Permutation Modules and
Rationality Problems 1

Lieven Le Bruyn*

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Abstract
In these lecture notes the main results concerning stable rationality of tori-invariants are recalled and proved. The Lenstra- and Saltman forest are introduced as conceptual tools in these investigations.

Key-Words
Representation theory, rationality problems, permutation lattices.

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* Research Associate of the NFWO

0. INTRODUCTION :

These notes are a transcript of lectures given as a third cycle course for the contact group algebra-geometry-number theory of the Belgian NSF (NFWO) in the fall of 1988. The initial goal of these lectures was to give an account of the negative solutions to the Emmy Noether problem for Abelian groups over $\mathbb{Q}$ by Hendrik Lenstra (1974) and for arbitrary groups over $\mathbb{C}$ by David Saltman (1984) as well as some of the surrounding theory. Unfortunately (for the audience) i became graduately more interested in the determination of the stable equivalence classes of tori-invariants and the connection between this problem and representation theory (integral and modular). For this reason, the present set of notes is not the most consistent possible. A totally rewritten version (preliminary title "the tori-junglebook") will be compiled shortly. As the notes of the next trimester will be junglebook-compatible a quick introduction is added at the end of these notes.

Let us give a brief survey of the results proved in this first set of notes :

In the first section we fix the main problems (1): The Noether-Saltman problem asking when lattice invariants are rational (2): The Noether-problem asking when permutation lattice invariants are rational and a seemingly generalization of both problems (3): The tori-problem asking when tori-invariants are rational. The historical root of this problem (the realizable Galois group problem) is recalled as well as the first results in the area (Fischer 1916 : permutation lattice invariants are rational over $\mathbb{C}$ for Abelian groups). Then, Saltman's procedure is given to derive counterexamples to the Noether-problem starting from certain non-rational lattice invariants. Further, some connections are given between the Noether-Saltman problem and the Merkurjev-Suslin theorem. In particular, stable rationality of the centers of generic crossed products (resp. generic division algebras) would imply Merkurjev-Suslin. And these problems are just particular Noether-Saltman settings. At this point we should warn the reader that Saltman gave non-stably rational centers of generic crossed products and this result will be explained in the next trimester. However, for the center of the generic division algebras no counterexamples are known but Mowgli conjectures that 8 by 8 matrices should be one. A thorough discussion of this example is also postponed until the next trimester.

In the second section we introduce a conceptual tool to visualize the tori-problem : the Lenstra forest. The G-lattices are classified by isomorphism according to their rank (by the Jordan-Zassenhaus result each rank has only a finite number of classes) and an edge is drawn between two lattices M and N whenever we have an exavt sequence of G-lattices

$$0 \to M \to N \to P \to 0$$

where P is a permutation lattice. It is easily checked that this defines an equivalence relation on the G-lattices. Moreover, we prove (following Masuda,Endo-Miyata,Voskresenskii and Colliot-Thelene,Sansuc) that the 'trees' (they can have loops and more roots !) in this forest correspond one-to-one to the stable equivalence classes of tori-invariants (and some weaker implications are given for lattice invariants).

In the third section we give an explicit description of the easiest forest of all namely that of a cyclic group of prime order. In this case one has as many trees as there are elements in the ideal classgroup of the p-th cyclotomic field giving a lot of non-rational tori-invariants. It also follows from the argument that all trees are isomorphic in this case.

In the fourth section we recall some of the basics of group cohomology theory which is indispensable before going into the forest. Proofs of all this material can be found in several excellent texts such as Serres 'Local Fields'.

In the fifth section we begin our main task i.e. counting the number of trees in the Lenstra forest of an arbitrary group. We follow here the approach by Coilliot-Thelene and Sansuc pending on the notion of flasque and coflasque resolutions of lattices. We introduce the Sansuc-semigroup (lattices modulo isomorphism upto adding permutation modules) and look at various subgroups : the Colliot-semigroup (spanned by the classes of flasque lattices) , the Coco-semigroup (spanned by the flasque-coflasque lattices) and the permutation classgroup $PCl(G)$ (the invertible elements in Sansuc). It turns out that the number of trees in the forest is equal to the number of elements in the Colliot-semigroup and the tree-invariant is the class of the end term in a flasque resolution.

In the sixt section we prove the Endo-Miyata result stating that the Colliot-semigroup is actually a group if and only if the group is metacyclic. Further we give Lenstras lunatic proof of the finiteness of the forest in the cyclic case and state the Endo-Miyata classification of all finite forests. We do not prove this result here but refer to the junglebook where it will be proved as a consequence of Dress' computation of the rank of the permutation classgroup.

In the seventh section we begin this Dress computation. Along the way we give the easy but elegant description due to Esther Beneish of invertible lattices as locally-locally permutation lattices.We give some applications of her results for tori- and lattice invariants. Using standard descent arguments, Dress reduces the computation of the rank of the permutation classgroup to the ranks of the classgroups over the p-adic integers and to character theory over $\mathbb{Q}$ and the p-adic fields.

In the next section we conclude this computation using some modular representation theory. The final result states that the rank is determined by the order of certain quotientgroups of the automorphism groups of the cyclic tops of hypo-elementary subgroups of G thereby reducing everything back to the group-level. In the junglebook a similar local-global investigation will be carried out for the flasque coflasque and coco-nuts.

In section nine of this first set of notes we introduce a refinement of the Lenstra forest namely the Saltman forest by replacing the role of permutation lattices by that of invertible lattices. Our motivation for this new forest is twofold : first its trees have an analogous rationality-interpretation as in the Lenstra forest by replacing stable-rationality by retract rationality and secondly the number of trees in the Saltman forest coincides with the number of elements in the Saltman semigroup $\Delta(G)$ wich is the quotiemt of the Colliot by the permutation classgroup. I.e. we have reduced the problem of counting the number of trees in the Lenstra forest (or equivalently, the number of stable equivalence classes of tori-invariants) to the study of the permutation classgroup (for which we have a pretty good picture by the Dress results) and to counting the number of trees is a finer forest.

In the last section we do some more flasque yoga. In particular we will show that every tree contains a coflasque and that those trees which contain additionally a flasque lattice do contain a coco-nut. More surprisingly we will prove an analoque of Horrocks method of killing cohomology of vectorbundles by monads. We prove that any lattice can be obtained as the cohomology of a short complex (a monad)

$$0 \to P_1 \to C \to P_2 \to 0$$

with P_i permutation lattices and C a coco-nut. In the junglebook we will derive more consequences of this monadology in particular in the study of self-dual lattices.

Moreover, we give a quick intro to G-jungles. Roughly speaking, the G-jungle is the Lenstra forest of G together with the dual forest. The edges of the Lenstra forest are coloured red, the dual edges i.e. coming from short exact sequences of the form

$$0 \to P \to N \to M \to 0$$

with P permutation are coloured blue and edges which are both red and blue are coloured green. The underlying theory of jungles can be found in chapter 7 of 'Winning ways for your mathematical games I" by Berlekamp,Conway and Guy.

In the second trimester the audience will leave me no choice but to work towards the counterexamples to the original Noether-problem. We will choose the approach via the relation modules and the unramified Brauer group as in Saltmans 'Multiplicative field invariant' paper. Further we will also have a closer look at the S_n-jungle and the place of the lattice describing the center of the generic division algebra.

The third trimester topics will probably center round applications of modular representation theory to the local jungle picture (which is still a bit misty at this time).

Finally it is a pleasure to thank the audience for forcing me to study these topics and at the same time giving me enough room to do some joy-walking in the forest and in particular Michel Van den Bergh and Christine Bessenrodt for daily discussions (in our office or via email) keeping me more or less on the right track. Also, i wonder whether someone can read these notes without playing Michelle Shockeds 'Anchorage' over and over again ; it was certainly impossible to write them without it.

1. THE NOETHER-SALTMAN PROBLEM :

In these notes we aim (1) to survey the known results on and (2) to develop some machinary which may be used in the study of :

(1.1) : The Noether-Saltman problem

Let G be a finite group and M a $\mathbb{Z}G$-lattice, i.e. a free $\mathbb{Z}$-module of finite rank with a (left) G-action. For a fixed basefield k let $k[M]$ denote the groupalgebra

$$k[M] = k[x_1, x_1^{-1}, ..., x_m, x_m^{-1}]$$

and let $k(M)$ denote its field of fractions. Clearly, G acts on $k(M)$ as a group of k-automorphisms. When is the field of invariants $k(M)^G$ rational over k,i.e. is isomorphic to $k(y_1, ..., y_m)$ as k-algebras ?

This problem extends the 'classical' rationality problem of Emmy Noether which is concerned with the special case of permutation lattices :

(1.2) : The Noether problem

Let G be a finite group acting faithfully on a finite set of indeterminates $x_1, ..., x_m$. When is $k(x_1, ..., x_m)^G$ rational over k ?

Without going into the (long) history of this problem,we will here merely scetch the context in which it first arose. This is one of the fundamental (and still open) problems in number theory :

(1.3) : Realizable Galois group problem

Let k be a number field and G a finite group. Is there a Galois extension K of k such that $Gal(K/k) \simeq G$?

In 1892 David Hilbert proved that the symmetric group on n letters , S_n, cal always be realized. His argument went roughly as follows : let S_n act on the polynomial algebra $k[x_1, ..., x_n]$ by permuting the variables. The classical theorem on symmetric functions asserts that the fixed algebra is the polynomial algebra on the elementary symmetric functions $\sigma_1, ..., \sigma_n$. Then, Hilbert showed that if k is a number field one can specialize the σ_i to elements $a_i \in k$ in such a way that $k[x_1, ..., x_n]$ specializes to a Galois extension $K = k[x_1, ..., x_n] \otimes_{k[\sigma_1,...,\sigma_n]} k$ with Galois group S_n.

The idea of Emmy Noether (1915) was to extend this to other groups by letting G act as a group of permutations on the indeterminates $x_1, ..., x_n$. In this generality it is fairly easy to give examples such that the fixed algebra is no longer a polynomial algebra. However, Noether observed that it would be sufficient to know that the field of invariants $k(x_1, ..., x_n)^G$ is rational over k :

(1.4) : Theorem (Emmy Noether 1915)

Let G be a finite group acting faithfully on a finite set of indeterminates $x_1, ..., x_n$ and let k be a number field. If the field of invariants $k(x_1, ..., x_n)^G$ is rational over k, then G can be realized as the Galois group of an extension of k.

proof : Let $k(x_1, ..., x_n)^G = k(y_1, ..., y_n)$ be a rational representation. Then, $k(x_1, ..., x_n) = k(y_1, ..., y_n)(\alpha)$ and we let $f(y_1, ..., y_n; z)$ be the minimal polynomial for α over $k(y_1, ..., y_n)$. By the irreducibility theorem of Hilbert we can specialize all the y_i to elements $a_i \in k$ such that the polynomial $f(a_1, ..., a_n; z)$ is irreducible over k. One can then verify that the splitting field of this polynomial is the required extension.

In view of this result it was clearly Emmy Noethers intention to show that the answer to (1.2) is always positive over numberfields. However, in 1969 Richard Swan gave a counterexample for $G = \mathbb{Z}/47\mathbb{Z}$ over $\mathbb{Q}$. Taking the basefield to be $\mathbb{Q}$ is essential in this example, for Ernst Fischer , a colleque and friend of Noether at that time in Erlangen , proved already in 1916 that the answer to Noethers problem is always positive in case the field k contains enough roots of unity and G is Abelian

(1.5) : Theorem (Ernst Fischer 1916)

Let G be a finite Abelian group of exponent e acting on a finite set of indeterminates $x_1, ..., x_n$. Let k be a field of characteristic not dividing e and containing primitive e-th roots of unity. Then, $k(x_1, ..., x_n)^G$ is rational over k.

proof : Let $V = \sum kx_i$ be an n-dimensional k-vectorspace on which G acts linearly. As G is abelian containing all e-th roots of unity, $kG = k \oplus ... \oplus k$ whence V is the sum of one-dimensional G- invariant subspaces $V = \oplus V_i$ with $V_i = ky_i$

and for all $\sigma \in G$ $\sigma.y_i = \chi_i(\sigma)y_i$ where the $\chi_i : G \to k^*$ are the characters of G. Then, $k(V) = k(y_1, ..., y_n)$ and the y_i are algebraically independent over k. Hence, the subgroup Y of the multiplicative group $k(x_1, ..., x_n)^*$ generated by the y_i is a free Abelian group with basis the y_i. Let $G^* = Hom(G, k^*)$ be the character group of G and define a morphism $\phi : Y \to G^*$ by $\phi(y_i) = \chi_i$. Then the kernel $M = Ker(\phi)$ is (as a subgroup of finite index in Y) free Abelian of rank n. Then, $k(M) \subset k(x_1, ..., x_n)^G$ and one verifies that $k[M] = k[Y]^G$ and clearly $k(M)$ is the field of fractions of $k[M]$. Hence, $k(M) = k(m_1, ..., m_n) = k(x_1, ..., x_n)^G$, done.

In particular, over the complex numbers $\mathbb{C}$ any invariant field $\mathbb{C}(x_1, ..., x_n)^G$ as in (1.2) is rational when G is Abelian. The corresponding question for non-Abelian groups G was only settled (negatively) in 1984 by David Saltman.

There is a standard procedure which enables us to construct from certain negative solutions to the Noether-Saltman problem a negative solution to the Noether problem. This procedure was outlined in 1987 by David Saltman in his 'Multiplicative Field Invariants' paper, but (as mentioned there) the underlying theory at least dates back to the Endo-Miyata paper of 1973 :

Suppose we have a faithful G-lattice M (i.e. no element of G acts trivially on M) , a permutation lattice P (i.e. a lattice having a free $\mathbb{Z}$-basis which is permuted by G) and an exact sequence of G-modules :

$$0 \to M \to P \to N \to 0$$

where N is a finite Abelian group with the induced G-action. Assume that the exponent of N is e and that k contains a primitive e-th root of unity. Then, the fieldextension

$$k(M) \subset k(P)$$

is a Kummer extension with Galois group $A = Hom(N, k^*)$. As G acts on N and also (trivially) on k^*, there is a natural G-action on A. Further, since the G-action on $k(M)$ extends to a G-action on $k(P)$ we have a Galois field tower :

$$k(M)^G \subset k(M) \subset k(P)$$

with Galois groups G (resp. A). Therefore, $Gal(k(P)/k(M)^G) = A \times_s G$ the semi-direct product of A with G.

If $V = kv_1 + ...kv_n$ is an n-dimensional vectorspace with a linear G-action , then G acts on $k(V) = k(v_1, ..., v_n)$ and the invariant field is denoted (as in the G-lattice case) by $k(V)^G$. Contrary to the lattice case (as we will see later) we have a lot of freedom in choosing a particular representation as long as we are interested in stable equivalence. Recall that two fields K and L are said to be stably equivalent (over k) if $K(x_1, ..., x_m) \simeq L(y_1, ..., y_n)$ as k-algebras. We say that G acts faithfully on a k-vectorspace V if the G-action induces a monomorphism $G \rightarrow GL_n(k)$.

(1.6) : Theorem (Endo-Miyata 1973)

Let the finite group G act faithfully on two k-vectorspaces V and W. Then, the invariant fields $k(V)^G$ and $k(W)^G$ are stably equivalent.

proof : We claim that $k(V \oplus W)^G$ is rational over $k(V)^G$.For, consider $W^{ext} = k(V)W \subset k(V \oplus W)$, then G acts semilinearly on W^{ext} and by Galois descent (see for example Knus and Ojanguren) one knows that $(W^{ext})^G$ is a $k(V)^G$-vectorspace having $k(V)^G$-dimension equal to the k-dimension of W. But then, any $k(V)^G$-basis of $(W^{ext})^G$ forms a trancendence basis of $k(V \oplus W)^G$ over $k(V)^G$.

Changing the roles of V and W in the foregoing argument one proves that $k(V \oplus W)^G$ is also rational over $k(W)^G$, done.

One can extend this result to reductive linear algebraic groups acting almost freely (i.e. such that the stabilizer group of a generic point is trivial) on finite dimensional vectorspaces. This extended result is known as the 'no- name-lemma' to invariant-theorists.

Anyway, an immediate consequence of (1.6) is the following :

(1.7) : Theorem (Saltman 1987)

Let $0 \rightarrow M \rightarrow P \rightarrow N \rightarrow 0$ be an exact sequence of G-modules where M is a faithful lattice, P a permutation lattice and N a finite Abelian group of exponent e. Let k be afield containing a primitive e-th root of unity and form $A = Hom(N, k^*)$ and $H = A \times_s G$. Then, $k(H)^H$ is stably equivalent to $k(M)^G$.

proof : Let $p_1, ..., p_n$ be a $\mathbb{Z}$-basis of P which is permuted by G and form the k-vectorspace $V = kp_1 + ... + kp_n$. Then , A acts on V by $a.p_i = \rho p_i$ for some root of unity ρ. Anyway, $A \times_s G$ acts k-linearly and faithful on V. As $k(P) = k(V)$ it follows from Galois theory that $k(V)^H = k(M)^G$.

Finally, let $W = k[H]$ with the natural H-action. Then, by the Endo-Miyata result $k(V)^H = k(M)^G$ and $k(W)^H = k(H)^H$ are stably equivalent.

Thus, even if the Noether-Saltman problem is a drastic generalization of the classical Noether problem it is closely connected to it and a better understanding of it may leed to finding the obstruction to the Noether problem. Moreover, there are several important rationality questions which can be phrased naturally in the Noether-Saltman setting. We will give two examples which are of great interest in the study of the Brauer group of fields. They both center round the problem of finding a purely algebraic proof of the celebrated

(1.8) : Theorem (Merkurjev-Suslin 1982)

If k is a field containing a primitive n-th root of unity such that the characteristic of k does not divide n. Then, the n-torsion part $Br(k)_n$ of the Brauer-group of k is generated by cyclic algebras.

The first approach is due to R. Snider (1979). For any finite group G he constructs a generic crossed product in the following way : form a free presentation of G

$$1 \to R \to F \to G \to 1$$

where F is a free group $< x_1, ..., x_g >$ and we assume that $g \geq 2$. Then, by factoring out the commutators $[R, R]$ in the first two terms of this sequence we obtain a free Abelian extension

$$1 \to A_G = R/[R, R] \to B_G = F/[R, R] \to G \to 1$$

Then, A_G is a $\mathbb{Z}G$-module which is called the relation module of G. Of course, A_G depends upon the particular presentation of G but different relation modules have closely related relation modules as we will see later. The free Abelian extension has the following generic property : let

$$1 \to C \to E \to G \to 1$$

be any extension of G with C Abelian, then there exists a morphism $\phi : B \to E$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \to & A_G & \to & B_G & \to & G & \to & 1 \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & C & \to & E & \to & G & \to & 1 \end{array}$$

is commutative.

Now, let $\Delta = (K, G, f)$ be a crossed product with group G such that $k \subset Z(\Delta)$,where the characteristic of k does not divide the order of the group G then we have an extension :

$$1 \to K^* \to E \to G \to 1$$

where E is the subgroup of Δ generated by K^* and the elements x_g such that $x_g^{-1} t x_g = g.t$ for all $t \in K$. Therefore, we have a morphism $\phi : B_G \to E$ which extends linearly to a k-algebra morphism :

$$\phi : k[B_G] \to \Delta$$

where $k[B_G]$ is the groupalgebra of B_G which is a domain since B_G is a torsion-free Abelian-by-finite group. Since $k[B_G]$ satisfies a polynomial identity it has a division ring of fractions $k(B_G)$ which Snider calls the generic crossed product with group G. The underlying idea is that any structural result we are able to prove for $k(B_G)$ will after specialization hold for any crossed product with group G and containing k in its center.

We will see later that A_G is a faithful G-lattice and that $k(B_G)$ is a crossed product with maximal subfield $k(A_G)$. Therefore, the center of the generic crossed product $k(B_G)$ is equal to the invariant field $k(A_G)^G$. Using these notations we have

(1.9) : Theorem (Robert Snider 1979)

If k is a field containing a primitive n-th root of unity , n being the order of a finite group G and prime to the characteristic of k. Assume that $Br(k)_n$ is generated by cyclic algebras. Then, if $k(A_G)^G$ is rational over k any crossed product with group G and center L containing k is similar to a product of cyclic algebras in $Br(L)$.

proof : In 1973 Spencer Bloch proved that whenever (a) k contains a primitive n-th root of unity (b) n is prime to the characteristic of k and (c) $Br(k)_n$ is generated by cyclic algebras, then $Br(k(x_1, ..., x_m))_n$ is also generated by cyclic algebras. Hence, if $k(A_G)^G$ is rational over k then the generic crossed product $k(B_G)$ is similar to a product of cyclic algebras. But then by genericity and specialization, any crossed product with group G over an overfield L of k is similar to a product of cyclic algebras in $Br(L)$.

The initial Brauer group assumption on k is satisfied if k is either algebraically closed, a number field or a functionfield over a finite field. Using the fact that every element in the Brauer group is determined by a crossed product, the Merkurjev-Suslin result would follow if all the lattice invariant fields $k(A_G)^G$ would be rational thereby establishing a link with the Noether-Saltman problem.

The second approach is due to C. Procesi (1981). He considers, instead of the generic crossed products the generic division algebras Δ_n. Here, Δ_n is the division ring of fractions of the subalgebra of $M_n(k[x_{ij}, y_{ij} : 1 \leq i,j \leq n])$ generated by the generic matrices $X = (x_{ij})_{i,j}$ and $Y = (y_{ij})_{i,j}$. Procesi proved that the center of Δ_n can be obtained as the field of invariants of a certain S_n- lattice :

Let B_n be the S_n-permutation lattice

$$B_n = \mathbb{Z}x_1 \oplus ... \oplus \mathbb{Z}x_n \oplus \mathbb{Z}y_{11} \oplus \mathbb{Z}y_{12} \oplus ... \oplus \mathbb{Z}y_{nn}$$

with S_n-action given by : $\sigma.x_i = x_{\sigma(i)}$ and $\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. Let $U_n = \mathbb{Z}u_1 \oplus ... \oplus \mathbb{Z}u_n$ be the standard permutation S_n-lattice. Then, we have an exact sequence of S_n-modules

$$0 \to A_n \to B_n \to U_n \to \mathbb{Z} \to 0$$

where $\alpha : B_n \to U_n$ is determined by $\alpha(x_i) = 1$ and $\alpha(y_{ij}) = u_i - u_j$ and $\beta : U_n \to \mathbb{Z}$ is given by $\beta(u_i) = 1$.

Procesi's crucial observation is that $k(A_n)^{S_n}$ is the center of the generic division algebra Δ_n. With roughly the same proof as that of Snider's theorem one obtains :

(1.10) : Theorem (Claudio Procesi 1981)

If k is a field containing a primitive n-th root of unity , n is prime to the characteristic of k and $Br(k)_n$ is generated by cyclic algebras. If $k(A_n)^{S_n}$ is rational over k, then any central simple L-algebra of dimension n^2 is similar to a product of cyclic algebras in $Br(L)$, L being an overfield of k.

Again, rationality of the lattice invariants $k(A_n)^{S_n}$ would imply the Merkurjev-Suslin result as well as some lifting properties for Azumaya algebras which are, at this moment, unknown. This problem is also connected to some moduli problems in algebraic geometry. More details on these two examples will be presented later.

2. THE LENSTRA FOREST

Throughout, G will be a finite group, k an arbitrary but fixed field and $\mathbb{Z}G$ denotes the integral groupring of G. Let us begin by structering the vast amount of G-modules in some subclasses :

(2.1) : Definitions

A G-module M is a module over the groupring $\mathbb{Z}G$, i.e. M is an Abelian group with a (left) G-action

A G-lattice M is a G-module such that M as an Abelian group is $\mathbb{Z}$-free (of finite rank)

A G-module M is faithful if $G \to Aut_{\mathbb{Z}}M$ is injective, i.e. no element of G acts trivially on M

A permutation G-module M is a G-lattice having a $\mathbb{Z}$-basis which is permuted by G, i.e. $M \simeq \oplus_i \mathbb{Z}G/H_i$ for some subgroups H_i of G

An invertible G-module M is a G-direct summand of a permutation G- module. (Lenstra calls them permutation-projective and in representation theory they are sometimes called trivial source modules)

We will be primarely be interested in (faithful) G-lattices. They are more tractable than the class of all G-modules by the following fundamental

(2.2) : Jordan-Zassenhaus Theorem

For any $n \in \mathbb{N}$, there are only finitely many isomorphism classes of G- lattices with $\mathbb{Z}$-rank $\leq n$

For a proof of this result we refer to R. Swan 'K-theory of finite groups and orders' pp 43-54. This result allows us to picture the isomorphism classes of G-lattices according to their $\mathbb{Z}$-rank (see the next chapter for a concrete example).

We will now introduce a new conceptual tool which visualizes the rationality result which we will prove below :

(2.3) : The Lenstra forest

In the picture of all isoclasses of G-lattices (vertically classified by their $\mathbb{Z}$-rank) we draw an edge between the classes $[M]$ and $[N]$ if and only if there is an exact sequence of G-modules

$$0 \to M \to N \to P \to 0$$

with P a permutation G-lattice.

We will see below that this relationship among the isomorphism classes is transitive so for clarity we will draw only the most necessary edges. The roots of the trees (in fact, they are more bushes since they can have more than one roots) are then pictured to lie a a hill supporting the Lenstra forest. For a concrete example see the next chapter.

Of course, there is an algorithm to draw any part of the forest : the permutation classes are easily recovered from the subgroup structure of G and then , inductively on the $\mathbb{Z}$-rank of M , we determine all extensions of permutation lattices by M and draw the corresponding edges.

In a later chapter we will similarly define the Saltman forest where the role of the permutation lattices is replaced by the invertible lattices. We will now indicate the importance of the Lenstra forest in our study of the Noether-Saltman problem

For any G-module M, let M^G denote the submodule consisting of all elements fixed by G. Then , $(-)^G$ is easily verified to be a left exact additive functor.The cohomology groups $H^i(G,-)$ of G are defined to be the right derived functors of $(-)^G$, i.e. whenever we have an exact sequence of G-modules

$$0 \to L \to M \to N \to 0$$

we have a long exact sequence of Abelian groups

$$0 \to L^G \to M^G \to N^G \to$$

$$\to H^1(G,L) \to H^1(G,M) \to H^1(G,N) \to$$

$$\to H^2(G,L) \to H^2(G,M) \to H^2(G,N) \to$$

$$\ldots$$

Later on we will devote more energy to cohomology groups and what they represent. At the moment we will just need them in order to define :

A G-module M is said to be **co-flasque** if $H^1(H,M) = 0$ for every subgroup H of G

In the rest of this chapter we will merely need the following fact which we will prove later :

(2.4) : Lemma
Any invertible G-lattice is co-flasque

This fact plays a crucial role in the proof of the following characterization of invertible G-modules :

(2.5) : Proposition For a G-module I the following are equivalent

(1) : I is an invertible G-module

(2) : If $M_1 \rightarrow M_2$ is a G-morphism such that $M_1^H \rightarrow M_2^H$ is epi for every subgroup H of G, then we have an epimorphism $Hom_G(I,M_1) \rightarrow Hom_G(I,M_2)$

(3) : If Q is a co-flasque G-module , then every exact sequence

$$0 \rightarrow Q \rightarrow M \rightarrow I \rightarrow 0$$

of G-modules splits

proof :

(1) implies (2) : We have $I \oplus J \simeq \oplus_i \mathbb{Z}G/H_i$ for some G- module J and subgroups H_i. But then

$$Hom_G(I,-) \oplus Hom_G(J,-) \simeq \oplus_i Hom_G(\mathbb{Z}G/H_i,-)$$

and since for every subgroup H of G and G-module M we have

$$Hom_G(\mathbb{Z}G/H_i, M) \simeq M^H$$

we obtain from (1) and the above an epimorphism

$$Hom_G(I,M_1) \oplus Hom_G(J,M_1) \rightarrow Hom_G(I,M_2) \oplus Hom_G(J,M_2)$$

from which (2) follows.

