Brauer Groups of Fields

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Abstract.
We present some recent methods and developments in the study of Brauer groups of fields arising from different approaches to the Merkurjev-Suslin theorem.

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Brauer Group, Skewfield, Schurgroup, Matrix Invariant, Rationality Problem.

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Introduction

In this short monograph we aim to highlight two evergreen branches of the theory of Brauer groups of fields.

The first part is rooted in the representation theoretic foundations for the Theory of the Brauer group, in particular we introduce the Schur subgroup of the Brauer group as well as the so-called Clifford-Schur subgroup that is related to projective representations of finite groups rather than to usual representations.

The second part deals with with generic division algebras and the links connecting this topic to the celebrated Merkurjev-Suslin theorem. This provides the opportunity to exhibit some recent methods and techniques not readily found in text-books; we touch upon : the rationality problem, permutation modules, invariants for actions of the projective linear group, vectorbundles over the projective plane, representation theory of hereditary algebras and a connection with matrixinvariants.

Both parts are related by Clifford algebras and the role they play in one of the possible formulations of the Merkurjev-Suslin theorem (see for example Problem 3 on page 22). The items we have included represent different approaches to this central theorem but without arriving at a non K-theoretical proof for the theorem, like petals of a flower that have been torn off leaving the heart naked. However we hope that the amalgam of recent techniques present here may provide enough new material for further hybridization and stimulate continuing interest in the Brauer group of a field.

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Contents

1. Brauer, Schur and Clifford-Schur Groups

1.1. Brauer Groups

Let k be a field. By an algebra over k we always mean a finite dimensional associative k-algebra with identity 1. If A is a k-algebra and $Z(A)$ is the centre of A then we say that A is **k-central** if $Z(A) = k$. We say that A is **simple** if it has no proper two-sided ideals and a k-central simple algebra A will be referred to by saying that A is a k-c.s.a. A k-c.s.a. A that is a division algebra will sometimes be called a **skewfield** over k.

1.1.1. Theorem (Wedderburn). If A is a k-c.s.a. then $A \cong M_n(D)$. for some skewfield D over k and both n and D are essentially uniquely determined by the isomorphism class of A.

Let A be a k-algebra, M a finitely generated A-module. Clearly $\mathrm{End}_A(M)$ is a subalgebra of $\mathrm{End}_k(M)$ called the **centralizer** of M. A left A-module M is **faithful** if $aM = 0$ with $a \in A$ yields $a = 0$. The homomorphism $l : A \to \mathrm{End}_K(M)$ defined by taking for $l(a)$ the left multiplication by a yields an isomorphism of k-algebras $A \cong A^l \subset \mathrm{End}_k(M)$ in case M is faithful.

Using the following lemma (Bourbaki N., Algèbre Ch. 8. Modules et anneaux semi-simples, Hermann, Paris, 1958, p. 26) it is very easy to prove Burnside's theorem that follows.

1.12. Lemma. Let M be a faithful finitely generated A-module and select generators of M as a module over $\mathrm{End}_A(M)$, say $m_1, \ldots, m_\mu$ then we have a monomorphism ${}_AA \to M^{(\mu)}$ given by $a \mapsto (am_1, \ldots, am_\mu)$.

1.1.3. Theorem (Burnside) If M is a simple left A-module such that $\mathrm{End}_A M = k.1_M$ then $A^l = \mathrm{End}_R(M)$.

Proof. From the lemma one easily derives that A^l is a simple k-algebra whenever M is a simple left A-module, then use the following. □

1.1.4. Theorem. (Density Theorem). Let A be a subring of $\text{End}_{\mathbb{Z}}(G)$, where G is an abelian group, such that G is simple as a left A-module. Then $D = \text{End}_A(G)$ is a division ring and A is a dense ring of linear transformations over the D-vector space G.

As a corollary to Burnside's theorem we obtain :

1.1.5. Corollary. Assume that k is algebraically closed. If M is a simple left A-module then $\text{End}_A(M) = k$ and consequently $A^k = \text{End}_k(M) \cong M_n(k)$ for some $n \in \mathbb{N}$. □

A left A-module M is **absolutely simple** if for every extension $l|k$ $M \otimes_k l$ is a simple left $A \otimes_k l$-module. As a consequence of Burnside's theorem we have :

1.1.6. Proposition. A simple left A-module M is absolutely simple if and only if $\text{End}_A(M) = k.1_M$ (cf. Curtis, Reiner, Representation Theory of Finite groups and Associative Algebras, Pure and Appl. Math. vol. 11, Interscience, New York 1962, pp. 102-103).

If A is a k c.s.a. then $A \otimes_k l$ is an l- c.s.a. for every field extension $l|k$. When we have that $A \otimes_k l = M_n(l)$ for some $n \in \mathbb{N}$ then we say that A is **split** by $l|k$. In view of Corollary 1.1.5. every k-c.s.a. may be split by $\overline{k}/k$, where $\overline{k}$ is an algebraic closure of k. Note that this implies that $\dim_k A$ is a square for each k-c.s.a. A. If we select a k-basis $\{u_1, \ldots, u_{n^2}\}$ for A then we may express a complete set of a matrix-units for $M_n(\overline{k})$ in terms of the $\overline{k}$-basis $\{u_1, \ldots, u_{n^2}\}$ for $M_n(\overline{k})$. The finite number of coefficients occuring in these expressions generate a finite dimensional subextension $K|k$ in $\overline{k}/k$ that obviously splits A. Any field K such that K/k splits A is called a **splitting field** for A (over k). The opposite algebra A^o of A is obtained by taking the abelian group $\underline{A}$ of A with new multiplication $a.b = ba$ for $a.b \in \underline{A}$. There is a k-algebra homomorphism $r : A^o \to A^r \subset \text{End}_k(A)$ mapping $b^o \in A^o$ to right multiplication by b. For $a, b \in A$ we have $l(a)r(b) = r(b)l(a)$ and A may be made into an $A \otimes_k A^o$-module (left as usual) if we put : $(a \otimes b^o)m = l(a)r(b^o)m = amb$ for $m \in A, a \in A, b^o \in A^o$. Clearly, putting $A^e = A \otimes_k A^o$, we have $Z(A) = \text{End}_{A^e}(A)$.

1.1.7. Theorem. Let A and B be simple k-algebras (even of infinite k-dimension) with centres $Z(A)$, resp. $Z(B)$.

1. If $\dim_k(A)$ and $\dim_k(B)$ are finite then the lattice of ideals in $A \otimes_k B$

is isomorphic to the lattice of ideals of $Z(A) \otimes_k Z(B)$. Then $A \otimes_k B$ is semisimple if and only if $Z(A) \otimes_k Z(B)$ is semisimple.

2. If A is a k-c.s.a. then $A \otimes_k B$ is simple with centre $Z(B)$ and if both A and B are k-c.s.a. then so is $A \otimes_k B$.

3. For any extension l/k, if A is a k-c.s.a. then $A \otimes_k l$ is a c.s.a. over l.

Two k-c.s.a.'s A and B are said to be **similar** or **Brauer equivalent** if there exist $n.m \in \mathbb{N}$ such that there is an isomorphism of k-algebras $M_n(A) \cong M_m(B)$. In view of Theorem 1.1.1. it is now clear that A is similar to B, denoted $A \sim B$, if and only if $A \cong M_q(D)$, $B \cong M_g(D)$ for some p and q in $\mathbb{N}$ and D a skewfield over k. The similarity classes of k-c.s.a.'s form a set $\mathrm{Br}(k)$ corresponding bijectively to the set of skewfields over k. If $A \sim A_1$ and $B \sim B_1$ then $A \otimes_k B \sim A_1 \otimes_k B_1$. Let us wite $[A]$ for the class of the k=c.s.a. A in $\mathrm{Br}(k)$. Putting $[A \otimes_k B]$ equal to $[A].[B]$ defines a commutative and associative operation in $\mathrm{Br}(k)$. Clearly $[k]$ is a unit element for this operation and $[A^\circ]$ is an inverse for $[A]$ (because $k = \mathrm{End}_{A^e}(A)$ and A is a simple k-algebra if and only if A is a simple left A^e-module hence application of the Burnside theorem proves the claim).
The abelian group $\mathrm{Br}(k)$ thus defined is called the **Brauer group** of k Extension of scalars by l/k determines a group morphism $\mathrm{Br}(k) \to \mathrm{Br}(l), [A] \mapsto [A \otimes_k l]$ called the **restriction** map (this makes more sense from the cohomological point of view). We obtain a functor $\mathrm{Br}(-)$ from the category of fields to abelian groups.

1.1.8. Examples.

a. If $k = \overline{k}$ then $\mathrm{Br}(\overline{k}) = 1$ (see Corollary I.1.5.).

b. If $k = \mathbb{R}$ then $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ because Frobenius showed that the quaternions $\mathbb{H}$ represent the only noncommutative division algebra central over $\mathbb{R}$.

c. If $k = \mathbb{Q}_p$, the p-adic numbers for the prime p, then Hasse showed that $\mathrm{Br}(\mathbb{Q}_p)$ is canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$. The image of $[A]$ in $\mathbb{Q}/\mathbb{Z}$ is called the **Hasse invariant** of A.

d. If k is a global field and $\underline{p}$ is a prime (finite of infinite) then we write $k_{\underline{p}}$ for the completion of k at $\underline{p}$. If A is a k-algebra then $A_{\underline{p}}$ is the completion $A \otimes_k k_{\underline{p}}$. The Hasse invariant of $A_{\underline{p}}$ is the **local Hasse invariant** of A at $\underline{p}$. One can show that $A_{\underline{p}}$ is split for almost all $\underline{p}$ and consequently we may define a canonical map $\mathrm{Br}(k) \to \oplus_f \mathrm{Br}(k_{\underline{p}})$. If $\underline{p}$ is a finite prime then $\mathrm{Br}(k_{\underline{p}})$

is identified with $\mathbb{Q}/\mathbb{Z}$. Viewing $\mathrm{Br}(\mathbb{R})$ as $(\frac{1}{2}\mathbb{Z})/\mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ we may define a map $\oplus_{\underline{p}}\mathrm{Br}(k\underline{p}) \to \mathbb{Q}/\mathbb{Z}$ by taking addition in $\mathbb{Q}/\mathbb{Z}$. It is a non-trivial fact, related to the reciprocity law in class field theory that we may determine $\mathrm{Br}(k)$ by the exact sequence :

$$1 \to Br(k) \to \oplus_{\underline{p}} Br(k_{\underline{p}}) \to \mathbb{Q} \to 0$$

Let us now return to the structure theory of k-c.s.a.

1.1.9. Proposition. Let A be a k-c.s.a. If M is an A-bimodule such that $\lambda m = m\lambda$ for all $\lambda \in k$, $m \in M$ then $M = M^{(A)} \otimes_k A$ where $M^{(A)} = \{m \in M, am = ma$ for all $a \in A\}$. In fact, the functor $(-)^{(A)} : A - \mathrm{bimod}_k \to k$-mod, $M \mapsto M^{(A)}$, defines an equivalence.

1.1.10. Corollary. If B is a k-algebra containing a k-c.s.a. A then $B \cong A \otimes_k C_B(A)$, where $C_B(A) = \{b \in B, ab = ba$ for all $a \in A\}$. Furthermore B is simple if and only if $C_B(A)$ is simple and B is a k-c.s.a. if and only if $C_B(A)$ is a k-c.s.a.

