^c, e] \to \mathrm{Sets}$$

which assigns to an algebra $A[\mathcal{P}]$ where $A \in \underline{\mathcal{C}}$, the set of all graded left ideals of $\Lambda[\mathcal{P}] \underset{R}{\otimes} A$ which are graded split projective $A[\mathcal{P}]$-modules of rank n.

IV.5.1. Lemma . If Λ is a smooth maximal order over a regular local domain R, then there is a natural one-to-one correspondence between $F_\Lambda(A)$ and $F_\Lambda^g(A)$.

Proof. We claim that the following maps establish the one-to-one correspondence :

$$\Psi_1 : F_\Lambda(A) \to F_\Lambda^g(A); L \to L \underset{(\Lambda \otimes A)}{\otimes} (\Lambda[\mathcal{P}] \otimes A)$$

$$\Psi_2 : F_\Lambda^g(A) \to F_\Lambda(A); M \to M_e$$

where e is the neutral element of the grading group. Since $\Lambda[\mathcal{P}] \underset{R}{\otimes} A$ is a strongly graded ring, Ψ_1 and Ψ_2 clearly define a one-to-one correspondence between left ideals of $\Lambda \otimes A$ and graded left ideals of $\Lambda[\mathcal{P}] \otimes A$. Further, since $\Lambda[\mathcal{P}] \underset{R}{\otimes} A$ is a graded Azumaya algebra over the graded regular domain $A[\mathcal{P}]$, $\mathrm{grgldim}(\Lambda[\mathcal{P}] \otimes A) < \infty$ whence $\mathrm{gldim}(\Lambda \underset{R}{\otimes} A) < \infty$ view of the equivalence of categories. Therefore, $\Lambda \otimes A$ is a regular order over the regular domain A. Because both projectivity and splitting are local conditions, we may as well assume that A is local regular.

Now, let $L \in F_\Lambda(A)$, then we claim that L is split as a left $\Lambda \otimes A$-module. Consider the exact sequence of left $\Lambda \otimes A$-modules :

$$0 \to L \to \Lambda \otimes A \to (\Lambda \otimes A)/L \to 0$$

Since $L \in F_\Lambda(A)$, this sequence splits as a sequence of A-modules. So, $(\Lambda \otimes A)/L$ is a left $\Lambda \otimes A$-module which is free as an A-module, but then, regularity of $\Lambda \otimes A$ entails that $(\Lambda \otimes A)/L$ is projective as left $\Lambda \otimes A$-module, finishing the proof of the claim.
It follows that $\Psi_1(L)$ is a graded split projective $\Lambda[\mathcal{P}] \otimes A$-module. Finally, an easy localization argument shows that $\Psi_1(L)$ has graded rank n. The proof that Ψ_2 maps elements of $F_\Lambda^g(A)$ to $F_\Lambda(A)$ is easy. □

Our strategy to represent the full subgenerator of F_Λ will be to represent first the functor F_Λ^g by a graded scheme which is relatively easy because $\Lambda[\mathcal{P}]$ is a graded Azumaya algebra, so we have to modify Grothendieck's arguments in the graded case. Subsequently, we will derive from this graded scheme a usual scheme which represents the subfunctor of F_Λ.

The actual computations are carried out for smooth maximal orders in quaternion-algebras, i.e. $p = 2$. However, using the structural results of the foregoing section the reader may easily verify that a similar approach is possible for smooth maximal orders in a p^2-dimensional division-algebra, p being a prime number. All graded schemes we will encounter have the property that their homogeneous part of degree e is a usual scheme. Let us give an example :

IV.5.2 Example : Let $R[\mathcal{D}, g]$ be a scaled Rees ring associated to the set of height one prime $\{p_1, \ldots, p_n\}$ and natural numbers $g = \{g_1, \ldots, g_n\}$. We will denote G to be $\mathbb{Z}^{(n)}$ and $H = g_1\mathbb{Z} \oplus \ldots \oplus g_n\mathbb{Z}$. There is a natural one-to-one correspondence between $\mathrm{Spec}(R)$ and $\mathrm{Spec}_g(R[\mathcal{D}])$ the set of all $\mathbb{Z}^{(n)}$-graded prime ideals of $R[\mathcal{D}, g]$ with the induced Zariski topology.

$$\begin{aligned}
\varphi_1 : \quad & \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}_g(R[\mathcal{D}, g]) \\
& m \longrightarrow \sum_{\sigma \in H} mR[\mathcal{D}, g]_\sigma + \sum_{\sigma \in G \setminus H} R[\mathcal{D}, g]_\sigma = m[\mathcal{D}] \\
\varphi_2 : \quad & \mathrm{Spec}(R[\mathcal{D}, g]) \longrightarrow \mathrm{Spec}(R) \\
& M \longrightarrow m = M_e
\end{aligned}$$

where $e = (0, \ldots, 0)$ is the neutral element. It is verified that these maps actually define a homeomorphism.
Moreover, for any $m \in \mathrm{Spec}(R)$ we have :

$$(R[\mathcal{D}, g]^g_{m[\mathcal{D},g]})_e \cong R_m$$

so the part of degree e of the affine graded spectrum of $R[\mathcal{D}, g]$ is isomorphic to $\underline{\mathrm{Spec}}(R)$.

We will now give a graded representation of F^g_Λ if Λ is smooth of case 1, i.e. $\mathcal{P} = \{P\}$ and $\dim_{R/m}(\Lambda/\Lambda m + P) = 2$ (quaternionic case). From Theorem IV.4.9. we retain there exists an étale extension S of R which splits Δ. First, we like to represent the functor

$$F^g_{\Lambda \otimes S} : \underline{G} \underset{R}{\otimes} S[\mathcal{P}] \longrightarrow \mathrm{Sets}$$

by a graded scheme over $\underline{\mathrm{SPEC}}^g(S[\mathcal{P}])$, i.e. we will define a graded scheme $\underline{X}^g$ such that for any S-algebra A in $\underline{C}$ there is a natural one-to-one correspondence between elements of $F^g_{\Lambda \otimes S}(A)$ and graded scheme

morphisms χ

$$\begin{tikzcd}
\mathrm{SPEC}^g(S[\mathcal{P}]) \arrow[rr, "\chi"] \arrow[dr, "\varphi_A"'] & & \underline{X}^g \\
& \mathrm{SPEC}^g S[\mathcal{P}] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where φ and φ_A are the structural morphisms. In the proof of Theorem IV.4.10. we have shown that

$$\Lambda[\mathcal{P}] \underset{R}{\otimes} S \cong M_2(S[\mathcal{P}])(0, 1)$$

So, if A is any $\mathbb{Z}$-graded $S[\mathcal{P}]$-algebra, then $(\Lambda[\mathcal{P}] \underset{R}{\otimes} S) \underset{S[\mathcal{P}]}{\otimes} A \cong M_2(A)(0, 1)$ hence we aim to represent the functor

$$G : \mathrm{gr\ Comm\ Alg}_{S[\mathcal{P}]} \longrightarrow \mathrm{Sets}$$

which assigns to a $\mathbb{Z}$-graded algebra A the set of all split projective graded left ideals of $M_2(A)(0, 1)$ of rank two. Take such a graded left ideal $L \in G(A)$, then $L = e_{11}L \oplus e_{22}L$ and since all matrix elements e_{ij} are homogeneous, it follows that $e_{11}L$ is a graded split projective rank one submodule of $A \oplus A(-1)$. Conversely, if M is a graded split projective one submodule of $A \oplus A(-1)$ then

$$L = M \oplus e_{21}M$$

is a graded split projective rank two left ideal of $M_2(A)(0, 1)$. So, it will suffice to represent

$$\mathrm{Grass}^g_1(0, -1) : \mathrm{gr\ Comm\ Alg}_{S[\mathcal{P}]} \longrightarrow \mathrm{Sets}$$

which assigns to a $\mathbb{Z}$-graded $S[\mathcal{P}]$-algebra A the set of all graded split rank one submodules of $A \oplus A(-1)$. As in the ungraded case, this can be achieved by representing the subfunctors $\mathrm{Gr}^g_i, i = 0, 1$, which assigns to A the set of all elements $M \in \mathrm{Grass}^g_1(0, -1)$ such that the gradation preserving composite morphism

$$A(-i) \xrightarrow{\varphi_i} A \oplus A(-i) \xrightarrow{u} M$$

is an isomorphism where, φ_i is the natural inclusion and u is the uniquely determined gradation preserving map for M. Suppose we have :

$$\begin{tikzcd} & A \oplus A(-i) \arrow[dl, shift left, "v"] \arrow[dr, "u"] & \\ A(-i) \arrow[ur, shift left, "\varphi_i"] \arrow[rr, "w"] & & M \end{tikzcd}$$

such that $u \circ \varphi_i$ is an isomorphism and let w be its inverse and $v = w \circ u$ which satisfies $v \circ \varphi_i = 1_{A(-1)}$. Conversely, suppose we have a gradation preserving morphism v which satisfies $v \circ \varphi_i = 1_{A(-i)}$, then it is clear that $M = A \oplus A(-1)/\mathrm{Ker}(v)$ is an element of $\mathrm{Gr}_i^g(A)$. One can therefore identify $\mathrm{Gr}_i^g(A)$ with the set of gradation preserving split morphism of φ_i, if one defines mappings

$$\alpha_i : \mathrm{HOM}_A(A \oplus A(-1), A(-i))_0 \to \mathrm{HOM}_A(A(-i), A(-i))_0$$
$$v \longrightarrow v \circ \varphi_i$$
$$\beta_i : \mathrm{HOM}_A(A \oplus A(-1), A(-i))_0 \to \mathrm{HOM}_A(A(-i), A(-i))_0$$
$$v \longrightarrow 1_{A(-i)}$$

then $\mathrm{Gr}_i^g(A)$ may be viewed as the kernel of the couple (α_i, β_i) and we claim that the functors

$$A_i : \text{gr Comm Alg}_{S[\mathcal{P}]} \longrightarrow \text{Sets}; A_i(A) = \mathrm{HOM}_A(A \oplus A(-1), A(-1))$$
$$B_i : \text{gr Comm Alg}_{S[\mathcal{P}]} \longrightarrow \text{Sets}; B_i(A) = \mathrm{HOM}_A(A(-i), A(-i))_0$$

are representable by graded schemes. Before proving this we need to define graded vector fibres.