(2) implies (3) : Let $0 \to Q \to M \to I \to 0$ be an exact sequence of G- modules and let H be a subgroup of G, then we have the long exact cohomology sequence

$$0 \to Q^H \to M^H \to I^H \to H^1(H,Q) = 0$$

i.e. $M^H \to I^H$ is epi for every subgroup H of G. By (2) we then have an epimorphism

$$Hom_G(I,M) \to Hom_G(I,I)$$

which implies that the sequence splits (just take an inverse image of $1_I \in Hom_G(I,I)$

(3) implies (1) : Consider the permutation G-lattice P and the G-epi

$$\psi : P = \oplus_{H<G}(\mathbb{Z}G/H \otimes I^H) \to I$$

where I^H is given trivial G-action (is epi for take $H = 1$). Then, for any subgroup H of G we have

$$\begin{array}{ccc} P^H & & \\ \uparrow & \searrow & \\ \mathbb{Z}G/H \otimes I^H & \to & I^H \end{array}$$

the lower map being surjective, so ψ^H is epi too. Let $J = Ker\psi$, then by the long exact cohomology sequence we have

$$0 \to J^H \to P^H \to I^H \to H^1(H,J) \to H^1(H,P) = 0$$

whence $H^1(H,J) = 0$ for every subgroup H of G , i.e. J is co-flasque. Then, by (3) ψ splits and $I \oplus J \simeq P$,done.

This result can be used to prove the promised transitivity of the relationship between the isomorphism classes of G-lattices

(2.6) : Proposition Let L, M and N be G-lattices such that there are exact sequences of G-modules $0 \to L \to M \to P_1 \to 0$ and $0 \to M \to N \to P_2 \to 0$ where P_i are permutation modules. Then there is an exact sequence of G-modules

$$0 \to L \to N \to P_1 \oplus P_2 \to 0$$

proof : We have the commutative exact diagram of G-modules

$$\begin{array}{ccccccccc} 0 & \to & L & \to & L & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & N & \to & P_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & P_1 & \to & R & \to & P_2 & \to & 0 \end{array}$$

Giving by the snake lemma the exact sequence

$$0 \to P_1 \to R \to P_2 \to 0$$

which splits by the foregoing result, whence $R = P_1 \oplus P_2$.

We will now extend our basic setting in order to contain tori-invariants. So, let l be a field containing k such that G acts faithfully as a group of k-algebra automorphisms on l. We have two specific situations in mind

(a) : l is a Galois extension of k with $G \subset Gal(l/k)$

(b) : $l = k(F)$ where F is a faithful G-lattice

Further, let M be any G-lattice, then G acts on the groupalgebra $l[M]$ by $g.(\sum_m \lambda_m m) = \sum_m g.\lambda_m g.m$ and hence also on the field of fractions $l(M)$. Note that the grouplaw on M is written additively although M is a G-submodule of the multiplicative group $l(M)^*$.

(2.7) : Proposition (Speiser 1919)

Let W be an l-vectorspace with a G-module structure such that $g.(\lambda w) = g.\lambda g.w$ for all $\lambda \in l, w \in W$. Then, W^G contains an l-basis for W.

proof : Let $n = \sum_{g \in G} g \in \mathbb{Z}G$ be the norm element. The idea to prove that $n.W \subset W^G$ contains an l-basis for W is to show that any l-linear map $\phi : W \to l$ which annihilates $n.W$ must be zero. Take such a ϕ and fix a $w \in W$. Then, for every $\lambda \in l$ we have

$$0 = \phi(n.\lambda w) = \sum_{g \in G} \phi(g.w) g.\lambda$$

by assumption on the G-action on W and l-linearity of ϕ. By the linear independence of the field isomorphisms $g \in G$. Hence, in particular, $\phi(w) = 0$ proving the claim.

(2.8) : Proposition (Masuda 1955)

Let P be a permutation G-lattice. Then, $l(P)^G$ is rational over l^G.

proof : Let $x_1, ..., x_m$ be the $\mathbb{Z}$-basis of P which is permuted by G. Consider the l-vectorspace $W = \sum_{i=1}^m l x_i \subset l(P)$. By definition of the G-action on $l(P)$, W satisfies the conditions of (2.7). Hence, we can find elements

$$\{y_1, ..., y_m\} \subset W^G \subset l(P)^G$$

such that $W = ly_1 + ... + ly_m$. But then, $l(y_1, ..., y_m) = l(P)$ and because the y_i are G-invariant $l(P)^G = l(y_1, ..., y_m)^G = l^G(y_1, ..., y_m)$, done.

This result illustrates the importance of permutation modules in rationality problems. The following results illustrates the usefulness of knowledge about the structure of the Lenstra forest :

(2.9) : Proposition (Lenstra 1974)

If $0 \to M_1 \to M_2 \to I \to 0$ is an exact sequence of G-lattices with I an invertible G-module. Then, $l(M_2)^G$ is isomorphic to $l(M_1 \oplus I)^G$ as l^G-algebras

proof : The injection $M_1 \to M_2$ induces an injection $l(M_1) \subset l(M_2)$ and we consider the exact sequence of G-modules

$$0 \to l(M_1)^* \to l(M_1)^* M_2 \to I \to 0$$

where λm is mapped to the image of m in I for all $\lambda \in l(M_1)^*$ and $m \in M_2$. For any subgroup H of G we have that $H^1(H, l(M_1)^*) = 0$ by Hilbert 90 , i.e. $l(M_1)^*$ is co-flasque and hence by (2.6.3) and the above sequence we know that

$$l(M_1)^* M_2 \simeq l(M_1)^* \oplus I$$

as G-modules. The obtained map $I \to l(M_1)^* M_2 \to l(M_2)^*$ can then be used to obtain an l-algebra isomorphism $l(M_1 \oplus I) \simeq l(M_2)$ which respects the G-action. Therefore the invariant-fields are isomorphic as l^G-algebras,done.

In his paper, Lenstra formulates the result only for I a permutation lattice, but the proof clearly extends to invertible G-modules. An important consequence of the foregoing two results is

(2.10) : Theorem (Lenstra 1974)

Let $0 \to M_1 \to M_2 \to P \to 0$ be an exact sequence of G-lattices with P a permutation lattice. Then, $l(M_2)^G$ is rational over $l(M_1)^G$.

proof : By (2.9) : $l(M_2)^G \simeq l(M_1 \oplus P)^G = l(M_1)(P)^G$. Replacing l by $l(M_1)$ in the formulation of the Masuda result shows that $l(M_1)(P)^G$ is rational over $l(M_1)^G$.

This result can be rephrased in Lenstra forest terminology indicating the importance of this object in the study of tori-invariants :

(2.11) : Corollary Let M and N be G-lattices such that their isoclasses $[M]$ and $[N]$ belong to the same tree in the Lenstra forest. Then, $l(M)^G$ and $l(N)^G$ are stably equivalent.

proof : Just apply (2.10) along a path in the Lenstra forest going from $[M]$ to $[N]$.

Lenstra's theorem is not directly applicable to the Noether-Saltman problem because of the field l on which G has to act faithfully. However, using the same method of proof one can get a similar result for lattice-invariants if we restrict to faithful lattices :

(2.12) : Theorem

Let $0 \to M_1 \to M_2 \to P \to 0$ be an exact sequence of G-lattices with P a permutation lattice and M_1 a faithful G-lattice. Then, $k(M_2)^G$ is rational over $k(M_1)^G$.

proof : Redo the first part of (2.9) but this time with the exact sequence

$$0 \to k(M_1)^* \to k(M_1)^* M_2 \to P \to 0$$

Because G acts faithfully on $k(M_1)$ we have again $H^1(H, k(M_1)^*) = 0$ by Hilbert 90 for every subgroup H of G . So, the sequence splits and as before we get a k-algebra isomorphism $k(M_2)^G \simeq k(M_1 \oplus P)^G$. Finally, we can apply the Masuda result with $l = k(M_1)$.

Let the faithful Lenstra forest be the Lenstra forest restricted to the faithful G-lattices. This can be easily obtained from the Lenstra forest by chopping down unfaithful trees and burrying the remaining unfaithful G-lattices under a heap of sand (they live close to the soil). Then, the foregoing result can be translated in forest terminology yielding our first result on the Noether-Saltman problem :

(2.13) : Corollary Let M and N be two faithful G-lattices such that $[M]$ and $[N]$ belong to the same tree in the faithful Lenstra forest, then $k(M)^G$ and $k(N)^G$ are stably equivalent.

In other words, in order to verify stable rationality of $k(M)^G$ for a faithful G-lattice M it suffices to verify it for any other faithful lattice in the same tree.

Returning to tori-invariants we know that $l(P)^G$ is rational over l^G for any permutation lattice P. Hence, for every lattice M belonging to the permutation-tree in the Lenstra forest (exercise : all permutation lattices belong to one tree) we have that $l(M)^G$ is stably rational over l^G. We will now show that the converse is also true i.e. if $l(M)^G$ is stably rational over l^G then M belongs to the permutation-tree. We first need an easy but useful lemma :

(2.14) : lemma (Swan 1969) Let K be a field containing k and $R_1, R_2 \subset K$ two affine k- algebras with field of fractions K. Then, there exist elements $0 \neq r_1 \in R_1$, $0 \neq r_2 \in R_2$ such that $R_1[\frac{1}{r_1}] = R_2[\frac{1}{r_2}]$

proof : Because R_1 is affine, there is a $0 \neq s_2 \in R_2$ such that $R_1 \subset R_2[s_2^{-1}]$. Now, $R_2[s_2^{-1}]$ is affine so there is a $0 \neq r_1 \in R_1$ such that $R_2[s_2^{-1}] \subset R_1[r_1^{-1}]$. Because $r_1 \in R_1$ we have $r_1 = ts_2^{-n}$ for some $n \in \mathbb{N}$ and $0 \neq t \in R_2$. Take $r_2 = ts_2$ then $R_2[r_2^{-1}] = R_2[t^{-1}][s_2^{-1}]$. Further, $t^{-1} = r_1^{-1}s_2^{-n} \in R_1[r_1^{-1}]$ whence $R_2[r_2^{-1}] \subset R_1[r_1^{-1}]$. Conversely, $r_1^{-1} = s_2^n t^{-1} \in R_2[t^{-1}] \subset R_2[r_2^{-1}]$ whence $R_1[r_1^{-1}] \subset R_2[r_2 - 1]$

In fact, this is nothing more than a restatement of birationality.

(2.15) Theorem (Voskresenskii 1970,Endo-Miyata 1973)

Let M be a G-lattice. Then, $l(M)^G$ is stably rational over l^G if and only if there is an exact sequence

$$0 \rightarrow M \rightarrow P_2 \rightarrow P_1 \rightarrow 0$$

of G-modules with P_1 and P_2 permutation G-lattices

proof : Assume there exists such an exact sequence then $[M]$ belongs to the permutation tree and so $l(M)^G$ is stably rational over l^G.

Conversely, assume $l(M)^G$ is stably rational over l^G i.e.

$$l(M)^G(x_1, ..., x_s) \simeq l^G(y_1, ..., y_{r+s})$$

and we let G act $l(M)(x_1, ..., x_s) = l(M) \otimes_{l(M)^G} l(M)^G(x_1, ..., x_s)$ (use Speiser) via the action on the first factor. We can consider two affine l-algebras in $l(M)(x_1, ..., x_s)$ namely $R_1 = l[M][x_1, ..., x_s]$ and $R_2 = l[y_1, ..., y_{r+s}]$. By the foregoing lemma, there exist elements $a_1 \in R_1$ and $a_2 \in R_2$ such that

$$R = R_1[a_1^{-1}] = R_2[a_2^{-1}]$$

and since G is finite we may even assume that $a_i \in R_i^G$. But then we have exact sequences of G-modules

$$0 \to R_1^* \to R^* \to P_1 \to 0$$

$$0 \to R_2^* \to R^* \to P_2 \to 0$$

where the P_i are permutation G-lattices. More precisely

$$P_i = \oplus_j \mathbb{Z} cl(P_j)$$

where the P_j are the prime ideals of the unique factorization domain R_i dividing $R_i a_i$ and $cl(P_i)$ is a formal element representing this ideal. Clearly, the G-action on P_i is induced by its action on the prime ideals (which are permuted by G since a_i is G-invariant). If we divide the first two terms of both sequences by l^* we get

$$0 \to M \to R^*/l^* \to P_1 \to 0$$

$$0 \to 0 \to R^*/l^* \to P_2 \to 0$$

giving the required exact sequence.

An immediate consequence of this result and (2.5) is

(2.16) : Corollary

Let M be a co-flasque G-lattice. Then, $l(M)^G$ is stably rational over l^G if and only if $M \oplus P_1 \simeq P_2$ for permutation G-modules P_1 and P_2.

Rephrasing the obtained results on tori-invariants in forest language we have

(2.17) : Corollary

Let M be a G-lattice, then $l(M)^G$ is stably rational over l^G if and only if $[M]$ is contained in the permutation tree in the Lenstra forest. In particular, if there are more trees there are non (stably) rational tori G- invariants.

In fact, more is true. Using the ideas of theorem (2.15) one can show that different trees correspond to different stable equivalence classes of tori invariants over l^G :

(2.18) : Theorem (Colliot-Thélène,Sansuc 1977)

Let M and N be two G-lattices such that the tori-invariants $l(M)^G$ and $l(N)^G$ are stably equivalent over l^G. Then, $[M]$ and $[N]$ belong to the same tree in the Lenstra forest

proof : By assumption we have

$$l(M)^G(x_1, ..., x_r) \simeq l(N)^G(y_1, ..., y_s)$$

as l^G-algebras. Tensoring both sides with l over l^G and using the Speiser result (2.7) we obtain an l-isomorphism

$$l(M)(x_1, ..., x_r) \simeq l(N)(y_1, ..., y_s)$$

Consider the following two affine l-algebras in this field

$$R_1 = l[M][x_1, ..., x_r] \text{ and } R_2 = l[N][y_1, ..., y_s]$$

with G acting trivially on the indeterminates. By Swan's lemma we can find elements $a_1 \in R_1^G$ and $a_2 \in R_2^G$ such that

$$R_1[a_1^{-1}] = R_2[a_2^{-1}] = R$$

As in the proof of (2.15) this gives rise to the following exact sequences of G-modules

$$0 \to l^*.M \to R^* \to P_1 \to 0$$

$$0 \to l^*.N \to R^* \to P_2 \to 0$$

with P_1 and P_2 permutation G-lattices. Dividing l^* out of the first two terms in both sequences we obtain the exact G-sequences

$$0 \to M \to R^*/l^* \to P_1 \to 0$$

$$0 \to N \to R^*/l^* \to P_2 \to 0$$

entailing that $[M]$ and $[N]$ both belong to the tree in the Lenstra forest containing $[R^*/l^*]$

Similarly, by restricting to faithful G-lattices (and the faithful Lenstra forest) we obtain our second result on the Noether-Saltman problem :

(2.18) : Corollary

If M is a faithful G-lattice and N a G-lattice. Then, $k(M \oplus N)^G$ is stably rational over $k(M)^G$ if and only if N is a direct neighbour of a permutation lattice.

Further, for any other G-lattice N' : if $k[M \oplus N]^G$ is stably equivalent to $k[M \oplus N']^G$ over $k[M]^G$, then $[N]$ and $[N']$ belong to the same tree in the Lenstra forest.

3. THE TINIEST FORESTS ON EARTH :

Surely, one can hardly think of an easier group to study than a cyclic group of prime order $C_p = \mathbb{Z}/p\mathbb{Z}$. Yet, already in this case some surprising things happen. In fact, using some easy integral representation theory we will be able to describe the full Lenstra forest in this case and use this knowledge to construct non (stably) rational tori invariants. The result we will prove in this section is

(3.1) : Theorem If $C_p = \mathbb{Z}/p\mathbb{Z}$, then the number of trees in the Lenstra forest is equal to the order of the ideal clasgroup of $\mathbb{Z}[\zeta_p]$, the ring of integers in the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$.

Moreover, each tree corresponds to a different stably equivalence class of tori invariants.

Let τ be a generator of $C_p = \mathbb{Z}/p\mathbb{Z}$ and let ζ_p be a primitive p-th root of unity. With $\mathbb{Q}(\zeta_p)$ we denote the p-th cyclotomic field. Then, it is well known that the ring of integers in $\mathbb{Q}(\zeta_p)$ is equal to

$$\mathbb{Z}[\zeta_p] = \mathbb{Z}[x]/(x^{p-1} + ... + x + 1)$$

which is a Dedekind domain having a free $\mathbb{Z}$-basis consisting of $1, \zeta, ..., \zeta^{p-2}$. Let us begin by describing some useful $\mathbb{Z}C_p$-lattices :

Let I be an invertible $\mathbb{Z}[\zeta_p]$-ideal in $\mathbb{Q}(\zeta_n)$. We can define a C_p -action on I by $\tau.i = \zeta_p i$ for all $i \in I$. Clearly, $\tau^p.i = \zeta_p^p i = i$ whence I is a $\mathbb{Z}C_p$-lattice of $\mathbb{Z}$-rank $p-1$. Further, two invertible ideals I and J are isomorphic as $\mathbb{Z}C_p$-lattices if and only if they are isomorphic over $\mathbb{Z}[\zeta_p]$ i.e. if they represent the same class in the ideal class group.

Starting from an invertible ideal I and an element $i_0 \in I$ we can construct a $\mathbb{Z}C_p$-lattice of $\mathbb{Z}$-rank p $I \oplus \mathbb{Z}y$ by defining the action to be $\tau.(i \oplus ny) = \zeta_p i + ni_0 \oplus ny$.We calculate

$$\begin{aligned} \tau^{p-1}.\tau(i \oplus ny) &= \tau^{n-2}.\tau(\zeta_p i + ni_0 \oplus ny) \\ &= \tau^{p-3}.\tau(\zeta^2 i + \zeta ni_0 + ni_0 \oplus ny) \\ ... &= (\zeta_p^p i + (\zeta_p^{p-1} + ... + \zeta_p + 1)ni_0 \oplus ny) = i \oplus ny \end{aligned}$$

showing that it is indeed a $\mathbb{Z}C_p$-lattice. A little computation shows that $(I, i_0) \simeq (I, i_0')$ whenever $i_0, i_0' \in I - (\zeta_p - 1)I$ so we will fix one of such elements i and we denote the isomorphism class by $[I, i]$. The following result classifies the $\mathbb{Z}C_p$-lattices :

(3.2) : Theorem (Diederichsen 1940,Reiner 1957)
Every $\mathbb{Z}C_p$-lattice M is isomorphic to a direct sum

$$M = [I_1, i_1] \oplus ... \oplus [I_r, i_r] \oplus I_{r+1} \oplus ... \oplus I_{r+s} \oplus \mathbb{Z}^t$$

Moreover, the isomorphism class of I is uniquely determined by the integers r, s, t and the ideal class of the product $I_1 ... I_{r+s}$ in $Cl(\mathbb{Z}[\zeta_p])$

proof : We give a scetch of the proof to show that nothing deep comes into it.Let n be the norm-element $1+\tau+...+\tau^{p-1} \in \mathbb{Z}C_p$, then clearly $\mathbb{Z}C_p/(n) \simeq \mathbb{Z}[\zeta_p]$. For any C_p-lattice M one defines $M_n = \{m \in M : n.m = 0\}$. Then, M_n is a $\mathbb{Z}C_p$-submodule of M such that M/M_n is $\mathbb{Z}$-torsionfree. Thus

$$M \simeq M_n \oplus X$$

as $\mathbb{Z}$ (but not necessarely $\mathbb{Z}C_p$-) modules. Since n annihilates M_n we can view M_n as a $\mathbb{Z}[\zeta_p]$-module which is torsion free (a norm argument). Further, $(\tau - 1)M \subset M_n$ and both have the same rank over $\mathbb{Z}[\zeta_p]$. From the module theory of Dedekind domains we get

$$M_n = \mathbb{Z}[\zeta_p]b_1 \oplus ... \oplus \mathbb{Z}[\zeta_p]b_{n-1} \oplus Ab_n$$

$$(\tau - 1)M = E_1 b_1 \oplus ... \oplus E_{n-1}b_{n-1} \oplus E_n A b_n$$

where A is an inveretible $\mathbb{Z}[\zeta_p]$-ideal in $\mathbb{Q}(\zeta_p)$ and the E_i are ideals in $\mathbb{Z}[\zeta_p]$ such that $E_{i+1} \subset E_i$. From $(\zeta_p - 1)M_n \subset (\tau - 1)M_n \subset (\tau - 1)M$ and the above we obtain that $(\zeta_p - 1)\mathbb{Z}[\zeta_p] \subset E_i \subset \mathbb{Z}[\zeta_p]$ for all i. Since $(\zeta_p - 1)\mathbb{Z}[\zeta_p]$ is a maximal ideal we therefore obtain an r such that $E_1 = E_2 = ... = E_r = \mathbb{Z}[\zeta_p]$ and $E_{r+1} = ... = E_n = (\zeta_p - 1)\mathbb{Z}[\zeta_p]$. Using these facts we deduce

$$L = (\tau - 1)M/(\zeta_p - 1)M_n = \mathbb{F}_p b_1 \oplus ... \mathbb{F}_p b_r$$

Since $(\tau - 1)M = (\zeta_p - 1)M_n \oplus (\tau - 1)X$ the map $\phi : X \to L$ given by $\phi(x) = (\tau - 1)x + (\zeta_p - 1)M_n$ is epi. We can choose a $\mathbb{Z}$-basis $x_1, ..., x_k$ of X such that $\phi(x_i) = (c_i mod p)b_i$ for $i \leq r$ and $\phi(x_i) = 0$ for $r + 1 \leq i \leq k$. But then, $(\tau - 1)x_i =$

$c_ib_i + (\zeta_p - 1)u_i$ for $i \leq r$ and $= (\zeta_p - 1)u_i$ for $r+1 \leq i \leq k$ where the $u_i \in M_n$. Take $y_i = x_i - u_i$ then

$$M = M_n \oplus \mathbb{Z}y_1 \oplus ... \oplus \mathbb{Z}y_k$$

and for $i \leq r$ we have $\tau.y_i = c_ib_i + y_i$ whereas $\tau.y_i = y_i$ for $r+1 \leq i \leq k$.But then, as $\mathbb{Z}[\zeta_p]b_i \oplus \mathbb{Z}y_i = (\mathbb{Z}[\zeta_p], c_i) = (\mathbb{Z}[\zeta_p], 1)$ for $i \leq r$ we have :

$$M = [\mathbb{Z}[\zeta_p], 1]^r \oplus \mathbb{Z}[\zeta_p]b_{r+1} \oplus ... \oplus \mathbb{Z}[\zeta_p]b_{n-1} \oplus Ab_n \oplus \mathbb{Z}y_{r+1} \oplus ... \oplus \mathbb{Z}y_k$$

Therefore, we have transformed any C_p-lattice in the prescribed form and the second statement follows from the fact that if

$$M = [I_1, i_1] \oplus ... \oplus [I_r, i_r] \oplus I_{r+1} \oplus ... \oplus I_{r+s} \oplus \mathbb{Z}^t$$

then,

$$M_n = I_1 \oplus ... \oplus I_{r+s} \simeq \mathbb{Z}[\zeta_p]^{r+s-1} \oplus (I_1 ... I_{r+s})$$

and the first part of the proof.

(3.3) : Corollary Let h_p be the order of the ideal classgroup of $\mathbb{Z}[\zeta_p]$, then the number of isomorphism classes of $\mathbb{Z}C_p$-lattices of $\mathbb{Z}$-rank n is given as

$$1 + h_p.\#\{(r, s, t) \in \mathbb{N}^3 : t + (p-1)s + pr = n; t \neq n\}$$

Next, each isomorphism class can be represented by a matrix

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

where $r, s, t \in \mathbb{N}$ and $0 \leq u \leq h_p - 1$. If I_u denotes a representant of the u-th ideal class , the corresponding C_p-lattice is

$$(\mathbb{Z}[\zeta_p], 1)^{\oplus r} \oplus \mathbb{Z}[\zeta_p]^{\oplus s-1} \oplus I_u \oplus \mathbb{Z}^{\oplus t}$$

if $s \neq 0$. If $s = 0$ and $r \neq 0$ the lattice is isomorphic to

$$(\mathbb{Z}[\zeta_p], 1)^{\oplus r-1} \oplus (I_u, i_u) \oplus \mathbb{Z}^{\oplus t}$$

and if $r = s = 0$ it is just $\mathbb{Z}^t$. We let $I_0 = \mathbb{Z}[\zeta_p]$ and clearly $\mathbb{Z}C_p \simeq (\mathbb{Z}[\zeta_p], 1)$. Therefore,

(3.4) : Corollary The permutation C_p-lattices are represented by the matrices $\begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix}$ with $r, t \in \mathbb{N}$

Further, it is easy to compute the isomorphism class of a direct sum of two C_p-lattices described by their matrices :

$$\begin{pmatrix} r_1 & s_1 \\ t_1 & u_1 \end{pmatrix} \oplus \begin{pmatrix} r_2 & s_2 \\ t_2 & u_2 \end{pmatrix} \simeq \begin{pmatrix} r_1 + r_2 & s_1 + s_2 \\ t_1 + t_2 & u_3 \end{pmatrix}$$

where $I_{u_3} \simeq I_{u_1}.I_{u_2}$. A direct consequence of this is

(3.5) : Corollary The invertible C_p-lattices are represented by the matrices $\begin{pmatrix} r & 0 \\ t & u \end{pmatrix}$ where $r, t \in \mathbb{N}$ and $0 \leq u \leq h_p - 1$

As we will see later in more detail, the isoclasses $[I_u, i_u]$ form the classgroup of the groupring $\mathbb{Z}C_p$ explaining somewhat the name 'invertible lattices'. Let us now turn to determining the edges

(3.6) : lemma There are at most h_p trees in the (faithful) Lenstra forest

proof : Note that there is just one unfaithful brach consisting of the isoclasses represented by $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ for all $n \in \mathbb{N}$ (i.e. the leftmost branch). So, the faithful Lenstra forest is the one obtained by deleting this one branch.

A first class of edges is obtained by adding a permutation lattice,i.e.

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \to \begin{pmatrix} r + r_1 & s \\ t + t_1 & u \end{pmatrix}$$

is an edge for all $r_1, t_1 \in \mathbb{N}$. Another class of edges comes from a reinterpretation of the definition of (I, i) , i.e. there is a non-split exact sequence of $\mathbb{Z}C_p$-modules

$$0 \to I \to (I, i) \to \mathbb{Z} \to 0$$

Adding this sequence a number of times and taking into account that the $\mathbb{Z}^n$ are permutation lattices we find edges

$$\begin{pmatrix} 0 & n \\ 0 & u \end{pmatrix} \to \begin{pmatrix} n & 0 \\ 0 & u \end{pmatrix}$$

Combining these two classes we see that a general isoclass represented by $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ lies in the same tree as $\begin{pmatrix} 0 & s \\ 0 & u \end{pmatrix}$ (by the first class of edges) , which lies in the same tree as $\begin{pmatrix} s & 0 \\ 0 & u \end{pmatrix}$ (by the second class) which lies in the tree of $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ (first class),done.

(3.7) : lemma The permutation tree consists precisely of the isoclasses represented by the matrices $\begin{pmatrix} r & s \\ t & 0 \end{pmatrix}$ for all $r, s, t \in \mathbb{N}$

proof : By the proof of the foregoing result each such class clearly belongs to the permutation tree. Now, any other isoclass represented by $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ lies in the same tree as $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. So it suffices to prove that $l((I_u, i_u))^G$ is not stably rational over l^G. Because (I_u, i_u) is an invertible C_p-lattice (whence co-flasque) we would have by (2.16) that

$$(I_u, i_u) \oplus P_1 \simeq P_2$$

for some permutation lattices P_i. But the left hand side is represented by $\begin{pmatrix} r+1 & 0 \\ t & u \end{pmatrix}$ which cannot be a permutation lattice unless $u = 0$,done.