On the other extreme we have a useful theorem about simple subalgebras of a k-c.s.a. :

1.1.11. Theorem (Skolem-Noether). Let B be a simple k-subalgebra of a k-c.s.a. A. Every isomorphism of k-algebras $\beta : B \to B' \subset A$ may be extended to an inner autorphism of A. Consequently if α is an automorphism of A fixing the elements of k then α is an inner automorphism of A. This result will be particurlarly useful when considering commutative subfields, and automorphisms of these, in k-c.s.a.. However, it is not clear whether there exist good subfields (in the sense of having enough k-automorphisms) in a given k-c.s.a. A commutative field l, $k \subset l \subset A$ is said be a **maximal subfield** if there is no properly larger commutative subfield in A. Obviously $\overline{k}$ is a maximal subfield of $M_n(\overline{k})$ so there need not exist maximal subfields of dimension n over k in a general k-c.s.a. For a skewfield D over k we can prove.

1.1.12. Proposition. If D is a skewfield over k of dimension n^2 then every maximal subfield l of D is n-dimensional over k. There always exists a maximal subfield l of D such that l/k is a separable extension.

1.1.13. Corollary. Every k-c..s.a. has a splitting field l such that l/k is a Galois extension. Note that it has been proved by S. Amitsur in 1972 that not every skewfield over k contains a Galois extension of k as a maximal subfield. Up to similarity we can obtain this result :

1.1.14, Proposition. If l/k splits a k-c.s.a. A then there is a $B \sim A$ such that l is a maximal subfield of B.

1.1.15. Corollary. Given a k-c.s.a. A then there exists a $B \sim A$ containing as a maximal subfield a Galois extension l/k splitting A.
The proofs usually provided for Proposition 1.1.12 and 1.1.14 depend heavily on the so-called double centralizer theorem :

I.1.16. Theorem. Let B be a simple subalgebra of the k-c.s.a. A

1. $C_A(B)$ is simple

2. $\dim_k B \dim_k C_A(B) = \dim_k A$

3. $C_A(C_A(B)) = B$

4. If B is central simple over k then $C_A(B)$ is k-c.s.a. and $A = B \otimes_k C_A(B)$.

This theorem may be extended in case A is not necessarily finite dimensional but B a finite dimensional subalgebra of A, in that case again $C_A(B)$ is simple and $C_A(C_A(B)) = B$.

The importance of Proposition I.1.14. is that, up to equivalence in the Brauer group, we may assume that the k-c.s.a.. A contains a maximal subfield l such that l/k is a Galois extension and $\dim_k A = (\dim_k l)^2$. For each $\sigma \in G = \mathrm{Gal}(l/k)$ we may select an invertible element u_σ of A such that for all $\lambda \in l$, $u_\sigma \lambda = \lambda^\sigma u_\sigma$ (by the Skolem-Noether theorem). One easily checks that $A = l\{u_\sigma, \sigma \in G\}$ and associativity of A yields a relation $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma\tau}$ for $\sigma, \tau \in G$ with $c(\sigma,\tau) \in l^*$ satisfying the 2-cocycle relation :

$$c(\sigma,\tau) \subset (\sigma,\tau,\gamma) = c(\tau,\gamma)^\sigma c(\sigma,\tau\gamma) \text{ for all } \quad \sigma,\tau,\gamma \in G$$

A change of l-basis $\{u_\sigma, \sigma \in G\} \to \{v_\sigma, \sigma \in G\}$ such that still $v_\sigma \lambda = \lambda^\sigma u_\sigma$ is necessarily one of the form $v_\sigma = f_\sigma u_\sigma$ for certain $f_\sigma \in l^*$. So if $v_\sigma v_\tau = d(\sigma,\tau)v_{\sigma\tau}$ then we arrive at $f_\sigma u_\sigma f_\tau u_\tau = f_\sigma f_\tau^\sigma c(\sigma,\tau)u_{\sigma\tau} = d(\sigma,\tau)f_{\sigma\tau}u_{\sigma\tau}$, hence $d(\sigma.\tau) = f_\sigma f_\tau^\sigma f_{\sigma\tau}^{-1} d(\sigma,\tau)$. Clearly $\mu(\sigma,\tau) = f_\sigma f_\tau^\sigma f_{\sigma\tau}^{-1}$ is a coboundary, i.e. a trivial 2-cocycle $\mu : G \times G \to l^*$. If we denote the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(l)$ by $\mathrm{Br}(l/k)$ that we may phrase the following theorem, then provides a cohomological interpretation of the Brauer group.

1.1.17. Theorem. Let l/k be a Galois extension with $\mathrm{Gal}(l/k) = G$ and let $c : G \times G \to l^*$ be a 2-cocycle then the crossed product $(l/k, G, c)$ is defined to be the k-algebra $l\{u_\sigma, \sigma \in G\}$ with multiplication defined by the rules : $u_\sigma \lambda = \lambda^\sigma u_\sigma$ for all $\sigma \in G, \lambda \in l$, $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma,\tau}$ for all $\sigma, \tau \in G$. The k-algebra $(l/k, G, c)$ is a k-c.s.a. Conversely if a k-c.s.a. A contains a Galois extension l/k such that $\dim_k A = (\dim_k l)^2$ then $A \cong (l/k, G, c)$ for some 2-cocycle c. Two crossed products $(l/k, G, c)$ and $(l/k, G, d)$ are isomorphic if and only if c and d are cohomological cocycles. The relative Brauer groups $\mathrm{Br}(l/k)$ is isomorphic to the second cohomology group $H^2(G, l^*)$, where $G = \mathrm{Gal}(l/k)$, and $\mathrm{Br}(k) = \varinjlim_G H^2(G, l^*)$ where G varies over the Galois groups of the Galois extensions of k.

As an immediate corollary of the above we may derive some important properties of the Brauer group of a field.

1.1.18. Corollary. The abelian group $\mathrm{Br}(k)$ is torsion.

Proof. If G is finite, $H^2(G, l^*)$ is torsion. □

1.1.19. Corollary. $\mathrm{Br}(k) = H^2(\overline{G}, k^*_{\mathrm{sep}})$ where G is the (profinite) Galois group of a separable closure k_{sep} of k. The fundamental relation between $\mathrm{Br}(k)$ and Galois cohomology motivates the introduction of some details on the cohomology of finite and profinite groups.

1.2. Some Cohomology Theory.

In the first part of this section we consider finite groups. Let M be a G-module, write M as a multiplicative group. The subgroup M^G of M consisting of all elements fixed by the action of all g in G is also defined to be the o-cohomology group $H^\circ(G, M)$. We may define a G-**norm** on M by $\eta_G(m) = \prod_{g\in G} gm$. The Tate-cohomology group $\hat{H}^\circ(G, M) = M^G/\eta_G(M)$. A 1-cocycle $f \in Z^1(G, M)$ is a map $f : G \to M$ satisfying $f(gh) = f(g).g(f(h))$ for all $g, h \in G$. A 1-coboundary is a 1-cocycle of the form $f(g) = m.g(m)^{-1}$ for some $m \in M$; the 1-coboundaries $B^1(G, M)$ form a subgroup of $Z'(G, M)$ and $H^1(G, M)$ is defined to be $Z^1(G, M)/B^1(G, M)$. A 2-cocycle $c \in Z^2(G, M)$ is a map $c : G \times G \to M$ satisfying $c(\rho, \sigma)c(\sigma\tau, \gamma) = \sigma c(\tau, \gamma)c(\sigma, \tau\gamma)$ for all $\sigma, \tau, \gamma \in G$; a 2-coboundary is a 2-cocycle of the form $c(\sigma, \tau) = f_\sigma.\sigma f_\tau.f_{\sigma\tau}^{-1}$; $H^2(G, M) = Z^2(G, H)/B^2(G, M)$ and the elements of the second cohomology group $H^2(G, M)$ are denoted by $[C]$ where $c \in Z^2(G, M)$ represents it.

As an exercise one may now try to define $Z^n(G, M), B^n(G, M)$ and $H^n(G, M)$ for all $n \in I\!N$. Let M_1 be the kernel of the norm map η_G; we define $\hat{H}^{-1}(G,M) = M_1/\{\text{values of the 1-coboundaries}\}$. For $n > 1$ we put $\hat{H}^n(G,M) = H^n(G,M)$.

1.2.1. Lemma. For an exact sequence of G-modules $1 \to N \to M \to P \to 1$, there are long exact sequences :

a. $\ldots \to \hat{H}^n(G,N) \to \hat{H}^n(G,M) \to \hat{H}^n(G,P) \to \hat{H}^{n+1}(G,N) \to \ldots$

b. $1 \to G^\circ(G,N) \to H^\circ(G,M) \to H^\circ(G,P) \to H^1(G,N) \to H'(G,M) \to \ldots$

For a cyclic group G the sequence **a.** is periodic of period 2; in case $n = 0,2$ the isomorphism $\hat{H}^\circ(G,M) \underset{\varphi}{\xrightarrow{\cong}} \hat{H}^2(G,M)$ may be explicited as follows. If $\overline{\alpha} \in M^G/\eta_G(M), G =< \sigma >\cong \mathbb{Z}/m\mathbb{Z}$, then φ takes $\overline{\alpha}$ to the class of the 2-cocycle : $c(\sigma^i,\sigma^j) = 1$ when $i+j < m$ and $c(\sigma^i,\sigma^j) = \alpha$ if $i+j \geq m$, where α represents $\overline{\alpha}$. The latter cocycle c is called the **cyclic cocycle** of σ and $\alpha \in M^G$. In the cyclic case we may define the **Herbrand quotient**

$$h(M) = \frac{|\hat{H}^\circ(G,M)|}{|G^1(G,M)|}$$

when both numbers are finite. For a short exact sequence $1 \to N \to M \to P \to 1$ such that $h(M)$ and $h(P)$ are defined then $h(N)$ is defined and it equals $h(M)h(P)$. In case M is finite then $h(M) = 1$ (this may be useful in the situation where number fields are being considered).

To a morphism of finite groups $G \to G'$ and a compatible G-module map $M' \to M$ there corresponds a morphism $H^n(G',M') \to H^n(G,M)$. In particular, for a normal subgroup H of G we have an action of G on $H^n(H,M)$ induced by the G-action on M and conjugation on H. If $H = G$ one can show that the action defined as above is trivial. We will write the operation in cohomology groups by addition from now on. If $|G| = \eta$ then $n(\hat{H}^m(G,M)) = 0$ and similarly if M is torsion of exponent n. In particular $\hat{H}^m(G,M) = 0$ whenever $(|G|, \exp(M)) = 1$.

If H is a subgroup of G then there is a canonical morphism $\text{res}_n : H^n(G,M) \to H^2(H,M)$. Moreover if H is a normal subgroup of G then we define the inflation map $\inf_m : H^m(G/H,M^H) \to H^m(G,M)$, $(\inf c)(g_1,\ldots,g_m) = c(\overline{g}_1,\ldots,\overline{g}_m)$ where $\overline{g}$ is the image of $g \in G$ in G/H.