Let A be a $\mathbb{Z}^{(n)}$-graded commutative ring and let E be a graded A-module. The tensor algebra $T(E) = \overset{\infty}{\underset{i=0}{\oplus}} (E^{(i)})$ is given the natural $\mathbb{Z}^{(n)}$-gradation, i.e.

$$T(E)_\gamma = \sum_{\sigma_1 + \ldots + \sigma_m = \gamma} E_{\sigma_1} \otimes \ldots \otimes E_{\sigma_m}$$

The symmetric algebra $S(E)$ over E is obtained by taking the quotient of $T(E)$ for the homogeneous twosided ideal generated by all elements $x \otimes y - y \otimes x$; $x, y \in h(E)$, i.e. $S(E)$ has a natural $\mathbb{Z}^{(n)}$-gradation.

By the universal property of $S(E)$ one can check that every gradation preserving A-linear morphism $E \to B$, B being a $\mathbb{Z}$-graded commutative A-algebra factorizes through $S(E)$. Furthermore, $S(E \oplus F) \cong S(E) \otimes S(F)$ a $\mathbb{Z}^{(n)}$-graded rings.

The graded vector fibre $V^g(E)$ of the $\mathbb{Z}^{(n)}$-graded A-module E is then defined to be the graded affine spectrum $\mathrm{Spec}^g S(E)$ which is a graded $\underline{\mathrm{SPEC}}^g(A)$-scheme representing the functor $\mathrm{HOM}_-(E \underset{S}{\otimes} -, -)_e$ where e is the neutral element of $\mathbb{Z}^{(n)}$.

IV.5.3. Example . Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}^{(n)}$, then there is an isomorphism of graded A-algebras

$$S(A(\sigma_1) \oplus \ldots \oplus A(\sigma_n)) \cong [t_1, \ldots, t_m]$$

where $\deg(t_i) = -\sigma_i$.

Clearly, the functors A_i are represented by the graded schemes $V^g(S[\mathcal{P}](i) \oplus S[\mathcal{P}](-i))$ whereas the B_i are represented by $V^g(S[\mathcal{P}])$. The maps α_i correspond to the graded scheme morphisms :

$$\alpha_0 \to f_0 : \underset{(\deg X=0, \deg Y=1)}{\mathrm{SPEC}^g S[\mathcal{P}][X,Y]} \to \mathrm{SPEC}^g S[\mathcal{P}][X]$$
$$\alpha_1 \to f_1 : \underset{(\deg X=-1, \deg Y=0)}{\mathrm{SPEC}^g S[\mathcal{P}][X,Y]} \to \mathrm{SPEC}^g S[\mathcal{P}][Y]$$

And these morphisms arise from the natural algebra inclusions. The β_i correspond to the morphisms :

$$\beta_0 \to g_0 : \underset{\deg X=0, \deg Y=1}{\mathrm{SPEC}^g S[\mathcal{P}][X,Y]} \longrightarrow \mathrm{Spec}^g S[\mathcal{P}][X]$$
$$\beta_1 \to g_1 : \underset{\deg X=-1, \deg Y=0}{\mathrm{SPEC}^g S[\mathcal{P}][X,Y]} \longrightarrow \mathrm{Spec}^g S[\mathcal{P}][Y]$$

coming from the graded algebra maps sending X to 1 (resp. Y to 1). Therefore, the functor Gr_0^g is represented by the kernel of the diagram below

$$\begin{tikzcd} \mathrm{SPEC}^g S[\mathcal{P}][X,Y] \arrow[r, "f_0"] & \mathrm{SPEC}^g S[\mathcal{P}][X] \\ & \mathrm{SPEC}^g S[\mathcal{P}][X,Y] \arrow[u, "g_0"] \end{tikzcd}$$

which is equal to $X^g(X-1) \cong \mathrm{SPEC}^g S[\mathcal{P}][Y]$ where $\deg(Y) = 1$. Similarly, Gr_1^g is represented by $\mathrm{SPEC}^g S[\mathcal{P}][X]$ with $\deg(X) = -1$. It rests us to glue the subfunctors to represent $\mathrm{Grass}_1^g(0,\dots,1)$. Let us calculate the fundamental modules for the subfunctors Gr_i^g. That is an element $M_0 \in \mathrm{Grass}_1^g(0,\dots,1)(S[\mathcal{P}][Y])$ resp. $M_1 \in \mathrm{Grass}_1^g(0,-1)(S[\mathcal{P}][X])$ such that for every graded commutative $S[\mathcal{P}]$-algebra A, the natural one-to-one correspondences between $\mathrm{HOM}(\mathrm{SPEC}^g(A), \mathrm{SPEC}^g S[\mathcal{P}][Y])$ and $\mathrm{Gr}_0^g(A)$ (resp. $\mathrm{HOM}(\mathrm{SPEC}^g(A), \mathrm{SPEC}^g S[\mathcal{P}][X])$ and $Gr_1^g(A)$) are given by assigning to a scheme morphism ψ, $\Gamma(\mathrm{SPEC}^g(A), \psi^*(\widetilde{M}_0))$ (resp. $\Gamma(\mathrm{SPEC}^g(A).\psi^*(M_1))$).
It is easy to verify that :

$$M_0 \cong S[\mathcal{P}][Y](0) \cong (S[\mathcal{P}][Y] \oplus S[\mathcal{P}][Y](-1))/(-Y,1)S[\mathcal{P}][Y]$$
$$M_1 \cong S[\mathcal{P}][X](-1) \cong (S[\mathcal{P}][X](1) \oplus S[\mathcal{P}][X])/(1,-X)S[\mathcal{P}][X](-1)$$

The open set of $\mathrm{SPEC}^g S[\mathcal{P}][Y]$ over which we have to glue $\mathrm{SPEC}^g S[\mathcal{P}][Y]$ with $\mathrm{SPEC}^g S[\mathcal{P}][X]$ is then the set for which the composition γ is an isomorphism :

$$\begin{tikzcd}
 & S[\mathcal{P}][Y](c) \oplus S[\mathcal{P}][Y](-1) \arrow[dr, "./(-Y,1)S[\mathcal{P}][Y]"] & \\
y \in S[\mathcal{P}][Y](-1) \arrow[ur, "y \vdash 0 \oplus y"] \arrow[rr, "\gamma"] & & M_0
\end{tikzcd}$$

i.e. $X^g(Y)$ is the desired open set. Similarly $X^g(X)$ is the open set of $\mathrm{SPEC}^g S[\mathcal{P}][X]$ over which one has to glue $\mathrm{SPEC}^g S[\mathcal{P}][X]$ with $\mathrm{SPEC}^g S[\mathcal{P}][Y]$. Now we have proved the following result.

IV.5.4. Theorem. The functor $F^g_{\Lambda\otimes S}$ is represented by a graded scheme $\mathrm{GRASS}_1^g(0.\dots,1)$ over $\mathrm{SPEC}^g(S[\mathcal{P}])$ which is obtained by glueing $\mathrm{SPEC}^g S[\mathcal{P}][Z]$, $\mathrm{dg}(Z) = 1$, together with $\mathrm{SPEC}^g S[\mathcal{P}][Z^{-1}]$ over $\mathrm{SPEC}^g S[\mathcal{P}][Z, Z^{-1}]$.
Note that the graded scheme $\mathrm{Grass}_1^g(0,-1)$ may be interpreted as the graded one dimensional projective space over $S[\mathcal{P}]$. We have seen above that the part of degree zero of a graded scheme is often a scheme. The part of the graded scheme $\mathrm{Grass}_1^g(0,-1)$ is the S-scheme obtained by gluing $\mathrm{Spec}(S[\mathcal{P}]_{\geq 0})$ with $\mathrm{Spec}(S[\mathcal{P}]_{\leq 0})$ over $\mathrm{Spec} S[\mathcal{P}]$. This scheme is never regular. For example, if $\Lambda = \mathbb{C}[X,-]$ then S can be taken to be

$\mathbb{C}[t]$ where $t = X^2$. In this case, the part of degree zero of $\mathrm{Grass}_1^g(0,-1)$ is the scheme obtained by gluing two affine cones over the complement of a ruling.