We can now finish the **proof of (3.1) :** Suppose we have an exact C_p-sequence

$$0 \rightarrow \begin{pmatrix} r_1 & s_1 \\ t_1 & u \end{pmatrix} \rightarrow \begin{pmatrix} r_2 & s_2 \\ t_2 & v \end{pmatrix} \rightarrow \begin{pmatrix} r_3 & 0 \\ t_3 & 0 \end{pmatrix} \rightarrow 0$$

Then by adding a factor I_w where $I_u.I_w \simeq \mathbb{Z}[\zeta_p]$ to the first two terms we get the exact C_p-sequence :

$$0 \rightarrow \begin{pmatrix} r_1 & s_1+1 \\ t_1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} r_2 & s_2+1 \\ t_2 & z \end{pmatrix} \rightarrow \begin{pmatrix} r_3 & 0 \\ t_3 & 0 \end{pmatrix} \rightarrow 0$$

where $I_v.I_w \simeq I_z$. Therefore, the middle factor has to belong to the permutation tree whence $z = 0$ or, equivalently $u = v$ finishing the proof.

A picture of a part of the Lenstra forest for C_p using the foregoing observations is given in the 'full forest' picture. We will now give another method of drawing the Lenstra forest using only the most necessary edges

(3.8) : lemma Starting from the vertex corresponding to the isoclass of G-lattices represented by $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ there are precisely $2 + min(s, t+1)$ elementary edges in the Lenstra forest . First, we have the trivial one obtained by adding $\mathbb{Z}C_p$ leading to the vertix determined by

$$\begin{pmatrix} r+1 & s \\ t & u \end{pmatrix}$$

And, the others lead to the vertices

$$\begin{pmatrix} r+k & s-k \\ t+1-k & u \end{pmatrix}$$

for all $0 \leq k \leq min(s, t+1)$

proof : Let us start from a arbitrary edge in the Lenstra forest

$$0 \to M \to N \to \mathbb{Z}^{\oplus a} \oplus \mathbb{Z}C_p^{\oplus b} \to 0$$

First of all, we can split off the $\mathbb{Z}C_p^{\oplus b}$ component from N by projectivity leading to an easier edge :

$$0 \to M \to N' \to \mathbb{Z}^{\oplus a} \to 0$$

and clearly there is a path from N' to N by adding $\mathbb{Z}C_p$ each time.

Next, we claim that we can restrict to the situation where $a = 1$. For, we have the following exact commutative diagram of C_p-lattices

$$\begin{array}{ccccccccc} 0 & \to & M & \to & N_1 & \to & \mathbb{Z}^{\oplus a-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & N' & \to & \mathbb{Z}^{\oplus a} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 \end{array}$$

where N_1 is the pullback of N' and $\mathbb{Z}^{\oplus a-1}$ over $\mathbb{Z}^{\oplus a}$. This leads to the following two exact sequences of C_p-modules

$$0 \to M \to N_1 \to \mathbb{Z}^{\oplus a-1} \to 0$$

$$0 \to N_1 \to N' \to \mathbb{Z} \to 0$$

showing that the edge from M to N' factors through N_1. Continuing in this matter we arrive at the first step edge

$$0 \to M \to N" \to \mathbb{Z} \to 0$$

and we know from (3.1) that N" must be represented by a matrix

$$\begin{pmatrix} * & * \\ * & u \end{pmatrix}$$

If $s = 0$ then $M \simeq \mathbb{Z}C_p^{\oplus r-1} \oplus (I_u, i_u) \oplus \mathbb{Z}^t$ and is thereby an invertible C_p-module. Therefore, $Ext^1_G(\mathbb{Z}, M) = H^1(G, M) = 0$ and the sequence splits i.e. $N_{a-1} \simeq M \oplus \mathbb{Z}$. If $s > 0$ we can split M into a direct sum

$$\begin{pmatrix} 0 & s \\ 0 & u \end{pmatrix} \oplus \begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix}$$

and we obtain the exact commutative diagram of C_p-modules

$$\begin{array}{ccccccc} 0 & \to & M & \to & N" & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & \mathbb{Z}^{\oplus t} \oplus \mathbb{Z}C_p^{\oplus r} & \to & H & \to & \mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where H is the pushout of N" and $\mathbb{Z}^{\oplus t} \oplus \mathbb{Z}C_p^{\oplus r}$. The lower exact sequence splits by invertibility of $\mathbb{Z}$ and co-flasqueness of $\mathbb{Z}^{\oplus t} \oplus \mathbb{Z}C_p^{\oplus r}$ and we arrive at the exact sequence

$$0 \to \mathbb{Z}[\zeta_p]^{\oplus s-1} \oplus I_u \to N" \to \mathbb{Z}^{\oplus t+1} \oplus \mathbb{Z}C_p^{\oplus r}$$

If N" is represented by the matrix $\begin{pmatrix} a & b \\ c & u \end{pmatrix}$ then we can split off the projective $\mathbb{Z}C_p^{\oplus r}$ component and obtain the exact sequence

$$0 \to \mathbb{Z}[\zeta_p]^{\oplus s-1} \oplus I_u \to N"_1 \to \mathbb{Z}^{\oplus t+1} \to 0$$

where N_1" is represented by the matrix $\begin{pmatrix} a-r & b \\ c & u \end{pmatrix}$ In Curtis-Reiner section 34, one finds a general result on such extensions for cyclic p-groups leading to the fact that N_1" can be any of the following C_p-lattices

$$\begin{pmatrix} a-r & s-a+r \\ t-a+r+1 & u \end{pmatrix}$$

where, of course $a \geq r$,$s+r \geq a$ and $t+r+1 \geq a$ entails that $r \leq a \leq r+min(s, t+1)$ and adding again the direct component $\mathbb{Z}C_p^{\oplus r}$ we see that N" can be any of the following C_p- lattices

$$\begin{pmatrix} a & s-a+r \\ t-a+r+1 & u \end{pmatrix}$$

and setting $a = r + k$ gives the desired result

An immediate consequence of this result is that all trees are isomorphic to the permutation-tree (with the unfaithful lattices removed).

The construction of the invertible (even projective) modules (I, i) is nothing but a special case of the general construction of projective modules over a ring Λ with the projective modules over Λ_1 and Λ_2 as building blocks whenever we have a pullback diagram

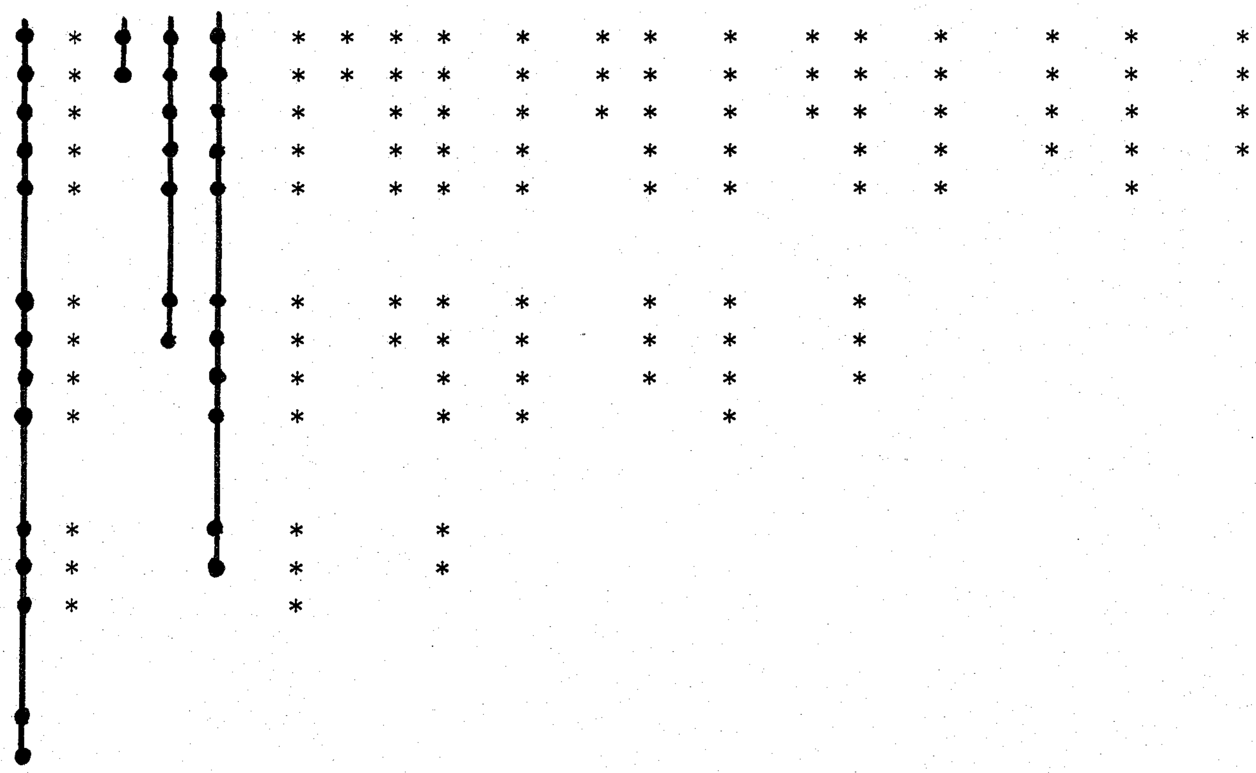
$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda_1 \\ \downarrow & & \downarrow \\ \Lambda_2 & \longrightarrow & \Lambda' \end{array}$$

with at least one of the two maps to Λ' being surjective, see e.g. Milnor 'Introduction to algebraic K-theory' Ch. 2. In our case we have the pull-back diagram

$$\begin{array}{ccc} \mathbb{Z}C_p & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}[\zeta_p] & \longrightarrow & \mathbb{Z}/p\mathbb{Z} \end{array}$$

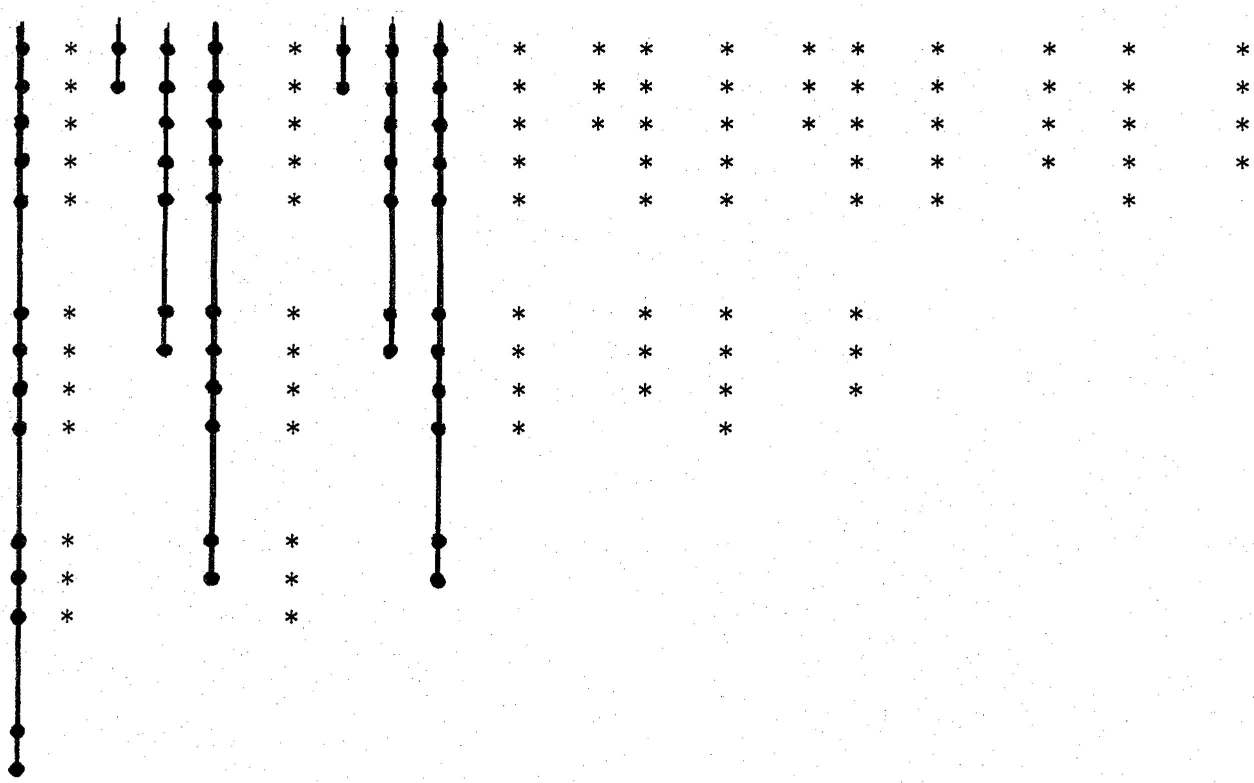
the upper map being the augmentation map and the lower being the quotient map by dividing out the maximal ideal $\mathbb{Z}[\zeta_p](\zeta_p - 1)$.

The Lenstra forest for $G = \mathbb{Z}/p\mathbb{Z}$



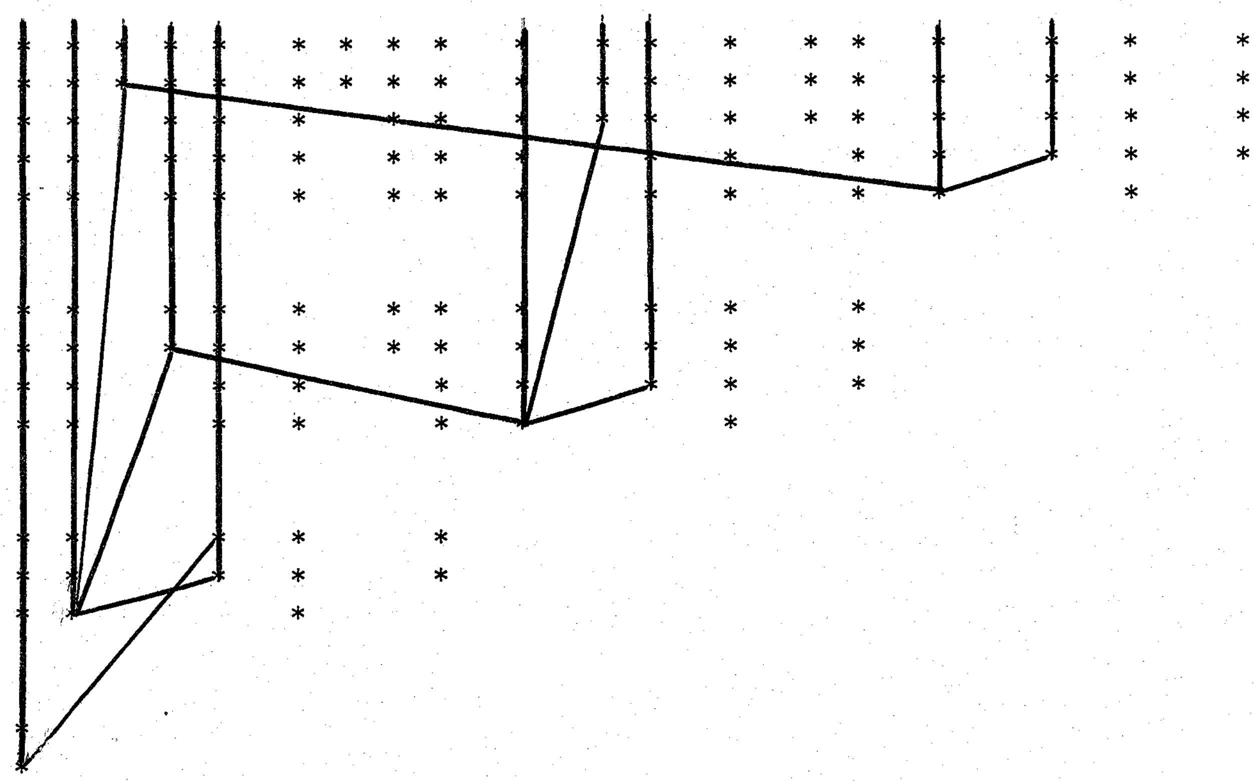
the permutation modules

The Lenstra forest for $G = \mathbb{Z}/p\mathbb{Z}$



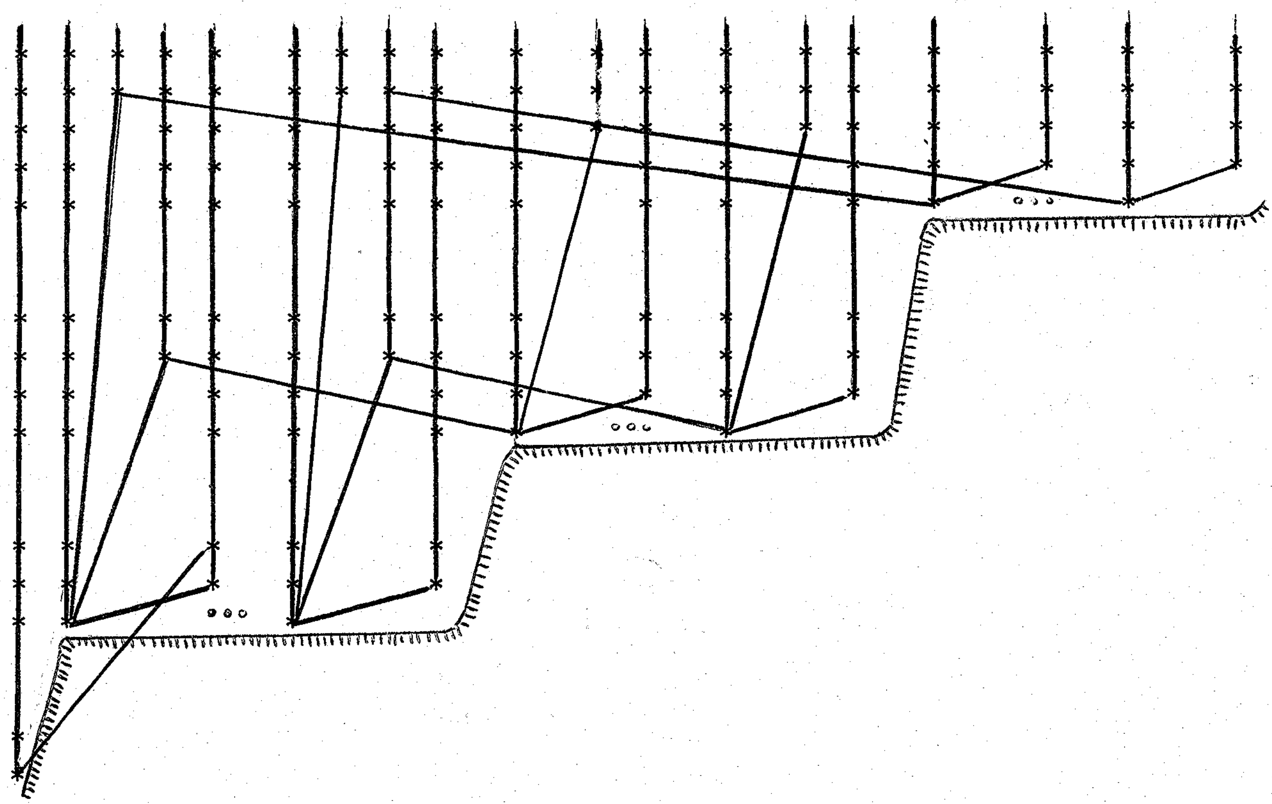
the invertible modules

The Lenstra forest for $G = \mathbb{Z}/p\mathbb{Z}$



the permutation tree

The Lenstra forest for $G = \mathbb{Z}/p\mathbb{Z}$



the full forest

4. COHOMOLOGICAL SURVIVAL KIT :

In order to study the structure of the Lenstra forest for arbitrary groups, it is indispensable to know some of the basic results in group (co)homology. Luckily, there exist several excellent reference texts (e.g. J.P. Serre 'Local Fields' ch VII,VIII,IX). For this reason, we will only mention some of the basic definitions and results and refer to the above mentioned work for full details :

In section 2 we have already defined the cohomology groups $H^i(G,M)$ of G with coefficients in a G-module M to be the right derived functors of the left exact additive functor $M \to M^G = \{m \in M \mid g.m = m \text{ for all } g \in G\}$. Since $M^G = Hom_G(\mathbb{Z}, M)$ where $\mathbb{Z}$ is given the trivial G-module action, we see that

$$H^i(G,M) \simeq Ext^i(\mathbb{Z}, M)$$

where the Ext-groups are the derived functors of the Hom-functor. This gives us a procedure to compute the cohomology groups : take a projective resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$

$$\ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are projective (or free) $\mathbb{Z}G$-modules.Then, for any G-module M define $V^i = Hom_G(P_i, M)$ then we obtain a cochain complex

$$V^0 \to V^1 \to V^2 \to \ldots$$

and $H^i(G,M) = Ker(V^i \to V^{i+1})/Im(V_{i-1} \to V_i)$ for $i > 0$ and $H^0(G,M) = Ker(V^0 \to V^1) = M^G$. We have seen before that a short exact sequence of G-modules gives rise to a long exact sequence of cohomology groups. Further, if M is an injective $\mathbb{Z}G$-module then $H^i(G,M) = 0$ for all $i \geq 1$. Maybe it is useful to give a concrete interpretation of the first two cohomology groups :

A 1-cocycle is a map $f : G \to M$ such that $f(g.h) = g.f(h) + f(g)$ and it is said to be a coboundary if there is some $m \in M$ such that $f(g) = g.m - m$ for all $g \in G$. The group $H^1(G,M)$ is then the quotientgroup of all 1-cocycles by the coboundaries. In particular, if G acts trivially on M we get that $H^1(G,M) = Hom(G,M)$. For example, $H^1(G,\mathbb{Z}) = Hom(G,\mathbb{Z}) = 0$ if G is a torsion group.

A 2-cocycle is a map $f : G \times G \to M$ satisfying

$$g.f(h,i) - f(g.h,i) + f(g,h.i) - f(g,h) = 0$$

It is also called a factor set. One encounters them in the study of extensions of G by M : let

$$1 \to M \to E \to G \to 1$$

be an exact sequence of groups such that M is normal in E. Since M is assumed to be Abelian, G acts on E by inner automorphisms. Take any section $s : G \to E$ then if $g,h \in G$ clearly $s(g).s(h)$ and $s(g.h)$ are in the same coset modulo M i.e. there is an $f(g,h) \in M$ such that

$$s(g).s(h) = f(g,h).s(g.h)$$

which allows us to define a composition law on E. Associativity then leads to the factor set condition on f. Further, if we choose a different section s the resulting f is modified by a coboundary. Thus, $H^2(G,M)$ classifies the (group)isomorphism classes of extensions of G by M.

Having defined cohomology, we now turn to homology. For any G-module M, let M_G be the quotient nodule M/DM where DM is the submodule generated by the elements $g.m - m$ for all $g \in G, m \in M$. Then $(-)_G$ is seen to be a right exact additive functor. The homology groups $H_i(G,M)$ of G with coefficients in M are defined to be the left derived functors of $(-)_G$.That is, whenever we have an exact sequence of G-modules

$$0 \to L \to M \to N \to 0$$

we have a long exact sequence of homology groups

$$\cdots$$

$$\to H_2(G,L) \to H_2(G,M) \to H_2(G,N) \to$$

$$\to H_1(G,L) \to H_1(G,M) \to H_1(G,N) \to$$

$$\to L_G \to M_G \to N_G \to 0$$

Again, since $M_G = \mathbb{Z} \otimes_{\mathbb{Z}G} M$ we see that

$$H_i(G,M) \simeq Tor_i^{\mathbb{Z}G}(\mathbb{Z}, M)$$

allowing us again to compute the homology groups starting from a projective (or free) resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$: take such a resolution and tensor it with M. The homology groups $H_i(G,M)$ are then the homology groups of the obtained complex. Note that $H_q(G,M) = 0$ for all $i \geq 1$ if M is a projective $\mathbb{Z}G$-module. It is fairly easy to verify that $H_1(G,\mathbb{Z}) = G/[G,G]$.

Homology and cohomology groups can be intertwined by means of the Tate cohomology groups : let $n = \sum_{g \in G} g \in \mathbb{Z}G$ be the norm element, then n induces an endomorphism on every G-module M which we denote by N : $N.m = \sum_{g\in G} g.m$. Further, if

$$0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

where $I_G = \sum_g \mathbb{Z}(g-1)$ is the augmentation ideal, then

$$I_G.M \subset KerN \text{ and } ImN \subset M^G$$

As $H_0(G,M) = M/I_G.M$ and $H^0(G,M) = M^G$ we obtain from the commutative diagram

$$\begin{array}{ccccccccc} & & & & H_0(G,M) & \to & ImN & \to & H^0(G,M) \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & KerN & \to & M & \to & ImN & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & I_G.M & \to & G.M & \to & 0 & & \end{array}$$

a map

$$N^* : H_0(G,M) \to H^0(G,M)$$

which allows us to define

$$TH_0(G,M) = Ker(N^*); TH^0(G,M) = Coker(N^*)$$

that is, if $._N M$ denotes the kernel of N acting on M, then we have from the above diagram that

$$TH_0(G,M) = ._N M/I_G.M; TH^0(G,M) = M^G/N.M$$

But then, for any exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

we have (by verification) the commutative exact diagram of Abelian groups

$$\begin{array}{ccccccccc}
 & & KerN_A^* & \to & KerN_B^* & \to & KerN_C^* & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H_1(G,C) & \to & H_0(G,A) & \to & H_0(G,B) & \to & H_0(G,C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(G,A) & \to & H^0(G,B) & \to & H^0(G,C) & \to & H^1(G,A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & CokerN_A^* & \to & CokerN_B^* & \to & CokerN_C^* & &
\end{array}$$

But then, there is a connecting morphism

$$\delta : TH^0(G,C) \to TH_0(G,A)$$

and we have the long exact sequence of Abelian groups

$$... \to H_1(G,C) \to TH_0(G,A) \to TH_0(G,B) \to TH_0(G,C) \to$$

$$\to TH^0(G,A) \to TH^0(G,B) \to TH^0(G,C) \to H^1(G,A) \to ...$$

This prompts us to define the Tate cohomology groups $TH^i(G,-)$:

$$TH^n(G,M) = H^n(G,M) \text{ if } n \geq 1$$

$$TH^0(G,M) = M^G/N.M$$

$$TH^{-1}(G,M) = ._NM/I_G.M$$

$$TH^{-n}(G,M) = H_{n-1}(G,M) \text{ if } n \geq 2$$

From the many important results on (Tate) cohomology, we mention here just a few for further reference :

(4.1) : Cup product theorem : Let A, B and C be G-modules, then there is a natural homomorphism (called the cup product)

$$TH^i(G,A) \otimes_{\mathbb{Z}} TH^j(G,B) \to TH^{i+j}(G, A \otimes_{\mathbb{Z}} B)$$

where $A \otimes B$ is given the diagonal G-action. The cup product is denoted $(a, b) \to a.b$ and satisfies :

$(a.b).c = a.(b.c)$ modulo identification $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

$a.b = (-1)^{dim(a).dim(b)} b.a$ modulo identification $A \otimes B = B \otimes A$ where $dim(a) = i$ if $a \in TH^i(G, A)$

(4.2) : Shapiro's lemma : Let H be a subgroup of G and M a $\mathbb{Z}H$-module. Then, there is a natural isomorphism

$$H^i(G, M \otimes_{\mathbb{Z}H} \mathbb{Z}G) \simeq H^i(H, M)$$

An immediate consequence of Shapiro's lemma is the promised proof of (2.4) :

(4.3) : lemma : Any invertible G-lattice I is co-flasque

proof : Recall that we must show that $H^1(H, I) = 0$ for any subgroup H of G. Since any invertible G-lattice is also an invertible H-lattice we may assume that $G = H$. Also, since cohomology preserves direct summands it suffices to prove that $H^1(G, \mathbb{Z}G/H) = 0$ for any subgroup H of G. By Shapiro's lemma we have

$$H^1(G, \mathbb{Z}G/H) = H^1(G, \mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G) \simeq H^1(H, \mathbb{Z}) = 0$$

because the action on $\mathbb{Z}$ is trivial and H is finite hence torsion.

