Matrixinvariants and complete intersections

Lieven Le Bruyn*

Yasuo Teranishi**

March 1988 - 88-07

Abstract.
In this note we show that the ring of matrixinvariants of m tuples n by n matrices is a complete intersection if and only if $(m,n) = (2,2),(2,3)$ or $(3,2)$. Moreover, in all other cases there are no locally complete intersection singularities.

Key Words and phrases.
Matrixinvariants, complete intersections

Subject Classification.
13H10, 16A38, 15A72, 16A46.

* University of Antwerp, UIA-NFWO
** University of Mannheim, BRD, and University of Nagoya, Japan

Matrixinvariants and complete instersections

Consider the vectorspace of m tuples of n by n matrices :

$$X_{m,n} = M_n(\mathbb{C}) \oplus ... \oplus M_n(\mathbb{C})$$

The linear group $GL_n(\mathbb{C})$ acts on $X_{m,n}$ by simultanous conjugation. The corresponding ring of polynomial invariants

$$\mathbb{C}[X_{m,n}]^{GL_n(\mathbb{C})}$$

will be denoted by $C(n,m)$ and is called the ring of matrixinvariants of m n by n matrices. C. Procesi has shown in [8] that $C(n,m)$ is generated by traces of products of the corresponding generic matrices and, as such, coincides with the center of the trace ring of m generic n by n matrices $R(n,m)$, which is also the ring of equivariant maps from $X_{m,n}$ to $M_n(\mathbb{C})$.

Apart from this general result, very little is known about the explicit structure of $C(n,m)$. In [7] it is shown that $C(2,2)$ is the polynomial algebra

$$\mathbb{C}[Tr(X_1), Tr(X_2), \mathrm{Det}(X_1), \mathrm{Det}(X_2), Tr(X_1X_2)]$$

The structure of $C(2,3)$ was determined by Formanek [O], see also [3] or [10] : consider the polynomial algebra.

$$\mathbb{C}[Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3),$$
$$Tr(X_1X_2), Tr(X_2X_3), Tr(X_3X_1)]$$

then $C(2,3)$ is a free module over this algebra of rank 2 generated by 1 and $Tr(X_1X_2X_3)$. Moreover, $Tr(X_1X_2X_3)$ satisfies the quadratic equation :

$$X^2 - AX + B = O$$

where

$$
\begin{aligned}
A =& Tr(X_1)Tr(X_2X_3) + Tr(X_2)Tr(X_1X_3) + Tr(X_3)Tr(X_1X_2) \\
& - Tr(X_1)Tr(X_2)Tr(X_3)
\end{aligned}
$$

$$
\begin{aligned}
B =& D(X_1)Tr(X_2X_3)^2 + D(X_2)Tr(X_1X_3)^2 + D(X_3)Tr(X_1X_2)^2 \\
& - Tr(X_1)Tr(X_2)Tr(X_1X_2)D(X_3) - Tr(X_1)Tr(X_3)Tr(X_1X_3)D(X_2) \\
& - Tr(X_2)Tr(X_3)Tr(X_2X_3)D(X_1) \\
& + Tr(X_1)^2D(X_2)D(X_3) + Tr(X_2)^2D(X_1)D(X_3) \\
& + Tr(X_3)^2D(X_1)D(X_2) \\
& - 4D(X_1)D(X_2)D(X_3) + Tr(X_1X_2)Tr(X_1X_2)Tr(X_2X_3)
\end{aligned}
$$

In general, $C(2,m)$ is a polynomial algebra in m variations over the center of the generic Clifford algebra for m-any quadratic forms of degree ≤ 4 see [2] and it can be expressed in terms of $SO_3(\mathbb{C})$-invariants, see [9].

The structure of $C(3,2)$ was determined in [10] and implicitly in [4] : consider the polynomial algebra $\mathbb{C}[Tr(X_1), Tr(X_1^2), Tr(X_1^3), Tr(X_2), Tr(X_2^2), Tr(X_2^3),$ $Tr(X_1X_2), Tr(X_1X_2^2), Tr(X_1^2X_2), Tr(X_1^2X_2^2)]$ then $C(3,2)$ is a free module of rank 2 over this algebra generated by 1 and $Tr(X_1X_2X_1^2X_2^2)$. Apart from these results only $C(4,2)$ is known, see [10].

What is known about the homological properties of $C(n,m)$? In view of the Hochster-Roberts result, $C(n,m)$ is a Cohen-Macaulay algebra and because it is the ring of invariants of the simple group $PGL_n(\mathbb{C})$ it is a unique factorization domain and hence Gorenstein, see for example [1]. In [5] it is shown that $C(n,m)$ is never regular except when $(m,n) = (2,2)$ which is, as we have seen above, a polynomial algebra. Recall that an algebra $\mathbb{C}[X_1,..,X_k]/I$ is said to be a complete intersection if the heigth of I concides with the mimimal number of generators of I. It follows from the above explicit descriptions that $C(2,3)$ and $C(3,2)$ are hypersurfaces and hence complete intersections. The main result of this note will assert that there are no others.