Now, let us briefly look at the second case, i.e. $\mathcal{P} = \{P, Q\}$ and $\dim_{R/m}(\Lambda/\Lambda m + P + Q) = 1$. In Theorem IV.4.9. we have seen that there is no étale extension of R which splits Σ. However, one can find an étale extension R_1 of R such that $\Lambda \otimes R_1 \cong R_1.1 \oplus R_1.i \oplus R_1 \oplus R_1 ij$ with $i^2 = p, j^2 = q$ where $P^2 = (p)$ and $Q^2 = (q)$. Moreover, there exists an extension $S = R_1[X]/(X^2 - p)$ which splits Σ and such that the ring $S(\Phi)$ as defined in the proof of Theorem IV.4.10. is a graded étale (even Galois) extension of $R_1[\mathcal{P}]$, in particular $S(\Phi)$ is graded regular. This entails that

$$(\Lambda \otimes R_1)[\mathcal{P}] \otimes S(\Phi) \cong M_2(S(\Phi))((0,0),(0,1))$$

Our first objective will be to represent the functor

$$F_S^g : \mathrm{gr\ Comm\ Alg}_{S(\Phi)} \longrightarrow \mathrm{Sets}$$

which assigns to any graded commutative $S(\Phi)$-algebra A the set of all split projective graded left rank two ideals of $M_2(S(\Phi))((0,0),(0,1))$. As in case 1, it is readily verified that this is equivalent to representing

$$\mathrm{Grass}_1^g((0,0),(0,-1)) : \mathrm{gr\ Comm\ Alg}_{S(\Phi)} \to \mathrm{Sets}$$

Extending the argument of case 1 to the $\mathbb{Z} \oplus \mathbb{Z}$-graded case, one obtains the following result :

IV.5.5. Theorem : The functor $\mathrm{Grass}_1^g((0,0),(0,-1))$ is represented by a graded scheme $\mathrm{GRASS}_1^g((0,0),(0,-1))$ over $\mathrm{SPEC}^g S[\mathcal{P}]$ which is obtained by glueing $\mathrm{SPEC}^g S[\mathcal{P}][Z]$, where $\deg Z = (0,1)$, with $\mathrm{SPEC}^g S[\mathcal{P}][Z^{-1}]$ over $\mathrm{SPEC}^g S[\mathcal{P}][Z, Z^{-1}], \dots$
Using graded Galois descent, it is then possible to find a graded scheme over $\mathrm{SPEC}^g R_1[\mathcal{P}]$ which represents the functor

$$f^g : \mathrm{gr\ Comm\ Alg}_{R_1[\mathcal{P}]} \longrightarrow \mathrm{Sets}$$

Having found a graded representations for the functors F^g_Λ, we like to use this information to represent F_Λ. We will restrict attention to case

1. If one calculates the explicit form of the graded scheme representing f^g, one can mimick easily the argument below.
If $A \in \mathcal{C}_S$, it follows from Lemma IV.5.1. that there is a natural one-to-one correspondence between elements of $F_{\Lambda \otimes S}(A)$ and graded $\mathrm{SPEC}^g S[\mathcal{P}]$- scheme morphisms from $\mathrm{SPEC}^g A[\mathcal{P}]$ to $\mathrm{GRASS}_1^g(0,-1)$. Any such scheme morphism φ is determined by graded $S[\mathcal{P}]$-algebra morphisms :

$$\varphi_1 : S[\mathcal{P}][Z] \longrightarrow A[\mathcal{P}]$$
$$\varphi_2 : S[\mathcal{P}][Z^{-1}] \longrightarrow A[\mathcal{P}]$$

such that their localization $(\varphi_1)_Z$ and $(\varphi_2)_{Z^{-1}}$ coincide. Clearly, φ_1 is completely determined by $\varphi_1(Z) \in A[\mathcal{P}]_1 = A_p$ so there is a one-to-one correspondence between $S[\mathcal{P}]$-algebra morphisms from $S[\mathcal{P}][Z]$ to $A[\mathcal{P}]$ and S-algebra morphisms from $S[X,Y]/(X-pY)$ to A.
Similarly, φ_2 is completely determined by $\varphi_2(Z^{-1}) \in A[\mathcal{P}]_{-1} = A$ so there is a one-to-one correspondence between graded $S[\mathcal{P}]$-algebra morphisms from $S[\mathcal{P}][Z^{-1}]$ to $A[\mathcal{P}]$ and S algebra morphisms from $S[X^{-1}]$ to A. Further, there is a one-to-one correspondence between graded $S[\mathcal{P}]$-algebra morphisms from $S[\mathcal{P}][Z,Z^{-1}]$ to $A[\mathcal{P}]$ and S-algebra morphisms from $S[X,X^{-1},Y]/(X-pY)$ to A. Therefore, if X denotes the S-scheme obtained by gluing $\mathrm{Spec}S[X,Y]/(X-p^y)$ with $\mathrm{Spec}S[X,X^{-1},Y]/(X-pY)$ then there is a natural one-to-one correspondence between graded $\mathrm{SPEC}^g S[\mathcal{P}]$-scheme morphisms from $\mathrm{SPEC}^g A[\mathcal{P}]$ to $\mathrm{GRASS}_1^g(0,-1)$ and $\mathrm{Spec}(S)$-scheme morphisms from $\mathrm{Spec}(A)$ to X. This concludes the proof of :

IV.5.6. Theorem. Let Λ be a smooth maximal order of case 1 over a regular local normal domain R. The étale local structure of the weak Brauer-Severi scheme, i.e. the scheme representing the scheme $\mathrm{Spec}S[Y]$ with $\mathrm{Spec}S[Z]$ over $\mathrm{Spec}S[Y,Z]/(1-pYZ)$.

IV.5.7. Example. The scheme X over $\mathrm{Spec}\mathbb{C}[t]$ associated to the smooth order $\Lambda = \mathbb{C}[[X,-]]$ has fibers which can be visualized as a family of conics, degenerating to a pair of distinct affine lines. The étale local structure of the full Brauer-Severi scheme was computed by M. Artin. Its fibers can be visualized as a family of conics, degenerating to a pair of projective lines meeting transversally at one point.
It is easy to give an open immersion of X in $BS_\Lambda \otimes \mathbb{A}^1_{\mathbb{C}}$.

The weak Brauer-Severi scheme misses one point corresponding to the left ideal of rank two L in

$$\Lambda\mathbb{C} \subset [t]/(t) \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ (t)/(t^2) & \mathbb{C} \end{pmatrix}$$

where

$$L = \begin{pmatrix} 0 & \mathbb{C} \\ (t)/(t^2) & 0 \end{pmatrix}$$

V. Two-dimensional Tame and Maximal Orders of Finite Representation Type.

V.1 Introduction :

In Chapter IV we have already touched upon the matter of orders of finite representation type. In this chapter we show how the methods developed there, maybe used to rederive some of the results in [3]. In particular, a variant of Theorem V.2.10 tells us that the representation theory of two dimensional tame orders is, in the case of finite representation type, determined by a rational double point together with a cyclic group action on its module category. This allows one to compute the ranks of the Cohen Macaulay modules. We then prove a structure theorem for tame orders with the above hypothesis. Each such tame order is of the form $\Lambda = k[[x,y]] *_c G$ where G acts linearly on the two dimensional space $V = kx + ky$ and c takes values in k. We then relate the C.M modules of Λ to the projective representations of G corresponding to c.

We also give a criterion for Λ to be a maximal in terms of c and G.

An important part of [3] and [4] is the classification theorem. This theorem describes the Brauer class of maximal orders of finite representation type in terms of ramification. Our methods can duplicate some parts of Artin's arguments. An important tool used by Artin however is the computation of the Brauergroup of the function field of a rational singularity. So far we have not been able to give a ring theoretic treatment of this result. We therefore follow Artin, stating the geometric results without proof.

Finally we give an overview of the recent paper [46]. In this paper two-dimensional orders of finite representation type are classified too. This time the classification is in terms of generators and relations. The methods in loc. cit. rely on representation theory and fall outside the scope of this book. Hence we state the results without proof. We also outline how the classification in [3,4] can be related to the classification in [46]. (The classifications are in terms of very different invariants.)

In this chapter all modules will be left modules and will be finitely generated unless otherwise specified.

To avoid confusion later on we redefine the notions of order and tensorproduct we will use here. An **R-order** is as introduced in II.5 i.e. a reflexive module over a central Krull domain R. A **tame R-order** is an R-order which is tame over its center in the usual sense. If Λ is an R-order and M, N are right and left reflexive Λ-modules then $M \perp_\Lambda N = (M \otimes_\Lambda N)^{**}$. If Λ is tame and I and J are fractional ideals then $I*J$ is the fractional ideal $(IJ)^{**}$. The biduals are always taken with respect to R.

V.2. Rational Double Points Associated to Two-dimensional Tame Orders of Finite Representation Type.

There are different divisorially graded rings that can be associated to a tame order in the way of Theorem IV.2.8. There is one however that deserves the epitet canonical. It is the analog of a well known construction in the commutative case.