If M and N are two G-lattices we will denote by $Hom(M, N) = Hom_{\mathbb{Z}}(M, N)$ given the natural G-module structure and by $M \otimes N = M \otimes_{\mathbb{Z}} N$ and by $M^* = Hom(M, \mathbb{Z})$ the G-module dual of M. We have the following canonical G-isomorphisms

$$(M^*)^* = M; Hom(M, N) = Hom(N^*, M^*) = M^* \otimes N; Hom(M, n)^* = Hom(N, M)$$

Further, if H is a subgroup of G, then $\mathbb{Z}G/H^* = \mathbb{Z}G/H$ entailing that all permutation modules are self-dual. By the cup-product theorem we have :

(4.4) : Duality lemma : For any G-lattice M we have a canonical isomorphism

$$TH^i(G, M) \simeq TH^{-i}(G, M^*)^*$$

Clearly, there are relations between the cohomology groups of G and of H and G/H when H is a normal subgroup of G. We will need only a very special result :

(4.5) : Inflation-Restriction lemma : Let M be a G-module and H a normal subgroup of G, then there is an exact sequence

$$0 \to H^1(G/H, M^H) \to H^1(G, M) \to H^1(H, M)$$

Further, we will often encounter cohomology groups for cyclic groups. In this case we have the following periodicity result :

(4.6) : Cohomology of cyclic groups : Let $G = \mathbb{Z}/n\mathbb{Z}$ be cyclic with generator τ. Define the elements $n = \sum_{i=0}^{n-1} \tau^i$ and $d = \tau - 1$. Then, for every G-module M we have

$$TH^i(G, M) = M^G/n.M \text{ for } i = 0 \bmod 2$$

$$TH^i(G, M) = {}_{n}M/d.M \text{ for } i = 1 \bmod 2$$

5. A FORESTERS GUIDE :

In this section we will introduce the Colliot-semigroup $Colliot(G)$ of a group G which can be viewed as a cataloque for the trees in the Lenstra forest. In particular we will associate to any G-lattice M an invariant $\phi(M) \in Colliot(G)$ such that $[M]$ and $[N]$ belong to the same tree if and only if $\phi(M) = \phi(N)$.

Let $Latt(G)$ be the set of (isoclasses) of all G-lattices. We define an equivalence relation on $Latt(G)$ as follows

$$M \sim N \text{ iff } M \oplus P_1 \cong N \oplus P_2$$

with P_1 and P_2 permutation G-lattices. The set of equivalence classes of $Latt(G)$ under this relation will be denoted by $Sansuc(G)$. The direct sum of G-lattices induces on $Sansuc(G)$ the structure of an Abelian semigroup which we call the Sansuc-semigroup of G. With $[M]_c$ we will denote the equivalence class of M in $Sansuc(G)$.

The invertible elements in $Sansuc(G)$ are easily characterized :

(5.1) : lemma The equivalence class $[M]_c \in Sansuc(G)$ has an inverse in $Sansuc(G)$ if and only if M is an invertible module

proof : Suppose that $[M]_c + [N]_c = [0]$, then there are permutation lattices P_i such that $M \oplus N \oplus P_1 \cong P_2$ entailing that M is a direct summand of a permutation lattice. The converse is also trivial.

This observation explains the terminology 'invertible modules'. The subset of $Sansuc(G)$ consisting of all equivalence classes of invertible modules forms an Abelian group which we will denote by $PCl(G)$ and call the permutation classgroup of G. Later on we will study this group in full detail.

Let us introduce a few new subclasses of G-lattices :

(5.2) : Definitions

A G-lattice M is said to be a stable permutation module if $[M]_c = [0]_c \in Sansuc(G)$ i.e. there exist permutation modules P_i such that $M \oplus P_1 \cong P_2$

A G-lattice M is said to be flasque if $TH^{-1}(H,M) = 0$ for all subgroups H of G

Later we will give examples of stable permutation modules which are not permutation modules. By the duality result we see that flasque modules are the duals of co-flasque modules. We have :

(5.3) : lemma The Tate cohomology groups $TH^1(G,M)$ and $TH^{-1}(G,M)$ depend only on the equivalence class $[M]_c \in Sansuc(G)$

proof : For any permutation lattice P we have by the duality result

$$TH^{-1}(G,M)^* = TH^1(G,P^*) = TH^1(G,P) = H^1(G,P) = 0$$

whence $TH^{-1}(G,P) = 0$ and as cohomology preserves direct sums we are done

Of fundamental importance is the notion of (co)flasque resolutions :

(5.4) : Definitions Let M be a G-lattice, then

A flasque resolution of M is an exact G-sequence

$$0 \to M \to P \to F \to 0$$

where P is a permutation module and F a flasque lattice

A coflasque resolution of M is an exact G-sequence

$$0 \to Q \to P \to M \to 0$$

where P is a permutation module and Q a coflasque lattice

(5.5) : Proposition

Every G-lattice M has a flasque (resp. coflasque) resolution

proof : By duality, it suffices to prove that every G-lattice M has a coflasque resolution. Consider

$$P = \oplus_{H<G}(\mathbb{Z}G/H \otimes M^H) \longrightarrow M \to 0$$

In the proof of (2.5.3) we have seen that the kernel of this map is coflasque,done.

We will now try to slice up $Sansuc(G)$ into some smaller and more manageable parts : let $Coflas(G)$ (resp. $Colliot(G)$) be the subset of $Sansuc(G)$ consisting of all classes of lattices which are $+1$ (resp. -1) cohomologically trivial i.e. of coflasque (resp. flasque) lattices. Lemma (5.3) saves us from worrying about well-definedness. Further, with $Coco(G)$ we denote the intersection $Coflas(G) \bigcap Colliot(G)$. Again, by the duality result we see that an invertible module is both flasque and coflasque. Thus, we have the following situation

$$PCl(G) \subset Coco(G) \subset Colliot(G) \subset Sansuc(G)$$

and the corresponding quotient semigroups will be denoted by

$$\Delta_1(G) = Colliot(G)/Coco(G)$$

$$\Delta_2(G) = Coco(G)/PCl(G)$$

and we clearly have the following :

(5.6) : lemma Both $\Delta_1(G)$ and $\Delta_2(G)$ are torsion-free Abelian semigroups

proof : Let $[M]_c \in Colliot(G)$ such that $[M^{\oplus n}] \in Coco(G)$, then , for any subgroup H in G we have

$$0 = TH^1(H, M^{\oplus n}) = TH^1(H, M)^{\oplus n}$$

and therefore $[M]_c \in Coco(G)$ whence $\Delta_1(G)$ is torsionfree. For $\Delta_2(G)$, just note that if $M^{\oplus n}$ is a direct summand of a permutation module, so is M.

In particular, if there are non-invertible flasque lattices, then $Colliot(G)$ is infinite. In the next section we will see that the condition $PCl(G) = Colliot(G)$ is equivalent to G being a metacyclic group (i.e. all Sylow p-subgroups are cyclic).

The duality $M \to M^* = Hom_G(M, \mathbb{Z})$ induces an involution on $Sansuc(G)$ which maps $PCl(G)$ to itself and maps $Coflas(G)$ isomorphically to $Colliot(G)$.

We will now investigate the relationship between flasque resolutions as G-lattices and as G/H-lattices, H being a normal subgroup of G.

(5.7) : lemma Let H be a normal subgroup of G and M a G-lattice, then :

(1) : If M is a permutation G-lattice , M^H is a permutation G/H-lattice

(2) : If M is a coflasque G-lattice, M^H is a coflasque G/H-lattice

(3) : If $0 \to Q \to P \to M \to 0$ is a coflasque resoltion of M in G-lattices, $0 \to Q^H \to P^H \to M^H \to 0$ is a coflasque resoltion of M^H in G/H-lattices

proof :

(1) : Take a $\mathbb{Z}$-basis of M which is permuted by G, them M^H has the orbitsums of H acting on this basis as a $\mathbb{Z}$-basis. Clearly, G/H permutes these orbitsums.

(2) : Take any subgroup G'/H of G/H. Then, by the inflation-restriction exact sequence we have

$$0 \to H^1(G'/H, M^H) \to H^1(G', M) = 0$$

whence M^H is a coflasque G/H-lattice

(3) : Since Q is coflasque, $H^1(H,Q) = 0$ entailing exactness of the sequence of H-invariants. Parts (1) and (2) then finish the proof.

(5.8) : lemma Let H be a normal subgroup of G and M a G/H-lattice, then :

(1) : M is a permutation G-module iff M is a permutation G/H-module

(2) : M is a stable permutation G-module iff M is a stable permutation G/H-module

(3) : M is an invertible G-module iff M is an invertible G/H-module

(4) : M is a coflasque G-module iff M is a coflasque G/H-module

(5) : M is a flasque G-module iff M is a flasque G/H-module

proof : (1) is easy from the fact that H acts trivially on M and (5.7.1). (2) and (3) follow easily from (1).

(4) : one direction follows from (5.7.2). Conversely, let G' be a subgroup of G then $G' \bigcap H$ is a normal subgroup og G' and we get the inflation restriction exact sequence

$$0 \to H^1(G'/(G' \bigcap H), M) \to H^1(G', M) \to H^1(G' \bigcap H, M) = Hom(G' \bigcap H, M)$$

Here, the last term is zero because M is a lattice (i.e. torsion free).Thus, $H^1(G', M) \simeq H^1(G'/(G' \bigcap H), M) = H^1(G'.H/H, M) = 0$ because M is a coflasque G/H-module. (5) follows by duality.

We are now in a position to relate flasque and coflasque resolution in G resp. G/H-modules for a G/H-lattice M :

(5.9) : Proposition : Let H be a normal subgroup of G and M a G/H-lattice, then

(1) : Every flasque (resp. coflasque) resolution of M in G/H-modules is one in G-modules

(2) : Every flasque (resp. coflasque) resolution of M in G-modules induces one in G/H-modules by taking H-invariants

proof : (1) follows from (5.8)

(2) : Take a flasque resoltion of M in G-lattices

$$0 \to M \to P \to F \to 0$$

then, as H acts trivially on M we get an exact sequence

$$0 \to M \to P^H \to F^H \to H^1(H,M) = Hom(H,M) = 0$$

by (5.7.1) P^H is a permutation G/H-lattice. Let us prove that F^H is a flasque G/H-lattice. Take any subgroup G'/H of G/H then we have a long exact sequence of Tate cohomology :

$$0 = TH^{-1}(G'/H, P^H) \to TH^{-1}(G'/H, F^H) \to TH^0(G'/H, M) \to TH^0(G'/H, P^H)$$

So it suffices to prove injectivity of the last map.Using the definition of TH^0 it is fairly easy to verify that

$$TH^0(G'/H, M) \simeq TH^0(G', M) \text{ and } TH^0(G'/H, P^H) \hookrightarrow TH^0(G', P)$$

and the commutativity of the diagram

$$\begin{array}{ccccc} 0 = TH^{-1}(G', F) & \to & TH^0(G', H) & \to & TH^0(G', P) \\ & & \| & & \uparrow \\ & & TH^0(G'/H, M) & \to & TH^0(G'/H, P^H) \end{array}$$

finishes the proof.

We will now investigate some properties of flasque resolutions :

(5.10) : Proposition :

(1) : Let $0 \to Q \to P \to M \to 0$ be a coflasque resolution of M. If P' is a permutation G-lattice, then every morphism $P' \to M$ factors through P

(2) : Let $0 \to M \to P \to F \to 0$ be a flasque resoltion of M. If P' is a permutation G-lattice, then every morphism $M \to P'$ factors through P

proof : (1) : We have the exact commutative diagram of G-modules

$$\begin{array}{ccccccccc} 0 & \to & Q & \to & P \times_M P' & \to & P' & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & Q & \to & P & \to & M & \to & 0 \end{array}$$

where $P \times_M P'$ is the pullback of P and P' along M. Because P' is invertible and Q coflasque, the upper sequence splits providing the required factorization

(2) : dual to (1).

The next result is an analoque of the Schanuel lemma for projective modules. It shows that flasque and coflasque resoltions are essentially unique :

(5.11) : Theorem (Voskresenskii 1975,Colliot-Thélène,Sansuc 1977)

(1) : Let $0 \to Q_1 \to P_1 \to M \to 0$ and $0 \to Q_2 \to P_2 \to M \to 0$ be two coflasque resolutions of M, then $[Q_1]_c = [Q_2]_c$

(2) : Let $0 \to Q_1 \to P_1 \to M \to 0$ and $0 \to Q_2 \to P_2 \to N \to 0$ be two coflasque resoltions with $[M]_c = [N]_c$, then $[Q_1]_c = [Q_2]_c$

(3) : Let $0 \to M \to P_1 \to F_1 \to 0$ and $0 \to M \to P_2 \to F_2 \to 0$ be two flasque resoltions of M, then $[F_1]_c = [F_2]_c$

(4) : Let $0 \to M \to P_1 \to F_1 \to 0$ and $0 \to N \to P_2 \to F_2 \to 0$ be two flasque resolutions with $[M]_c = [N]_c$, then $[F_1]_c = [F_2]_c$

proof : (1) : We have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & Q_1 & \to & P_1 \times_M P_2 & \to & P_2 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & Q_1 & \to & P_1 & \to & M & \to & 0 \end{array}$$

Then, the upper sequence splits yielding $P_1 \times_M P_2 \cong Q_1 \oplus P_2$. Replacing the roles of Q_1 and Q_2 one finds similarly that $P_1 \times_M P_2 \cong Q_2 \oplus P_1$ finishing the proof

(2) : If $[M]_c = [N]_c$ then by definition $M \oplus P_3 \cong N \oplus P_4 = X$. But then, $0 \to Q_1 \to P_1 \oplus P_3 \to X \to 0$ and $0 \to Q_2 \to P_2 \oplus P_4 \to X \to 0$ are two coflasque resolutions of X and therefore by (1) we have $[Q_1]_c = [Q_2]_c$

(3) and (4) are proved dually.

This result allows us to define two maps

$$
\begin{array}{ccccc}
 & & Sansuc(G) & & \\
 & \phi_G \swarrow & & \searrow \kappa_G & \\
Colliot(G) & & & & Coflas(G)
\end{array}
$$

Where $\phi_G([M]_c) = [F]_c$ if $0 \to M \to P \to F \to 0$ is a flasque resolution of M and $\kappa_G([M]_c) = [Q]_c$ if $0 \to Q \to P \to M \to 0$ is a coflasque resolution of M.

First of all we want to get rid off the subscript $._G$. So, let us investigate for a moment the behaviour of our sub-semigroups of $Sansuc(G)$ under subgroups and quotientgroups : let H be a subgroup of G and let

$$Res : Latt(G) \to Latt(H)$$

be the restriction i.e. viewing a $\mathbb{Z}G$-module as a $\mathbb{Z}H$-module via the natural inclusion. Then it is clear that if M is a permutation (resp. stable permutation,invertible,flasque,coflasque) G-lattice , then $Res(M)$ is a permutation (resp. stable permutation,invertible, flasque, coflasque) H-lattice. In particular, the restriction induces a well-defined morphism

$$Res : Sansuc(G) \to Sansuc(H)$$

of semigroups which induces morphisms

$$Coflas(G) \to Coflas(H); Colliot(G) \to Colliot(H); Coco(G) \to Coco(H)$$

$$PCl(G) \to PCl(H); \Delta_1(G) \to \Delta_1(H); \Delta_2(G) \to \Delta_2(H)$$

Now, let us consider induction

$$Ind : Latt(H) \to Latt(G)$$

which maps an H-lattice M to $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$.Then, it is clear (using Shapiro's lemma) that if M is a permutation (resp. stable permutation,invertible,flasque,coflasque) H-lattice, then $Ind(M)$ is a permutation (resp. stable permutation,invertible,flasque ,coflasque) G-lattice. Therefore, the induction induces a well-defined morphism of semigroups

$$Ind : Sansuc(H) \to Sansuc(G)$$

and this morphism induces morphisms between the corresponding $Coco, PCl, \Delta_i$. Now, consider an H-lattice M , then the H-lattice $Res \circ Ind(M)$ contains M as a direct summand. Because direct summands of flasque (resp. coflasque,invertible) modules have the similar property, we see that the morphisms

$$Res \circ Ind : \Delta_i(H) \to \Delta_i(H)$$

are injective entailing that $Ind : \Delta_i(H) \to \Delta_i(G)$ is injective, too.

Let us now turn to quotientgroups G/H for a normal subgroup H of G. By (5.8) we see that if M and N are G/H-lattices , then $[M]_c = [N]_c$ in $Sansuc(G)$ if and only if $[M]_c = [N]_c \in Colliot(G/H)$ inducing a natural semigroup injection :

$$\pi_H : Sansuc(G/H) \hookrightarrow Sansuc(G)$$

and again this map induces the equalities

$$PCl(G/H) = PCl(G) \bigcap Sansuc(G/H)$$

$$Coflas(G/H) = Coflas(G) \bigcap Sansuc(G/H)$$

$$Colliot(G/H) = Colliot(G) \bigcap Sansuc(G/H)$$

In particular, we obtain from these observations the following : if M is a G-lattice on which a normal subgroup H of G acts trivially, then $\phi_{G/H}([M]_c)$ coincides with $\phi_G([M]_c)$ via the natural embedding π_H and similarly, $\kappa_{G/H}([M]_c)$ coincides with $\kappa_G([M]_c)$. Therefore, we will forget the subscripts from now on and denote for any G- lattice M : $\phi(M) = \phi_G([M]_c)$ and $\kappa(M) = \kappa_G([M]_c)$. Further, it follows from (5.9) that if M is a G-lattice on which the normal subgroup H acts trivially, then $\phi(M) = [F]_c = [F^H]_c$ if $0 \to M \to P \to F \to 0$ is a flasque resolution of M and that $\kappa(M) = [Q]_c = [Q^H]_c$ if $0 \to Q \to P \to M \to 0$ is a coflasque resolution of M.

Before we will prove that $\phi(M)$ is an invariant for the tree in the Lenstra forest containing $[M]$, we will demonstrate some lemmas which are handy in computing $\phi(M)$ and $\kappa(M)$:

(5.12) : lemma

(1) : $\phi : Sansuc(G) \to Colliot(G)$ and $\kappa : Sansuc(G) \to Coflas(G)$ are surjective

(2) : If M is any G-lattice, then $\phi(M)^* = \kappa(M^*)$

(3) : $\kappa : Colliot(G) \to Coflas(G)$ is an isomorphism of semigroups with inverse ϕ

(4) : On $PCl(G)$ the involution sending an element to its inverse coincides with both ϕ and κ

proof : (1) : Let $0 \to Q \to P \to F \to 0$ be a coflasque resolution of a flasque module F (or, a flasque resolution of a coflasque module Q), then $\phi(Q) = F$ and $\kappa(F) = Q$ proving surjectivity. This also proves (3) and (4). In order to prove (2) : just dualize a flasque resolution of M.

(5.13) : lemma Consider an exact sequence of G-lattices

$$0 \to M \to N \to I \to 0$$

with I an invertible G-lattice, then

(1) : $\phi(M) = \phi(N) + [I]_c$

(2) : If I is a permutation G-lattice then $\phi(M) = \phi(N)$

proof : (1) : Let $0 \to N \to P \to F \to 0$ be a flasque resolution of N and consider the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & P & \to & Coker & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & N & \to & P & \to & F & \to & 0 \end{array}$$

from which we obtain the exact sequence

$$0 \to I \to Coker \to F \to 0$$

then this sequence splits as I in invertible and F flasque (this is just a dual version to (2.5)) whence $Coker = F \oplus I$. But then,from

$$0 \to M \to P \to F \oplus I \to 0$$

and flatness of $F \oplus I$ we obtain $\phi(M) = [I]_c + [F]_c = [I]_c + \phi(N)$

(2) follows easily from (1).

(5.14) : lemma Consider any exact sequence of G-lattices

$$0 \to M \to P \to N \to 0$$

where P is apermutation lattice. Then,

(1) :$\phi(M) = \phi(\kappa(N))$
(2) : $\kappa(N) = \kappa(\phi(M))$

proof : We have the commutative exact diagram

$$\begin{array}{ccccccccc} & & 0 & \to & \kappa(N) & = & \kappa(N) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & H & \to & P_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & P & \to & N & \to & 0 \end{array}$$

where H is the pullback of P_1 and P along N. This gives us the exact sequence

$$0 \to \kappa(N) \to H \to P \to 0$$

which splits because P is invertible and $\kappa(N)$ co-flasque. Thus,

$$0 \to M \to P \oplus \kappa(N) \to P_1 \to 0$$

is exact and therfore by (5.13) we have

$$\phi(M) = \phi(P) + \phi(\kappa(N)) = \phi(\kappa(N))$$

(2) is proved dually.

Finally, we can state and prove the main result of this section :

(5.15) : Theorem (Colliot-Thélène,Sansuc 1977)
If M and N are G-lattices, the following are equivalent
(1) : $[M]$ and $[N]$ belong to the same tree in the Lenstra forest
(2) : $\phi(M) = \phi(N)$ in $Colliot(G)$
(3) : there exist two exact sequences of G-lattices

$$0 \to M \to E \to P_1 \to 0$$

$$0 \to N \to E \to P_2 \to 0$$

with P_1 and P_2 permutation G-lattices

proof :
(1) $\Rightarrow$ (2) : iterated use of (5.13.2) along a path in the tree from $[M]$ to $[N]$

(2) $\Rightarrow$ (3) : Take flasque resolutions of M and N :

$$0 \to M \to P_M \to F_M \to 0$$

$$0 \to N \to P_N \to F_N \to 0$$

then by (2) we have permutation lattices P_i such that

$$F_M \oplus P_1 \cong F_N \oplus P_2 = F$$

But then we have exact sequences

$$0 \to M \to P_M \oplus P_1 \to F \to 0$$

$$0 \to N \to P_N \oplus P_2 \to F \to 0$$

And we can take the pullback diagrams

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & P_M \oplus P_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & P_N \oplus P_2 & \to & F & \to & 0 \end{array}$$

and similarly we have

$$\begin{array}{ccccccccc} 0 & \to & M & \to & E & \to & P_N \oplus P_2 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & P_M \oplus P_1 & \to & F & \to & 0 \end{array}$$

finishing the proof

(3) $\Rightarrow$ (1) : $[M]$ and $[N]$ belong to the tree in the Lenstra forest containing $[E]$

Concluding, studying the number of trees in the Lenstra forest (and hence, equivalently, the stable equivalence classes of tori-invariants $l(M)^G$ over l^G) is reduced to the study of the Abelian semigroup $Colliot(G)$. This study naturally breaks up in two parts : (a) the study of the permutation classgroup $PCl(G)$ and (b) the study of the torsionfree Abelian semigroup $\Delta(G) = Colliot(G)/PCl(G)$.

6. A TOUR OF THE FINITE FORESTS :

In this section we will list all groups G such that the Lenstra forest contains only finitely many trees. In the foregoing section we have seen that there are always infinitely many trees if $Colliot(G) \neq PCl(G)$. Therefore, we will first characterize all finite groups G for which the Colliot-semigroup is actually a group (and hence coincides with the permutation class group).Or, in forest language : for which finite groups does every tree in the Lenstra forest contain an invertible module ?

We begin with an easy lemma which reduces the study of invertible G-lattices to the p-Sylow subgroups G_p of G. Recall that a Sylow p-subgroup of G is a p-group (i.e. each element has order a power of p) which is not contained in any larger p-subgroup of G. All Sylow p-subgroups of G are conjugated and have order p^m if the order of G is $p^m.s$ with $(p,s) = 1$.

(6.1) : lemma : Let M be a G-lattice. Equivalent are

(1) : M is an invertible G-lattice

(2) : M is an invertible G_p-lattice for every Sylow subgroup G_p of G

proof :

(1) $\Rightarrow$ (2) is easy because any permutation G-lattice is a permutation H-lattice for all subgroups H of G , see also the restriction map explained in the last section.

(2) $\Rightarrow$ (1) : Take a flasque resolution of M as a G-lattice

$$0 \to Q \to P \to M \to 0$$

Then, for every Sylow p-subgroup G_p of G and (2.5) we know that this sequence splits as G_p-lattices. But then it also splits as G-lattices from standard theory. So, $M \oplus Q \cong P$, done.

Clearly, it suffices to take one representant for each p dividing the order of the group G.This result implies that

$$PCl(G) = \bigcap Res^{-1}_{G_p}(PCl(G_p))$$

where $Res_{G_p} : Sansuc(G) \to Sansuc(G_p)$ is the restriction map defined in the previous section.

From group theory we recall that a finite group G is said to be metacyclic if and only if all of its Sylow subgroups are cyclic. For example : any group G of squarefree order is metacyclic.

We will now present the promised characterization of all groups G such that each tree in the Lenstra forest contains an invertible lattice. We recall that J_G is defined to be I_G^* where I_G is the kernel of the augmentation map $\mathbb{Z}G \to \mathbb{Z}$.

(6.2) : Theorem (Endo-Miyata 1973)

The following statements are equivalent

(1) : G is metacyclic

(2) : Every coflasque G-lattice is invertible

(3) : $Colliot(G) = PCl(G)$

(4) : $\phi(J_G) \in PCl(G)$

proof :

(1) $\Rightarrow$ (2) : Since invertibility is checked on the Sylow p-subgroups of G by (6.1) we may assume that $G = C_{p^l}$ the cyclic p-group of order p^l. We intend to prove that any coflasque C_{p^l}-lattice is invertible by induction on l.

Let M be C_{p^l}-coflasque and define

$$M' = \{m \in M : \Phi_{p^l}(\tau)m = 0\}$$

where $\Phi_{p^l}(x) = x^{p.p^{l-1}} + x^{(p-1).p^{l-1}} + ... + x^{p^{l-1}} + 1$ is the cyclotomic polynomial of degree p^l. As any nontrivial subgroup C' of C_{p^l} contains $\tau^{p^{l-1}}$ we see that $(M')^{C'} = 0$ for otherwise M' would have an element m' such that $\tau^{p^{l-1}}.m = m$ whence $\Phi_{p^l}(\tau).m = pm = 0$ which is impossible because M is a lattice (i.e. has no $\mathbb{Z}$-torsion). But then we obtain from the periodicity result on cohomology of cyclic groups (4.6) that

$$TH^2(C', M') \simeq TH^0(C', M') = 0$$

Then, if we let $M"$ be the cokernel of the inclusion of M' in M we obtain from the long exact cohomology sequence that

$$H^1(C', M") = TH^1(C', M") = 0$$

for all subgroups C' of C_{p^l}, i.e. $M"$ is a coflasque C_{p^l}-lattice. However, by definition $M"$ is also a $C_{p^{l-1}}$-lattice whence by the induction hypotheses we may assume that $M"$ is an invertible $C_{p^{l-1}}$-lattice (and hence also an invertible C_{p^l}-lattice). Therefore

$$M' \oplus N \cong P$$

as C_{p^l}-lattices where P is a permutation lattice. But then, we obtain from (5.13) that

$$\phi(M) = \phi(M') - [M"]_c$$

Hence, M is invertible iff $\phi(M)$ is invertible iff $\phi(M')$ is invertible. But, by definition we can view M' as a torsionfree $\mathbb{Z}[\zeta_{p^l}]$-module where ζ_{p^l} is a primitive p^l-th root of unity. By the theory of Dedekind domains (and finiteness of the classgroup) we can find a $k \in \mathbb{N}$ such that

$$M'^{\oplus k} \simeq \mathbb{Z}[\zeta_{p^l}]^{\oplus m}$$

for some $m \in \mathbb{N}$. Therefore, it suffices to prove invertibility of $\phi(\mathbb{Z}[\zeta_{p^l}])$. But, $\mathbb{Z}[\zeta_{p^l}]$ is equal to M_0' where $M_0 = \mathbb{Z}C_{p^l}$. Then repeating the first part of the proof with M replaced by M_0 we find that

$$\phi(\mathbb{Z}[\zeta_{p^l}]) = \phi(\mathbb{Z}C_{p^l}) - [\mathbb{Z}C_{p^l}"]_c = [\mathbb{Z}C_{p^l}"]_c$$

and $M_0"$ is invertible , done.