1.2.2. Lemma. (The inflation-restriction sequence). Let H be a normal subgroup of G and let M be a G-module. If $H^i(H,M) = 0$ for $1 \leq i \leq$

$n-1$ then we may define the **transgression** map, $\text{trans}_n : H^n(H,M)^G \to H^{n+1}(G/H, M^H)$ and we obtain an exact sequence :

$$0 \to H^n(G/H, M^H) \underset{\inf_n}{\longrightarrow} H^n(G,M) \underset{\text{res}_\eta}{\longrightarrow} H^n(H,M)^G \underset{\text{trans}_n}{\longrightarrow} H^{n+1}(G/H, M^H) \underset{\inf_{n+1}}{\longrightarrow} H^{n+1}(G,M)$$

Let us just recall a particular case of the above sequence i.e. the Hochschild-Serre sequence. Suppose H is central in G, $H \subset Z(G)$, i.e. we view G as a central extension of G/H given by $1 \to H \to G \xrightarrow{\pi} G/H \longrightarrow 1$. We define the map $t = \text{trans}_1 : H^1(H,M)^G \longrightarrow H^2(G/H, M^H)$ by fixing a factor set $f : G \times G^1 \to H$ determining the central extension G of G/H and putting $t(\varphi)(\sigma,\tau) = \varphi(f(\sigma,\tau))$ for $\sigma,\tau \in G$, where $\varphi \in H^1(H,M)^G$. If we assume that G acts trivially on M then we obtain an exact sequence :

$$(HS) 0 \to \text{Hom}(G/H, M) \underset{\inf}{\longrightarrow} \text{Hom}(G,M) \underset{\text{res}}{\longrightarrow} \text{Hom}(H,M) \xrightarrow{t} H^2(H,M) \underset{\inf}{\longrightarrow} H^2(G,M)$$

Finally let us also mention that the corestriction $\text{cor}_n : H^n(H,N) \to H^n(G,M)$ may be defined for every subgroup H of G and every $n \in I\!N$. Here it suffices to define cor_2. Let T be a transversal for H in G, i.e. set of right coset representatives and define :

$$(\text{cor}_2 f)(\sigma,\tau) = \prod_{t \in T} t^{-1} f(t\sigma\varphi(t\sigma)^{-1}, \varphi(t\sigma)t\varphi(t\sigma\tau)^{-1})$$

where $\varphi : G \to T$ maps a $g \in G$ to its representive in T. It is known that cor o res is just multiplication by $|G : H|$.

If G_1, G_2 are finite groups and Π is a $G_1 \times G_2$-module such that G_1 acts trivially then :

1.2.3. Theorem.

$$H^n(G_1 \times G_2, M) \cong \coprod_{p+g=n} H^P(G_1, H^q(G_2, M)) \qquad (*)$$

for any any $n \geq 0 (p, q \geq 0)$.

The isomorphism in this theorem is not functorial in M. Of particular interest to us is the case $n = 2, 10$:

$$\begin{aligned} &H^2(G_1 \times G_2, M) \\ &\quad \cong H^0(G_1, H^2(G_2, M)) \oplus H^1(G_1, H^1(G_2, M)) \oplus H^2(G_1, H^0(G_2, M)) \\ &\quad = H^2(G_2, M) \oplus \mathrm{Hom}(G_1, H^1(G_2, M)) \oplus H^2(G_1, M^{G_2}) \end{aligned}$$

1.2.4. Corollary. If both G_1 and G_2 operate trivially then (*) is functorial in M and it reduces to :

$$H^2(G_1 \times G_2, M) \cong H^2(G_1, M) \oplus H^2(G_2, M) \oplus P(G_1 \times G_2, M)$$

where $P(G_1 \times G_2, M)$ is the abelian group of pairings into M (i.e. bimultiplicative maps $G_1 \times G_2 \to M$).

In general one can only say that the left hand side of (*) has a normal series in which the composition factors are the groups appearing in the right hand side of (*).

Let us now consider profinite groups. An inverse system of topological groups is a family G_i, indexed by a directed set I, together with continuous homomorphisms $\pi_{ij} : G_j \to G_i$ for every pair $i \leq j$ satisfying $\pi_{ij}\pi_{jk} = \pi_{ik}$. The inverse limit $\varprojlim_{i \in I} G_i$ is the subgroup of $\prod_{i \in I} G_i$ consisting of the $(s_i)_{i \in I} \in \prod_{i \in I} G_i$ such that $\pi_{ij} s_j = s_i$ for $i \leq j$. The group $\varprojlim G_i$ is a closed subgroup of $\prod_{i \in I} G_i$ and hence it is compact if all the G_i are compact. Now in case all G_i are finite groups with the discrete topology then $\varprojlim_{i \in I} G_i$ is called a **profinite** group. A morphism of inverse systems $G_i \to G_i', i \in I$, yields a (continuous) morphism $\varprojlim_{i \in I} G_i \to \varprojlim_{i \in I} G_i'$. Consider a profinite group G.

Any open normal subgroup H of G has finite index in G since G is compact (i.e. the covering of G by cosets of H may be obtained from a finite covering). We obtain : $G = \varprojlim_U G/U$ where U runs over all open normal subgroups of G.

If H is a closed subgroup of G then $H \cong \varprojlim_U H/H \wedge U$ and $G/H \cong \varprojlim_U G/UH$ in case H is also normal. Consequently if G is profinite and H is a closed subgroup of G then H and G/H are profinite too.

1. $\hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

2. $\hat{\mathbb{Z}} = \varprojlim_m \mathbb{Z}/m\mathbb{Z} \cong \prod_{P\in \mathrm{Spec}\hat{\mathbb{Z}}} \hat{\mathbb{Z}}_p$. The latter isomorphism stems from the Chinese Remainder Theorem, since it yields

$$\mathbb{Z}/m \cong \prod_{p^{n(p)}|m} \mathbb{Z}/p^{n(p)}$$

The profinite groups appear very naturally in the infinite Galois theory. Let L/k be a Galois extension with $[L:k]$ not necessarily finite and let $\{K_i, i \in I\}$ be a family of finite Galois subextensions of L/k such that $L = U_i K_i$. Then we have defined an inverse system of finite Galois groups $\mathrm{Gal}(K_i/k)$ together with the restriction maps $\mathrm{Gal}(K_j(k) \to \mathrm{Gal}(K_i/k)$. We obtain an isomorphism $\mathrm{Gal}(L/k) \cong \varprojlim_i \mathrm{Gal}(K_i/k)$ determined by the restriction maps $\mathrm{Gal}(L/k) \longrightarrow \mathrm{Gal}(K_i/k)$. There is a bijective order reversing correspondence between the closed subgroups of $\mathrm{Gal}(L/k)$ and the subextensions K/k of L/k. For example $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$. In connection with the Brauer group theory and its cohomological interpretation we need to introduce a few facts about the cohomology of profinite groups.

Let G be a profinite group and M a **discrete** G-module in the sense that any stabilizer subgroup fixing a point of M is open in G. For each open normal subgroup U of finite index we have a cohomological group $H^n(G/U, M^U)$ for $n \geq 0$, together with a canonical homomorphism : $\pi_{UV} : H^n(G/V, M^V) \longrightarrow H^n(G/U), M^U)$ where $U \subset V$.
Put $H^n(G,M) = \varinjlim_U H^n(G/U, M^U)$ (direct limit here !). One may also obtain a "direct" definition of the group $H^n(G,M)$ by considering only continuous cochains $f : G^n \to M$ and now repeating the earlier construction of cohomology groups. The cohomology groups in the profinite case share a lot of good properties with the finite case. For example, to a short exact sequence we do have associated a long exact cohomology sequence as in Lemma 1.2.1.b. Furthermore the equivalents of Lemma 1.2.2. and Theorem 1.2.3. do hold. In view of its importance in connection with the Schur subgroup of the Brauer group it is useful to introduce cohomological dimensional (cf. J.P. Serre, Cohomological Galoisienne).

Let G be a profinite group, p a prime number. The cohomological p-dimension of G, $\mathrm{cd}_p G$, is the smallest n such that the p-primary component $H^q(G,A)_p$ of $H^q(G,A)$ is zero whenever A is a torsion G-module and $q > n$.

If no such n exists then we put $\text{cd}_p(G) = \infty$. One can show that $\text{cd}_p(G)$ is also the smallest $u \in \mathbb{Z}$ such that the $H^q(G, A) = 0$ for $q > n$ and A being a p-primary torsion group. The strict cohomological p-dimension $scd_p(G)$ is defined as before but without the torsion assumptions on A; similar for the strict cohomological dimension. In fact we know that $\text{scd}_p(G) - \text{cd}_p(G)$ or else $\text{scd}_p(G) = 1 + \text{cd}_p(G)$.

1.2.6. Lemma. If $\text{cd}_p(G)$ is finite then $\text{cd}_p H = \text{cd}_p G$ for open subgroup H of G.

1.2.7. Examples.

1. $\text{cd}_p\hat{\mathbb{Z}} = \text{cd}\hat{\mathbb{Z}} = 1$ and $\text{scd}_p\hat{\mathbb{Z}} = \text{scd}\hat{\mathbb{Z}} = 2$ for all p.

2. $\text{cd}_p\hat{\mathbb{Z}}_p = 1$ and $\text{cd}_q\hat{\mathbb{Z}}_p = 0$ if $q \neq p$.

Similarly $\text{scd}_p(\hat{\mathbb{Z}}_p) = 2$ and $\text{scd}_q\hat{\mathbb{Z}}_p = 0$ if $q \neq p$.

1.3. Schur Groups.

If the order of the group G is invertible in the field k then the group algebra kG is semisimple i.e. $kG = A_1 \oplus \ldots \oplus A_m$ where $l_i = Z(A_i)$. An extension l/k splits kG lG decomposes as a direct sum of matrix rings over l.

1.3.1. Brauer Splitting Theorem.
If $n^{-1} = |G|^{-1} \in k$ then we may construct a splitting field the kG by adjoining roots of unity to k. As a consequence of this theorem it is clear that each $l_i = Z(A_i), i = 1, \ldots, m$ is a subcyclotomic extension i.a. a subextension of a cyclotomic extension $k(\omega)/k$. In case $l_i = k$ we have $A_i \in \text{Br}(k)$. It is still a basic problem to determine the Schur index of an irreducible representation (or character) of a finite group G by the character table of G. The Schur index is $\sqrt{(D_i : k)}$ where D_i is the division algebra in $[A_i]$ and A_i is the simple component of KG belonging to the representation. We now define the Schur subgroup $S(k)$ in $\text{Br}(k)$ as the subgroup (check !) consisting of those k-c.s.a.'s that appear as epimorphic images of group rings kG for some finite groups G.
We may refer to Yamada [5] for a treatment of the basic results concerning the Schur group. Most results exist when $k = \mathbb{Q}$ or a numberfield or an extension of a p-adic field.

1.3.1. Theorem. Let $k_c/\mathbb{Q}$ be the largest subextension of $k/\mathbb{Q}$ which is subcyclotomic, then the restriction $S(k_c) \to S(k)$ is onto.

The cohomological description of the Schur group is a consequence of the Brauer-Witt theorem (although our presentation of the result in cohomological terms is not completely identical to the classical formulation). We say that a 2-cocycle f is a **cyclotomic cocycle** if there is a cyclotomic extension $k(\omega)/k$ such that the class $[f]$ is in $H^2(k(\omega)/k)$ and the values of f are roots of unity (these are then necessarily in the roots of unity group $\mu(k(\omega))$). A cyclotomic cohomology class is one containing a cyclotomic cocycle. The cyclotomic classes in $H^2(k(\omega)k)$ is a subgroup denoted by $H^2_c(k(\omega)/k)$ and it is the image of the cohomology group $H^2(\text{Gal}(k(\omega)/k), \mu(k(\omega)))$ under the canonical map into $H^2(k(\omega)/k) = H^2(\text{Gal}(k(\omega)/k), k(\omega)^*)$. That $H^2_c(k(\omega)/k)$ is a subgroup of the Schur group is clear and the same holds for $\varinjlim_{n\in\mathbb{N}} H^2_c(k(\omega_n)/k)$. The Brauer-Witt theorem deals with the converse of this.