V.2.1 Definition : Let Λ be an R_0-order and assume furthermore that R_0 is factorial and contains a field k. Then we define the reflexive canonical module as $\omega_{\Lambda/k} = Hom_{R_0}(\Lambda, R_0)$. This is a Λ-Λ bimodule in a natural way.

V.2.2 Lemma : Let Λ be a tame R_0 order.

(a) $\omega_{\Lambda/k}$ is independent of R_0.

(b) $\omega_{\Lambda/k}$ is $Z(\Lambda)$ commuting.

(c) If Λ is tame then $\omega_{\Lambda/k} = P_1^{*(1-e_1)} * P_2^{*(1-e_2)} * \cdots P_n^{*(1-e_n)} \perp \omega_{Z(\Lambda)/k}$ where the P_i represent all the ramified divisors in $X^{(1)}(\Lambda)$ and the e_i are the corresponding ramification indices (See chapter IV).

Proof : (b) is clear. (a) and (c) follow from local computations using [46]. □

V.2.3 Corollary : If Λ is a tame order with the above assumptions and if $\omega_{\Lambda/k} \cong \Lambda$ as bimodules then Λ is a reflexive Azumaya algebra over a commutative ring R with $\omega_{R/k} \cong R$.

From the above lemma it follows that we can represent $\omega_{\Lambda/k}$ by a two sided fractional ideal I in $Q(\Lambda)$. If I represents a torsion element in $Cl_c(\Lambda)$, i.e. if $I^{*n} = a\Lambda$, where we take n minimal, then we can form $\Lambda_1 = \Lambda[I,n]_! = \Lambda[I]/(1-aX^n)$ where $\Lambda[I] = \oplus_{i=-\infty}^{\infty} I^{*i}X^i$ does not depend on the choice of I but does depend on a. In important cases however Λ_1 will also be independent of a (i.e. in the case that R_0 is complete local with an algebraically closed residue class field of char. zero). We have therefore chosen to exclude a from the notation. We will define $\Lambda[\omega_{\Lambda/k}] = \Lambda[I,n]_!$.

V.2.4 Proposition : Let Λ be tame with assumptions as above and assume furthermore that k has char. zero. Then $\Lambda_1 = \Lambda[\omega_{\Lambda/k}]$ is a tame order and $\omega_{\Lambda_1/k} \cong \Lambda_1$ as Λ_1-Λ_1 bimodules.

Proof : That Λ_1 is tame follows as in [38] once we have shown that Λ_1 is prime. This follows from the fact that we can choose a not to be a p'th power for all $p|n$. Then $Q(\Lambda_1) = Q(\Lambda)\otimes_Q(Z(\Lambda))Q(Z(\Lambda))[X]/(1-aX^n)$ which is prime. That $\omega_{\Lambda_1/k} \cong \Lambda_1$ follows by direct computations. □

Let us now restrict to the following situation. R_0 will be a powerseries ring in two variables over an algebraically closed field of characteristic zero. Λ will be a tame R_0-order of finite representation type, i.e. Λ will have a finite number of indecomposable reflexive left Λ-modules. In this case the prerequisites for defining $\Lambda[\omega_{\Lambda/k}]$ are clearly present and thus we put $\Lambda_1 = \Lambda[\omega_{\Lambda/k}]$ (Corollary V.2.3 and Proposition V.2.4 then already tell us that Λ_1 is a very special tame order but in this case we can do more).
The following proposition is basically due to Artin.

V.2.5 Proposition : $\Lambda_1 \cong End_{R_1}(M)$ where R_1 is the complete local ring of a rational double point and M is a reflexive R_1-module.

Proof : Put $R_1 = Z(\Lambda_1)$. R_1 being a direct summand of Λ_1 clearly

has finite representation type. Since R_1 is C.M. $\omega_{R_1/k}$ coincides with the usual dualizing module (see [23]). Hence R_1 is also Gorenstein because by V.2.3 and V.2.4 $\omega_{R_1/k} \cong R_1$. So R_1 is the complete local ring of a rational double point. To finish the proof of the proposition it suffices to show that the reflexive Brauergroup $\beta(R_1)$ is zero. Recall that the reflexive Brauergroup of a commutative ring S is defined as the similarity classes of reflexive Azumaya algebras with composition $\perp_S$. A and B are called similar if there are reflexive S-modules M and N such that $A \perp_S End_S(M) \cong B \perp_S End_S(N)$ as S algebras. □

V.2.6 Lemma : Let S be a two-dimensional complete local ring containing an algebraically closed residue classfield of char. zero. If S has finite representation type then $\beta(S) = 0$.

Proof : It is well known that S may be written as S'^G where S' is a formal power series ring in two variables and G acts linearly without pseudo reflections (see [6]) for an almost completely ring theoretical proof). Hence S'/S is unramified in codimension one and therefore reflexive Galois. So we can use the Chase Harrison Rosenberg sequence [11,12].

$$Cl(S') \to H^2(G, S'^*) \to \beta(S') \to H^1(G, Cl(S'))$$

Since $Cl(S') = 0$ we obtain $\beta(S) = H^2(G, S'^*)$ and since S' is complete of char. zero $H^2(G, S'^*) \cong H^2(G, k^*)$. Now note that since G acts without pseudo reflections, G must be contained in $Sl_2(k)$. Hence the following lemma finishes the proof. □

V.2.7 Lemma : ([3]). Assume that G is a subgroup of $Sl_2(k)$. Then $H^2(G, k^*) = 0$.

Proof : It is clear that we may replace G with its Sylow p-subgroups. Now a small p-group is either cyclic or binary dihedral. It is well known and easy to prove that in both these cases one has $H^2(G, k^*)$=0. □

Using the notation of Proposition V.2.5 we will call R_1 (the local ring of) the rational double point associated to Λ.

V.2.8 Remark : To complete his classification in [4] Artin needs a much stronger theorem than V.2.6. This theorem will be stated in Section 3. The methods to prove this theorem fall far outside the scope of this book however.

It is useful to analyse the connection between Λ and Λ_1 a little more closely. This will be useful if one tries to employ the method outlined in section V.4 and V.5 to compare the classifications in [3,4] and [46] of maximal orders of finite representation type. So let Λ be a tame R_0 order of finite representation type and $R = Z(\Lambda)$. Using Lemma V.2.2 we know that

$$\omega_{\Lambda/k} = P_1^{*(1-e_1)} * \cdots * P_k^{*(1-e_k)} \perp \omega_{R/k}$$

where the $(P_i)_i$ are the ramified prime divisors in $X^{(1)}(\Lambda)$ and the $(e_i)_i$ are the ramification indices. Let $P = P_1^{*(1-e_1)} * \cdots * P_k^{*(1-e_k)}$ and define $e = lcm e_i$. Then if m is the smallest positive integer such that $\omega_{\Lambda/k}^{\perp m} \cong \Lambda$ we must necessarily have $e|m$. Hence let $ef = m$. Since $P^{*e} = (p\Lambda)^{**}$ for some divisorial R-ideal p : $\omega_{\Lambda/k}^{\perp} = p \perp e_R \omega_{R/k}^{\perp e} \perp_R \Lambda$ Hence f is the smallest number such that $(p \perp_R \omega_{R/k}^{\perp e})^{\perp f} \cong R$ as R-R bimodules. Define $J = p \perp_p \omega_{R/k}^{\perp e}$ Let $R_2 = R \oplus J \oplus J^{\perp 2} \oplus \cdots \oplus J^{\perp(f-1)}$ where as usual the multiplication is defined via a bimodule morphism $J^{\perp f} \to R$. It is easy to see that R_2 is independent of the particular choice of the isomorphism. Let $\Lambda_2 = \Lambda \perp_R R_2$. It is then easy to see that $\omega_{\Lambda_2} \cong \omega_\Lambda \otimes_R R_2$ as bimodules. Furthermore one verifies that $\Lambda_1 \cong \Lambda_2[\omega_{\Lambda_2/k}]$. Hence we have proved the following.

V.2.9 Lemma : Let Λ be a tame R_0-order of finite representation type and $R = Z(\Lambda)$. Let e be the lcm of the ramification indices of Λ/R. Then there is an unramified extension R_2/R such that if we define $\Lambda_2 = \Lambda \perp_R R_2$ then $\omega_{\Lambda_2/k}^{\otimes e} \cong \Lambda_2$ and $\Lambda_1 \cong \Lambda_2[\omega_{\Lambda_2/k}]$

The structure of C.M. modules over rational double points is well understood. See e.g. [5]. Now by Proposition V.2.5 we know that Λ_1 is reflexive Morita equivalent with R_1. Furthermore Λ_1 is reflexive Galois over Λ. It is therefore tempting to try to describe the reflexive modules in terms of reflexive modules over R_1. This can sometimes be done. In fact it was the method employed by Artin to determine the reflexive modules in case $Z(\Lambda)$ is regular.