$(2) \Rightarrow (3)$: trivial

$(3) \Rightarrow (4)$: As $\phi(J_G) \in Col(G)$ this is trivial

$(4) \Rightarrow (1)$: Consider an exact sequence of G-lattices

$$0 \to Q \to F \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

where the last map is the augmentation and F is a free $\mathbb{Z}G$-lattice. Then, after splitting this sequence up into $0 \to Q \to F \to I_G \to 0$ and $0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ we obtain from the long exact cohomology sequences that for any subgroup G' of G

$$TH^n(G', Q) \simeq TH^{n-1}(G', I_G) \simeq TH^{n-2}(G', \mathbb{Z})$$

In particular, taking $n = 1$ we obtain that $TH^1(G', Q) \simeq TH^{-1}(G', \mathbb{Z}) \simeq TH^1(G', \mathbb{Z})^* = 0$ whence Q is coflasque and the exact sequence $0 \to Q \to F \to I_G \to 0$ is a coflasque resolution of I_G i.e. $\kappa(I_G) = [Q]_c$ or by duality $\phi(J_G) = [Q^*]_c$. By assumption, Q^* (and hence Q) is invertible. Thus, we can find a G-lattice N and a permutation G-lattice P such that $Q \oplus N \cong P$. This implies that $TH^2(G', Q) \simeq TH^0(G', \mathbb{Z})$ embeds into $TH^2(G', P)$.Let us compute both groups :

$$TH^0(G', \mathbb{Z}) = \mathbb{Z}/(\sum_{g \in G'} g).\mathbb{Z} = \mathbb{Z}/ \mid G' \mid \mathbb{Z}$$

on the other hand

$$TH^2(G',P) = TH^2(G',\oplus\mathbb{Z}G'/H) = \oplus TH^2(G',\mathbb{Z}G/H)$$

where the H are subgroups of G'. But then by Shapiro's lemma we obtain

$$TH^2(G',P) = \oplus TH^2(H,\mathbb{Z}) = \oplus Hom(H,\mathbb{Q}/\mathbb{Z})$$

and this last group contains only elements of order the exponent of the H's. Combining the two computations we conclude that if we take G' to be a Sylow p-subgroup of G we see that the order of G' must be equal to its exponent, yielding that G' is cyclic,done.

In particular, if G is a cyclic group the foregoing result implies that every tree in the Lenstra forest contains an invertible G-lattice. We will now give some results due to H. Lenstra (1974) showing that there are only a finite number of trees in the cyclic forest as well as a computation of the tori-invariants of invertible lattices.

Let us begin with some general remarks in the case when G is a finite Abelian group , say

$$G = \mathbb{Z}/m_1\mathbb{Z} \times ... \times \mathbb{Z}/m_k\mathbb{Z}$$

Since every subgroup H of G is normal, we can view $\mathbb{Z}[G/H]$-modules as $\mathbb{Z}G$-modules. We want to study in particular the cyclic factorgroups $C_m = G/H$ of G. For any such $C_m = \mathbb{Z}/m\mathbb{Z}$ we can define the ring $\mathbb{Z}[\zeta_m]$ as follows : let τ be a generator of C_m and let

$$\Phi_m(x) \in \mathbb{Z}[x]$$

be the m-th cyclotomic polynomial. The easiest way to define them is by

$$\Phi_m(x) = \prod_{d|m}(x^{m/d}-1)^{\mu(d)}$$

where $\mu(d)$ is the Möbius-function, i.e. $\mu(d) = 0$ if d is non-squarefree and $\mu(d) = (-1)^{p_d}$ where p_d is the number of primedivisors of d if d is squarefree. The first few examples of such polynomials are

$$\Phi_1(x) = x-1; \Phi_2(x) = x+1; \Phi_3(x) = x^2+x+1; \Phi_4(x) = x^2+1$$

Then, we can form the ideal

$$\Phi_m(\tau)\mathbb{Z}C_m \subset \mathbb{Z}C_m$$

in the integral groupring of C_m and define the quotient ring to be

$$\mathbb{Z}[\zeta_m] = \mathbb{Z}C_m / \Phi_m(\tau)\mathbb{Z}C_m$$

which clearly coincides with the ring of integers in the m-th cyclotomic field $\mathbb{Q}(\zeta_m)$ where ζ_m is a primitive m-th root of unity. Via the natural epimorphism

$$\mathbb{Z}G \to \mathbb{Z}C_m \to \mathbb{Z}[\zeta_m]$$

we can view any $\mathbb{Z}[\zeta_m]$-lattice as a G-lattice. Conversely, if M is a G-module, we define

$$F_{G,C_m}(M) = (M \otimes_G \mathbb{Z}[\zeta_m])/(\text{ additive torsion })$$

Then, F_{G,C_m} defines a functor from G-modules to $\mathbb{Z}[\zeta_m]$-lattices which is the left adjoint to the functor defined by the epimorphism $\mathbb{Z}G \to \mathbb{Z}[\zeta_m]$

(6.3) : Proposition Let $CF(G)$ be the set of all cyclic factorgroups of a group G and let G/H be a factorgroup of G. Then, there is a natural inclusion $CF(G/H) \hookrightarrow CF(G)$ such that we have for every G/H-module M

(1) : If $C_m \in CF(G/H)$ then $F_{G,C_m}(M) \cong F_{G/H,C_m}(M)$ as $\mathbb{Z}[\zeta_m]$-modules

(2) : If $C_m \in CF(G) - CF(G/H)$, then $F_{G,C_m}(M) = 0$

proof : (1) : Because M is a G/H-module, we have $M \simeq M \otimes_G \mathbb{Z}G/H$ as G-modules. But then

$$F_{G/H,C_m}(M) = (M \otimes_G \mathbb{Z}G/H \otimes_{G/H} \mathbb{Z}[\zeta_m]/(\text{ torsion }) = M \otimes_G \mathbb{Z}[\zeta_m]/(\text{ torsion })$$

which is $F_{G,C_m}(M)$,done.

(2) : Because $C_m \in CF(G) - CF(G/H)$ we can find an element $g \in G$ such that

$$\begin{array}{ccc} & G & \\ \phi \swarrow & & \searrow \psi \\ G/H & & C_m \end{array}$$

$\phi(g) = 1$ and $\psi(g) \neq 1$. Then, g acts trivially on M i.e. $(\psi(g)-1).(M \otimes_G \mathbb{Z}[\zeta_m]) = 0$. Because $\psi(g) - 1$ is nonzero in $\mathbb{Z}[\zeta_m]$ it is the divisor of a positive natural number (just take the norm of the cyclotomic extension) but then $M \otimes_G \mathbb{Z}[\zeta_m]$ is torsion, i.e. $F_{G,C_m}(M) = 0$,done.

Therefore, up to $\mathbb{Z}[\zeta_m]$-isomorphism, F_{G,C_m} does not depend upon the choice of G. For this reason, we will write F_{C_m} instead of F_{G,C_m} from now on.

(6.4) : Proposition Given an exact sequence of G-modules

$$0 \to M \to N \to N/M \to 0$$

with N/M a torsion group. Then, for every $C_m \in CF(G)$ we have that $F_{C_m}(M)$ is isomorphic to the image of M under the map $N \to F_{C_m}(N)$

proof : If we denote by J_m the kernel of the natural epimorphism $\mathbb{Z}G \to \mathbb{Z}[\zeta_m]$, then for every G-module L there is a surjection $L \to F_{C_m}(L)$ with kernel $KL = \{x \in L : \exists k \in \mathbb{Z} - 0 \mid k.x \in J_m.L\}$. By assumption, N/M is torsion and hence $KM = \{m \in M : \exists k \in \mathbb{Z} - 0 \mid k.m \in J_m.M\} = M \bigcap \{n \in N : \exists k' \in \mathbb{Z} - 0 \mid k'.n \in J_m.N\}$ whence $F_{C_m}(M) = M/KM = M/(M \bigcap KN) \hookrightarrow N/KN = F_{C_m}(N)$

An immediate but important consequence of (6.3) is

(6.5) : Proposition Let P be a permutation G-lattice. Then, for every $C_m \in CF(G)$ we have that $F_{C_m}(P)$ is free as a $\mathbb{Z}[\zeta_m]$-lattice

proof : By assumption $P \cong \oplus_i \mathbb{Z}G/H_i$ for some subgroups H_i of G. For any of these factors we have

$$F_{C_m}(\mathbb{Z}G/H_i) = \mathbb{Z}[\zeta_m] \text{ if } C_m \in CF(G/H_i)$$

or 0 otherwise,done.

From now on, we restrict attention to the case when $G = C_m = \mathbb{Z}/m\mathbb{Z}$ a cyclic group with generator τ. For any $m \in N$ we denote with $Div(m)$ the set of all positive divisors of m. Then, for every $d \in Div(m)$ there is a uniquely determined factorgroup C_d of G of order d. For any subset $\mathcal{C} \subset Div(m)$ we define

$$\Phi_{\mathcal{C}}(x) = \prod_{d \in \mathcal{C}} \Phi_d(x)$$

For example, $\Phi_{Div(m)}(x) = x^m - 1$. Further, for any G-module M we define

$$M_{\mathcal{C}} = M/\Phi_{\mathcal{C}}(\tau).M$$

$$M^i_{\mathcal{C}} = M_{\mathcal{C}}/(\text{ torsion })$$

(6.6) : lemma Let M be a projective $\mathbb{Z}G$-module and $d \in Div(m)$. Then, $M_{Div(d)}$ is an invertible G-lattice

proof : Clearly,

$$M_{Div(d)} = M/(\tau^d - 1).M = M \otimes_G \mathbb{Z}C_d$$

M being G-projective, $M_{Div(d)}$ is a projective $\mathbb{Z}C_d$-module hence a direct summand of $\mathbb{Z}C_d^{\oplus n}$ for some n which is clearly a permutation G-lattice

Luckily, we can extend this result to invertible lattices :

(6.7) : lemma Let M be an invertible G-lattice and $d \in Div(m)$. Then, $M^i_{Div(d)}$ is an invertible G-lattice

proof : For any G-module M, $M^i_{Div(d)}$ is a C_d-lattice. Since direct sums are preserved, it suffices to verify the result for any factorgroupring $\mathbb{Z}C_n$. But then, $\mathbb{Z}C_n^{i_{Div(d)}} = \mathbb{Z}C_a$ where $a = gcd(n,d)$ and they are still permutation lattices.

(6.8) : lemma If M is a projective $\mathbb{Z}G$-module and $\mathcal{C}$ and $\mathcal{C}'$ are subsets of $Div(m)$ such that $\mathcal{C} \bigcap \mathcal{C}' = \emptyset$. Then,

$$0 \to M_{\mathcal{C}} \to M_{\mathcal{C}\cup\mathcal{C}'} \to M_{\mathcal{C}'} \to 0$$

is an exact sequence of G-modules

proof : The map $M_{\mathcal{C}\cup\mathcal{C}'} = M/\Phi_{\mathcal{C}}(\tau)\Phi_{\mathcal{C}'}(\tau)M \to M_{\mathcal{C}'} = M/\Phi_{\mathcal{C}}(\tau)M$ is the natural one and the map from $M_{\mathcal{C}} = M/\Phi_{\mathcal{C}}(\tau)M \to M_{\mathcal{C}\cup\mathcal{C}'}$ is given by multiplication with $\Phi_{\mathcal{C}'}$. Since everything in sight preserves direct summands and M is $\mathbb{Z}G$-projective it is enough to check exactness for $M = \mathbb{Z}G$ which is easy.

Again, there exists an extension to invertible G-modules

(6.9) : lemma Let M an invertible G-lattice, $\mathcal{C}$ a subset of $Div(m)$ and $d \in Div(m)$ such that $Div(d) \bigcap \mathcal{C} = \emptyset$. Then,

$$0 \to M^i_{\mathcal{C}} \to M^i_{\mathcal{C}\cup Div(d)} \to M^i_{Div(d)} \to 0$$

is an exact sequence of G-modules

proof : Similar.

Next, we introduce a delightful lunatic technical tool due to Lenstra :

(6.10) : Definition The Lenstra graph $\mathcal{L}_m$ of a natural number $m \in \mathbb{N}$ is the graph with vertices all partitions of $Div(m)$ (which is the set of all positive divisors of m) and there is an edge between the partitions P and P' if and only if there exists a $d \in Div(m)$ and a subset $D \in P$ such that $Div(d) \subset D$, $Div(d) \neq D$ and P' is the partition $P - \{D\} \cup \{Div(d), D - Div(d)\}$

Let us give a few examples of Lenstra graphs :

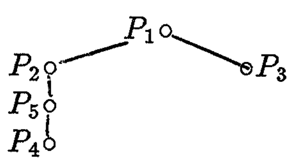
Take $m = p$ a prime number, then $Div(m) = \{1, p\}$. There are just two partitions $P_1 = \{\{1\}, \{p\}\}$ and $P_2 = \{\{1, p\}\}$. Take $d = 1$ i.e. $Div(d) = \{1\}$ then in P_2 we have $Div(1)$ is properly contained in $1, p$ and P_1 is the partition $\{Div(1), \{1, p\} - Div(1)\}$ i.e. $\mathcal{L}_p$ is

$$P_1 \circ - - - - - - - \circ P_2$$

Take $m = p^2$, then $Div(m) = \{1, p, p^2\}$ giving rise to the five partitions : $P_1 = \{\{1\}, \{p\}, \{p^2\}\}; P_2 = \{\{1, p\}, \{p^2\}\}; P_3 = \{\{1, p^2\}, \{p\}\}$; $P_4 = \{\{p, p^2\}, \{1\}\}$ and $P_5 = \{\{1, p, p^2\}\}$. For $d = 1$ we have $Div(d) = \{1\}$ and thus the useful elements D of the partition P with corresponding partition P' are

P	D	P'
P_2	$1, p$	P_1
P_3	$1, p^2$	P_1
P_5	$1, p, p^2$	P_4

and for $d = p$; $Div(d) = \{1, p\}$ giving only for P_5 a useful $D = \{1, p, p^2\}$ with corresponding partition $P' = P_2$. Hence the Lenstra grapg $\mathcal{L}_{p^2}$ has the following shape



For $m = p^3$ we have $Div(m) = \{1, p, p^2, p^3\}$ and the partitions : $P_1 = \{\{1\}, \{p\}, \{p^2\}, \{p^3\}\}; P_2 = \{\{1, p\}, \{p^2\}, \{p^3\}\}; P_3 = \{\{1, p\}, \{p^2, p^3\}\}$; $P_4 = \{\{1, p^2\}, \{p\}, \{p^3\}\}; P_5 = \{\{1, p^2\}, \{p, p^3\}\}; P_6 = \{\{1, p^3\}, \{p\}, \{p^2\}\}$; $P_7 = \{\{1, p^3\}, \{p, p^2\}\}; P_8 = \{\{p, p^2\}, \{1\}, \{p_3\}\}; P_9 = \{\{p, p^3\}, \{1\}, \{p^2\}\}$; $P_{10} = \{\{p^2, p^3\}, \{1\}, \{p\}\}; P_{11} = \{\{1, p, p^2\}, \{p^3\}\}; P_{12} = \{\{1, p, p^3\}, \{p^2\}\}$; $P_{13} = \{\{1, p^2, p^3\}, \{p\}\}; P_{14} = \{\{p, p^2, p^3\}, \{1\}\}; P_{15} = \{\{1, p, p^2, p^3\}\}$.

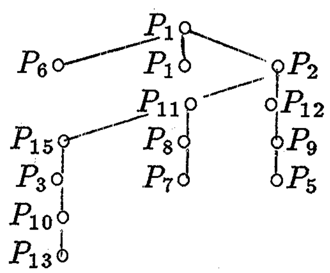
For $d = 1$ we have the following useful D in P with corresponding partition P'

P	D	P'
P_2	$1, p$	P_1
P_3	$1, p$	P_{10}
P_4	$1, p^2$	P_1
P_5	$1, p^2$	P_3
P_6	$1, p^3$	P_1
P_7	$1, p^3$	P_8
P_{11}	$1, p, p^2$	P_8
P_{12}	$1, p, p^3$	P_9
P_{13}	$1, p^2, p^3$	P_{10}
P_{15}	$1, p, p^2, p^3$	P_{14}

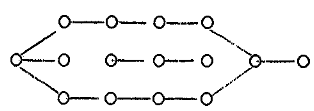
For $d = p$; we get the following information

P	D	P'
P_{11}	$1, p, p^2$	P_2
P_{12}	$1, p, p^3$	P_2
P_{15}	$1, p, p^2, p^3$	P_3

And finally for $d = p^2$ we have $P = P_{15}$ with $D = \{1, p, p^2, p^3\}$ and corresponding partition $P' = P_{11}$. This leads to the following picture of the Lenstra graph $\mathcal{L}_{p^3}$



As a rather pleasant exercise we suggest the reader to draw the Lenstra graph for $m = p.q$. The result should be agraph isomorphic to



Observe that in all these examples the Lenstra graph is connected. This is a general fact

(6.11) : Proposition (Lenstra 1974)

For any natural number $m \in \mathbb{N}$ we have that the Lenstra graph $\mathcal{L}_m$ is connected

proof : As in the above examples we call P_1 the trivial partition consisting of all singletons. In order to prove the result it is sufficient to show that for any partition P of $Div(m)$ there is a path from P to P_1.

We use induction on m and, for fixed m, we use induction on $n(P) = \#P_1 - \#P$. If $n(P) = 0$, then $P = P_1$ and the required path trivially exists. Now, assume $n(P) > 0$ and let $e \in Div(m)$ be minimal with respect to the existence of a class $D \in P$ such that $e \in D$ and $\#D > 1$. Clearly, $e < m$ for otherwise $P = P_1$ and $n(P) = 0$. Thus, we can apply our induction on m : there exists a path

$$G_0 = i : i \in Div(e) - -G_1 - -G_2 - -.. - -G_b = Div(e)$$

where all the G_i are partitions of $Div(e)$. We will now lift this path in $\mathcal{L}_e$ to one in $\mathcal{L}_m$: for all $0 \leq j \leq b$ let D_j be the unique class in G_j containing e. For $0 \leq j \leq 2b+1$ we will now define partitions of $Div(m)$ as follows : for $0 \leq j \leq b$ let

$$H_j = \{C \in P \mid C \cap Div(e) = \emptyset\} \cup \{D \cup D_j\} \cup \{G_j - \{D_j\}\}$$

and for $b+1 \leq j \leq 2b+1$ define

$$H_j = \{C \in P \mid C \cap Div(e) = \emptyset\} \cup \{D - \{e\}\} \cup \{G_{2b+1-j}\}$$

Then, one can verify fairly easily (for an example : see below) that the path

$$H_0 = P - -H_1 - -H_2 - -... - -H_{2b+1}$$

is a well defined path from P to H_{2b+1} in $\mathcal{L}_m$ and by construction

$$H_{2b+1} = \{D - \{e\}, \{e\}\} \cup P - \{D\}$$

whence $n(H_{2b+1}) = n(P) - 1$ and by induction on $n(P)$ we can find a path in $\mathcal{L}_m$ from H_{2b+1} to P_1, done.

Let us give an example of the construction : assume we want to construct a path from $P = \{\{p, p^2\}, \{1\}\}$ in $\mathcal{L}_{p^2}$ to P_1. Then, $e = p$, $D = \{p, p^2\}$ and we look at a path in $\mathcal{L}_p$ from $\{\{1\}, \{p\}\}$ to $\{\{1, p\}\}$. We have seen that there is one such path with $G_0 = \{\{1\},\{p\}\}$, $G_1 = \{\{1,p\}\}$. The corresponding D_j are $D_0 = \{p\}$, $D_1 = \{1,p\}$. Then, the constructed H_j for $0 \leq j \leq 3$ are

$$H_0 = \{\{1\},\{p,p^2\}\}; H_1 = \{\{1,p,p^2\}\}$$

$$H_2 = \{\{p^2\},\{1,p\}\}; H_3 = \{\{p^2\},\{1\},\{p\}\}$$

which is precisely the path $P_4 - -P_5 - -P_2 - -P_1$ obtained before.

The Lenstra graph will turn out to be of crucial importance in the proof of the next couple of results. Recall the definition of the $M_{\mathcal{C}}$ and $M^i_{\mathcal{C}}$ for anu G-module M and any subset $\mathcal{C} \subset Div(m)$. We can clearly extend these definitions to any partition P of $Div(m)$

$$M(P) = \oplus_{\mathcal{C}\in P} M_{\mathcal{C}} \text{ and } M^i(P) = \oplus_{\mathcal{C}\in P} M^i_{\mathcal{C}}$$

(6.12) : Proposition Let P and P' be partitions of $Div(m)$

(1) : If M is a projective $\mathbb{Z}G$-lattice, then $l(M(P))^G \simeq l(M(P'))^G$

(2) : If M is an invertible G-lattice, then $l(M^i(P))^G \simeq l(M^i(P'))^G$

proof : (1) : By the very existence of a path in $\mathcal{L}_m$ from P to P' we may as well assume that P and P' are adjacent. This means that there exists an element $d \in Div(m)$ and $D \in P$ such that $Div(d)$ is properly contained in D and

$$P' = \{Div(d), D - Div(d)\} \cup \{\mathcal{C} \mid \mathcal{C} \in P - \{D\}\}$$

By (6.8) there is an exact sequence of $\mathbb{Z}G$-lattices

$$0 \to M_{D-Div(d)} \to M_D \to M_{Div(d)} \to 0$$

and if we add to the first two terms of this sequence the module $N = \oplus_{\mathcal{C}\in P-D} M_{\mathcal{C}}$, then we get the exact sequence

$$0 \to N \oplus M_{D-Div(d)} \to M(P) \to M_{Div(d)} \to 0$$

By (6.6) we know that $M_{Div(d)}$ is an invertible G-lattice. Therefore, by (2.9) we have an isomorphism of l^G-algebras

$$l(M(P))^G \simeq l(N \oplus M_{D-Div(d)} \oplus M_{Div(d)})^G \simeq l(M(P'))^G$$

(2) is similar but using (6.7) and (6.9) instead.

This allows us to calculate the tori-invariants of any invertible C_m-lattice :

(6.13) : Theorem (Lenstra 1974) Let $G = C_m$ be the cyclic group of order m and let M be an invertible G-lattice. Then,

$$l(M)^G \simeq l(\oplus_{C_d} F_{C_d}(M))^G$$

as l^G-algebras, C_d ranges over the cyclic factorgroups of C_m

proof : Consider the following two 'trivial' partitions of $Div(m)$

$$P_1 = \{\{d\} : d \in Div(m)\} \text{ and } P_* = \{Div(m)\}$$

then we calculate

$$M^i(P_1) = \oplus_{d|m}(M/\Phi_d(\tau).M)/(torsion) = \oplus_{C_d} F_{C_d}(M)$$

$$M^i(P_*) = (M/(\tau^m - 1).M)/(torsion) = M$$

and the foregoing result finishes the proof.

Recall that the $F_{C_d}(M)$ are finitely generated torsion-free (whence projective) modules over the ring of integers $\mathbb{Z}[\zeta_d]$ in the d-th cyclotomic field $\mathbb{Q}(\zeta_d)$. Further, since any tree in the Lenstra forest contains an invertible lattice if $G = C_m$ a cyclic group, we have reduced the problem of stable equivalence classes of tori-invariants (over l^G) to that of tori-invariants of sums of invertible ideals in certain cyclotomic fields.

We now come to a major result which is miraculous in the sense that it gives an equivalence between rationality and stable rationality for a large class of examples. Note that in general there do exist stable rational non-rational fields.

(6.14) : Theorem (Lenstra 1974) Let $G = C_m$ be the cyclic group of order m and let M be an invertible C_m-lattice. Then, the following statements are equivalent

(1) : $l(M)^G$ is rational over l^G

(2) : $l(M)^G$ is stably rational over l^G

(3) : $[M]$ lies in the permutation tree in the Lenstra forest

(4) : For every cyclic factorgroup C_d of C_m we have that the $\mathbb{Z}[\zeta_d]$-module $F_{C_d}(M)$ is free

proof : (1) $\Rightarrow$ (2) : obvious

(2) and (3) are equivalent by (2.17)

(2) $\Rightarrow$ (4) : Since M is an invertible C_m-lattice we can find by (2.16) permutation lattices P_1 and P_2 such that

$$M \oplus P_1 \simeq P_2$$

By (6.5) this implies that $F_{C_d}(M) \oplus F_1 \cong F_2$ where F_1 and F_2 are free $\mathbb{Z}[\zeta_d]$-modules. This entails that $F_{C_d}(M)$ is free as a $\mathbb{Z}[\zeta_d]$-module

(4) $\Rightarrow$ (1) : We can find a permutation C_m-lattice

$$N = \oplus_{C_d} \mathbb{Z} C_d^{e_d}$$

such that $F_{C_d}(M) = F_{C_d}(N)$ for all cyclic factorgroups C_d of C_m (use (6.3) and freeness of all $F_{C_d}(M)$). But then,

$$\begin{array}{rcl} l(M)^G & \cong & l(\oplus_{C_d} F_{C_d}(M))^G \\ & & \| \\ l(N)^G & \cong & l(\oplus_{C_d} F_{C_d}(N))^G \end{array}$$

and finally, $l(N)^G$ is rational over l^G by (2.8),done.

An immediate consequence of this result is :

(6.15) : Corollary There are only finitely many trees in the Lenstra forest of a finite cyclic group C_m of order m.

proof : By finiteness of the classgroups of the ring of integers $\mathbb{Z}[\zeta_d]$ for all cyclic factorgroups C_d of C_m we can find a natural number n such that for a given invertible C_m-lattice M :

$$F_{C_d}(M^{\oplus n}) = F_{C_d}(M)^{\oplus n}$$

is a free $\mathbb{Z}[\zeta_d]$-module for all C_d. But then, by the foregoing result $[M]_c^{\oplus} = 0$ because it lies in the permutation tree. Since the trees are classified by $PCl(C_m)$ by (meta)cyclicity we are done.

The above mentioned rationality results can be extended to arbitrary finite Abelian groups if we put restrictions on the invertible lattices

(6.16) : Corollary Let G be a finite Abelian group and M a G-lattice of the form $M = \oplus_{G/H} M_{G/H}$ where each $M_{G/H}$ is an invertible G/H-lattice , the G/H ranging over the cyclic factorgroups of G. Then, there is an l^G-isomorphism

$$l(M)^G = l(\oplus_{C_d \in CF(G)} F_{C_d}(M))^G$$

proof : Let G/H be any cyclic factorgroup of G. We can apply (6.13) to G/H, the lattice $M_{G/H}$ and the field l^H and obtain

$$l^H(M_{G/H})^{G/H} \cong l^H(\oplus_{C_d} F_{C_d}(M_{G/H}))^{G/H}$$

where C_d ranges over the cyclic factorgroups of G/H and the isomorphism is one of $(l^H)^{G/H} = l^G$-algebras. Since H acts trivially on $M_{G/H}$ this isomorphism can be rewritten as

$$l(M_{G/H})^G \cong l(\oplus_{C_d} F_{C_d}(M_{G/H}))^G$$

Tensoring this isomorphism with l over l^G gives us a G-action preserving isomorphism (using Spoeiser's result)

$$l(M_{G/H}) \simeq l(\oplus_{C_d} F_{C_d}(M_{G/H}))$$

and combining all G/H-terms we obtain a G-action preserving isomorphism

$$l(M) \cong l(\oplus_{C_d \in CF(G)} F_{C_d}(M))$$

as l-algebras from which the result follows.