1.3.3. Theorem. (Brauer-Witt): Let k be a field of characteristic 0 then $S(k) = \varinjlim_n H^2_c(k(\omega_n)/k)$.

Proof. Cf. $[Y]$ the proof is based on the Brauer induction theory in representation theory of finite groups.

1.3.4. Proposition (Bernard). In $[A] \in S(k)$ has order m then k must contain a primitive m^{th} root of unity. We may provide a more general result in connection with the theory of the Clifford-Schur group, see further.

1.3.5. Examples.

a. $\text{Br}(\mathbb{C}) = S(\mathbb{C}) = 1$.

b. $\text{Br}(\mathbb{R}) = S(\mathbb{R}) = \mathbb{Z}/r\mathbb{Z}$

c. $S(\mathbb{Q}_p) = \mathbb{Z}/(p-1)\mathbb{Z}$ if p is odd, $S(\mathbb{Q}_2) = \mathbb{Z}/2\mathbb{Z}$

d. Let k be a local field. We may assume that k is a subcyclotomic extension of $\mathbb{Q}_p$, i.e. an abelian extension (this is a consequence of the local version of the Krocki-Weber theorem). Galois cohomology theory of local fields entails that corestriction yields a monomorphism $S(k) \hookrightarrow S(\mathbb{Q}_p)$. If p is odd then the recipcocity map of local class theory may be used to determine the image, if $p = 2$ then the a very complicated argument leads to $|S(k)|$ is either 1 or 2. Let us write Q_p^{ab} for the abelian closure of $\mathbb{Q}_p$ obtained by adjoining all roots of unity to $\mathbb{Q}_p$. Then : $S(k) = \text{Gal}(\mathbb{Q}_p^{ab}(k)_{\text{tors}}$.

$S(\mathbb{Q}) = \text{Br}_2(\mathbb{Q}) =$ the 2-torsion part of $\text{Br}(\mathbb{Q})$.

In the last example the theorem of uniform distribution, due to Bernard and

M. Schacher plays an important part, so let us just mention it here.

1.3.6. Theorem. Let k be a finite abelian extension of $\mathbb{Q}$ and let p be a prime rational or ∞. If $[A] \in S(k)$ then the Hasse invariants of $[A]$ at the primes of k lying over the given p are all determined by one of them, the Hasse invariants all have the same order and the local Schur indices at the primes lying over p are equal.

The Schur group is obviously strongly linked to the representation theory of finite groups in a similar way one may define a Clifford-Schur subggroup of the Brauer group that is connected to projective representations of finite groups. Again these subgroups of the Brauer group are well-understood for $k = \mathbb{Q}$ or k a field containing enough roots of unity, but in general there remain several intriging problems concerning the relations $S(k) \subset CS(k) \subset Br(k)$.

First let us describe the **k-rational subgroup of** $\mathrm{Br}(k)$. Let $\mathrm{Br}_{\mathrm{rat}}(k)$ be the subgroup of $\mathrm{Br}(k)$ generated by all crossed products $(l/k, G, c)$ such that $c(\sigma, \tau) \in k^*$ for all $\sigma, \tau \in G = \mathrm{Gal}(k/k)$. We need an elementary fact :

1.3.7. Lemma. Let k be a field, then $k^* = \mu(k) \oplus F$ where F is a free abelian group and $\mu(k)$ is the group of roots of unity contained in k^*.

Proof. If k is a number field, let R be a number ring of integers in k. Then $U(R) = \mu \oplus$ free abelian group by the Dirichlet unit theorem (if char $k = p \neq 0$ then $k = R$ is a finite field and then $\mu(k) = k^*$. So we have :

$$0 \to U(R) \to k^* \to \oplus_p \mathbb{Z}$$

the sum over all primes p of R. Since subgroups of free abelian groups are free, k^* has the form we claimed. Now we proceed by induction on the transcendence degree of k over the prime subfield. So we may assume that k is a finite extension of $K(H)$ and the lemma holds for K (moreover we may assume that K is algebraically closed within k). Now there is an exact sequence $0 \to K^* \to k^* \to \oplus_p \mathbb{Z}$, the sum being over all discrete valuations of k/K. □

1.3.8. Lemma. (Brauer). If $\alpha \in \mathrm{Br}(k)$ then $\alpha = [(l/k, G, c)]$ where all $c(\sigma, \tau) \in l$ are roots of unity. If α has exponent m then we can arrange that $c(\sigma, \tau)^m = i$ for all $\sigma, \tau \in G$.

Proof. Put $\overline{G} = \mathrm{Gal}(\overline{k}_{\mathrm{sup}}/k)$ and $\alpha \in H^2(\overline{G}, \overline{k}^*_{\mathrm{sep}}), \mu = \mu(\overline{k}^*_{\mathrm{sep}})$. We have an exact sequence $0 \to \mu \to \overline{k}^* \overline{\mathrm{k}}^* / \mu \to 0$. As a $\overline{G}$-module, $\overline{k}^* / \mu$ is torsion free and m-divisible. Thus $H^n(\overline{G}, \overline{k}^*/\mu) = 0$ (except p = char$k$-part). Hence α is in the image of $H^2(\overline{G}, \mu)$, proving the first assertion. Furthermore $H^2_{p}, (\overline{G}, \mu| \hookrightarrow H^2_{p'}(\overline{G}, \overline{k}^*)$ is injective. Taking m-th powers defines an exact sequence $0 \to \mu_m \to \mu \xrightarrow{\cdot m} \mu \to 0$ yielding $H^2(\overline{G}, \mu_m) \to H^2(\overline{G}, \mu) \xrightarrow{\cdot m} H^2(\overline{G}, \mu)$, so α in the image of $H^2(\overline{G}, \mu_m)$! □

Now following a suggestion of D. Saltman we may prove.

1.3.9. Theorem. $\mathrm{Br}_{\mathrm{rat}}(k)$ is generated by all Brauer classes of cyclic algebras and algebras of exponent dividing $m = |\mu(k^*)|$.

Proof. $\mathrm{Br}_{\mathrm{ret}}|k|$ is the image of $H^2(\overline{G}, k^*)$ in $H^2(\overline{G}, \overline{k}^*_{\mathrm{rep}})$. Note that, since every k-c.s.a. A may be descended to a c.s.a. over a finitely generated field over the prime field, B say, such that $A = B \otimes k$, we may assume that $k^* = \mu \oplus F$ where F is free abelian and μ is cyclic of order m.

Now $H^2(\overline{G}, k^*) = H^2(\overline{G}, \mu) \oplus H^2(\overline{G}, F)$.

Since $H^2(\overline{G}, \mathbb{Z}) = H^1(\overline{G}, \mathbb{Q}/\mathbb{Z})$, the group $H^2(\overline{G}, F)$ is generated by cyclic algebras. Since it is also clear that all cyclic algebras are in $\mathrm{Br}_{\mathrm{rat}}(k)$ it will now suffice to observe that all $\alpha \in \mathrm{Br}(k)$ of exponent dividing m are indeed in $\mathrm{Br}_{\mathrm{rat}}(k)$ (because conversely we see that every element in $H^2(\overline{G}, \mu)$ has exponent dividing m). But by the lemma $\alpha = (l/h, G, c)$ where $c(\sigma, \tau) \in \mu$.□

For a commutative ring R and finite groups G_1 and G_2 we have the canonical embedding $H^2(G_1, U(K)) \times H^2(G_2, U(R)) \to H^2(G_1 r G_2, 0(R))$ where $U(R)$ is the group of units of R. These maps define an inductive system of abelian groups and we denote its inductive limit by $\mathcal{G} = \varinjlim_{G} H^2(G, U(R))$.

We identify $H^2(G, U(R))$ as a subgroup of $\mathcal{G}$.

For any given subgroup $\mathcal{H}$ of $\mathcal{G}$ we define th $\mathcal{H}$-Schur group in $\mathrm{Br}(R)$, $S_{\mathcal{H}}(R)$, to be the set of classes in $\mathrm{Br}(R)$, represented by Azumaya algebras over R that are epimorphic images of twisted group rings RG^c with $[c] \in H^2(G, U(R))$ contained in $\mathcal{H}$. Note that $RG^c = \oplus_{\sigma \in G} R u_\sigma$ with $u_\sigma u_\tau = c(\sigma, \tau) u_{\sigma\tau}$ for all σ, τ. If R is a field K then the matrix ($n \times n$ size) representations of KG^c correspond to the projective representations $G \to PGL_n(K)$ factorizing over a set map $G \to GL_n(K), \sigma \mapsto \rho_\sigma$ satisfying $\rho_\sigma \rho_\tau = c(\sigma, \tau) \rho_{\sigma\tau}$ for the given $[c]$.

A second construction depends on a class $\mathcal{P}$ of finite groups that is closed under finite products.

We will write $CS_{\mathcal{P}}(R)$ for the set of classes in $\text{Br}(K)$ represented by Azumaya algebras that are epimorphic images of RG^c for some $G \in \mathcal{P}$ and some $[c] \in H^2(G, U(R))$.

1.3.10. Lemma. The sets $S_{\mathcal{H}}(R)$ and $CS_{\mathcal{P}}(R)$ are subgroups of $\text{Br}(R)$. If $\mathcal{H}(\mathcal{P})$ is generated in $\mathcal{G}$ by the images of $H^2(G, U(R))$ for $C \in \mathcal{P}$ then $CS_{\mathcal{P}}(R)$ is a subgroup of $S_{\mathcal{H}(\mathcal{P})}(R)$.

Proof. Let $[A] \in S_{\mathcal{H}}(R)$ be given by an epimorphism π, $\pi : RG^c \to A$, where $[c] \in \mathcal{H}$. Let $(-)^\circ$ denote the opposite ring, then $\pi^\circ; (RG^c)^\circ \to A^\circ$ is an epimorphism of rings. Putting $RG^c = \oplus_{\sigma \in G} Ru_\sigma$ with $u_\sigma u_\tau = C(\sigma,\tau)u_{\sigma\tau}$ for $\sigma, \tau \in G$ we may obtain the opposite $(RG^c)^\circ$ by using the transformation $u_\sigma \mapsto u_{\sigma^{-1}}$ for $\sigma \in G$, in defining $c^\circ(\sigma,\tau) = c(\tau^{-1},\sigma^{-1})$ for $\sigma,\tau \in G$. Hence $[A^\circ] = [A]^{-1} \in S_{\mathcal{H}}(R)$, because $c(\tau^{-1},\sigma^{-1}) = c(\tau^{-1},\tau)c(\sigma^{-1},\sigma)c(\tau^{-1}\sigma^{-1},\sigma\tau)^{-1}c(\sigma,\tau)^{-1}$, i.e. $c(\tau^{-1},\sigma^{-1}) \sim c(\sigma,\tau)^{-1}$ in $H^2(G,U(R))$. Obviously, if $[A],[B] \in S_{\mathcal{H}}(R)$ are given by : $\pi_A : RG^c \to A, \pi_B : RH^d \to B$, respectively, then the canonical morphism $R(G \times H)^{(c,d)} \to A \otimes_R B$, determines $[A].[B]$ in $\text{Br}(R)$, hence $[A].[B] \in S_{\mathcal{H}}(R)$. Note that the isomorphism of R-algebras $\Psi : RG^c \otimes_R RH^d \to R(G \times H)^{(c,d)}$ is of course defined by $\Psi(u_g \otimes v_h) = w_{(g,h)}$ for $g \in G, h \in H$. By restriction to groups G, H in $\mathcal{P}$ a similar proof yields that $CS_{\mathcal{P}}(R)$ is a subgroup of $\text{Br}(R)$. The final statement is obvious. □

1.3.11. Examples. If $\mathcal{H} = 1$ then the $\mathcal{H} = 1$ then the $\mathcal{H}$-Schur group reduces to the Schur group $S(R)$ in $\text{Br}(R)$. In case $\mathcal{H} = \mathcal{G}$ then the $\mathcal{H}$-Schur group is called the Clifford-Schur subgroup of $\text{Br}(R)$, denoted by $CS(R)$. For $\mathcal{H} \subset H_1$ we have : $S(R) \subset S_{\mathcal{H}}(R) \subset S_{\mathcal{H}_1}(R) \subset CS(R)$. If $\mathcal{P}$ is the class of finite abelian groups, then we write $CS_{\text{ab}}(R)$ for $CS_{\mathcal{P}}(R)$ as defined above. In case $\mathcal{P}$ is the class of nilpotent finite groups we write $CS_{\text{nil}}(R)$ for $CS_{\mathcal{P}}(R)$. Finally, if $\mathcal{P}$ is the class of p-groups then we write $CS_p(R)$; note that $CS_p(R) \neq CS(R)_p$ a priori.