Before we come to the proof, we recall some results of [5]. Let $V_{m,n}$ be the variety corresponding to $C(n,m)$, then it is well known that $V_{m,n}$ parametrizes the isomorphism classes of semi-simple n-dimensional representations of

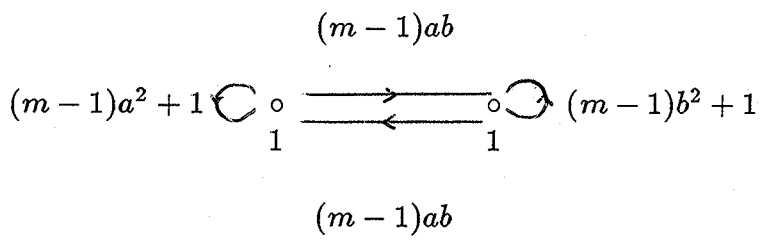
$\mathbb{C} < X_1, ..., X_m >$. A point $\xi \in V_{m,n}$ is said to be of representation type $\tau = (e_1, k_1; ...; e_r, k_r)$ if the corresponding isomorphism class of semi-simple representations is build from r distinct simple components of dimensions k_i occuring with multiplicity e_i. If τ is such a representation type, then $V_{m,n}(\tau)$ is defined to be the subset of $V_{m,n}$ consisting of all points of representation type τ. In [5] it is shown that the sets $V_{m,n}(\tau)$ form a finite stratification into locally closed smooth subvarieties. Moreover $V_{m,n}(\tau)$ lies in the closure of $V_{m,n}(\tau')$ if and only if τ is a degeneration of τ' If $\xi \in V_{m,n}$ is of representation type $(e_1, k_1; ...; l_r, k_r)$ then one forms the quiver Δ_ξ consisting of r vertices $(x_1, ..., x_r)$ and $(m-1)k_i^2 + 1$ loops in vertex x_1 and $(m-1)k_i k_j$ directed edges from x_i to x_j. Let d_ξ be the dimension vector $(e_1, ..., e_r)$. Then it is proved in [5] that the étale locale structure of $V_{m,n}$ near ξ is that of the variety $V(\Delta_\xi, d_\xi)$ of semi-simple representations of the quiver Δ_ξ of dimension vector d_ξ near the origin. Moreover, the coordinate ring of $V(\Delta_\xi, d_\xi)$ is generated by traces of oriented cycles in the quiver Δ_ξ, see [6].

Let V be the variety corresponding to the algebra $\mathbb{C}[x_1, ..., x_k]/I$, then V is said to be locally a complete intersection in v is $I\mathbb{C}[x_1, .., x_k]_m$ is generated by a regular sequence of length the heigth of I for all maximal ideals m lying over v. It is clear that the subset of all points $v \in V$ s.t. V is locally a complete intersection in v forms an open subvariety $V^{c.i.}$. We are now in a position to state the main result :

THEOREM : If $(m,n) \neq (2,2), (2,3)$ or $(3,2)$, then the locally complete intersection locus $V_{m,n}^{c.i}$ coincides with the open subvariety of regular points $V_{m,n}^{\text{reg}} = V_{m,n}[(1,n)]$.

Proof: Since being locally a complete intersection can be expressed in homological terms, it is being preserved under étale extensions so we only need to study the étale local structure of $V_{m,n}$ near a point ξ. Suppose $V_{m,n}^{c.i}$ is strictly larger than $V_{m,n}[(1,n)]$, which is precisely the nonsingular locus when $(m,n) \neq (2,2)$, see [5,Th II.3.4.]. Then by the stratification result mentioned before, $V_{m,n,}^{c.i}$ must contain a point ξ corresponding to a semi-simple representation having two distinct simple components, i.e. ξ is of type $(1, a; 1, b)$

with $a+b=n$. The corresponding quiver Δ_ξ is



and the dimension vector $d_\xi = (1,1)$. The coordinate ring of $V(\Delta_\xi, d_\xi)$ is then a polynomial algebra in $(m-1)(a^2+b^2)+2$ variables over

$$\mathbb{C}[W] = \mathbb{C}[t_{ij} : 1 \leq i,j \leq (m-1)ab]/I_2$$

where I_2 is the ideal generated by the determinants of all 2 by 2 minors of the generic matrix $(t_{ij})_{i,j}$. It is well known that W is a complete intersection in the origin if and only if $(m-1)ab = 2$. By étale decent this finishes the proof.

In view of the explicit descriptions of $C(2,2), C(2,3)$ and $(3,2)$ given before and the above theorem, we obtain immediatly :

COROLLARY: The ring of matrixinvariants $C(m,n)$ is a complete intersection in and only if $(m,n) = (2,2),(2,3)$ or $(3,2)$. In particular, it is a complete intersection if and only if the corresponding ring of equivariant maps $R(n,m)$ is of finite global dimension.

The last statement follows from [5].

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