And similarly we have the following extension of (6.14) :

(6.17) : Theorem (Lenstra 1974)

Let G be a finite Abelian group and M a G-lattice of the form $M = \oplus_{G/H} M_{G/H}$ where each G/H is a cyclic factorgroup of G and $M_{G/H}$ is an invertible lattice. Then, the following statements are equivalent

(1) : $l(M)^G$ is rational over l^G

(2) : $l(M)^G$ is stably rational over l^G

(3) : $[M]$ belongs to the permutation tree in the Lenstra forest

(4) : For every cyclic factorgroup C_d of G we have that the $\mathbb{Z}[\zeta_d]$-module $F_{C_d}(M)$ is free

proof : Similar to (6.14).

This ends our excursion in the cyclic forests. It is perhaps surprising that the Lenstra forest provides us with some grip on all C_m-lattices even when the representation theory of them is wild e.g. if $m = p^3$. Anyway, let us now turn to the problem of classifying all finite forests :

From the theory of groups we recall the following characterization of metacyclic groups

(6.18) : Theorem The following are equivalent

(1) : G is a metacyclic group of order g

(2) : G is generated by two elements a and b with defining relations

$$a^m = 1, b^n = 1, b^{-1}.a.b = a^r$$

where the numbers m, n and r satisfy (a): $m.n = g$, (b): $r^n \equiv 1 \bmod m$ and (c): $((r-1).n, m) = 1$

proof : See e.g. M. Hall 'The theory of groups' theorem 9.4.3

The dihedral group of order $2.n$ is the group generated by two elements a and b with defining relations : $a^n = 1, b^2 = 1$ and $b.a = a^{-1}.b$.Further, the generalized quaternion group of order $4.n$ is the group generated by two elements a and b with defining relations : $a^{2n} = 1, b^2 = a^n$ and $b.a.b^{-1} = a^{-1}$. Using this group-lingo we can now state the characterization of all finite forests :

(6.19) : Theorem (Endo-Miyata 1973)

There are only finitely many trees in the Lenstra forest of G if and only if G is one of the following classes of groups :

(1) : G is a cyclic group

(2) : G is a dihedral group of order $2.p^r$ with p an odd prime

(3) : G is the direct product of a cyclic group of order q^s and a dihedral group of order $2.p^r$ where p and q are odd primes and p is a primitive $q^{s-1}(q-1)$-th root of unity modulo q^s

(4) : G is a generalized quaterniongroup of order $4p^r$ where p is an odd prime congruent to 3 modulo 4

This list of groups can be seen to be equivalent to the following :

G is either a cyclic group or a direct product of a cyclic group of order n and a group generated by two elements a and b and defining relations

$$a^{p^r} = b^{2^s} = 1; b^{-1}.a.b = a^{-1}$$

where p is an odd prime which is prime in $\mathbb{Z}[\zeta_{n.2^s}]$ and $(2p, n) = 1$

We will not give the proof of this result , here. Later on, we will compute the rank of the permutation classgroup of G in terms of grouptheoretical information. It is clear that the classification of all groups G such that the Lenstra forest contains only finitely many trees can then be recovered from the structure theory of metacyclic groups and the rank $= 0$ version of this computation.

7. THE LOCALS ARE QUITE HELPFUL

Before we can attack the Colliot-semigroup of an arbitrary finite group with a reasonable chance of success, we have to arm ourselves with some of the basic tools of integral representation theory. In particular, we want to know how much information is already contained in the local picture. In the case of integral grouprings 'local' always has a twofold meaning : local with respect to the group means that we restrict to Sylow p-subgroups of G whereas local with respect to the ring means that we localize $\mathbb{Z}$. We have already seen a group-local-global result (6.1) saying that invertibility can be checked by restricting to the Sylow subgroups. One of our first aims in this section is to look at the ring-local behaviour of invertible lattices. First we have to learn some of the lingo :

For a fixed group G we denote by $\pi(G)$ the set of primes dividing the order $\#G$. For any G-lattice M we will denote by $M_{\pi(G)} = \cup_{p\in\pi(G)} M_p$. An important,though fairly coarse invariant of a G-lattice is its genus

(7.1) : Definition : Two G-lattices M and N are said to lie in the same genus , which fact we will denote by $M \vee N$, if and only if $M_{\pi(G)} \cong N_{\pi(G)}$

This is not the usual definition of genera, but it is equivalent to it as the next result shows

(7.2) : lemma : Let M and N be two G-lattices,equivalent are

(1) : M and N lie in the same genus

(2) : $M_p \cong N_p$ for all $p \mid \#G$

(3) : $M_p \cong N_p$ for all primes p

proof : see e.g. Gruenberg's 'Relation modules for finite groups' Cor. 4.12 p.22

Working over $\mathbb{Z}_{\pi(G)}$ (rather than over $\mathbb{Z}$) has some advantedges. For example we can use the following cancellation result

(7.3) : Proposition : Let π be a finite set of prime numbers and let L,M and N be $\mathbb{Z}_\pi G$-modules which are $\mathbb{Z}$-torsionfree. Then, if $L \oplus M \cong L \oplus N$ then $M \cong N$

proof : see e.g. Gruenberg ,theorem 4.4 p.19

Important consequences of the Jordan-Zassenhaus theorem are

(7.4) : Theorem : (1) : There are only finitely many isomorphism classes in each genus

(2) : Let M and N be G-lattices, then $M \vee N$ if and only if there is some $n \in \mathbb{N}$ such that $M^{\oplus n} \cong N^{\oplus n}$

proof : see Swan-Evans, theorem 6.11 p.114.

Note that the same result holds if we replace $\mathbb{Z}$ by any intermediate ring $\mathbb{Z} \subset R \subset \mathbb{Q}$. Moreover, if R is semi-local we can take $n = 1$.

Most of our definitions on G-lattices have extensions to the case of RG-modules. Here R will be $\mathbb{Z}_\pi$,$\mathbb{Z}_p$,$\hat{\mathbb{Z}}_p$,$\mathbb{Q}$ or $\hat{\mathbb{Q}}_p$ where $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers i.e. the completion of $\mathbb{Z}_p$ with respect to the p-adic topology and $\hat{\mathbb{Q}}_p$ is its field of fractions :

(7.5) : Definitions :

(1) : An RG-lattice M is an RG-module which is free as an R-module

(2) : A permutation RG-lattice P is isomorphic to $\oplus_i RG/H_i$ for some subgroups H_i of G

(3) : An invertible RG-lattice is an RG-direct summand of a permutation RG-lattice

(4) : With $Latt(R,G)$ we will denote the set of isoclasses of RG-lattices

(5) : With $Sansuc(R,G)$ we denote the equivalence classes of $Latt(R,G)$ for the relation $M \sim_R N$ iff $M \oplus P_1 \cong N \oplus P_2$ for permutation RG-lattices P_i. The direct sum turns $Sansuc(R,G)$ into an Abelian semigroup

(6) : With $PCl(R,G)$ we denote the group of invertible elements in the semigroup $Sansuc(R,G)$. It is easy to see that these are precisely the classes of invertible RG-lattices

The next lemma , due to Andreas Dress , shows that invertibility also allows a ring-local-global criterium :

(7.6) : lemma A G-lattice M is invertible if and only if M_p is an invertible $\mathbb{Z}_pG$-lattice

proof : One direction is trivial. Conversely, let

$$M_p \lhd P(p) = \oplus_{i=1}^{j_0} \mathbb{Z}_p G/H_i^{(p)}$$

then, for some $(s,p) = 1$ we have

$$M_{MC(s)} \lhd P(p)_{MC(s)} = \oplus \mathbb{Z}_{MC(s)} G/H_i^{(p)}$$

where $MC(s)$ is the multiplicatively closed set $\{1, s, s^2, ...\}$. Then $s = q_1^{e_1}...q_k^{e_k}$ and for each q_j we can repeat the foregoing argument and find an s_j s.t. $(s_j, q_j) = 1$ and

$$M_{MC(s_j)} \lhd P(q_j)_{MC(s_j)} = \oplus_{i=1}^{j_j} \mathbb{Z}_{MC(s_j)} G/H_i^{(q_j)}$$

But then $\mathbb{Z} = \mathbb{Z}s + \sum \mathbb{Z}s_j$ from which we deduce a splitting

$$M \lhd \oplus_{j=0}^{k} \oplus_{i=1}^{j_j} \mathbb{Z}G/H_i^{(q_j)}$$

done.

The following result is well-known, see e.g. Reiner 'Maximal Orders' :

(7.7) : Proposition

(1) : A $\mathbb{Z}_pG$-lattice M is isomorphic to (resp. a direct summand of) a $\mathbb{Z}_pG$-lattice N if and only if the same holds for $\hat{\mathbb{Z}}_p \otimes M$ and $\hat{\mathbb{Z}}_p \otimes N$

(2) : A $\hat{\mathbb{Z}}_pG$-lattice M is of the form $\hat{\mathbb{Z}}_p \otimes N$ for some $\mathbb{Z}_pG$-lattice N if and only if the $\hat{\mathbb{Q}}_pG$-lattice $\mathbb{Q} \otimes M$ is of the form $\hat{\mathbb{Q}}_p \otimes V$ for some $\mathbb{Q}G$-lattice V

Therefore, we have the pullback-diagram of semi-groups (with the direct sum as composition law)

$$\begin{array}{ccc} Latt(\mathbb{Z}_p, G) & \hookrightarrow & Latt(\hat{\mathbb{Z}}_p, G) \\ \downarrow & & \downarrow \\ Latt(\mathbb{Q}, G) & \hookrightarrow & Latt(\hat{\mathbb{Q}}_p, G) \end{array}$$

and because permutation-lattices are defined over any coefficient ring, we have also a pullback diagram of semigroups

$$\begin{array}{ccc} Sansuc(\mathbb{Z}_p, G) & \hookrightarrow & Sansuc(\hat{\mathbb{Z}}_p, G) \\ \downarrow & & \downarrow \\ Sansuc(\mathbb{Q}, G) & \hookrightarrow & Sansuc(\hat{\mathbb{Q}}_p, G) \end{array}$$

which entails in particular a pullback diagram of the groups of invertible elements

$$\begin{array}{ccc} PCl(\mathbb{Z}_p, G) & \hookrightarrow & PCl(\hat{\mathbb{Z}}_p, G) \\ \downarrow & & \downarrow \\ PCl(\mathbb{Q}, G) & \hookrightarrow & PCl(\hat{\mathbb{Q}}_p, G) \end{array}$$

The point is that this reduces the study of $\mathbb{Z}_pG$-lattices to character theory (over $\mathbb{Q}$ and $\hat{\mathbb{Q}}_p$) and modular representation theory (over $\hat{\mathbb{Z}}_p$). As we move along , we will recall some of the (many) results in these areas.For example :

(7.7) : Green's indecomposability theorem : Let G be a p-group and H any subgroup, then $\hat{\mathbb{Z}}_pG/H$ is an indecomposable $\hat{\mathbb{Z}}_pG$-lattice

proof : This is a special case of Green-correspondence, see e.g. Benson 'Modular Representation Theory' thm 2.12.2.

An immediate consequence of the above mentioned results is :

(7.8) : Theorem (Esther Beneish 1988) Let G be a p-group. Then, any invertible G-lattice lies in the same genus as a permutation lattice

proof : By (7.2) it suffices to prove that I_p is a permutation $\mathbb{Z}_pG$-lattice for each invertible G-lattice I. Clearly, $\hat{\mathbb{Z}}_p \otimes I_p$ is a direct summand of a permutation $\hat{\mathbb{Z}}_pG$-lattice $\oplus_i \hat{\mathbb{Z}}_pG/H_i$. But by (7.7) any of the terms is indecomposable. By the Krull-Schmidt theorem we obtain that $\hat{\mathbb{Z}}_p \otimes I_p$ must be the direct sum of some of these terms,so $\hat{\mathbb{Z}}_p \otimes I_p$ is a permutation $\hat{\mathbb{Z}}_pG$-lattice, say

$$\hat{\mathbb{Z}}_p \otimes I \cong \oplus_j \hat{\mathbb{Z}}_pG/H_j$$

But then by (7.6.1) $I_p \cong \oplus \mathbb{Z}_pG/H_j$, done.

And, by the group-local-global principle for invertible lattices we can generalize this result to any group G :

(7.9) : Theorem (Esther Beneish 1988) A G-lattice I is invertible if and only if $Res_{G_p}(I)$ lies in the same genus as a permutation G_p-lattice for every Sylow-subgroup G_p of G

proof : Trivial from (6.1) and (7.8).

Clearly, these results already imply that a ring-local-global principle fails for permutation lattices : just take an invertible not-permutation lattice for a cyclic group C_p (e.g. $p = 23$). There are some other interesting direct consequences for lattice invariants which are not mentioned in her paper. They depend upon the following result :

(7.10) : Theorem (Roiter's replacement theorem)

(1) : If M lies in the same genus as N and F is any faithful$G$-lattice, then there exists a G-lattice F' in the same genus as F such that $M \oplus F \cong N \oplus F'$

(2) : If M lies in the same genus as N, then there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \to M \to N \to U \to 0$$

where $U = U_1 \oplus ... \oplus U_r$, each U_i is a simple $\mathbb{F}_{p_i}G$-module and $p_1, ..., p_r$ are distinct primes and prime to the order of G

proof : See e.g. Gruenberg th.5.9 and 5.15

(7.11) : Definition We say that a field K is a k-rational factor of degree m of a field L if there exists an affine k-algebra R with field of fractions K and an affine k-algebra S with field of fractions a rational fieldextension of degree m of L such that R is a retract of S i.e. we have a triangle

$$\begin{tikzcd} & S \arrow[dr] & \\ R \arrow[ur] \arrow[rr, phantom, "="] & & R \end{tikzcd}$$

(7.12) : Proposition Let I be a faithful invertible lattice of a p-group G, then

(1) : the lattice invariants $k(I)^G$ are a k-rational factor of degree the $\mathbb{Z}$-rank of I of any lattice invariants $k(F)^G$ where F is a faithful G-lattice

(2) : the lattice invariants $k(I)^G$ are stably equivalent to a Noether setting $k(H)^H$ where H is afinite group having G as a p-Sylow subgroup provided k contains enough roots of unity

proof : (1) : I lies in the same genus as a permutation lattice P. Then, by Roiter's replacement result we can find a G-lattice F' such that

$$I \oplus F' \cong P \oplus F$$

Now, $k[I]$ is clearly a retract of $k[I \oplus F']$ preserving the G-action,thus $k[I]^G$ is a retract of $k[I \oplus F']^G = k[P \oplus F]^G$. By faithfulness of I, P and F we have that $k(I)^G$ is a rational factor of $k(P \oplus F)^G$ which is a rational extension of $k(F)^G$,done.

(2) : Clear from (7.10.2) and (1.7)

Having given a nice local picture of invertible lattice, our next aim is to study the permutation classgroup (or at least its rank) using local data. We will follow here Andreas Dress' paper closely. To begin with, we have the following consequence of the Jordan-Zassenhaus theorem

(7.13) : Proposition The permutation classgroup $PCl(G)$ is finitely generated as Abelian group

proof : $G_0(\mathbb{Z}G)$ is the Abelian group with generators the isoclasses of f.g. $\mathbb{Z}G$-lattices and relations $[M] = [M'] + [M"]$ whenever we have a G-exact sequence $0 \to M' \to M \to M" \to 0$. By the splitting property of invertible lattices we see that $PCl(G)$ is a subquotient of $G_0(\mathbb{Z}G)$. By Swan-Evans theorem 3.8 we known that $G_0(\mathbb{Z}G)$ is finitely generated, hence so is $PCl(G)$

We will often need the following grouptheoretical result :

(7.14) : Theorem (Andreas Dress 1971)

Let G be a finite group and $\mathcal{H}$ a family of subgroups of G which is subconjugately closed, i.e. if H_1 and H_2 are subgroups of G such that $g.H_1.g^{-1} \subset H_2$ then if $H_2 \in \mathcal{H}$,so is H_1. Further, let $\chi : \mathcal{H} \to \mathbb{Z}$ be a map such that $\chi(g.H.g^{-1}) = \chi(H)$ for all $g \in G$ and all $H \in \mathcal{H}$.

Then, there exist two G-sets S and T such that :

(1) : $\#(G).\chi(H) + \#(S^H) = \#(T^H)$ for all $H \in \mathcal{H}$

(2) : $(S \cup T)^H =$ for all subgroups H not contained in $\mathcal{H}$

proof : See Dress' lecture notes on representation theory of groups at the University of Bielefeld.

A first and important consequence of this result is the following extension of Artin's induction theorem (the case when $R = \mathbb{Q}$) and Swan's theorem on projective $\mathbb{Z}G$-lattices (the case when $R = \mathbb{Z}$). Recall that a permutation lattice RG/H is a projective RG-lattice iff $\#H$ is a unit in R.

(7.15) : Theorem (Andreas Dress 1975)

Let $\mathbb{Z} \subset R \subset \mathbb{Q}$. Then, for any finitely generated projective RG-lattice M there exist permutation lattices $P_1 = RS$, $P_2 = RT$ where S and T are G-sets such that

(1) : for some $n \in \mathbb{N}$: $M^{\oplus n} \oplus P_1 \cong P_2$

(2) : for a subgroup H of G : $(S \cup T)^H \neq$ then H is cyclic of order a unit in R

Moreover, if R is semilocal then n can be chosen to be $\#G$

proof : Let χ_M be the rational character afforded by the $\mathbb{Q}\,G$-module $\mathbb{Q} \otimes M$. Since $\chi_M(g)$ is an algebraic integer lying in $\mathbb{Q}$ we have $\chi_M(g) \in \mathbb{Z}$ for all $g \in G$. Further, by [Curtis-Reiner,Exercise 15.3,p.399] we have $\chi_M(g) = \chi_M(h)$ if the cyclic group $< g >$ is conjugated to $< h >$.

Now, define $\mathcal{H}$ to be the set of all cyclic subgroups of G with order a unit in R, then $\mathcal{H}$ is clearly subconjugately closed. Moreover, by the exercise mentioned above, all requirements of (7.14) are satisfied. Thus, we can find G-sets S and T such that

(1) : $\#G.\chi_M(g) + \#(S^{<g>}) = \#(T^{<g>})$ for all $g \in G$ with order a unit in R

(2) : $(S \cup T)^H =$ for all subgroups H of G which are either non-cyclic or have non-invertible order in R

By [CR,Th. 32.15,p.679] we have that $\chi_M(g) = 0$ for all g whose order is not-invertible in R. Therefore, we have for all $g \in G$ the equality

$$\#G.\chi_M(g) + \#S^{<g>} = \#T^{<g>}$$

But then the permutation lattices RS and RT are both of the form

$$\oplus RG/ < g_i >$$

where $g_i \in G$ with invertible order in R, i.e. RS and RT are projective RG-modules. But then by the above equality, RT and $N = M^{\oplus \#G} \oplus RS$ are two projective RG-lattices having the same rational characte, i.e. $\mathbb{Q} \otimes RT \cong \mathbb{Q} \otimes N$ and therefore RT and N lie in the same genus by [CR,Th.32.1 p 671] i.e.

$$R_m T \cong R_m \otimes N$$

for every maximal ideal m of R. Since there is at most one isomorphism class in the genus of a semi-local ring (a consequence of [CR,31.6,p 645]) we are done in the semi-local case. In the general case we know by an R-analoque of (7.4.2) that there is a natural number $n \in \mathbb{N}$ s.t.

$$M^{\oplus n \# G} \oplus RS^{\oplus n} \cong RT^{\oplus n}$$

finishing the proof.

In the special case $R = \mathbb{Q}$ any $\mathbb{Q}G$-lattice is clearly projective entailing by the theorem that $PCl(\mathbb{Q}, G)$ is finite and each element has order a divisor of $\#G$. This also follows from the following refinement of

(7.16) : Artin's Induction Theorem : Each rational character χ of G can be expressed in the form

$$\chi = \sum a_C \chi_{G/C}$$

where the sum is taken over all cyclic subgroups C of G and the coefficients a_C are given by the formula

$$a_C = \frac{1}{\#G/C} \sum_{C < C'} \mu(\#C'/C)\chi(z)$$

where C' ranges over all cyclic subgroups containing C , z is agenerator of C' and μ is the Möbius function

proof : Curtis-Reiner 15.4 p.378

Thus, we have full control over $PCl(\mathbb{Q}, G)$. Next, we will try to connect the study of the permutation classgroup to the local classgroups, i.e. we want to study the natural morphism

$$\text{loc} \ : PCl(G) \to \prod_{p | \#G} PCl(\mathbb{Z}_p, G)$$

sending a class $[M]_c$ to the product of classes $[\mathbb{Z}_p \otimes M]_c$.

In this study we will have to know when two permutation $\mathbb{Z}_pG$-lattices are isomorphic. If G is a p-group this is a triviality by Krull-Schmidt and Green's indecomposability result. But, if G is arbitrary this question is not that easy and has some connections with number theory.

Let us give a quick example of non-uniqueness of decomposition : suppose G is a group containing an Abelian subgroup $H = \mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. Let $H_1, ..., H_{q+1}$ denote the subgroups of order q. Then one has an isomorphism of $\mathbb{C}\, H$-lattices

$$\mathbb{C}\, H \oplus \mathbb{C}^{\oplus q} \cong \mathbb{C}\, H/H_1 \oplus ... \mathbb{C}\, H/H_{q+1}$$

which can be verified by decomposing both sides into linear factors. By representation theory we can replace $\mathbb{C}$ in the above isomorphism by $\hat{\mathbb{Z}}_p$ for any prime $p \neq q$ (and hence by descent by $\mathbb{Z}_p$) and obtain the isomorphism

$$\mathbb{Z}_p H \oplus \mathbb{Z}_p^{\oplus q} \cong \mathbb{Z}_p H/H_1 \oplus ... \oplus \mathbb{Z}_p H/H_{q+1}$$

Now tensor this with $\mathbb{Z}_p G$ over $\mathbb{Z}_p H$ and obtain

$$\mathbb{Z}_p G \oplus \mathbb{Z}_p(G/H)^{\oplus q} \cong \mathbb{Z}_p G/H_1 \oplus ... \oplus \mathbb{Z}_p G/H_{q+1}$$

Giving an aboundacy of examples of non-unique decomposition of permutation $\mathbb{Z}_p G$-lattices. As an example of the usefulness of knowing isoclasses of permutation lattices we mention here

(7.17) : Proposition (Roggenkamp-Scott 1980)

Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group G and p a fixed prime number. Let $H_1, ..., H_n$ be subgroups of G. If we have an isomorphism of permutation $\mathbb{Z}_p G$-lattices

$$\oplus_{i=1}^n \mathbb{Z}_p G/H_i^{\oplus a_i} \cong \oplus_{i=1}^n \mathbb{Z}_p G/H_i^{\oplus b_i}$$

for some $a_i, b_i \in \mathbb{N}$. Then, we have an isomorphism

$$\oplus_{i=1}^n Cl(K^{H_i})_p^{\oplus a_i} \cong \oplus_{i=1}^n Cl(K^{H_i})_p^{\oplus b_i}$$

where $Cl(K^H)_p$ denotes the p-torsion part of the classgroup of the ring of integers in the fixed field K^H

proof : See LNM 882 p.256

Hence, knowing the isoclasses of permutation $\mathbb{Z}_p G$-lattices for all primes p gives us relations between the classgroups of intermediate fields. In view of this application it is perhaps surprising that we do have a fairly precise result.

Recall that a group H is said to be p-**hypoelementary** if $H/O_p(H)$ is a cyclic group where $O_p(H)$ is the largest normal p-subgroup of H. With $Hyp_p(G)$ we denote the set of all p-hypoelementary subgroups of G.

(7.18) : Theorem (Andreas Dress 1972)
Two G-sets S and T give rise to isomorphic permutation $\mathbb{Z}_pG$-lattices, i.e. $\mathbb{Z}_pS \cong \mathbb{Z}_pT$ if and only if $\#S^H = \#T^H$ for all $H \in Hyp_p(G)$

proof : See LNM 342,p 224-234

As an application of this result we will now show that the morphism loc defined above recognizes torsion elements

(7.19) : lemma Let M be a G-lattice. If $M_p = \mathbb{Z}_p \otimes M$ defines a torsion class in $PCl(\mathbb{Z}_p, G)$ for all primes $p \mid \#G$, then M defines a torsion class in $PCl(G)$

proof : Replacing M by $M^{\oplus m}$ for a suitable $m \in \mathbb{N}$ we may assume that for each prime $p \mid \#G$ we have two G-sets S_p and T_p such that

$$M_p \oplus \mathbb{Z}_pS_p \cong \mathbb{Z}_pT_p$$

Taking the rational character χ_M of $\mathbb{Q} \otimes M$ on both sides we obtain for all elements $g \in G$

$$\chi_M(g) + \#S_p^{<g>} = \#T_p^{<g>}$$

Now, define $\mathcal{H} = \cup_{p|\#G} Hyp_p(G)$ and a map $\chi : \mathcal{H} \to \mathbb{Z}$ by

$$\chi(M) = \#T_p^H - \#S_p^H$$

if $H \in Hyp_p(G)$. Of course, we have to show that this map is well defined i.e. does not depend upon the choice of p. So, let $H \in Hyp_p(G) \bigcap Hyp_q(G)$, then H is readily seen to be cyclic, i.e. $H =< g >$ but then

$$\#T_p^H - \#S_p^H = \chi_M(g) = \#T_q^H - \#S_q^H$$

Further one checks that $\chi(H) = \chi(g.H.g^{-1})$ for all $g \in G$ and $H \in \mathcal{H}$ and that $\mathcal{H}$ is subconjugately closed. Then, we can apply (7.14) and obtain two G-sets S and T such that

$$\#G.\chi(H) + \#S^H = \#T^H$$

for all $H \in \mathcal{H}$ or, replacing the definition of $\chi(H)$

$$\#G.\#T_p^H + \#S^H = \#G.\#S_p^H + \#T^H$$

for all $H \in Hyp_p(G)$. Then, by (7.17) we obtain that

$$(\bullet) : \mathbb{Z}_p S \cup \cup_{i=1}^{\#G} T_p \cong \mathbb{Z}_p T \cup \cup_{i=1}^{\#G} S_p$$

On the other hand, from $M_p \oplus \mathbb{Z}_p S_p \cong \mathbb{Z}_p T_p$ we derive

$$(\bullet\bullet) : M_p^{\oplus \#G} \oplus \mathbb{Z}_p S \cup T \cup \cup_{i=1}^{\#G} S_p \cong \mathbb{Z}_p S \cup T \cup \cup_{i=1}^{\#G} T_p$$

Combining $(\bullet)$ and $(\bullet\bullet)$ and using the semi-local cancellation result (7.3) we find

$$M_p^{\oplus \#G} \oplus \mathbb{Z}_p S \cong \mathbb{Z}_p T$$

which is valid for all primes $p \mid \#G$, i.e. $M^{\oplus \#G} \oplus \mathbb{Z}S$ lies in the same genus as $\mathbb{Z}T$, but then by (7.4.2) we can find an integer $n \in \mathbb{N}$ such that

$$M^{\oplus n.\#G} \oplus \mathbb{Z}S^{\oplus n} \cong \mathbb{Z}T^{\oplus n}$$

That is, M defines a torsion class in $PCl(G)$,done.