1.3.12. Lemma. The subgroup $CS(R)$ of $\text{Br}(R)$ consists of those classes of Azumaya algebras may be represented by an R-Clifford system of some finite group.

Proof. If $[A] \in CS(R)$ is given by $\pi_A : RG^c \to A$, then A is by definition a Clifford system. Conversely if $A = \sum_{\sigma \in G} Ru_\sigma$ with $u_\sigma u_\tau = \lambda(\sigma,\tau)u_{\sigma\tau}$ for $\sigma,\tau \in G$ and with $\lambda(\sigma,\tau) \in U(R)$, then A is an epimorphic image of $\hat{A} = \oplus_{\sigma \in G} R\omega_\sigma$ with multiplication $\omega_\sigma \omega_\tau = \lambda(\sigma,\tau)u_{\sigma\tau}$. The associativity of

A entails that λ represents an element of $H^2(G, U(R))$ and thus $\hat{A}$ is just RG^λ. □

The Schur subgroup is a complicated invariant to study, for rings it remains today a rather obscure object that has hardly been investigated. Even when a commutative ring does contain enough roots of unity it is not always possible to describe algebras representing a class in the Schur subgroup in terms of cyclic crossed product algebras because there is no good equivalent of the Mercurjev-Suslin theorem. On the other hand one may evoke results from [5] and we use a theory of projective characters that will allow to carry the general theory of the Clifford-Schur groups at least as far as the existing theory of the Schur group. In fact the $CS(R)$ is much closer to $\mathrm{Br}(R)$ than $S(R)$ in general; it is a nice problem to give a characterization of domains (even fields) for which $\mathrm{Br}(S) = CS(R)$.

The following characterization of Azumaya algebra classes in $CS(R)$ extends that classical result for fields relating $H^2(G, k^*)$ and $H^1(G, PGL_n(k))$. The constructive treatment we propose applies to commutative rings in a desirable generality.

1.3.13. Proposition. Let R be any commutative ring. Every $[A] \in CS(R)$ may be represented by an Azumaya algebra over R that is an epimorphic image of a skew group ring over $M_n(R)$ for some finite group of order n. In case $\mathrm{Pic}(R) = 1$, every such epimorphic image of a skew group ring over $M_n(R)$ is necessarily representing an element in the Clifford-Schur subgroup.

Proof. Recall that for any group G and any ring T a skew group ring $T^*_\varphi G$ is given by a group morphism $\varphi : G \to \mathrm{Aut}T$ and $T^*_\varphi G = \oplus_{\sigma\in G} T\omega_\sigma$ with $\omega_\sigma\omega_\tau = \omega_{\sigma\tau}$ for all $\sigma, \tau \in G$ but $\omega_\sigma t = \varphi_\sigma(t)\omega_\sigma$, where φ_σ denotes $\varphi(\sigma)$, for all $t \in T$. Now consider $[A] \in CS(R)$ represented by $\pi_A : RG^c \to A$ for some given $[c] \in H^2(G, U(R))$. Consider $M_n(RG^c)$ where $n = |G|$ and write :$RG^c = \oplus_{\sigma\in G} Ru_\sigma$, where $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma\tau}$ for all $\sigma, \tau \in G$. Right multiplication by u_σ^{-1} in RG^c determines an R-linear map $m_\sigma : RG^c \longrightarrow RG^c$ that may be given by an invertible matrix M_σ in $M_n(R)$, (using the basis $u_\sigma, \sigma \in G$, in RG^c). From $u_\sigma u_\tau = c(\sigma,\tau)u_{\sigma\tau}$ it follows that $M_\sigma M_\tau = c(\sigma,\tau)^{-1}M_{\sigma\tau}$ for all $\sigma, \tau \in G$, (note : the appearance of $c(\sigma,\tau)^{-1}$ is due to the use of right multiplications !). Define a group morphism $\varphi_c : G \to \mathrm{Aut}_R M_n(R), \sigma \to \varphi_\sigma$, where φ_σ is given by $\varphi_\sigma(x) = M_\sigma x M_\sigma^{-1}$ for every $x \in M_n(R)$. Now construct the R-algebra $D = M_n(R) *_{\varphi_c} G = \oplus_{\sigma\in G} M_n(R)U_\sigma$, where for each $\sigma \in G : U_\sigma r = \varphi_\sigma(r)U_\sigma$ for $r \in M_n(R)$. Put $W_\sigma = U_\sigma M_\sigma^{-1}$. Then : $W_\sigma x W_\sigma^{-1} = U_\sigma M_\sigma^{-1} x M_\sigma U_\sigma^{-1} = \varphi_\sigma(M_\sigma^{-1} x M_\sigma) = *\varphi_\sigma^{-1}(x) = x$. Therefore we obtain : $D = \oplus_{\sigma \in G} M_n(R) W_\sigma$ where W_σ centralizes $M_u(R)$ in D.

Now we calculate :

$$\begin{aligned} W_\sigma W_\tau &= U_\sigma M_\sigma^{-1} U_\tau M_\tau^{-1} = U_\sigma U_\tau \varphi_\tau^{-1}(M_\sigma^{-1}) M_\tau^{-1} \\ &= U_{\sigma\tau} M_\tau^{-1} M_\sigma^{-1} M_\tau M_\tau^{-1} = W_{\sigma\tau} M_{\sigma\tau} M_\tau^{-1} M_\sigma^{-1} \end{aligned}$$

where $M_{\sigma\tau} M_\tau^{-1} M_\sigma^{-1} = c(\sigma,\tau)$ (see above).
Consequently $D = \oplus_{\sigma \in G} M_n(R) W_\sigma \cong M_n(\oplus_{\sigma\in G} R W_\sigma) \cong M_n(RG^c)$. Clearly $M_n(\pi_A) : M_n(RG^c) \to M_n(A)$, where $[M_n(A)] = [A]$, provides the epimorphism claimed to exist in the first part of the proposition. In this generality we have associated to a given $[c]$ such that $\pi_A : RG^c \to A$ represents $[A]$ in $CS(R)$ a group morphism $\varphi_c : G \to \mathrm{Aut}_R M_n(R)$ representing $[A]$ by an epimorphic image of $M_n(R) *_{\varphi_c} G$. Conversely let there be given a group morphism : $\varphi : G \to \mathrm{Aut}_R M_u(R)$ and consider $B = \oplus_{\tau\in G} M_n(R) u_\tau$, with $u_\tau x = \varphi_\tau(x) u_\tau$ for $x \in M_n(R), \tau \in G$. Since $\mathrm{Pic}(R) = 1$ we may apply the Skolem-Noether theorem for Azumaya algebras over a centre with trivial Picard group (cf. [ ]) and obtain units v_τ in $M_n(R)$ such that $\varphi_\tau(x) = v_\tau x v_\tau^{-1}$ for all $x \in M_n(R)$. Clearly, $v_\tau^{-1} u_\tau$ commute with the whole $M_n(R)$. Putting $\omega_\tau = v_\tau^{-1} u_\tau$ for $\tau \in G$ we then calculate :

$$\begin{aligned} \omega_\sigma \omega_\tau &= v_\sigma^{-1} u_\sigma v_\tau^{-1} u_\tau = v_\sigma^{-1} \varphi_\sigma(v_\tau^{-1}) u_\sigma u_\tau \\ &= v_\sigma^{-1} v_\sigma v_\tau^{-1} v_\sigma^{-1} u_{\sigma\tau} = v_\tau^{-1} v_\sigma^{-1} v_{\sigma\tau} \omega_{\sigma\tau} \end{aligned}$$

Put $c(\sigma,\tau) = v_\tau^{-1} v_\sigma^{-1} v_{\sigma\tau}$. It is easily checked that $c(\sigma,\tau) \in U(R)$ because each $c(\sigma,\tau)$ commutes with the whole $M_n(R)$ and it is equally obvious that c is a 2-cocycle. So we use this $[c]$ for the class $[c_\varphi]$ associated to φ, $[c_\varphi] \in H^2(G, U(R))$. Therefore $\oplus_{\sigma\in G} k\omega_\sigma \cong kG^{c_\varphi}$ and it commutes with $M_n(R)$ in the ring $M_n(R) *_\varphi G$. Hence $M_n(R) *_\varphi G \simeq M_n(R) \otimes_R RG^{c_\varphi} \cong M_n(RG^{c_\varphi})$. It is clear from looking at these instructions that φ is the automorphism obtained from c_φ and c the 2-cocyclic obtained from φ_c (taking care of some inverses in the right way).

Elements in the centre $Z(RG^c)$ of a twisted group ring RG^c which are homogeneous in the G-gradation of RG^c represent special ray-classes (cf. [ ]) that one can get rid of by reducing the group G first. Recall that $\sigma \in G$ is said to be c-regular if for all τ commuting with σ we have $c(\sigma,\tau) = c(\tau,\sigma)$.

The inverse and all conjugates of a c-regular element are again c-regular. A **ray class** is the class of conjugates of a c-regular element and a **ray class sum** is the sum of the elements u_σ in $RG^c = \oplus_{\sigma \in G} Ru_\sigma$ where σ varies over a ray class. We recall, cf. [ ] :

1.3.14. Proposition. If R is a connected commutative ring (i.e. 0 and 1 are the only idempotent elements in R), then the centre $Z(RG^c)$ is freely generated over R by the ray class sums.

It is clear that a c-regular element r of G that is contained in $Z(G)$ corresponds to a basic elements u_σ that is central in RG^c. Moreover it is equally clear that $Z(G)_{\text{reg}} = \{\sigma \in Z(G), \sigma \text{ is } c\text{-regular in } G\}$ is a subgroup of G and that $RZ(G)^c_{\text{reg}}$ is central in RG^c. We now immediately obtain the following useful result :

1.3.15. Proposition. If R is a connected commutative ring then we may assume, for an $[A] \in CS(R)$, that it is given by an epimorphism $\Psi : (RG^c \to A$ where $Z(G)_{\text{reg}} = \{e\}$, e the unit of G.