One always needs some extra information however. We can compare this with the methods we will outline in section V.5 There one finds the ranks of the Cohen Macaulay modules essentially as a free extra result. After choosing a character of $H = \mathbb{Z}/n\mathbb{Z}$ we let $H = \mathbb{Z}/n\mathbb{Z}$ act on Λ_1/Λ. This makes the extension reflexive H-Galois (the reflexive equivalent of II.5.15.). We then use the following result :

V.2.10 Proposition : Let Γ/Λ be a reflexive Galois extension of R_0-orders for a finite group G. Let M be an indecomposable reflexive Γ-module and define $H_M = \{\alpha \in G |_\alpha M \cong M\}$ and $|H_M| = h_M$. Then the decomposition of M as Λ-module consists of h_M indecomposable non-isomorphic Λ-modules $M_1, \ldots, M_{h_M}$ and $\Gamma \perp_\Lambda M_i = \oplus_{\overline{\alpha} \in G/H}\ {}_\alpha M$.

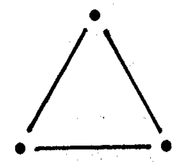
Proof : This is a standard application of extension and restriction of scalars (see for example [45]). □

From this proposition it should be clear that if one knows the action of H on Λ_1-ref one can compute the ranks of the indecomposable Λ-module. Since Λ_1 is reflexive Morita equivalent to R_1 and one knows the indecomposable reflexive R_1-modules we have to determine the action of H on R_1-ref. If σ is a generator of H the functor $M \to {}_\sigma M$ can also be written as $M \to {}_\sigma(\Lambda)_1 \otimes_{\Lambda_1} M$. ${}_\sigma(\Lambda)_1$ is an invertible $\Lambda_1 - \Lambda_1$ bimodule and hence it corresponds to a reflexive invertible $R_1 - R_1$ bimodule I'_σ. By Lemma II.5.3.5 we see that I'_σ induces the action σ on $Z(\Lambda)$. Hence $I'_\sigma = {}_\sigma(I_\sigma)_1$. So we see that σ acts on R_1-ref through $M \to_\sigma (I_\sigma \perp_{R_1} M)$.
For H to generate a H-action one easily verifies that the condition $I_\sigma \perp \sigma(I_\sigma) \perp \cdots \perp \sigma^{n-1}(I_\sigma) = R_1$ must be satisfied.

We give a simple example :
Let

$$\Lambda = \begin{pmatrix} R & R & R \\ pq & R & R \\ pq & pq & R \end{pmatrix}$$

where $R = k[[x,y]]$, $p = (x)$, $q = (y)$. Then $R_1 = k[[x,y,z]]/(z^3 - xy)$. Hence R_1 corresponds to the rational double point A_2 with extended Dynkin diagram



If I is the reflexive R_1 ideal (z,y) then the indecomposable reflexives of R_1 are given by R_1,I,I^{*2}.
Clearly if $\sigma \in H$ then ${}_\sigma M \cong M$ if $M \in \{R_1, I, I^{*2}\}$. Hence the action of H on R_1-ref is determined by the invertible bimodule I_σ for σ a generator of H. If $I_\sigma = I, I^2$ then Proposition V.2.10 implies that Λ has only one indecomposable reflexive module which is clearly impossible since there are three non-isomorphic projective Λ-modules. Hence I_σ is R and consequently there are 9 indecomposable reflexive Λ-modules all of rank three over R.

It does not take a big effort to actually find these modules. They are given by :

$$\begin{pmatrix} R \\ R \\ R \end{pmatrix} \quad \begin{pmatrix} R \\ R \\ p \end{pmatrix} \quad \begin{pmatrix} R \\ p \\ p \end{pmatrix}$$
$$\begin{pmatrix} R \\ R \\ q \end{pmatrix} \quad \begin{pmatrix} R \\ R \\ pq \end{pmatrix} \quad \begin{pmatrix} R \\ p \\ pq \end{pmatrix}$$
$$\begin{pmatrix} R \\ q \\ q \end{pmatrix} \quad \begin{pmatrix} R \\ q \\ pq \end{pmatrix} \quad \begin{pmatrix} R \\ pq \\ pq \end{pmatrix}$$

V.3. Skew Group Rings and Projective Representations Associated to Two-Dimensional Tame Orders of Finite Representation Type

As in the previous section let Λ be a tame R_0-order where R_0 is a formal powerseries ring in two variables over an algebraically closed field of characteristic zero. We will show that Λ is reflexive Morita equivalent to $S *_c G$ where $S \cong k[[x,y]]$, G acts linearly on $kx + ky$ and c takes values in k.
As before let $R = Z(\Lambda)$, $\Lambda_1 = \Lambda[\omega_{\Lambda/k}]$ and $R_1 = Z(\Lambda_1)$. Since R_1 is the complete local ring of a rational double point (Proposition V.2.5, $R_1 = S^G$ where $S = k[[x,y]]$ and G acts without pseudo reflections. S/R_1 is unramified in codimension one and hence it is reflexive Galois. Then $\Gamma = \Lambda_1 \perp_{R_1} S$ is a reflexive Azumaya algebra. Furthermore from II.5.1.13 it follows that Γ/Λ_1 is reflexive Galois.

V.3.1 Lemma : There is a group G acting on S such that $S^G = R$.

Proof : As in [3 lemma 4.2]. Let K, K_1, L be resp. the quotient fields of R, R_1, S. It suffices to show that we can extend the automorphisms in H to L since L/K will then be Galois. If $G = Gal(L/K)$ then G will leave S invariant since S is the integral closure of R in L. Let $\sigma \in H$. We denote by S^σ the R_1-algebra isomorphic to S as a ring but with an embedding $R_1 \longrightarrow \sigma R_1 \subset S$. Then $S \perp_{R_1} S^\sigma$ is unramified in codimension one over S and is therefore totally split (by purity of the branch locus). But this also means that $L \perp_{K_1} L^\sigma$ is totally split which implies that σ extends to L by classical Galois theory.

Now let us continue with the proof of Proposition V.3.2 below. Lemma V.3.1 has given us a group G acting on S. We claim that we can extend the action of G to Γ. To this end we simply define $\sigma(\gamma\otimes s) = \overline{\sigma}(\gamma)\otimes\sigma(s)$ where $\overline{\sigma}$ is the image of σ in H. Since Γ/Λ is reflexive G-Galois. Γ is reflexive Morita equivalent to $\Lambda * G$. On the other hand since $\beta(S) = 0$ and every reflexive S-module is free we see that $\Lambda = M_n(S)$. It follows then from Proposition II.5.3.7 that Λ is reflexive Morita equivalent to $S *_c G$ where $c \in H^2(G, S^*)$.
It is now well known that we can make the following simplifications :

(1) The action of G on $k[[x,y]]$ can be linearized in x and y.

(2) The cocycle c can be given values in k^* since $H^2(G,S^*) \cong H^2(G,k^*)$.

So let us summarize what we have shown :

V.3.2 Proposition : With Λ as above, Λ is reflexive Morita equivalent to $k[[x,y]] *_c G$ where G acts linearly and faithfully on $kx + ky$ and c takes values in k.

Proof : The action of G on $k[[x,y]]$ and hence on $kx + ky$ must be faithful since otherwise $Z(\Lambda) \not\subset k[[x,y]]$. □

In view of the above proposition it makes sense to study the indecomposable reflexive modules of $k[[x,y]] *_c G$. This can be done by studying the projective representations of G [25]. Recall that a projective representation of k is a homomorphism $\alpha : G \to PGL_n(k)$. The exact sequence $0 \to k^* \to GL_n(k) \to PGL_n(k) \to 0$ induces a map $\delta : Hom(G, PGL_n(k)) = H^1(G, PGL_n(k)) \to H^2(G,k^*)$. If $\delta(\alpha) = c$ then we say that α corresponds to c. If we form the skew group ring k_cG then α may be used to define a left k_cG module structure on k^n. Hence projective representations of G correspond to representations of k_cG for some c.
A projective representation is said to be **irreducible** if it corresponds to an irreducible representation of k_cG.

V.3.3 Proposition : Let $\Lambda = k[[x,y]] *_c G$ with assumptions as in Proposition V.3.2. Then the indecomposable Λ-modules are in one-one correspondence with the projective representations corresponding to c.

Proof : Since gl. dim. $\Lambda = 2$ every reflexive Λ-module is projective. Projective Λ-modules are in one-one correspondence with projective $k_cG = \Lambda/rad\Lambda$-modules. These are by definition the projective representation of G corresponding to c. □

Suppose now that $\Lambda = k[[x,y]] *_c G$ as above. It is then a natural question to ask whether there is a criterion for Λ to be a maximal order in terms of c and G. Such a criterion was found by Artin in [3]. Here we give a modified proof not using any geometry or sophisticated cohomology.
Let $S = k[[x,y]]$, $R = Z(\Lambda) = S^G$. We will use the following criterion for Λ to be a maximal order :
If $p \in X^{(1)}(R)$ then Λ_p is a maximal order if and only if $rad(p\Lambda)$ is a prime ideal if and only if $\Lambda' = \Lambda/rad(p\Lambda)$ is a prime ring if and only if $A' = Q(\Lambda/rad(p\Lambda))$ is a prime ring.
It is easily verified that $rad(p\Lambda) = rad(pS) *_c G$ and hence $\Lambda' = \Lambda/rad(p\Lambda) = (S/rad(pS)) *_c G$. If p does not ramify in S/R then G acts X-outer on $S/rad(pS)$. Hence Λ' is prime.
This means that we can restrict attention to elements in $X^{(1)}(R)$ that do ramify in S. It is easy to see that the primes in S, that are ramified, come from lines of pseudo reflections in $V = kx+ky$. Furthermore, lines of pseudo reflections, that correspond to the same height one prime in R, are conjugate.
Let $p_L \in X^{(1)}(S)$ correspond to a line of pseudo reflection L and define the following groups : H_L is the stabilizer of L, H_L^0 is the subset of H_L that leaves L pointwise fixed. Let L' be a H_L complement of L in V. Then H_L acts faithfully on $L \oplus L'$ and hence $H_L \subset k^* \times k^*$. This implies that H_L is abelian. Furthermore H_L^0 acts faithfully on L' and therefore $H_L^0 \subset k^*$. This implies that H_L^0 is cyclic. Let $H_L' = H_L/H_L^\circ$. Then by the definition of H_L°, H_L' must act faithfully on L. This again implies that H_L' is cyclic.
Now $A' = Q(S/\cap_{\sigma\in G}\, p_L) *_c G$ will be Morita equivalent to $A'' = Q(S/p_L) *_{c'} H_L$ where c' is the restriction of c to H_L. Since $H^2(H_L^\circ, k^*) = 0$ we may assume that $\sigma\tau = \tau\sigma$ in A'' if $\sigma, \tau \in H_L^\circ$.