We can now relate the rank of the permutation classgroup to the ranks of the local classgroups

(7.20) : Theorem (Andreas Dress 1975)

For any finite G we have an isomorphism

$$\mathbb{Q} \otimes PCl(G) \cong \prod_{p \mid \#G} \mathbb{Q} \otimes PCl(\mathbb{Z}_p, G)$$

or,alternatively

$$rkPCl(G) = \sum_{p \mid \#G} rkPCl(\mathbb{Z}_p, G)$$

proof : By the foregoing result we know that $[M]_c$ is torsion in $PCl(G)$ iff $[M_p]_c$ is torsion in $PCl(\mathbb{Z}_p, G)$ or in other words

$$\mathbb{Q} \otimes PCl(G) \to \prod_{p \mid \#G} \mathbb{Q} \otimes PCl(\mathbb{Z}_p, G)$$

is a monomorphism. In order to prove that it is epimorphic, let us take for any $p \mid \#G$ an invertible Z_pG-lattice M_p. Tensoring with $\mathbb{Q}$ and using (7.15) or (7.16) we can find G-sets S_p and T_p such that

$$(\mathbb{Q} \otimes M_p)^{\oplus \#G} \oplus \mathbb{Q} S_p \cong \mathbb{Q} T_p$$

We can now define for every $p \mid \#G$

$$N_p = M_p^{\oplus \#G} \oplus \mathbb{Z}_p S_p \oplus (\oplus_{q \mid \#G, q \neq p} \mathbb{Z}_p T_q)$$

That is : $[N_p]_c = \#G.[M_p]_c$ in $Pcl(\mathbb{Z}_p, G)$. Now define the G-set

$$T = \cup_{p \mid \#G} T_p$$

then by definition of N_p we have that

$$\mathbb{Q} \otimes N_p \cong \mathbb{Q} T$$

for every prime p dividing the order of G. But then we can find a $\mathbb{Z}G$-lattice N such that

$$\mathbb{Z}_p \otimes N \cong N_p \text{ for } p \mid \#G$$

$$\mathbb{Z}_p \otimes N \cong \mathbb{Z}_p T \text{ for } (p, \#G) = 1$$

For, we can identify $\mathbb{Q} \otimes N_p$ with $\mathbb{Q} T$ for all p dividing $\#G$ and consider the intersection

$$N = (\cap_{p \mid \#G} N_p) \cap \mathbb{Z}[\frac{1}{\#G}]T$$

The claim is then easily verified because localization commutes with finite intersections. By lemma (7.6) we know that N is an invertible G-lattice and under the natural map loc $[N]_c$ is mapped to $\prod_{p \mid \#G}(\#G.[M_p]_c)$. But then $\frac{1}{\#G} \otimes [N]_c \in \mathbb{Q} \otimes PCl(G)$ is mapped to $\prod_{p \mid \#G}[M_p]_c$ proving surjectivity, done.

An immediate consequence of this result is

(7.21) : Proposition (Hendrik Lenstra 1974) If G is a p-group, then the permutation classgroup is finite

proof : By (7.20) $rkPCl(G) = rkPCl(\mathbb{Z}_p, G) = 0$ because any invertible $\mathbb{Z}_pG$-lattice is a permutation $\mathbb{Z}_pG$-lattice. Then, using (7.13) $PCl(G)$ is finite

Having reduced the computation of the rank of the permutation classgroup to those of the local classgroups, we now want to compute their ranks in terms of those of $PCl(\hat{\mathbb{Z}}_p, G)$ and $PCl(\hat{\mathbb{Q}}_p, G)$. By Artin-induction one can often restrict to induction from cyclic subgroups and then the follwowing result comes in handy

(7.22) : lemma Let G be acyclic group and V an irreducible $\hat{\mathbb{Q}}_pG$-lattice. Then, there exist invertible $\hat{\mathbb{Z}}_pG$-lattices I_1 and I_2 such that $\hat{\mathbb{Q}}_p \otimes I_1 \oplus V \cong \hat{\mathbb{Q}}_p \otimes I_2$

proof : Let τ be a generator of G of order $p^n.m$ where $(p,m) = 1$. Let V be an irreducible $\hat{\mathbb{Q}}_pG$-lattice, then there exists a $p^n.m$-th root of unity ζ such that $V = \hat{\mathbb{Q}}_p(\zeta)$. However, ζ does not need to be primitive. But, $\hat{\mathbb{Q}}_p(\zeta) = \hat{\mathbb{Q}}_p(\zeta_1) \otimes \hat{\mathbb{Q}}_p(\zeta_2)$ where ζ_1 is a p^n-th root of unity and ζ_2 is an m-th root of unity, $\zeta = \zeta_1.\zeta_2$ and τ^i acts on an element $x_j \in \hat{\mathbb{Q}}_p(\zeta_j)$ by $\tau^i.x_j = \zeta_j^i x_j$.

We claim that the result holds for V if it holds for $\hat{\mathbb{Q}}_p(\zeta_1)$ and $\hat{\mathbb{Q}}_p(\zeta_2)$. That is, assume that $\hat{\mathbb{Q}}_p(\zeta_1) \oplus \hat{\mathbb{Q}}_p \otimes J_1 \cong \hat{\mathbb{Q}}_p \otimes J_2$ and that $\hat{\mathbb{Q}}_p(\zeta_2) \oplus \hat{\mathbb{Q}}_p \otimes J_1' \cong \hat{\mathbb{Q}}_p \otimes J_2'$ for invertible $\hat{\mathbb{Z}}_pG$-lattices J_i, J_i'. But then

$$\hat{\mathbb{Q}}_p(\zeta_1) \otimes \hat{\mathbb{Q}}_p(\zeta_2) \oplus \hat{\mathbb{Q}}_p \otimes (J_1 \otimes J_2' \oplus J_1' \otimes J_2) \cong \hat{\mathbb{Q}}_p \otimes (J_1 \otimes J_1' \oplus J_2 \otimes J_2')$$

whence $\hat{\mathbb{Q}}_p(\zeta) \oplus \hat{\mathbb{Q}}_p \otimes I_1 \cong \hat{\mathbb{Q}}_p \otimes I_2$ where I_i are clearly invertible $\hat{\mathbb{Z}}_pG$-lattices (tensor-products of invertibles are invertible). Thus, we may assume that $V = \hat{\mathbb{Q}}_p(\zeta)$ for ζ either a p^n-th or an m-th root of unity. We may also assume that V is faithful i.e. we may restrict to the case where G is a p-group or has order prime to p. In the second case, any $\hat{\mathbb{Z}}_pG$-lattice is projective whence invertible, so choosing a $\hat{\mathbb{Z}}_pG$-lattice $M \subset V$ gives $\hat{\mathbb{Q}}_p \otimes M \cong V$ by irreducibility and we are done. In the first case, we have for a p^n-th root of unity ζ that $[\hat{\mathbb{Q}}_p(\zeta) : \hat{\mathbb{Q}}_p] = [\mathbb{Q}(\zeta) : \mathbb{Q}]$ and $tr_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta) = -1$ if $n = 1$ and 0 otherwise. But then

$$V \oplus \hat{\mathbb{Q}}_pG/G_0 \cong \hat{\mathbb{Q}}_pG$$

where G_0 is a subgroup of order p and we are done.

(7.23) : Theorem (Andreas Dress 1975)

There is an exact sequence of Abelian groups

$$0 \to \mathbb{Q} \otimes PCl(\mathbb{Z}_p, G) \to \mathbb{Q} \otimes PCl(\hat{\mathbb{Z}}_p, G) \to \mathbb{Q} \otimes PCl(\hat{\mathbb{Q}}_p, G) \to 0$$

or in other words

$$rkPCl(\mathbb{Z}_p) = rkPCl(\hat{\mathbb{Z}}_p, G) - rkPCl(\hat{\mathbb{Q}}_p, G)$$

proof : The starting point is of course the pullback diagram

$$\begin{array}{ccc} PCl(\mathbb{Z}_p, G) & \hookrightarrow & PCl(\hat{\mathbb{Z}}_p, G) \\ \downarrow & & \downarrow \\ PCl(\mathbb{Q}, G) & \hookrightarrow & PCl(\hat{\mathbb{Q}}_p, G) \end{array}$$

By the Artin induction theorem, $PCl(\mathbb{Q}, G)$ is a finite group whence $\mathbb{Q} \otimes PCl(\mathbb{Q}, G) = 0$. This implies exactness of the sequence

$$0 \to \mathbb{Q} \otimes PCl(\mathbb{Z}_p, G) \to \mathbb{Q} \otimes PCl(\hat{\mathbb{Z}}_p, G) \to \mathbb{Q} \otimes PCl(\hat{\mathbb{Q}}_p, G)$$

Since any permutation $\mathbb{Q}G$ (hence $\hat{\mathbb{Q}}_pG$) character is $\mathbb{Q}$-generated by those of cyclic subgroups of G by the Artin-induction theorem, we see that any direct factor or even more generally any element of $\mathbb{Q} \otimes PCl(\hat{\mathbb{Q}}_p, G)$ can be written as the sum of elements of the form

$$r \otimes (\hat{\mathbb{Q}}_pG \otimes_{\hat{\mathbb{Q}}_pC} V)$$

for some $r \in \mathbb{Q}$, C a cyclic subgroup of G and V a $\hat{\mathbb{Q}}_pC$-lattice. The surjectivity of the map $\mathbb{Q} \otimes PCl(\hat{\mathbb{Z}}_p, G) \to \mathbb{Q} \otimes PCl(\hat{\mathbb{Q}}_p, G)$ now follows from (7.22),done.

8. THE HYPO-OBSTRUCTION :

In this section we will conclude our computation of the size of the permutation classgroup $PCl(G)$ of an arbitrary finite group. It will turn out that non-finiteness is caused by nontriviality of certain quotients of the automorphismgroups of the cyclic tops of hypoelementary subgroups of G.

Combining the theorems (7.20) and (7.23) gives us

$$rkPCl(G) = \sum_{p|\#G} rkPCl(\hat{\mathbb{Z}}_p, G) - rkPCl(\hat{\mathbb{Q}}_p, G)$$

We will start the computation of the right hand side by connecting the permutation classgroups to certain representation rings. In the sequel let R denote either $\hat{\mathbb{Z}}_p$ or $\hat{\mathbb{Q}}_p$ (or $\mathbb{Q}$ for that mather). Since we have the Krull-Schmidt result for RG-lattices, we can define $A_R(G)$ to be the free Abelian group generated by the isoclasses of indecomposable RG-lattices (of course, the addition is given by the direct sum).Actually, the tensorproduct defines a ringstructure on $A_R(G)$ (see e.g. Benson's Modular Representation Theory LNM 1081 2.2) but we do not really need this here. Similarly, we define $A_R(G, Perm)$ (resp. $A_R(G, Triv)$) for the subgroups (actually,subrings) of $A_R(G)$ consisting of the permutation (resp. invertible) RG-lattices. Note that at this point we give in to the modular tradition of calling invertible lattices trivial source modules (see later). Then, we clearly have an exact sequence of Abelian groups

$$0 \to A_R(G, Perm) \to A_R(G, Triv) \to PCl(R, G) \to 0$$

which tells us that the rank of $PCl(G)$ is equal to

$$\sum_{p|\#G} (rkA_{\hat{\mathbb{Z}}_p}(G, Triv) - rkA_{\hat{\mathbb{Z}}_p}(G, Perm) + rkA_{\hat{\mathbb{Q}}_p}(G, Triv) - rkA_{\hat{\mathbb{Q}}_p}(G, Perm))$$

and in order to calculate all these terms explicitly we will mobilize the full power of modular representation theory.

Let us begin with the easier $\hat{\mathbb{Q}}_p$-terms. Because any $\hat{\mathbb{Q}}_p G$-lattice is projective and hence invertible we have that $A_{\hat{\mathbb{Q}}_p}(G, Triv) = A_{\hat{\mathbb{Q}}_p}(G) = K_0(\hat{\mathbb{Q}}_p G)$. Here, $K_0(RG)$ is the Grothendieck group (actually ring) of finitely generated projective RG-lattices modulo exact sequences, i.e. equipped with the direct sum as addition rule. In case R is a field K we can compute the rank of $K_0(KG)$ by means of Berman-Witt theory, which we will now briefly recall.

An element $g \in G$ is said to be **K-regular** if its order is a unit in K. Two K-regular elements $g, h \in G$ are said to be **K-conjugate** iff there is an $x \in G$ such that $x.g.x^{-1} = h^i$ where i is apositive integer satisfying the following property : let ζ be a primitive $\# < g >$-th root of unity in an algebraic closure of K, then there must be a K-automorphism of $K(\zeta)$ sending ζ to ζ^i. So, in particular we have that i and $\# < g >$ are coprime and that $\# < g >= \# < h >$. Using this lingo we have

(8.1) : Berman-Witt theorem The rank of the Grothendieck ring $K_0(KG)$, or equivalently, the number of isoclasses of simple KG-lattices, is equal to the number of K-conjugacy classes of K-regular elements

proof : For characteristic zero see Curtis-Reiner Th. 21.5 p.494 and for characteristic p Th. 21.25 p.508

We can rephrase this result in the following way. Let $\mathcal{C}$ be the set of all cyclic subgroups $C =< g >$ of G where g is K-regular and define for each C

$$A_G(C) = N_G(C)/C_G(C)$$

where $N_G(C)$ (resp. $C_G(C)$) denotes the normalizer (resp. centralizer) of C in G. Then, we can view $A_G(C)$ as a subgroup of the automorphismgroup $Aut(C)$ of C, i.e. the group of units of the ring $\mathbb{Z}/\#C\mathbb{Z}$. We can identify C with the multiplicative group of all $\#C$-th roots of unity in an algebraic closure $K^\sim$ of K. Let $B_K(C)$ be the subgroup of $Aut(C)$ consisting of those automorphisms induced by K-automorphisms of $K^\sim$. The Berman-Witt theorem can now be restated as

$$rkK_0(KG) = \sum_{C\in\mathcal{C}}^{\diamond}(Aut(C) : A_G(C)B_K(C))$$

where $\sum_{C\in\mathcal{C}}^{\diamond}$ means that the sum is taken over different conjugacy classes of C's in $\mathcal{C}$. Thus, we get our first explicit result

(8.2) : Proposition Let $\mathcal{C}$ be the set of all cyclic subgroups in G, then

$$(1) : rkA_{\hat{\mathbb{Q}}_p}(G, Triv) = \sum_{C \in \mathcal{C}}^{\diamond} (Aut(C) : A_G(C)B_K(C))$$

$$(2) : rkA_{\hat{\mathbb{Q}}_p}(G, Perm) = \sum_{C \in \mathcal{C}}^{\diamond} 1$$

$$(3) : rkPCl(\hat{\mathbb{Q}}_p, G) = \sum_{C \in \mathcal{C}}^{\diamond} [(Aut(C) : A_G(C)B_K(C)) - 1]$$

proof : (1) : follows from the above reformulation of the Berman-Witt theorem.

(2) : By descent $rkA_{\hat{\mathbb{Q}}_p}(G, Perm) = rkA_{\mathbb{Q}}(G, Perm)$ which is by the Artin-induction theorem the number of conjugacy classes of cyclic subgroups in G. See also C-R Exercise 15.4 p. 399

(3) : follows from (1) and (2).

Next, let us consider the $\hat{\mathbb{Z}}_p$-terms.By descent and (7.18) it is easy to see that

$$rkA_{\hat{\mathbb{Z}}_p}(G, Perm) = \sum_{H \in Hyp_p(G)}^{\diamond} 1$$

This also follows from Conlon's theorem, see e.g. Benson Th.2.13.6 p.67.

Now, we turn to the more difficult task of computing the rank of $a_{\hat{\mathbb{Z}}_p}(G, Triv)$ or, equivalently, the number of isoclasses of indecomposable invertible $\hat{\mathbb{Z}}_pG$-lattices.

We say that a subgroup D of G is a **vertex** of an indecomposable $\hat{\mathbb{Z}}_pG$-lattice M if M is a direct summand of $Ind_H^G \circ Res_H^G(M)$ and this does not hold for any proper subgroup of D. A **source** of M is then an indecomposable $\hat{\mathbb{Z}}_pD$-lattice N such that M is a direct summand of $Ind_H^G(N)$.

For any subgroup H of G, $g \in N_G(H)$ and N a $\hat{\mathbb{Z}}_pH$-lattice, we denote by N^g the lattice N with H-action induced via the automorphism on $\hat{\mathbb{Z}}_pH$ induced by conjugation with g. We have :

(8.3) : Proposition Let M be an indecomposable $\hat{\mathbb{Z}}_pG$-lattice,then

(1) : the vertices of M are p-subgroups of G which are all conjugated

(2) : let D be a vertex of M and N_1, N_2 two $\hat{\mathbb{Z}}_pD$-lattices which are sources of M, then there is a $g \in N_G(D)$ such that $N_1 \cong N_2^g$

(3) : an indecomposable $\hat{\mathbb{Z}}_pG$-lattice M is invertible iff for each vertex D of M the source of M is the trivial $\hat{\mathbb{Z}}_pD$-lattice $\hat{\mathbb{Z}}_p$

proof : See Benson Prop. 2.5.1 p.38 for (1) and (2) and Prop. 2.6.1 p.39 for (3). Remark that this also explains the terminology 'trivial source modules'.

Let $\mathcal{P}$ be the set of p-subgroups of G and for any $D \in \mathcal{P}$ let $c_p(G,D)$ denote the number of isoclasses of indecomposable $\hat{\mathbb{Z}}_pG$-lattices with vertex D and trivial source $\hat{\mathbb{Z}}_p$. Then, from the above results we get the reduction

$$rkA_{\hat{\mathbb{Z}}_p}(G,Triv) = \sum_{D\in\mathcal{P}}^{\diamond} c_p(G,D)$$

Now, we are in a position to apply the crucial

(8.4) Green-correspondence theorem There is a one-to-one correspondence between indecomposable $\hat{\mathbb{Z}}_pG$-lattices with vertex D and indecomposable $\hat{\mathbb{Z}}_pN_G(H)$-lattices with vertex H. Under this correspondence invertibles are mapped to invertibles.

proof : See Benson Th.2.12.2 p.61

Hence, for any $D \in \mathcal{P}$ we have $c_p(G,D) = c_p(N_G(D),D)$ i.e. we have to count the isoclasses of indecomposable $\hat{\mathbb{Z}}_pN_G(D)$-lattices with vertex D and source $\hat{\mathbb{Z}}_p$. In other words, we have to count isoclasses of direct factors of the groupring $\hat{\mathbb{Z}}_pN_G(D)/D$, i.e. the indecomposable projective $\hat{\mathbb{Z}}_pN_G(D)/D$-lattices which by the theory of projective covers (see C-R Th.18.2 p.430) correspond one-to-one to the isoclasses of simple $\mathbb{F}_pN_G(D)/D$-lattices. Rephrasing

$$c_p(G,D) = rkK_0(\mathbb{F}_pN_G(D)/D)$$

which we can calculate again by the Berman-Witt theory explained before. Therefore

$$(\bullet) : rkK_0(\mathbb{F}_pN_G(D)/D) = \sum_{\overline{H}\in\overline{\mathcal{C}_p}}^{\diamond} (Aut(\overline{H}) : A_{N_G(D)/D}(\overline{H})B_{\mathbb{F}_p}(\overline{H}))$$

where $\overline{\mathcal{C}_p}$ is the set of all cyclic subgroups $\overline{H} =< g >$ of $N_G(D)/D$ of order prime to p. Of course, we prefer to lift this information to the G-level.

The $N_G(D)/D$-conjugacy class of $\overline{H}$ lifts uniquely to a G-conjugacy class of a p-hypoelementary subgroup H of G having $D = O_p(H)$ as a normal p-Sylow subgroup (i.e. the cyclic top $H/O_p(H)$ has order prime to p).

For any $H \in Hyp_p(G)$ we denote by $A^p_G(H)$ the subgroup of the automorphism group of the cyclic top $Aut(H/O_p(H))$ consisting of those automorphisms induced by conjugation with elements of $N_G(H) \subset N_G(O_p(H))$. Further, with $B^p(H)$ we denote $B_{\mathbb{F}_p}(H/O_p(H))$ as before. But then, we can reformulate ($\bullet$) as

$$rkK_0(\mathbb{F}_pN_G(D)/D) = \sum_{H \in Hyp_p(G), O_p(H)=D}^{\diamond} (Aut(H/O_p(H)) : A^p_G(H)B^p(H))$$

and adding up all relevant terms we obtain

(8.5) : Proposition

$$(1) : rkA_{\hat{\mathbb{Z}}_p}(G, Perm) = \sum_{H \in Hyp_p(G)}^{\diamond} 1$$

$$(2) : rkA_{\hat{\mathbb{Z}}_p}(G, Triv) = \sum_{H \in Hyp_p(G)}^{\diamond} (Aut(H/O_p(H)) : A^p_G(H)B^p(H))$$

$$(3) : rkPcl(\hat{\mathbb{Z}}_p, G) = \sum_{H \in Hyp_p(G)}^{\diamond} [(Aut(H/O_p(H)) : A^p(H)B^p(H)) - 1]$$

One extra bit of information is needed, namely : how are the terms in (8.2) and (8.5) related in case H happens to be a cyclic subgroup of G ? In this case, consider the canonical epimorphism

$$Aut(H) \mapsto Aut(H/O_p(H))$$

then $A_G(H)$ is mapped onto $A^p_G(H)$ and $B_{\hat{\mathbb{Q}}_p}(H)$ onto $B_{\hat{\mathbb{Q}}_p}(H/O_p(H)) = B^p(H)$. The kernel of this epimorphism is $Aut(O_p(H)) = B_{\hat{\mathbb{Q}}_p}(O_p(H))$ which is contained in $B_{\hat{\mathbb{Q}}_p}(H)$. These remarks prove

$$(Aut(H) : A_G(H)B_{\hat{\mathbb{Q}}_p}(H)) = (Aut(H/O_p(H) : A^p_G(H)B^p(H))$$

for all cyclic subgroups H of G. A combination of all the results so far proves our promised calculation of the rank of the permutation classgroup

(8.6) : Theorem (Andreas Dress 1975)

$$(1) : rkPCl(\mathbb{Z}_pG) = \sum_{H \in Hyp_p(G)-\mathcal{C}}^{\diamond} [(Aut(H/O_p(H)) : A_G^p(H)B^p(H)) - 1]$$

$$(2) : rkPCl(G) = \sum_{p|\#G} \sum_{H \in Hyp_p(G)-\mathcal{C}}^{\diamond} [(Aut(H/O_p(H)) : A_G^p(H)B^p(H)) - 1]$$

This result is rather powerfull. For example, we get immediatly Lenstra's result that the permutation classgroup of a cyclic group is finite. Moreover, we can easily characterize the Abelian groups having finite permutation classgroup

(8.7) : Corollary Let G be a finite Abelian group. Then $PCl(G)$ is infinite if and only if there is a prime p such that G_p is not cyclic and there is a cyclic subgroup C of G of order n prime to p such that $Aut(\mathbb{Z}/n\mathbb{Z})$ is not generated by the multiplication map with p mod n

proof : If the rank of $PCl(G)$ is not zero we have a $p \mid \#G$ and a non-cyclic p-hypoelementary subgroup H such that $Aut(H/O_p(H))$ is not generated by $A_G^p(H)B^p(H)$. Recall that $A_G^p(H)$ consisted of autos induced by conjugation so $A_G^p(H) = 1$. Further,$B^p(H) = B_{\mathbb{F}_p}(H/O_p(H))$ which was the subgroup of automorphisms induced by $\mathbb{F}_p$-automorphisms of the algebraic closure $\overline{\mathbb{F}_p}$ which are known to be powers of the Frobenius morphism, i.e. $B^p(H)$ is the subgroup of $Aut(H/O_p(H))$ generated by the multiplication map by p mod n. Taking C to be the cyclic subgroup generated by a lift of the generator of the cyclic group $H/O_p(H)$ the result follows.

For example, if $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ then $PCl(G)$ is infinite giving examples of invertible G-lattices which do not lie in the same genus as a permutation G-lattice.Also, it is easy to produce metacyclic groups having infinite PCl. For example, take the non-Abelian group of order 21 which contains a normal 7-Sylow subgroup. Then the only non-cyclic hypoelementary subgroup is the group itself i.e. $G/O_7(G) = \mathbb{Z}/3\mathbb{Z}.B^7(G) = 1$ since 7 mod 3 is 1 and $A_G^7(G)$ is also trivial. Therefore the rank of the permutation classgroup of the non-Abelian group of order 21 is 1.

We leave it as an exercise to you that the Endo-Miyata classification of the finite forests follows from the Dress result and the classification of metacyclic groups.

9. THE SALTMAN FOREST :

Now that we understand the permutation classgroup, we conclude our study of the Colliot-semigroup $Colliot(G)$ (and hence of the number of trees in the Lenstra forest) by investigating the torsion free Abelian semigroup

$$\Delta(G) = Colliot(G)/PCl(G)$$

It is easy to see that elements of $\Delta(G)$ classify the trees in a coarser forest

(9.1) : The Saltman forest :
In the picture of all isoclasses of G-lattices (classified vertically according to their $\mathbb{Z}$ rank) we draw an edge between the classes $[M]$ and $[N]$ if and only if there exists an exact sequence of G-lattices

$$0 \to M \to N \to I \to 0$$

with I an invertible G-lattice.

Again, this defines an equivalence relation on the isoclasses of G-lattices which is transitive :

(9.2) : lemma Let L, M and N be G-lattices such that there exist exact sequences of G-lattices $0 \to L \to M \to I_1 \to 0$ and $0 \to M \to N \to I_2 \to 0$ where the I_i are invertible G-lattices. Then, there is an exact sequence of G-lattices

$$0 \to L \to N \to I_1 \oplus I_2 \to 0$$

proof : We have the following exact commutative diagram of G-lattices

$$\begin{array}{ccccccccc}
0 & \to & L & \to & L & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & N & \to & I_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I_1 & \to & R & \to & I_2 & \to & 0
\end{array}$$

Which gives us the exact sequence of G-lattices

$$0 \to I_1 \to R \to I_2 \to 0$$

and since I_2 in invertible and I_1 coflasque this sequence splits by (2.5) giving $R = I_1 \oplus I_2$

(9.3) : Proposition There is a one-to-one correspondence between

(1) : trees in the Saltman forest

(2) : elements of the torsion-free Abelian semigroup $\Delta(G)$

proof : Suppose we have an exact sequence of G-lattices

$$0 \to M \to N \to I \to 0$$

with I an invertible G-lattice, then by (5.13.1) we obtain

$$\phi(M) = \phi(N) + [I]_c$$

i.e. under the composite map $\sigma : Latt(G) \to Sansuc(G) \to Colliot(G) \to \Delta(G)$ the classes $[M]_c$ and $[N]_c$ are mapped to the same element. This shows that $\sigma(M) = \sigma(M') \in \Delta(G)$ whenever $[M]$ and $[N]$ belong to the same tree in the Saltman forest.

Conversely, assume $\sigma(M) = \sigma(N)$ then by definition

$$\phi(M) + [I_1]_c = \phi(N) + [I_2]_c \in Colliot(G)$$

where I_i are invertible G-lattices.But then we have invertible G-lattices J_i such that $I_i \oplus J_i = P_i$ for some permutation lattice P_i. From the essential uniqueness of flasque resolutions we see that ϕ is additive and therefore :

$$\phi(M \oplus J_1) = \phi(M) + \phi(J_1) = \phi(M) + [I_1]_c = \phi(N) + [I_2]_c = \phi(N \oplus J_2)$$

Then, by (5.15) $M \oplus J_1$ and $N \oplus J_2$ belong to the same tree in the Lenstra forest (and hence a fortiori in the Saltman forest). Finally, the J_i being invertible $[M]$ and $[N]$ belong to the same tree in the Saltman forest.