Proof. By the foregoing $RZ(G)^c_{\text{reg}}$ is central in RG^c and also $RG^c = (RZ(G)^c_{\text{reg}}(G/Z(G)_{\text{reg}})^{\overline{c}}$ where $\overline{c}$ is defined in the obvious way but noting that $\overline{c}$ also takes its values in $U(R)$. Since $Z(A) = R$, the epimorphism Ψ takes $RZ(R)^c_{\text{reg}}$ to R i.e. the u_σ with $\sigma \in Z(G)_{\text{reg}}$ are specialized to some $x_\sigma \in U(R)$ and it follows that we may factorize Ψ through $RG^c \to R(G/Z(G)_{\text{reg}})^{\overline{c}} \longrightarrow A$. If necessary we continue the same argument for $G/Z(G)_{\text{reg}}$ and $\overline{c}$, the result will follow eventually. □

1.3.16. Remarks 1. The proposition extends Hilfsatz 1 of H. Oplolka, F. Lorenz, [ ], which is given for fields using algebraic closure and absolutely irreducible projective representations. Our proof shows that this elementary observation is valid very generically and that is really just a matter of determining the centre.

2. To c we may associate the pairing $f : G \times G \to U(R)$, $(\sigma, \tau) \mapsto c(\sigma,\tau)c(\tau,\sigma)^{-1}$. Since the order of $[c]$ divides $n = |G|$, the values $f(\sigma,\tau) \in U(R)$ are n-th roots of unity. In fact, in view of the foregoing proposition, the restriction of f to $G \times Z(G)$ has the property that $f(-,\tau)$ is not trivial $G \to U(R)^*$. So after reduction to the case where $Z(G)_{\text{reg}} = \{e\}$ it follows that R contains the m-th roots of unity where m is the exponent of $Z(G)$. Note that, for $[c] = 1$ we arrive at the reduction of $RG \to A$

to $R(G/Z(G))^{\overline{c}} \to A$ and no condition on roots of unity. We refer to $Z(G)/Z(G)_{\text{reg}}$ as the **root-group** of the cocycle c. Proposition 2.2. may then be rephrased as follows : if $A \in CR(R)$ is given by $RG^c \to A$ then it may be given by $RG_1^{c_1} \to A$ where G_1 has trivial root group in the sense that it equals $Z(G_1)$. For example if $R = \mathbb{Q}$ then the only p group that can appear is $(\mathbb{Z}.z\mathbb{Z})^t, t \in \mathbb{N}$.

3. If G is abelian then after the reduction discussed in the proposition we obtain that $Z(RG^c) = R$ because $Z(RG^c)$ is G graded for an abelian group G. When $|G|^{-1} \in R$ then a nontrivial ideal of RG^c intersects $R = Z(RG^c)$ nontrivially it follows that the R-central Azumaya algebra A is necessarily isomorphic to RG^c. It would not be so restrictive to restrict attention to Q-algebras R but in order to allow rings of integers (over $\mathbb{Z}$) etc... we include.

1.3.17 Proposition. Let R be a connected commutative ring and let $RG^c \longrightarrow A$ represent $[A] \in CS(R)_{ab}$, where G is an abelian finite group and RG^c being an Azumaya algebra, then $A \cong RG^c$.

Proof. In the proof of Proposition 1.3.15. we have seen that $RG^c \to A$ factorizes through $RG^c \to R(G/Z(G)_{\text{reg}})^{\overline{c}} \longrightarrow A$ and hence we may assume that $Z(G)_{\text{reg}} = \{e\}$ without destroying the Azumaya condition put on RG^C. Consequently we may assume that $Z(RG^c) = R$. If p is a prime ideal of R then $(R/p)G^{\overline{c}} \to A/pA$ represents an element of $CS(R/P)_{\text{ab}}$ and $Z((R)p/G^{\overline{c}}) = R/p$. If $|G| \in p$, let G_p be a p-Sylow subgroup of G for $p = \text{char}(R/p)$. For $\sigma \in G_p$ we obtain $u_\sigma u_\tau = \frac{c(\sigma,\tau)}{c(\tau,\sigma)} u_\tau u_\sigma$. Since $u_\sigma^p \in U(R)$ it follows that $\left(\frac{c(\sigma,\tau)}{c(\tau,\sigma)}\right)^p = 1$, hence $\overline{c}(\sigma,\tau) = \overline{c}(\tau,\sigma)$ or u_σ mod pRG^c is central in $(R/p)G^{\overline{c}}$, i.e. it is in R/p. The latter contradicts the fact that for any $\mu \in R$, $u_\sigma - \mu$ cannot be in pRG^c. Consequently the reduction to the case Greg $= \{e\}$ allows to assume that $G_p = \{e\}$ for every prime p of R such that $|G| \in p$ i.e. to the case $|G|^{-1} \in R$ cf. Remark 1.3.16(3).

1.3.12 Corollary. If the class of A in $CS(R)_{ab}$ is represented by $RG^c \to A$ where RG^c is an Azumaya algebra, then the root-group of c has invertible order in $U(R)$.

1.3.19. Corollary. Let m be the order of the torsion part of $U(R)$, where R is a connected commutative ring, then :

1. $CS(R)_{ab} \subset \text{Br}(R)_m$

2. $CS(R)_{\text{nil}} \subset \prod_m \text{Br}(R)_p$

3. If R is a numberfield k then equality holds in **1.** and **2.**.

Proof. 1. If $[A] \in CS(R)_{ab}$ is represented by $RG^c \longrightarrow A$ then first we may assume that $G_{\text{reg}} = \{e\}$ in view of Proposition 1.3.15. If n is the exponent of G then Remark 1.3.16(2) entails that $n|m$. The exponent of $[A]$ in $\text{Br}(R)$ is a divisor of n hence of m.

2. If $[A] \in CS(R)_{\text{nil}}$ is represented by $\pi : RG^c \to A$ where G is nilpotent then $RG^c = RG^{c_1}_{p_1} \otimes_R \ldots \otimes_R RG^{c_t}_{P_t}$ where each c_i is the restriction of G to G_{p_i} and G is the product of its Sylow subgroups $G_{p_i}, i = 1, \ldots, t$. Indeed, if $\sigma \in G_{p_1}, \tau \in G_{p_2}$ then $u_\sigma u_\sigma = \frac{c(\sigma,\tau)}{c(\tau,\sigma)} u_\tau u_\sigma$ yields that $\frac{c(\sigma,\tau)}{c(\tau,\sigma)}$ is a root of unity for a p_1-power as well as for some p_2-power, hence $c(\sigma, \tau) = c(\tau, \sigma)$ or u_σ commutes with u_τ. The decomposition of A into a product of algebras from $\text{Br}(R)_p$ is then obtained by taking $A_{p_i} = \pi(RG^{c_i}_{p_i})$, $i = 1, \ldots, t$. Note that A_{p_i} does indeed define a class in $\text{Br}(R)_{p_i}$ with $p_i|m$ because $Z(G_{p_i})$ is a non-trivial p_i-group the exponent of which has to be a divisor of m.

3. In case $R = k$, the converse of **1.** follows immediately from the Hasse-Brauer-Noether theorem because this allows to replace A by a cyclic algebra $(l/k, H, d)$ where l/k has Galois group H and $d \in H^2(H, k^*), H$ a cyclic group. Since k contains the m-th roots of unity, it follows that $l = kH^e$ for some $[e] \in H^2(H, k^*)$. Therefore $A = l * H$ may be obtained as an epimorphic image of a twisted group ring kE^c where E is a suitable central extension of $H^*_{(e,d)}H$ (see further for the actual construction of E and $H_{(e,d)} * H$). The converse of **2.** follows in a similar way.

The statement of Corollary 1.3.19(3) has been proved in [ ], there we also find :

1.3.20. Proposition. Let k be a number field then $CS(k) = \text{Br}(k)$. For an abelian group G we define $H^2_{\text{sym}}(G, U(R))$ to be the subgroup of $H^2(G, U(R))$ consisting of the symmetric 2-cocyclies, i.e. those satisfying the relation $c(\sigma, \tau) = c(\tau, \sigma)$ for $\sigma, \tau \in G$. We write $\text{Br}_{\text{sym}}(R)$ for the subgroup of the Brauer group consisting of algebra classes represented by a crossed product algebra $(S/R, G, c)$ where S/R is a Galois extension with abelian Galois group G and c represents $a[c] \in H^2_{\text{sym}}(G, U(R))$. Some caution is due here because over general commutative rings it is not necessarily true that an Azumaya algebra split by some Galois extension S/R is automatically equivalent to a crossed product $(S/R, \text{Gal}(S|R), c)$ i.e. in general $\text{Br}(R)_{\text{sym}}$ may be different from $\varinjlim_{G} H^2_{\text{sym}}(G, U(R))$. For local rings, more generically for rings R with trivial Picard group $\text{Pic}(R)$, the forementioned problem does not present itself.

1.3.21. Proposition. Let R be a connected commutative ring with $\text{Pic}(R) = 1$ and assuming that R contains primitive n-th roots of unity for all n, then $\text{Br}_{\text{sym}}(R) \subset CS(R)$.

If R is moreover a field k then :

$$\text{Br}(k) = CS(k) = CS_{\text{nil}}(k) = CS_{ab}(k)$$

Proof. The proof of the second statement follows from the Merkurjev-Suslin theorem because every element of $\text{Br}(k)$ may be represented by a product of (symbols) cyclic algebras and the following proof of the first statement will entail that the latter algebras are in $CS_{\text{ab}}(k)$.

Now consider $[A] \in Br\text{Br}_{\text{sym}}(R)$, say $A = (S/R, H, d)$ where H is abelian and $[d] \in H^2(H, O(R))$. The presence of roots of unity in R yields, by a result of G. Bergman (by Kummer theory in the case of fields), yields that S/R is H-strongly graded (in the sense of [ ] or [ ]) and hence $\delta = RH^e$ for some $[e] \in H^2(H, U(R))$ because $\text{Pic}(R) = 1$ cf. loc. cit.. Define an action of H on itself by defining a group morphism $\chi : H \longrightarrow \text{Aut}(H)$ as follows : $\chi(\sigma)(\tau) = \deg_H \sigma(u_\tau)$ where $\delta = RH^e = \oplus_{\tau \in H} Ru_\tau$ and $\sigma(u_\tau)$ is the image under the Galois action (that is homogeneous here). On $H \times H$ we now define a group structure $E = H *_\chi H$ by the multiplication rule :

$$(\sigma, \tau)(\sigma', \tau') = (\sigma\chi(\tau)(\sigma'), \tau\tau')$$

Since e takes values in $U(R)$ we have : $\sigma(u_\tau) = u_{\chi(\sigma)(\tau)}$. In a straightforward way one may calculate that α given by : $\alpha[(\sigma, \tau), (\sigma_1, \tau_1)] = e(\sigma, \chi(\tau)(\sigma_1))d(\tau, \tau_1)$, determines a 2-cocycle $[\alpha] \in H^2(E, U(R))$. An R-algebra epimorphism $\pi : RE^\alpha \longrightarrow A$ may now be defined by $(\sigma, \tau) \mapsto u_\sigma v_\tau$, where $A = \oplus_{\tau \in H} \delta u_\tau$ with $v_\tau v_{\tau_1} = d(\tau, \tau_1)$ (in fact $A \cong RE^\alpha$). □

1.3.22. Corollary. If k is either a numberfield, or an algebraic extension of a p-adic field or a field containing all roots of unity, then $(CS(R)_{\text{nil}})_p = CS_p(R)$.

Proof. Follows easily from the foregoing results.

For a prime p, a divisor of the order, m, of the torsion part of $k(R)$ we do not know whether $CS_p(R) = CS(R)_p$. Obviously, in view of Corollary 1.3.19.(3), this equality does hold when $R = k$ is a number field. The main problem here comes down to the following :

Problems 1. Let k be a field containing the n-th roots of unity. Let Δ be a skewfield over k such that $H_n(\Delta)$ is a twisted group ring kG^c is then Δ a twisted groupring kH^d where H is related to G in a suitable sense, in fact the following relation would then have some possibility of existing :

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow 1$$

2. Suppose Char$k = 0$ and k contains all roots of unity. Is it true that $CS(k)_p = CS_p(k)$. Note that $Br(k)_p = CS(k)_p$ does hold here. Just as in **1.** the problem seems to be created by the necessity of obtaining all algebras representing elements of $CS(k)_p$ as epimorphic images of twisted group rings (i.e. not only up to equivalence in the Brauer group).