The cocycle c' will then give rise to a pairing $\omega : H_L^\circ \times H_L' \to k^*$ following the rule $\forall \sigma \in H_L^\circ, \tau \in H_L : \tau\sigma = \omega(\sigma, \overline{\tau})\sigma\tau$.
It is clear that ω is well defined. Hence $A'' = (Q(S/p_L)H_L^\circ) *_{c''} H_L'$ where H_L' acts on $A''' = S/p_L * H_L^\circ$ through ω. c'' is some cocycle irrelevant for the sequel.
Now we have one more step to make. Let h_L°, h_L' be generators of H_L°, H_L' respectively. Then $\omega(h^\circ, h_L')$ is some root of unity, say of order t. Clearly $t \leq \mathrm{Min}(|H_L^\circ|, |H_L'|)$. Furthermore $A''' = \oplus_{i=1}^{|H_L^\circ|} Q(S/p_L)$. Then the action of h_L' on the components of Γ is given by cycles of length t.
There are now two possibilities :

(1) $t < |H_L^\circ|$. In that case H_L' does not act transitively on the components of Γ and hence A'' will not be a prime ring.

(2) $t = |H_L^\circ|$. In this case A'' will be Morita equivalent to $Q(S/p_L) *_{d'} H_L''$ for some subgroup H_L'' of H_L'. Now remember that $Q(S/p_L)$ is the function field of L and H_L' acts faithfully on L. Hence H_L'' acts faithfully on L and $Q(S/p_L) *_{d'} H_L''$ will be a prime ring.
Hence we have proved :

V.3.4 Theorem : [3, Theorem 4.16]. Let $\Lambda = k[[x,y]] *_c G$ with assumptions as in Proposition V.3.2 Then Λ will be a maximal order if and only if for every line of pseudo reflection of order $t(= |H_L^\circ|)$ the restriction of c to H_L has order t.

V.4 Classification of Two Dimensional Maximal Orders of Finite Representation Type.

In this section we summarize Artin's classification of two dimensional maximal orders finite representation type [3]. The basic technical tool is the computation of the Brauergroup of the function field of a 2 dimensional rational singularity. Unfortunately we have no ring theoretical proof for this theorem and hence we will state it without proof.
In this section we keep the notation and conventions of the previous sections, i.e. $R_0 = k[[x,y]]$, k algebraically closed of char. zero, R is an integrally closed Noetherian local ring, finite as a module over R_0. K is the quotient field of R.

Now we introduce some new notation. For $n \in \mathbb{Z}$ let μ_n denote the group of n'th roots of unity in k. There is a natural map $\mu_{mn} \to \mu_n : x \to x^m$. We then define $\mu = \varprojlim_n \mu_n$. This is a profinite group, non canonically isomorphic to $\hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$.
Fix a certain prime $p \in X^{(1)}(R)$ and define $\overline{K} = Q(R/p)$. Then $\overline{K}$ is isomorphic to $k((u))$. Furthermore it is easy to see that there is a canonical isomorphism between $Gal(\overline{K})$ and μ. Now let $[A] \in Br(K)$ and choose a maximal order Λ in A. Define $\overline{L} = Q(Z(\Lambda/rad(p\Lambda))$. It is well known that $\overline{L}/\overline{K}$ is cyclic of order e_p where e_p is the ramification index of p in Λ and furthermore there is a canonical element of $Gal(\overline{L}/\overline{K})$ obtained by conjugation with the uniformising element

of Λ_p. Hence there is a canonical homomorphism $Gal(\overline{K}) \to \mathbb{Z}/e_p\mathbb{Z}$ where e_p is the ramification index of p in Λ. It is easy to see that this homomorphism is independent of the choice of Λ in A. $[A]$ defines an element of

$$Hom(Gal(\overline{K}), \mathbb{Z}/e_p\mathbb{Z}) \cong Hom(\mu, \mathbb{Z}/e_p\mathbb{Z}) \subset Hom(\mu, \mathbb{Q}/\mathbb{Z}) \cong^{def} \mu(-1)$$

Let's denote this element by $\mu_p([A])$. By taking the sum over all $p \in X^{(1)}(R)$ we obtain a map $\phi : Br(K) \to \oplus_{p\in X^{(1)}(R)}\mu(-1) : [A] \to \oplus_{p\in X^{(1)}(R)}\mu_p([A])$. (Note that this direct sum is well defined since Λ is ramified only in a finite number of height one primes.)

V.4.1 Theorem : [4] Assume that R is the complete local ring of a rational singularity. Then there is an exact sequence

$$0 \to Br(K) \longrightarrow \phi \oplus_{p\in X^{(1)}(R)} \mu(-1) \longrightarrow \Sigma\mu(-1) \to 0$$

where Σ denotes the sum map.

For us it is sufficient to know that if R is of finite representation type then R is the local ring of a rational singularity.

In the sequel we follow [4] closely.
Denote by $C_{n,q}$ the cyclic subgroup of $GL_2(k)$ generated by

$$\sigma = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^q \end{bmatrix}$$

where ζ is a primitive n'th root of unity and $(n,q) = 1$.

V.4.2 Theorem : Let Λ be a maximal order of finite representation type with center R.

(i) There is a cyclic subgroup $G_0 = C_{n,q}$ of GL_2 such that R is isomorphic to the fixed ring R'^{G_0} of the power series ring $R' = k[[u,v]]$ under the action of G_0.

(ii) Let $\Lambda' = \Lambda \perp_R R'$. Then Λ' is a tame order over R' and with suitable choice of u, v its ramification data has one of the forms listed

in table V.4.4. The integers n, q satisfy the congruence relations listed in that table.

V.4.3 Theorem : Suppose given a triple consisting of

(a) a choice of ramification data (V.4.4).

(b) a cyclic group $G_0 = C_{n,q}$ satisfying the congruences V.4.4.

(c) a generator of the cyclic group $\mu_e(-1) = Hom(\mu_e, \mathbb{Q}/\mathbb{Z})$.

Let $R = k[[u,v]]^{G_0}$ and let K be the field of fractions of R. Then there is a unique class $[D] \in Br(K)$ of period e such that the maximal R-orders D have finite representation type and such that the ramification data and cyclic group they determine as in Theorem V.4.2 are the given ones.

V.4.4 Table :

I_d	$e = d$	$f = uv$	
II_d	$e = 2$	$f = v(v - u^d)$	$q \cong d(\text{mod } n)$
III_d	$e = 2$	$f = u(v^2 - u^d)$	$2q \cong d(\text{mod } n)$ and $q \not\cong (1/2)d$, if d is even
IV	$e = 3$	$f = v(v - u^2)$	$q \cong 2 \ (\text{mod } n)$
V	$e = 2$	$f = v(v^2 - u^3)$	$2^q = 3 \ (\text{mod } n)$

Here e denotes the ramification index of the ramified primes in $X^{(1)}(R)$. As we will see below, these are all equal. The divisors of f generate these ramified primes.

Proof of theorem V.4.2 :

Since R has finite representation type $R = R'^{G_0}$ where G_0 acts without pseudo reflections. As above let $\Lambda' = \Lambda \perp_R R'$. Λ' will be of finite representation type by IV.2.5 and II.5.1.13. Λ/R and Λ'/R' will be ramified in finite sets of height one primes : $\{p_1, \ldots, p_k\}$ and $\{p'_1, \ldots, p'_l\}$ respectively. Furthermore R' is factorial and hence $p'_i = (f_i)$, $f_i \in R'$. As in V.2 we can construct $\Lambda'_1 = \Lambda'[\omega_{\Lambda'/k}]$ and define $R'_1 = Z(\Lambda'_1)$. Let $H = \mathbb{Z}/n\mathbb{Z}$. We let H act on Λ'_1/Λ' via the primitive root of unity ζ (i.e. $\overline{n}a_{\overline{m}} = \zeta^n a_{\overline{m}}$).
To summarize we have the following diagram :

$$\begin{tikzcd} \Lambda \arrow[r] & \Lambda' \arrow[r] & \Lambda'_1 \\ R \arrow[u] \arrow[r] & R' \arrow[u] \arrow[r] & R'_1 \arrow[u] \end{tikzcd}$$

V.4.5 Lemma : Λ'_1/Λ is reflexive Galois.