For this reason we will call $\Delta(G)$ the Saltman-semigroup of G. Having defined σ on G-lattices, we will now define σ on maps between G-lattices, the idea being that σ is 'allmost' a functor.

(9.4) : Definition Let M, N be two G-lattices and $f, f' : M \to N$ two $\mathbb{Z}G$-morphisms. We say that f and f' are equivalent and denote $f \sim f'$ iff $f - f'$ factors through a permutation lattice, i.e.

$$\begin{tikzcd} M \arrow[rr, "f-f'"] \arrow[dr, "g"'] & & N \\ & P \arrow[ur, "h"'] & \end{tikzcd}$$

where P is a permutation lattice

The next lemma is a triviality :

(9.5) : lemma Let $g : K \to M$, $f, f' : M \to N$ and $h : N \to L$ be G-maps between G-lattices. Then,

(1) : If $f \sim 0$ then $f \circ g \sim 0$ and $h \circ f \sim 0$

(2) : if $f \sim 0$ and $f' \sim 0$ then $f + f' \sim 0$ and $-f \sim 0$

(3) : $1_M \sim 0$ if and only if M is invertible

proof : For (1) it suffices to look at the diagram

$$\begin{tikzcd} K \arrow[r] & M \arrow[rr] \arrow[dr] & & N \arrow[r] & L \\ & & P \arrow[ur] & & \end{tikzcd}$$

For (2) suppose that f (resp. f') factors through the poermutation lattice P_f resp $P_{f'}$. Then, $f + f'$ clearly factors through $P_f \oplus P_{f'}$ in the obvious way

(3) : If the identity factors through a permutation lattice, then M is a direct summand of this permutation lattice,done.

Using this equivalence relation, we can define a funny category $Flas(G)$ whose objects are the flasque G-lattices and with morphisms the equivalence classes of G-morphisms. Our goal is to define a 'quasi-functor'

$$\sigma : Latt(G) \to Flas(G)$$

We will need the following

(9.6) : lemma Let F be a flasque lattice and I an invertible lattice. Let $i_F : F \to F \oplus I$ and $p_F : F \oplus I \to F$ be the natural injection and projection. Then, in the category $Flas(G)$ $[i_F]$ is an isomorphism with inverse $[p_F]$.

proof : Consider the composite

$$i_F \circ p_F : F \oplus I \to F \oplus I$$

which induces the identity on the F-component and the zero map on the I-component.Then, $1_{F\oplus I} - i_F \circ p_F$ induces the zero map on F and the identity on I. Therefore, it factors through I via p_I and i_I. I being invertible it also factors through a permutation lattice yielding that $[i_F \circ p_F] = [i_F] \circ [p_F] = [1_{F\oplus I}]$,done.

Now, we will define $\sigma(f)$ where $f : M \to N$ is a G-map between two G-lattices.Consider flasque resolutions of M and N . Then, by (5.10.2) we can complete the diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & P_M & \to & F_M & \to & 0 \\ & & \downarrow & & & & & & \\ 0 & \to & N & \to & P_N & \to & F_N & \to & 0 \end{array}$$

with a map $F : P_M \to P_N$ which induces a map $g : F_M \to F_N$. Because $\sigma(M) = F_M$ (upto isomorphism in $Flas(G)$ by the foregoing lemma) and $\sigma(N) = F_N$ (upto iso in $Flas(G)$), it is natural to define $\sigma(f)$ to be the equivalence class of the map $g : F_M \to F_N$. Of course, we have to investigate how $\sigma(f)$ depends upon (1) the particular choice of the extension F of f and (2) the choice of the flasque resolution.

(1) : Suppose we have two extensions $F_1, F_2 : P_M \to P_N$ of f, then since $F_1 - F_2 \mid M = 0$ it induces a morphism

$$h : F_M \to P_N$$

such that if we compose h with the epimorphism $P_N \to F_N$ we obtain $g_1 - g_2$. Therefore, $g_1 - g_2$ factors through a permutation lattice P_N whence $[g_1] = [g_2]$ in $Flas(G)$.

(2) : Suppose we have chosen other flasque resolutions of M and N say $0 \to M \to P'_M \to F'_M \to 0$ and $0 \to N \to P'_N \to F'_N \to 0$. Then, we can form the pushout-diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & P_M & \to & F_M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & P'_M & \to & P_M \times^M P'_M & \to & F_M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F'_M & \to & F'_M & \to & 0 & & \end{array}$$

then we obtain from (2.5) two split exact sequences leading to

$$P_M \times^M P'_M \cong P_M \oplus F'_M \cong P'_M \oplus F_M$$

And, replacing the roles of M by N we similarly obtain

$$P_N \times^N P'_N \cong P_N \oplus F'_N \cong P'_N \oplus F_N$$

The extensions of f are denoted by $F : P_M \to P_N$ and $g : F_M \to F_N$ (resp. $F' : P'_M \to P'_N$ and $g' : F'_M \to F'_N$).Now, adding P'_M to the last two tems of the first flasque resolution of M (resp. P'_N to that of N) and defining $H = F \oplus F' : P_M \oplus P'_M \to P_N \oplus P'_N$ we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & P_M \oplus P'_M & \to & P_M \times^M P'_M & \to & 0 \\ & & \downarrow f & & \downarrow H & & \downarrow h & & \\ 0 & \to & N & \to & P_N \oplus P'_N & \to & P_N \times^N P'_N & \to & 0 \end{array}$$

Now we remember the decompositions of $P_M \times^M P'_M$ and $P_N \times^N P'_N$ and obtain the commutative diagram

$$\begin{array}{ccccc} F_M & \leftarrow & P_M \times^M P'_M & \to & F'_M \\ \downarrow g & & \downarrow h & & \downarrow g' \\ F_N & \leftarrow & P_N \times^N P'_N & \to & F'_N \end{array}$$

where the horizontal maps are p_{F_M} and $p_{F'_M}$ (resp. p_{F_N} and $p_{F'_N}$) which are all isomorphisms in $Flas(G)$. Concluding we see that $[g]$ and $[g']$ are determined upto automorphisms in $Flas(G)$ explaining the 'quasi-functorial' behaviour of σ. We do not care too much about this auto-dependence since we will be primarely interested in the question whether a map $f : M \to N$ satisfies $\sigma(f) = [0]$.

(9.9) : lemma Let M be a G-lattice, equivalent are

(1) : $\sigma(M) = 0 \in \Delta(G)$

(2) : $\sigma(1_M) = [0]$

proof : (2) $\Rightarrow$ (1) : Take a flasque resolution of M

$$0 \to M \to P_M \to F_M \to 0$$

then,clearly $\sigma(1_M) = [1_{F_M}]$ which is [0] by assumption,i.e. F_M is a direct factor of a permutation lattice i.e. an invertible lattice. Then, $\sigma(M) = [F_M]_c = 0 \in \Delta(G)$

$(1) \Rightarrow (2)$: By assumption, we have a flasque resolution of M

$$0 \to M \to P_M \to I_M \to 0$$

where I_M is an invertible lattice. But then, $\sigma(1_M) = [1_{I_M}] = [0]$ since I is adirect factor of a permutation lattice.

(9.8) : Theorem (Saltman 1984)

Let $f : M \to N$ be a G-morphism. Then, $\sigma(f) = [0]$ if and only if there exist an exact commutative diagram

$$\begin{array}{ccccccccc} & & M & \to & P_1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N & \to & N' & \to & P_2 & \to & 0 \end{array}$$

where P_1 and P_2 are permutation lattices. Or, in forest-lingo : $[f]$ becomes eventually $[0]$ if we move up in the tree of $[N]$ in the Lenstra forest

proof : $\Leftarrow$: Suppose we have such an exact commutative diagram. Take flasque resolutions of M and N'

$$0 \to M \to P_M \to F_M \to 0$$

$$0 \to N' \to P_{N'} \to F_{N'} \to 0$$

Then we can form the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & N' & \to & P_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P_{N'} & \rightleftharpoons & P_{N'} & \to & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & H_N & \to & F_{N'} & \to & 0 & & \end{array}$$

giving (by the snake-lemma) an exact G-sequence

$$0 \to P_2 \to H_N \to F_{N'} \to 0$$

which splits by $(2.5)^*$, i.e. $H_N = P_2 \oplus F_{N'}$. Taking the leftmost column in the above diagram we get the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & P_M & \to & F_M & \to & 0 \\ & & \downarrow f & & \downarrow F' & & \downarrow g & & \\ 0 & \to & N & \to & P_{N'} & \to & P_2 \oplus F_{N'} & \to & 0 \\ & & \downarrow i & & \| & & \downarrow p_{F_n} & & \\ 0 & \to & N' & \to & P_{N'} & \to & F_{N'} & \to & 0 \end{array}$$

Denoting $f' = i \circ f$ and $g' = p_{F_N} \circ g$ we have by definition of σ that $\sigma(f) = [g]$ upto iso in $Flas(G)$ and $\sigma(f') = [g']$ upto iso in $Flas(G)$. The last column in the above diagram yields

$$[g] = [0] \text{ iff } [g'] = [0]$$

(use that p_{F_N} is an iso in $Flas(G)$). Using the quasi-functorial properties of σ we have that the triangle

$$\begin{array}{ccccc} & & M & & \\ & f' \swarrow & & \searrow & \\ N' & & \leftarrow & & P_1 \end{array}$$

maps to the tringle

$$\begin{array}{ccccc} & & \sigma(M) & & \\ & \sigma(f') \swarrow & & \searrow & \\ \sigma(N') & & \leftarrow & & 0 \end{array}$$

entailing that $\sigma(f') = [g'] = [0]$ whence $\sigma(f) = [g] = [0]$,done.

$\Rightarrow$: Conversely, suppose $\sigma(f) = [0]$ that is the exact diagram obtained by taking flasque resolutions of M and N

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_M & \longrightarrow & F_M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow F & & \downarrow g & & \\ 0 & \longrightarrow & N & \longrightarrow & P_N & \longrightarrow & F_N & \longrightarrow & 0 \end{array}$$

has the property that g factors through a permutation lattice P. But then we can form the pullback diagram (the colums are not exact !)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P_M & \longrightarrow & F_M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow F & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & P_N \times_{F_N} P & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & P_N & \longrightarrow & F_N & \longrightarrow & 0 \end{array}$$

which proves the result by taking $P_M = P_1$,$N' = P_N \times_{F_N} P$ and $P_2 = P$,done.

Our main motivation to study the Lenstra forest is that different trees correspond to different stable equivalence classes of tori-invariants $l(M)^G$ over l^G,see (2.18). Therefore, it is only natural to ask whether the Saltman forest also corresponds to some (weaker) rationality property. Before giving the formal definition of retract rationality we will motivate the concept by investigating the connection between tori-invariants $l(M)^G$ and $l(N)^G$ if $[M]$ and $[N]$ lie in the same tree in the Saltman forest. So, by iteration we may assume that we have an exact G-sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow I \longrightarrow 0$$

where I is an invertible G-lattice. By (2.9) we have an isomorphism of l^G-algebras

$$l(N)^G \cong l(M \oplus I)^G$$

Howver, if I is an invertible lattice not contained in the permutation tree (they exist e.g. for C_p if the ideal classgroup of the p-th cyclotomic field is not trivial) then $l(M)^G$ and $l(M \oplus I)^G$ are not stable equivalent over l^G.But, I being a direct factor of a permutation lattice P we have a commutative G-diagram

$$\begin{tikzcd} & M \oplus P \arrow[dr] & \\ M \oplus I \arrow[ur] \arrow[rr, "1_{M\oplus I}"] & & M \oplus I \end{tikzcd}$$

This induces a triangle of l^G-algebras

$$\begin{tikzcd} & l[M \oplus P]^G \arrow[dr] & \\ l[M \oplus I]^G \arrow[ur] \arrow[rr] & & l[M \oplus I]^G \end{tikzcd}$$

and $l(M \oplus P)^G$ is rational over $l(M)^G$ by (2.8), say $l(M \oplus P)^G \cong l(M)^G(x_1, ..., x_n)$. By (2.14) we can find an element $a \in l[M \oplus P]^G$ and $b \in l[M]^G[x_1, ..., x_n]$ such that $l[M \oplus P]^G[\frac{1}{a}] = l[M]^G[x_1, ..., x_n][\frac{1}{b}]$ giving rise to the triangle

$$\begin{tikzcd} & l[M][x_1, ..., x_n][\frac{1}{b}] \arrow[dr] & \\ l[M \oplus I]^G \arrow[ur] \arrow[rr] & & l[M \oplus I]^G[\frac{1}{b'}] \end{tikzcd}$$

which motivates the following

(9.9) : Definition (Saltman 1980) A field extension $K \subset L$ is said to be retract rational if there exists an affine K-algebra R with field of fractions L and nonzero elements $f \in K[x_1, ..., x_n]$ and $r \in R$ such that there is a triangle

$$\begin{tikzcd} & K[x_1, ..., x_n][\frac{1}{f}] \arrow[dr] & \\ R \arrow[ur] \arrow[rr, hook] & & R[\frac{1}{r}] \end{tikzcd}$$

Clearly, we first have to verify that this property depends only upon the field-extension and not on the particulat choice of the affine subalgebra :

(9.10) : lemma : If $K \subset L$ is retract rational and S is any affine K-algebra with field of fractions L, then there exist elements $g \in K[x_1, ..., x_n]$ and $s \in S$ such that we have a diagram

$$\begin{tikzcd} & K[X_1, ..., x_n][\frac{1}{f}] \arrow[dr] & \\ S \arrow[ur] \arrow[rr, hook] & & S[\frac{1}{s}] \end{tikzcd}$$

proof : Easy : use definition of retract rationality and Swan's lemma.

Of course one can extend the notion of retract rationality for other maps than the identity morphism. We say that a map factors rationally if there is such a triangle with top rational. We have the following characterization of such maps

(9.11) : Theorem (Saltman 1984) Let $f : M \to N$ be a G-map. Then, the induced map $\phi : l(M)^G \to l(N)^G$ factors rationally over l^G if and only if $\sigma(f) = [0]$

proof : $\Rightarrow$: Assume that ϕ factors rationally, i.e. we have a triangle

$$\begin{tikzcd} & l^G[x_1, ..., x_n][\frac{1}{f}] \arrow[dr, "\delta"] & \\ l[M]^G \arrow[ur, "\psi"] \arrow[rr] & & l[N]^G[\frac{1}{s}] \end{tikzcd}$$

which we can tensor up with l and applying Speiser gives us a triangle

$$\begin{tikzcd} & l[x_1, ..., x_n][\frac{1}{f}] \arrow[dr, "\delta'"] & \\ l[M] \arrow[ur, "\psi'"] \arrow[rr] & & l[N][\frac{1}{s}] \end{tikzcd}$$

Now, we can redo the classical tricks to obtain a sequence

$$0 \to N \to N' = l(N)[\frac{1}{s}]^*/l^* \to P_2 \to 0$$

where P_2 is a permutation lattice (coming from a factorozation of s) and $l[x_1, ..., x_n][\frac{1}{f}]^*/l^* = P_1$ is also a permutation lattice. That is we have the diagram

$$\begin{array}{ccccccccc} & & M & \to & P_1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & N & \to & N' & \to & P_2 & \to & 0 \end{array}$$

which entails $\sigma(f) = [0]$

$\Leftarrow$: Then, by (9.8) we have a diagram as above which gives rise to the triangle

$$\begin{array}{ccccc} & & l[P_1]^G & & \\ & \nearrow & & \searrow & \\ l[M]^G & & \rightarrow & & l[N']^G \end{array}$$

Here, the top field of fractions is rational over l^G and the right bottom field is rational over $l(N)^G$ and then we are done by the following lemma, the proof of which we give give cadeau.

(9.12) : lemma (a): Let K and L be two F fields which are stable equivalent over F. Then if L/F is retract rational so is K/F

(b) : Let S, T and T' be affine F-algebras where $T \subset T'$ and the fields of fractions give a rational field extension. If $\phi : S \rightarrow T$ os an F-algebra map such that the composition $S \rightarrow T \hookrightarrow T'$ factors rationallt, then so does ϕ

So, the tori-invariants $l(M)^G$ are retract rational over l^G if and only if $[M]$ belongs to the permutation tree in the Saltman forest. In the next trimester we will have a closer look at retract rationality for tori- and lattice invariants.

10. MORE FLASQUE YOGA :

In this last section we collect some material related to flasque and coflasque resolutions which we found only after completing the foresters guide. Further, we will give a brief intro to the G-jungle which is obtained from the Lenstra forest by colourfully adjoining the dual forest.

In (5.11) we gave an analoque to the Schanuel lemma for flasque and coflasque resolutions. Benson (LNM 1081 lemma 1.4.2.1) found a generalization of Schanuels lemma which has also a flasque analoque :

(10.1) : lemma (1) : Consider the exact G-sequences

$$0 \to Q \to P \to M \to 0$$

$$0 \to Q' \to N \to M \to 0$$

where Q and Q' are coflasque lattices, P in an invertible lattice (for example the upper sequence can be a flasque resolution of M) and the epi $N \to M$ factors through a permutation lattice. Then,

$$Q' \oplus P \cong N \oplus Q$$

(2) : Consider the exact G-sequences

$$0 \to M \to P \to F \to 0$$

$$0 \to M \to N \to F' \to 0$$

where F and F' are flasque lattices, P is invertible and $M \to N$ factors through a permutation lattice. Then,

$$F' \oplus P \cong N \oplus F$$

proof : (1): Consider the pullback diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Q \arrow[d] \arrow[r, equal] & Q \arrow[d] & \\
0 \arrow[r] & Q' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "P'"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q' \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

where $\alpha : N \to P$ exist by the factorizing assumption on $\sigma : N \to M$. By proposition (5.10) we know that the map $P' \to M$ factors through a map $\beta : P' \to P$ (the proof of (5.10) only uses invertibility). So, we can complete the diagram

$$
\begin{tikzcd}
0 \arrow[r] & Q' \arrow[r, "\zeta"] \arrow[dd, equal] & X \arrow[rr, shift left] \arrow[dd, "\delta"'] & & P \arrow[r] \arrow[dd] \arrow[ll, shift left, "\gamma"] & 0 \\
 & & & P' \arrow[ur, "\beta"] \arrow[dr] & & \\
0 \arrow[r] & Q' \arrow[r, "\epsilon"] & N \arrow[rr, "\sigma"] \arrow[ur, "\alpha"] & & M \arrow[r] & 0
\end{tikzcd}
$$

where γ exists because P is invertible and Q' is coflasque. By commutativity of the diagram e have

$$(1_N - \alpha\beta\gamma\delta) \subset Ker(\sigma) = Q'$$

so we can apply ϵ^{-1} to it and form a map

$$\theta = \alpha\beta\gamma + (1_N - \alpha\beta\gamma\delta)\epsilon^{-1}\zeta$$

which gives the required splitting for δ i.e.

$$N \oplus Q \cong X \cong Q' \oplus P$$

(2) : is proved dually.

In the foresters guide we have seen that the Colliot-semigroup (i.e. isoclasses of flasque lattices upto adding permutation lattices) parametrize the different trees in the Lenstra forest. This may suggest that every tree contains a flasque lattice. We will see below that this is not the case in general. However, each tree does contain a coflasque lattice

(10.2) : lemma Every tree in the Lenstra forest contains a coflasque lattice. In particular, all lattices in $[\kappa\phi M]_c$ lie in the same tree as M

proof : Consider the following pullback diagram obtained from a flasque resolution of M and a coflasque resolution of a representant of $[\phi M]_c$

$$\begin{array}{ccccccccc} & & & & \kappa\phi M & = & \kappa\phi M & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & R & \rightarrow & P & \rightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & P_M & \rightarrow & \phi M & \rightarrow & 0 \end{array}$$

The middle vertical sequence splits because P_M is invertible and $\kappa\phi M$ is coflasque, i.e. $R = P_M \oplus \kappa\phi M$ and then the middle horizontal sequence gives the result.

In fact, we have the following characterization of trees containing a flasque lattice

(10.3) : lemma Every tree in the Lenstra forest which contains a flasque lattice contains also a coco-nut

proof : Let F be a flasque lattice. Then, by the argument in the proof of the foregoing result (replacing M by F) we obtain an exact G-sequence

$$0 \rightarrow F \rightarrow P_F \oplus \kappa\phi F \rightarrow P \rightarrow 0$$

with P_F and P permutation lattices and $\kappa\phi F$ coflasque.But then the long exact sequence of Tate cohomology gives us that $TH^{-1}(H, \kappa\phi F) = 0$ for every subgroup H of G. So, $\kappa\phi F$ is also a flasque lattice,done.

Perhaps this is a good place to make a few comments about the Sansuc,Colliot,Coco and Saltman semigroups introduced before. In the next trimester we will see that these semigroups are, in general, not cancellative. However, we will see that it is always possible to write them as a disjoint union of cancellative genus-closed sub semi-groups.

From (10.3) it follows that there are infinitely many trees not containing a flasque lattice whenever $Coco(G) \neq Colliot(G)$. Let us give a drastic example of such a situation due to Colliot-Thélène and Sansuc :

(10.4) : Example The Lenstra forest of the Klein Vierer-group V_4 contains infinitely many trees and all flasque lattices lie in the permutation tree

proof : As V_4 is not metacyclic, $PCl(V_4) \neq Colliot(V_4)$ and hence there are infinitely many trees in the forest. Let M be an arbitrary V_4-lattice and consider the endomorphism σ on M determined by multiplication with $1+s$ ($V_4 =< s,t >$). Then we have the sequence

$$0 \to Ker(\sigma) \to M \to Im(\sigma) \to 0$$

First, s acts trivially on $Im(\sigma)$ i.e. $Im(\sigma)$ is really a $V_4/ < s >=< t >$-lattice. Secondly, $Ker(\sigma)^{V_4} = 0$ for otherwise $Ker(\sigma)$ would contain two-torsion elements. Then , the exact cohomology sequence gives us an epi

$$TH^{-1}(V_4, M) \to TH^{-1}(V_4, Im(\sigma)) \to 0$$

Now, restrict attention to the case when M is a flasque V_4-lattice. Then, $TH^{-1}(V_4, Im(\sigma)) = 0$ and decomposing $Im(\sigma)$ into indecomposable $< t >$-lattices as in section 3 this implies that $Im(\sigma)$ can only contain components $\mathbb{Z}$ and $\mathbb{Z} < t >$ yielding that $Im(\sigma)$ is a permutation V_4-lattice. But then, the above sequence gives us

$$\phi(Ker(\sigma)) = \phi(M)$$

Now, decompose $Ker(\sigma)$ as a $< t >$-lattice (by restriction) into indecomposables i.e. as a sum of copies of $\mathbb{Z}$, $\mathbb{Z} < t >$ and $Ker(\mathbb{Z} < t > \to Z$. s acts on $Ker(\sigma)$ as $-1_{Ker(\sigma)}$ so this decomposition is really one as V_4-lattices. Now, use the fact that the $< t >$-indecomposables mentioned above are (with the given s-action) iso to the V_4-lattices $Ker(\mathbb{Z}V_4/ < t > \to \mathbb{Z})$, $Ker(\mathbb{Z}V_4 \to \mathbb{Z}V_4/ < s >)$ and $Ker(\mathbb{Z}V_4/ < st > \to \mathbb{Z})$ which all lie in the permutation tree. Therefore, $\phi(Ker(\sigma)) = \phi(M) = 0$ i.e. M lies also in the permutation tree.

This example also shows that it is perfectly possible for tori-invariants $l(M)^G$ to be rational over l^G and yet $l(M^*)^G$ is not stable rational over l^G.

In the study of moduli spaces of vectorbundles over projective spaces it is often useful to write a bundle as the cohomology bundle of a three term complex (a monad) consisting of bundles which are better understood. This idea of Horrocks (supported by the Beilinson spectral sequence) has many applications. It is perhaps a bit surprising that we also have monads for G-lattices (at least if we consider permutation lattices and cocos to be well understood lattices)

(10.5) : Proposition Any G-lattice M is the cohomology lattice of a monad (i.e. a three term complex consisting of a mono and an epi)

$$P_1 \hookrightarrow C \mapsto P_2$$

where P_1 and P_2 are permutation lattices and C is a Coco (i.e. flasque and coflasque)

proof: Consider the dual lattice M^*, then by (10.2) M^* lies in the same tree as the coflasque lattice $\kappa\sigma M^*$, i.e. using (5.15) and adding a permutation lattice if necessary to $\kappa\phi M^*$ we find an exact sequence

$$0 \to M^* \to \kappa\phi M^* \to P_1 \to 0$$

with P_1 a poermutation lattice.Dualizing this sequence gives us

$$0 \to P_1 \to \phi\kappa M \to M \to 0$$

where the middle term is a flasque lattice. But then by (10.4) $\phi\kappa M$ lies in the same tree as a coco-nut C , i.e. we have an exact sequence

$$0 \to \phi\kappa M \to C \to P_2 \to 0$$

which gives rise to the exact diagram ($\bullet$)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_1 & \to & \phi\kappa M & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & P_1 & \to & C & \to & Coker & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & P_2 & = & P_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

from which we deduce that M is the cohomology of the monad

$$(\diamond) : P_1 \hookrightarrow C \mapsto P_2$$

we call the diagram ($\bullet$) the display of the monad ($\diamond$).

So, roughly speaking, all G-lattices are understood if we know all maps from permutation lattices to cocos and back. For more applications of this technique we refer you to the junglebook. There, we will also develop the jungle-cohomology (initiated by work of James Arnold) which is an analoque of Tate cohomology but replacing the role of projective lattices by permutation lattices. This theory leads to jungle-versions of the Carlson concepts of complexity and cohomology variety.

Now, it is about time to introduce these jungles :

(10.6) : The G-jungle s the picture obtained by classifying all isoclasses of G-lattices according to their $\mathbb{Z}$-rank and drawing

(a) : a red edge between $[M]$ and $[N]$ iff there is an exact G-sequence

$$0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$$

with P a permutation lattice

(b) : a blue edge between $[M]$ and $[N]$ iff there is an exact G-sequence

$$0 \rightarrow P \rightarrow M \rightarrow N \rightarrow 0$$

with P a permutation lattice

(c) : a green edge between $[M]$ and $[N]$ iff there is a blue and red edge between $[M]$ and $[N]$

In other words, forgetting the blue edges we have the Lenstra forest and forgetting the red edges we have the dual forest. Again, we have a hill supporting this jungle and by transitivity we may restrict attention to drawing those edges coming from extensions by transitive permutation lattices. If we have the full jungle picture, it is trivial to harvest the flasque,coflasque and coconuts as follows :

- the flasques are the ones with precisely k green edges over it and all the others red (k is the number of transitive permutation lattices)
- the coflasques are the ones with k green edges over it and the others all green
- the coconuts are the ones with only k green edges over it (and no others)

Clearly, the duality operator forces the jungle to be highly symmetric.

For the underlying theory and notation on jungles we refer to "Hackenbush" which is chapter seven of "Winning ways for your mathematical plays 1" by Elwyn Berlekamp,John Conway and Richard Guy. In particular we mention here the definition of the purple mountain which is that part of the jungle consisting of the blue-red edges connected to the hill and the remaining part is called the green jungle. The green jungle can be very big (e.g. in the C_p-case where it is the whole jungle) or extremely small (e.g. in the V_4-case where it is just the top of the permutation tree). The diligent reader will already feel that it makes a lot of difference to rationality problems whether the lattice lies in the purple mountain (the ugly case) or in the green jungle (the good case).

Unfortunately, there are some trendy animals living in the jungle. We leave you with the following boasting Baloo example : the tori-invariants of the Leech lattice under any of the Conway sporadic groups is not rational...