3. For a field as in Proposition 1.3.21. k say, the equality $\mathrm{Br}(k) = CS(k)_{ab}$ entails the Merkurjev-Suslin theorem. Now $CS(k) = CS(k)_{ab}$ may be proved here without referring to the Merkurjev-Suslin theorem so one might hope to obtain a non-K-theoretic approach to this result by finding a proof of $\mathrm{Br}(k) = CS(k)$ not depending on it.

2. Generic Division Algebras and the Merkurjev-Suslin Theorem

2.0. Introduction.

The best known general result on the generation of Brauer groups of a field is the following one proved by Merkurjev and Suslin in [36] :

2.0.0. Theorem (Merkurjev-Suslin)
If k is a field containing a primitive n-th root of unity, then the n-torsion part of the Brauer group of $k, \mathrm{Br}_n(k)$, is generated by cyclic algebras.

The proof of this result depends on fairly heavy K-theory, see the survey papers of Van der Kallen [47] and Soulé's Bourbaki talk [45] for some of the details. The original proof can be found in [36] and a simplified version in [47].

Starting with the work of Amitsur, ring theorists have tried to prove the conjectural generation of the Brauer group by cyclic algebras via the so-called "generic division algebras". Using a result of S. Bloch, the rationality (or stable rationality) of the center of these generic division algebras would imply the Merkurjev-Suslin results as well as some (at this moment unknown) results on Azumaya algebras of local rings. Moreover, recent results show the importance of this (stable) rationality problem for as different field as algebraic geometry and the representation theory of finite dimension wild hereditary algebras.

We will outline the known results on this problem focussing on the more recent ones. For this reason we will content ourselves with giving precise references for the proofs of many of the more classical results.

2.1. The generic Division Algebras.

For the moment, k will be an arbitrary (but usually infinite) field. For all integers $n, m \in \mathbb{N}$ we consider the polynomial algebra

$$\mathcal{P}_{m,n} = k[x_{ij}(k) : 1 \le i, j \le n, 1 \le k \le m]$$

that is, in as many variables as there are entries in m n by n matrices. The fields of fractions of $\mathcal{P}_{m,n}$ will be denoted by $\mathcal{K}_{m,n}$. The ring of m generic n by n matrices, $\mathbb{G}_{m,n}$, is the k-subalgebra of $M_n(\mathcal{P}_{m,n})$ generated by the so called generic matrices

$$X_l = (x_{ij}(l))_{i,j} \in M_n(\mathcal{P}_{m,n})$$

for all $1 \leq l \leq m$. Another interpretation of $\mathbb{G}_{m,n}$ can be given as follows : consider the quotient of the free k-algebra in m variables

$$\mathbb{H}_{m,n} = k < x_1, \ldots, x_m > / I_n$$

where I_n is the T-ideal (i.e. an ideal closed under all k-endomorphisms of $k < x_1, \ldots, x_m >$, see [P, p. 43]) of all identities of $M_n(k)$. Then, there is a natural morphism

$$\mathbb{H}_{m,n} \longrightarrow \mathbb{G}_{m,n}$$

by sending the variable x_i to the generic matrix X_i. This is easily seen to be an isomorphism. This entails that $\mathbb{G}_{m,n}$ has the following universal property. Suppose $k \subset l$ and Δ is a central simple l-algebra of rank n^2 which is generated as l-algebra by $\delta_1, \ldots, \delta_m$. Then there is a morphism

$$\pi : \mathbb{G}_{m,n} \longrightarrow \Delta$$

by sending X_i to δ_i and $\pi \otimes l$ is an epimorphism. Using this universal property it is then fairly easily to prove the next result :

2.1.1. Theorem (Amitsur,)
For all $n, m \in \mathbb{N}, \mathbb{G}_{m,n}$ is a domain.

For a proof see [40, p.63]. Since $\mathbb{G}_{m,n}$ is obviously a ring satisfying a polynomial identity we can apply Posner's result [22] to obtain that $\mathbb{G}_{m,n}$ is a left and right Öre-domain and hence has a classical ring of quotients $\Delta_{m,n}$ which is a division algebra of rank n^2 over its center $Z_{m,n}$, see also [40, p. 63].

The division algebra $\Delta_{m,n}$ is called the generic division algebra for m n by n matrices.

2.1. The rationality problem :

Recall that a field $L \supset k$ is said to be rational over k iff there exist $x_1, \ldots, x_k \in L$ s.t. $L = k(x_1, \ldots, x_k)$ the purely trancendental field of trancendence degree k.

A field $L \supset k$ is said to be stably rational over k iff a rational extension of L is rational over k, i.e. there exist $k, l \in \mathbb{N}$ s.t. $L(y_1, \ldots, y_k) \cong k(x_1, \ldots, x_l)$.

Finally, a field $L \supset k$ is said to be unirational if L is contained in a rational field over k, i.e. $L \subset k(x_1, \ldots, x_k)$.

Clearly, one has the following implications

$$\text{rational} \Rightarrow \text{stably rational} \Rightarrow \text{unirational}$$

The converse implication unirational $\stackrel{?}{\Longrightarrow}$ stably rational was called the Lüroth problem and stably rational $\stackrel{?}{\Longrightarrow}$ rational was called the Zariski problem. Both are now settled negatively, the Lüroth problem by a.o. M. Artin-D. Mumford and the Zariski problem only recently by J. L. Colliot-Thélène and J.J. Sansuc.

One of the main open problems in p.i.theory and in the study of Brauer groups is :

Question 1. For which $m, n \in \mathbb{N}$, is $Z_{m,n}$ (stably) rational k ?

In the next section we will illustrate the importance of this question with respect to the Brauer group. Other applications will be given later.
Clearly, $Z_{m,n} \subset \mathcal{K}_{m,n}$ hence $Z_{m,n}$ is obviously unirational. An easy but very useful result on this question is :

2.1.1. Proposition (Procesi).
For any $n \in \mathbb{N}, m \geq 2 : Z_{m,n}$ is rational over $Z_{z,n}$.

Proof : Let $\{u_1, \ldots, u_n z\}$ be a basis of $\Delta_{2,n}$. We claim that the $n^2(m-2)$ elements

$$\mathrm{Tr}(X_i u_j) \qquad 3 \leq i \leq m; 1 \leq j \leq n^2$$

form a trancendence basis of $Z_{m,n}$ over $Z_{2,n}$. Later on, we will independently see that $\mathrm{trdeg} Z_{m,n} = (m-1)n^2 + 1$ for all m, n. Therefore, it is sufficient to verify that this set generates $Z_{m,n}$. Let $L = Z_{2,n}\{Tr(X_i u_j) : 3 \leq i \leq m, 1 \leq j \leq n^2\} \subset K_{m,n}$ and consider $L\Delta_{2,n} \subset \Delta_{m,n}$. Then, $L\Delta_{2,n} \cong L \otimes_{Z_{2,n}} \Delta_{2,n}$ is a central simple algebra over L. For every generic matrix we have

$$X_i = \sum_{j=1}^{n^2} \alpha_{ij} u_j$$

with $\alpha_{ij} \in Z_{m,n}$ (this is because $Z_{m,n}\Delta_{2,n} = \Delta_{m,n}$ by simplicity) but then

$$\mathrm{Tr}(X_i u_k) = \sum_{j=1}^{n^2} \alpha_{ij} \mathrm{Tr}(u_j u_k)$$

for all i and k. Since $\mathrm{Tr}(X_i u_k) \in L$ and $\mathrm{Tr}(u_j u_k) \in L$ (because they are in $Z_{z,n}$) we obtain that all $\alpha_{ij} \in L$. Hence, $X_i \in L\Delta_{2,n}$ for all $1 \leq i \leq m$ whence $L\Delta_{z,n} = \Delta_{m,n}$ and so $L = Z_{m,n}$. □

This result reduces question 1 to the special case of two matrices. Since clearly $Z_{1,n}$ is rational with trancendence basis the coefficients of the characteristic polynomial of X_1, a naive approach to question 1 would be to prove that $Z_{2,n}$ is rational over $Z_{1,n}$. However, as we will see later this cannot be true even for $n = 4$.

2.3. Stable rationality implies Merkurjev-Suslin

In this section we will clarify the main motivation for trying to answer question 1 positively. As mentioned before this uses a result of S. Bloch proved in 1973 but published only in 1981 [15]. Bloch states his result in terms of K-theory, but we need the following formulation :

2.3.1. Theorem (Bloch, 1973)

Let k be a field such that the n-torsion part of the Brauer group is generated by cyclic algebras.
Assume $\mathrm{char}(k) \not| n$ and k contains a primitive n-th root of unity. Let L be rational over k, then the n-torsion part of $\mathrm{Br}(L)$ is generated by cyclic algebras.

We recommend the Fein-Schacher proof of this result [18, p. 54] which is based on the Auslander-Brumer-Faddeev theorem and a result of Rosset on the relation between the trace map on K_2 and the corestriction on the Brauer group. The initial assumption on k is satisfied if k is a local, global or algebraically closed field.

Now, assume that $Z_{m,n}$ is stably rational over k which is a field satisfying the assumptions of theorem 2, then by applying Bloch's theorem twice we obtain that the n-torsion part if $\mathrm{Br}(Z_{m,n})$ is generated by cyclic algebras. In particular, $\Delta_{m,n}$ is Brauer equivalent to a tensor product of cyclic algebras. That is, for some h

$$(*) \qquad M_h(\Delta_{m,n}) \cong S_1 \otimes S_2 \otimes \ldots \otimes S_u$$

with S_i cyclic. This fact plays a crucial role in the proof of the following result.

2.3.2. Theorem (Procesi, 1981). Let k be a field satisfying the assumptions of theorem 2 and assume that $Z_{m,n}$ is stably rational over k. Then, or any field $l \supset k$ the n-torsion part of $\text{Br}(l)$ is generated by cyclic algebras.

The proof consists in an easy specialization argument, the most difficult part of it consists in translating the italian of [41, p. 4]. Thus, stably rationality of $Z_{m,n}$ gives a purely ringtheoretical proof of the Merkurjev-Suslin result. Moreover, it would give new information such as :

2.3.3. Theorem. (Saltman, [43]).

Let R be a local ring with residue field l containing a field k satisfying the assumptions of theorem 2. If $Z_{m,n}$ is stably rational over k, then every division algebra Λ over l can be lifted to an Azumaya algebra Δ over R, i.e. $\Lambda \otimes l \cong \Delta$.

Note that the Merkurjev-Suslin result implies this result only upto equivalence in the Brauer groups. Further, Saltmans result only presupposes so called "retract rationality" of $\Delta_{m,n}$. For a proof and definition of retract rationality we refer the reader to [43].

2.4. S_n-modules and $Z_{m,n}$

There are two important realizations of $Z_{m,n}$ as the invariant field of a group action on a rational field. In a later section we will see that $Z_{m,n}$ is the field of $GL_n(k)$- (really of $PGL_n(k)$-) invariants acting on $\mathcal{K}_{m,n}$.
In this section we will see that $Z_{m,n}$ is also the invariant field of a finite group action on a rational field. The group is the symmetric group S_n on n letters and the rational field turns out to be a splitting field of $\Delta_{m,n}$.