Proof : We already know that Λ'/Λ and Λ'_1/Λ' are reflexive Galois so we will try to apply II.5.1.14. This means that we have to lift the action of the elements of G_0 to Λ'_1. Since $\omega_{\Lambda'/k} \cong Hom_{R_0}(\Lambda', R_0)$ we can define an action on G_0 on $\omega_{\Lambda'/k}$ and consequently on all powers $\omega^{\perp m}_{\Lambda'/k}$. Let $n = |H|$ and let f denote the $\Lambda' - \Lambda'$ bimodule isomorphism $f : \omega^{\perp n}_{\Lambda'/k} \to \Lambda'$ used for the definition of Λ'_1. Now look at the following diagram :

$$\begin{tikzcd} \omega^{\perp n}_{\Lambda'/k} \arrow[r, "\sigma^{\perp n}"] \arrow[d, "f"] & \omega^{\perp n}_{\Lambda'/k} \arrow[d, "f"] \\ \Lambda' \arrow[r, "\sigma"] & \Lambda' \end{tikzcd}$$

where $\sigma \in G_0$. If this diagram were commutative then σ would lift to Λ'_1 in the obvious way. In general however there will be $a_\sigma \in R'^*$ such that $(f^{-1} \circ \sigma \circ f)(-) = a_\sigma(\sigma^{\otimes' n}(-))$. The $(a_\sigma)_\sigma$ determine an element of $H^1(G, R'^*)$ and since R' is complete, $H^1(G, R'^*) \cong H^1(G, k^*)$. This means that we can change f by an element of R'^* such that $a_\sigma \in k^*$. Assume that this is the case. Then take b_σ such that $b^n_\sigma = a_\sigma$ and define $\sigma^*(u) = b_\sigma \sigma(u)$ for $u \in \omega_{\Lambda'/k}$. Then σ^* extends to a ring automorphism of Λ'_1 which we also denote by σ^*. It is easily verified that the conditions of Proposition II.5.14 are satisfied and hence Λ'_1/Λ will be G-Galois for some finite group G with $G/H \cong G_0$.

V.4.6 Lemma : The number of ramified prime divisors in Λ'/R' is either 2 or 3.

Proof : As in IV.2.8 we conclude that the ring $R'' = R'[s_1, \ldots, s_l]/(s_i^{e_i} - f_i)$ has finite representation type. Since R'' is a complete intersection, it is also Gorenstein and hence it is a rational double point. Since a rational double point has embedding dimension 3, we deduce that $l \leq 3$. The fact that $l \geq 2$ follows from the fact that $l \geq k$ and $k \geq 2$ by Theorem V.4.1. □

V.4.7 Lemma : The group G_0 is cyclic.

Proof : Since $2 \leq k \leq l \leq 3$ by Lemma V.4.6, there is an i such that $rad(p_i R')$ is prime. Since G_0 is small, G_0 acts faithfully on $Q(R'/p_i R')$. Hence G_0 is cyclic since $Q(R'/p_i R')$ is a powerseries ring in one variable. □

V.4.8 Lemma : The ramification indices in Λ'/R' are equal. If $l = 2$ then the group G_0 fixes the primes p'_1, p'_2. If $l = 3$ then G_0 fixes one of the p_i's and interchanges the other two. In this case the ramification indices are equal to 2. In either case $k = 2$.

Proof : The assertion for $l = 2$ follows from V.4.1. Assume that $l = 3$. The ring R'' (constructed as in the proof of Lemma V.4.6) is a rational double point and hence has embedding dimension 3. Therefore two of the variables u, v, s_i can be eliminated and since $e_i > 1$ we can eliminate u,v. This means that two of the elements f_i are local parameters. Adjusting coordinates and reindexing one obtains $s_1^{e_1} = u$, $s_2^{e_2} = v$, $s_3^{e_3} = f(u, v)$. We substitute into the third equation and write this neutrally as $\phi(s_1^{e_1}, s_2^{e_2}, s_3^{e_3})$. Reindexing again if necessary we may assume $e_1 \leq e_2 \leq e_3$. Since this locus defines a rational double point, $e_1 = 2$ and the coefficient of s_1^2 is not zero [26, Theorem 4]. Changing coordinates again we may assume that the equation has the form $s_1^2 = g(s_2^{e_2}, s_3^{e_3})$.
Again since this is a rational double point the locus $g(s_2^{e_2}, s_3^{e_3})$ can have at most a triple point with an infinitely near double point [26, 2.6, 2.7]. Therefore the possible values of the e_i are $(2,2,n)$ and $(2,3,r)$ where n is arbitrary and $r = 3$, 4, 5. The exact sequence V.4.1 rules out all values except $(2,2,2)$. □

Let $f = f_1 f_2 f_3$. Then $R'_1 = R'[w]/(w^e - f)$. This is a rational double point. This happens precisely if f is as in table V.4.4. Note that the case where l is 3 is contained in III_{2d}

V.4.9 Proposition : The coordinates u, v can be chosen in such a way that the action of G_0 is linear and diagonal and such that the ramification locus has one of the standard forms listed in table V.4.4.

Proof : This is a case by case verification using the following idea : Since R_1' is a rational double point $R_1' = S$, $S = k[[u',v']]^{G^*}$ and G^* acts faithfully without pseudo reflections. Again one can show that there is an action for a finite group G on S such that $S^G = R$. This action maybe taken to be linear on $ku' + kv'$. The subgroup G' of G that stabilizes R' is generated by pseudo reflections. Groups generated by pseudo reflections have been classified. The invariants of S under G' are generated by homogeneous polynomials in u', v' of known degrees. One can then use this information together with the shape of f (see table V.4.4) to bring the action of G_0 in the desired form.

Proof of theorem V.4.3 : Let $f = f_1 f_2$ of $f_1 f_2 f_3$ and let $G_0 = C_{n,q}$. The restrictions on n and q imply that G_0 fixes one of the f_i and fixes the other or interchanges the other two depending on whether $l = 2$ or 3. Define $R = R_1^{G_0}$ and let $p_i = (f_i) \cap R$. Then there are two different $p_i's$. Then theorem V.4.1 implies that the classes in the Brauergroup $Br(K)$, which have period e, are in bijective correspondence with generators of the cyclic group $\mu_e(-1)$. Choose such an $[A]$ in $Br(K)$ and let Λ be a maximal K-order in A. Then the ramification of $\Lambda' = \Lambda \otimes_R R'$ is given by (f_i). Hence Λ' has finite representation type and therefore Λ has finite representation type. □

To finish this section we include a word on the rational double points associated to the algebras in table V.4.4. A first step in their determination is the computation $J = p^{\perp(1-e)} \perp \omega_{R/k}^{\perp e}$ (see Lemma V.2.9). J determines an element of $Cl(R)$. We can use the exact sequence

$$0 \to H^1(G_0, R'^*) \to Cl(R) \to Cl(R')$$

since R'/R is unramified. Hence $Cl(R) = H^1(G_0, R'^*) = H^1(G_0, k^*) \cong Hom(G_0, k^*) \cong \mu_n$ using the given generator $\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix}$ of G_0. It is easily verified that $\omega_{R/k}$ corresponds to $\sigma(f)/f$. Hence J corresponds to $\zeta^{(1+q)e}\left(\frac{\sigma(f)}{f}^{(1-e)}\right)$. This quantity can be computed for each of the algebras listed in table V.4.4. From this, one can compute the associated rational double points for each of the algebras in table V.4.4.

V.4.10 Example : Let Λ be a tame order with type III_d with

$d = 2k$ =even. Then $\sigma(f)/f = \zeta^{2q+1}$. Hence $\zeta^{(1+q)e}(\sigma(f)/f)^{1-e} = \zeta^{2q+2-2q-1} = \zeta$. Therefore J has order n and hence $R_2 \cong R'$ (see Lemma V.2.9). Hence the rational double point associated to Λ is given by the local ring $R'(\sqrt{f})$ which corresponds to the rational double point D_{d+2}.

V.5. Appendix
Recent Work of I. Reiten and the Second Author.

In the recent paper [46] tame and maximal orders of finite representation type were classified also. The methods employed are representation theoretic and hence they fall outside the scope of this book. We will however introduce the necessary notation to state one of the main classification results. From this it will be clear that the classification is in totally different invariants compared to table V.4.4.
The job of matching up the two classifications has not been completed yet. We outline the method that - in principle - should do it. However we do not carry out the procedure since the computations involved seem to be tedious.

V.5.1 The main classification result.

Let us briefly recall some definitions. A **quiver** Q is a directed graph. Q maybe formally defined as a quadruple (Q_0, Q_1, b, e) where Q_0, Q_1 are sets denoting resp. the vertices and the arrows of Q. $b, e : Q_1 \to Q_0$ are maps assigning to an arrow its beginning and its end. A priory, there are no restrictions on Q_0, Q_1, b, e i.e. we allow loops and multiple arrows. With Q one can form the **path algebra** kQ. kQ has a k-basis consisting of directed paths in Q. Composition of paths is concatenation if this is possible and zero otherwise. kQ becomes a k-graded algebra by giving the arrows degree one.
A pair (Q, τ) where Q is a quiver and $\tau : Q_0 \to Q_0$ is a bijection is called a **stable translation quiver** if for each vertex q in Q_0 there is a bijective map between the arrows ending in q and the arrows starting in τq. If $q \in Q_0$ let us denote with B_q, E_q the vector spaces with basis $\{f \in Q_1 | b(f) = q\}$ and $\{f \in Q_1 | e(f) = q\}$. Suppose that (Q, τ) is a stable translation quiver. We say that $w \in (kQ)_2$ is a quadratic relation ending in $q \in Q_0$ if $w = \sum_{i=1}^{n} a_i Y_i$ such that every Y_i is a path of length two starting in τq and ending in q, and if w induces a nondegenerate pairing between the vector spaces B_q^* and E_q^*. We say that $W \subset (kQ)_2$ is a set of quadratic relations in Q if W contains exactly one quadratic relation ending in q for every $q \in Q_0$.
If Δ is a tree then there is a natural way to construct a translation quiver $\mathbb{Z}\Delta$ from Δ. Fix an arbitrary orientation on Δ. The vertices of $\mathbb{Z}\Delta$ are (x, n) where $x \in \Delta_0$ and $n \in \mathbb{Z}$ for each arrow $\alpha : x \to y$ in Δ there are, for every n, arrows $(y, n) \to (x, n+1)$. The translation is

given by $\tau(x,n) = (x, n-1)$. It is an interesting exercise to show that Δ does not depend on the chosen orientation on Δ.
Let Λ be a two-dimensional tame R_0-order where R_0 is as before a two-dimensional power series field over an algebraically closed field of char. zero. Assume that Λ has finite representation type. Then from chapter IV we know that Λ is reflexive Morita equivalent to an order Λ' of global dimension 2. We can also assume that Λ' is basic (i.e. $\Lambda'/rad\Lambda' \cong k \oplus \cdots \oplus k$). The following proposition is a very short extract from [46].

V.5.1.1 Proposition : Let Λ be basic and of global dimension two. Then $\Lambda = \widehat{kQ/G}$ where $Q \cong \mathbb{Z}\Delta/G$ with $\Delta = A_{\infty,\infty}$, D_n, E_6, E_7, E_8. G is a group acting freely on $\mathbb{Z}\Delta$ such that $\mathbb{Z}\Delta/G$ is finite, W is a set of quadratic relations on Q and $\hat{}$ denotes completion at the positive part.

V.5.1.2 Remark : It is easy to see that Q is uniquely determined by $\widehat{kQ/W}$. This is not the case for W. In [46] I.Reiten and the second author classify exactly which $W's$ occur and they describe when they are "equivalent" i.e. give rise to the same algebra. In V.5.2. it will be useful to know which quivers occur for $End_S(M)$ where S is the local ring of a rational double point. This is answered in the following proposition [46].

V.5.1.3 Proposition : Assume that Λ is a basic tame order of global dimension two and $\Lambda \cong \widehat{kQ/W}$ as above. Then $\Lambda \cong End_S(M)$ where S is the local ring of a rational double point and M is a reflexive S module if $Q = \Delta'$ where Δ is an extended Dynkin diagram corresponding to S and Δ' is obtained from Δ by replacing $[\bullet \longrightarrow \bullet]$ by $[\bullet \rightleftarrows \bullet]$ and taking the identity for τ.

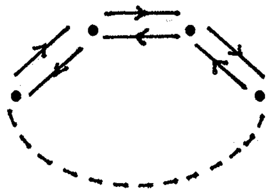
V.5.2 The dualizing module

Let Λ be a tame R_0-order. In V.2 we have defined $\omega_{\Lambda/k}$ as the bimodule $Hom_{R_0}(\Lambda, R_0)$. An exercise in local duality however shows that (as M. Artin pointed out to us)

$$\omega_{\Lambda/k} = \lim_{l} Hom(Ext^2_{\Lambda-mod}(\Lambda/rad^n\Lambda, \Lambda), k) \qquad (*)$$

This description of $\omega_{\Lambda/k}$ is ideal if Λ is of the form $\widehat{kQ/W}$, i.e. in the case that Λ is basic of global dimension two. In that case ω_Δ will be left projective and hence $\omega_{\Lambda/k}$ is an invertible bimodule. Since Λ is basic , $\omega_{\Lambda/k}$ is right free and hence $\omega_{\Lambda/k} = {}_f\Lambda_1$ where f is some ring automorphism of Λ defined up to an inner automorphism of Λ. Furthermore since $\omega_{\Lambda/k}$ is R_0 commuting we see that f must be the identity on $Z(\Lambda)$. The order of $\omega_{\Lambda/k}$, i.e. the smallest n such that $\omega_{\Lambda/k}^{\otimes n} \cong \Lambda$ as bimodules, is given by the smallest n such that f^n is inner, i.e. such that $f^n(-) = u - u^{-1}$. As we will see below f is obtained from a graded automorphism of kQ/W. From this one can easily see that one can take u to be in $(kQ/W)_0$. Then $\Lambda[\omega_{\Lambda/k}] \cong \Lambda[X]$ where $aX = Xf(a)$ and $X^n = u^{-1}$. We now want to change X by an element of $(kQ/W)_0$ to obtain $X^n = 1$. It is easy to see that this is possible if and only if $f(u) = u$. Since f is trivial on $Z(\Lambda)$, $f = v - v^{-1}$ where v is some element of the quotient field of Λ.
Then $f^n = v^n - v^{-n}$ and hence $v^n u^{-1} \in Q(Z(\Lambda))$. From this it easily follows that $f(u) = u$. Hence we may assume that $X^n = 1$ and thus $\Lambda[\omega_{\Lambda/k}] \cong \Lambda * \mathbb{Z}/n\mathbb{Z}$. As in [RR] it follows that $\Lambda * \mathbb{Z}/n\mathbb{Z}$ is Morita

equivalent to $k\widehat{Q'/W'}$ where Q' and W' can be obtained from Q and W by a well defined procedure. This means that we can compute the rational double point associated to Λ once we have computed the automorphism f. This can be done from (*) but the computation is long and tedious. We just state the result. We also restrict ourselves to the case with single arrows. This is not such a big limitation since there are very few double arrow cases. They are all of the form



The double arrow cases can be treated in a similar way as the single arrow cases but the notation is more complicated.

Assume that (Q,τ) is a stable translation quiver with single arrows. If q, $q' \in Q_0$ then we denote with $X_{qq'}$ the arrow starting in q and ending in q', if it exists. If W is a quadratic relation then W is of the form

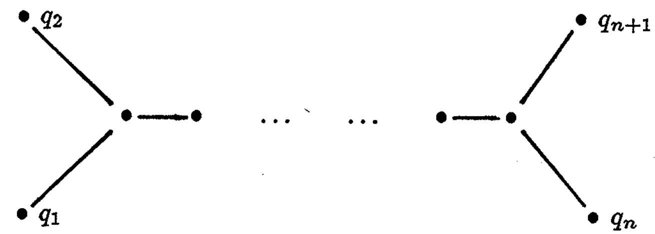
$$W = \sum_{q' \to q} a_{q'q} X_{\tau q q'} X_{q'q}$$

where $a_{q'q} \in k^*$

V.5.2.1 Proposition : Let $k\widehat{Q/W}$ as in Proposition V.5.1.1 Then $\omega_{\Lambda/k} \cong {}_f\Lambda_1$ where a possible choice of f is given by

$$f(X_{\tau q' \tau q}) = -\frac{a_{q'q}}{a_{\tau q q'}} X_{q'q}$$

V.5.2.2 Example : In [46 p. 4.16] the maximal orders are classified. Case (2) is given by $\Lambda = k\widehat{Q/W}$ where $Q = \mathbb{Z}\tilde{D}_m/(\tau^t \rho_6)$ with m even, $(\frac{m-2}{2}, t) = 1$, $\frac{m-2}{2} - t$ is odd and W is arbitrary. ρ_6 acts on $\tilde{D}_m$ as follows



$\rho_6(q_1) = q_n, \rho_6(q_m) = q_2, \rho_6(q_2) = q_{m+1}, \rho_6(q_{m+1}) = q_1$. The relations on Q may be taken to be of the form :

$$\sum_{q' \to q} X_{\tau q q'} X_{q'q}$$

i.e. all $a_{q'q}$ are one. Then $f(X_{\tau q' \tau q}) = -X_{q'q}$.

Since the period of τ is even we can replace $X_{\tau^{2k}q'\tau^{2k}q}$ with $-X_{\tau^{2k}q'\tau^{2k}q}$ for all k. Then $f(X_{\tau q' \tau q}) = X_{q'q}$. Hence the order of $\omega_{\Lambda/k}$ is the order of τ which is $4t$.

Applying [45] we find that $\Lambda[\omega_{\Lambda/k}] = k\widehat{Q'/W'}$ where Q' is

Hence the rational double point corresponding to Λ is D_m. Comparing this with example V.4.10 it is tempting to assume that Λ is in the category III_d, d even. This is indeed the case since one can show that no other algebra in the classification of [46] gives rise to a dualizing module with even period and with an associated rational double point D_m, m even.

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