Consider the first generic matrix X_1. The characteristic roots of X_1 are algebraically independent over k. For, consider the specialization given by sending X_1 to

$$\overline{X}_1 = \begin{pmatrix} x_{11}(1) & & 0 \\ & \ddots & \\ 0 & & x_{nn}(1) \end{pmatrix} \qquad \text{i.e.} \quad \overline{X}_{ij}(1) = \delta_{ij} x_{ij}(1)$$

then the characteristic polynomial of X_1 specializes to that of $\overline{X}_1$ and since the coefficients of the characteristic polynomial of $\overline{X}_1$ are independent over k, the same is true for X_1.

So, call the eigenvalues of X_1 $\{u_1, \ldots, u_n\}$ and consider the subfield $F_{m,n} = Z_{m,n}(u_1, \ldots, u_n) \hookrightarrow \overline{K}_{m,n}$ the algebraic closure of $K_{m,n}$. One can then prove the following important result :

2.4.1. Theorem (Procesi)

(1) : $F_{m,n}$ is a splitting field for $\Delta_{m,n}$

(2) : $F_{m,n}/F_{m,n}$ is Galois with $\text{Gal}(F_{m,n}/Z_{m,n}) \cong S_n$

(3) : $F_{m,n}$ is rational over k of trdeg $= (m-1)n^2 + 1$

For a proof of this result, see [P. , p. 9 5]. Since by proposition 1 one can restrict attention to the special case $m = 2$ we will now give the precise description of $F_{2,n}$ due to Formanek [19] :
Let $K = \overline{K}_{2,n}$, then there exists an invertible matrix $T \in M_n(K)$ with all its entries lying in $\mathcal{K}_{1,n}(u_1, \ldots, u_n)$ (where the u_i's are the characteristic roots of X, as above), such that

$$TX_1T^{-1} = \text{diag}(x_1, \ldots, x_n)$$

Further, the diagonal entries of $TX_1T^{-1} \stackrel{(N)}{=} X$ and all entries of $T.X_2.T^{-1} \stackrel{(N)}{=} Y$ are algebraically idempotent over k. We will use the following notation :

$$X = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} ; Y = \begin{pmatrix} y_{11} & \cdots & y_{in} \\ \vdots & & \\ y_{ni} & \cdots & y_{nn} \end{pmatrix}$$

and $L = k(x_1, \ldots, x_n; y_{11}, \ldots, y_{nn})$. With $\mathcal{G}$ we will denote the k-subalgebra of $M_n(L)$ generated by X and Y. As before, $\mathcal{G}$ can be shown to be a domain with classical ring of quotients Δ which is a division algebra with centre Z. Then, it is clear that the map $X_1 \vdash X; Y_1 \vdash Y$ induces k-algebra isomorphisms between $\mathcal{G}_{2,n}$ and $\mathcal{G}$, between $\Delta_{2,n}$ and Δ and between $Z_{2,n}$ and Z.

Of fundamental importance will be a "nearly" short exact sequence of S_n-modules

$$(**) \qquad 1 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} U \xrightarrow{\beta} V \longrightarrow 1$$

where $B = < x_1, \ldots, x_n; y_{11}, \ldots, y_{nn} >$ is the free Abelian group of rank $n^2 + n$ written multiplicatively which becomes an S_n-module via the action

$\sigma(X_i) = X_{\sigma(i)}; \sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. Let $U =< u_1, \ldots, u_n >$ be the free Abelian group of rank n with the permutation action of S_n and $V =< v >$ the free Abelian group of rank one with trivial S_n-action.
One defines the S_n-module homomorphisms :

$$\alpha : B \to U; \alpha(X_i) = 1, \alpha(y_{ij}) = u_i u_j^{-1}$$
$$\beta : U \to V; \beta(u_i) = v$$

and defines $A = \text{Ker}\alpha$, then (**) is seen to be exact.

Moreover, A is easily seen to be free Abelian of rank $n^2 + 1$ (it is a f.g. torsionfree Abelian gp) and the subfield $F(A) \subset L$ is rational over k of trancendence degree $n^2 + 1$.
Further, A is generated by $X_1, \ldots, X_n$ and all the monomials of the form $y_{i,i_2} y_{i_2 i_2} \cdots y_{i_q i_1}$ with $q \geq 1$.

Clearly, the S_n-action on A induces an action of S_n as k-auto-morphisms on $k(A)$ and one can prove [19, p. 206] :

2.4.2. Thereom 6 : (Fomanek d'après Procesi).

$$Z = k(A)^{S_n}$$

2.5. The easy cases $n = 2$ or $n = 3$
Theorem 6 allows us to prove rationality by hand for small values of n. This has been done by Procesi and Formanek.

2.5.1. Proposition : (Procesi).
$Z_{2,2}$ (and hence $Z_{m,2}$) is rational over k.

Proof. $k(A) = kok(x_1, x_2, y_{11}, y_{22}, y_{12}y_{21})$ with the obvious $S_2 = \mathbb{Z}/2\mathbb{Z}$ action. It is easy to verify that

$$k(A)^{S_2} = k(x_1 + x_2, x_1x_2, y_{11} + y_{22}, y_{11}y_{22}, y_{12}y_{21})$$

which is rational over k.

Actually, Procesi proved rationality in another way [P, p. 99]. He proved that $Z_{2,2} = (Tr(X_1), T_2(Y_2), D(X_1), D(X_2), D(X_1 + X_2))$. It is an easy excercise to see that the two approaches give the same field.

2.5.2. Proposition : (Formanek, 1979).
$Z_{2,3}$ (and hence $Z_{m,3}$) is rational over k.

For a full proof we refer to [19, p. 208]. We will here merely describe A and $k(A)^{S_3}$:

$$A =< X_1, X_2, X_3, y_{11}, y_{22}, y_{33}, y_{23}y_{32}, y_{13}y_{31}, y_{12}y_{21}, y_{12}y_{23}y_{31} >$$

is the Abelian group on ten generators with the obvious S_3-action.

Define :

$$a_1 = X_1 + X_2 + X_3, \quad a_2 = X_1X_2 + X_2X_3 + X_3X_1, \quad a_3 = X_1X_2X_3$$
$$a_4 = y_{11} + y_{22} + y_{33}, \quad a_5 = X_1y_{11} + X_2y_{22} + X_3y_{33}$$
$$a_6 = X_1^2y_{11} + X_2^2y_{22} + X_3^2y_{33}$$
$$b = y_{23}y_{32} + y_{13}y_{31} + y_{12}y_{21}$$
$$v_1 = y_{23}y_{32}b^{-1}, v_2 = y_{13}y_{31}b^{-1}, v_3 = y_{12}y_{21}b^{-1}$$
$$a_7 = X_1v_1 + X_2v_2 + X_3v_3, \quad a_8 = X_1^2v_1 + X_2^2v_2 + X_3^2v_3$$
$$p_{12}y_{23}y_{31}b^{-1}, \qquad q = y_{13}y_{32}y_{21}b^{-1}$$
$$a_9 = p + q, a_{10} = (X_1^2X_2 + X_2^2X_3 + X_3^2X_1)P + (X_1X_2^2 + X_2X_3^2 + X_3X_1^2)q$$

Then, Formanek proves that $k(A)^{S_3} = k(a_1, \dots, a_{10})$ and is therefore rational.

In both cases, we see that $k(A)^S$ is even rational over the rational function-field over the characteristic roots of X. In the next sections we will see that this fails already to be true for $n = 4$.

2.6. Stable rationality and permutation modules.
Another advantage of theorem 6 is that one can invoke the fairly extensive theory on permutation modules and rationality problems. In this section we will briefly recall these results. In the next section we will apply them in order to show that $Z_{2,4}$ connot be stably rational over $Z_{1,4}$.

Let G be a finite group. A G-lattice is a G-module which is a finitely generated free Abelian group. If $M =< X_1, \dots, X_n >$ is a G-lattice, then G acts on its group ring [N] and on its field of fractions $k(M)$. Such an action is called a lattice action.
A G-lattice is called a permutation module if it has a $\mathbb{Z}$-basis which is permuted by G. A G-lattice M is called permutation-projective if there is a G-lattice N s.t. $M \oplus N$ is a permutation module.

The work of Masuda [33], Endo-Miyata [17], Swan [46], Voshresenskii [49] and Lenstra [31] shows how results on G-lattices give info on the (stable) rationality of fixed fields of lattice actions. In the following theorem we will give some of their results :

Theorem (Masuda, Endo-Miyata, Swan, Voshresenskii, Lenstra). Let G be a finite group acting faithfully on a field k, then

(1) If P is a permutation module, then $k(P)^G$ is rational over k^G

(2) If P is a permutation-projective G-lattice and if $1 \to M \to N \to P \to 1$ is an exact sequence of G-lattices, then $k(N)^G \cong k(M \oplus P)^G$ as k^G-algebras.

(3) If P is a permutation module over G and $1 \to M \to N \to P \to 1$ an exact sequence of G-lattices, then $k(M)^G$ is rational over $k(M)^G$.

(4) If M is a G-lattice, then $k(M)^G$ is stably rational over k^G if and only if there exists an exact sequence of G-lattices $1 \to M \to P \to Q \to$ where P and Q are permutations modules.

How can this machinery be used in our setting. Consider again the exact sequence of S_n-lattices (**)

$$1 \to A \to B \xrightarrow{\alpha} U \xrightarrow{\beta} V \longrightarrow 1$$

If we denote $R =< X_1, \ldots, X_n >$ the standard S_n-permutation module, then it is clear that $A = P \oplus A_0, B = P \oplus B_0$ where

$$1 \longrightarrow A_0 \longrightarrow B_0 \xrightarrow{\alpha_0} U \xrightarrow{\beta} V \cong \mathbb{Z} \longrightarrow 1 \qquad (***)$$

$B_0 =< y_{11}, y_{12}, \ldots, y_{nn} >, \alpha_0 = \alpha | B_0$ and $A_0 = \alpha | B_0$ and $A_0 = \text{Ker}\alpha_0$. So, we can take as the field l in the foregoing theorem $l = k(P)$, then $l^{S_n} \cong Z_{1,n}$ the field generated by the characteristic roots of X and $Z_{z,n} \cong l(A_0)^{S_n}$.

By Theorem 7.(4) we then see that $Z_{2,n}$ is stably rational over $Z_{1,n}$ if and only if there is an exact sequence of S_n-lattices

$$1 \longrightarrow A_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 1$$

with P_i an S_n-permutation module. Note that (***) fails this objective only because of V a trivial S_n-module. Nevertheless, this seemingly minor obstruction turns out to be unsurmountable.

2.7. $Z_{2,4}$ is not stably rational over $Z_{1,4}$

In this section we will show.

2.1.1. Proposition 4 : (Snider, unpublished)
$Z_{2,4}$ cannot be stably rational over $Z_{1,4}$.

In fact, a similar result holds for all n non-square free (Saltman, 1986). Later on, this observation will be used to invalidate Maruyama's proof of the stable rationality of the moduli space of stable vectorbundles on $\mathbb{P}_2$ with Chern-classes $c_1 = 0, c_2 = n$.

Proof of Proposition 4 : Assume $Z_{2,4}$ to be stably rational over $Z_{1,4}$, then there is an exact sequence of S_4-lattices

$$(1) \qquad 1 \longrightarrow A_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 1$$

with P_i an S_4-permutation module. Let $G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \subset S_4$ be the group $\{(1)(2)(3)(4), (12)(34), (13)(24), (14)(23)\}$. Since G acts transitively on $U =< u_1, u_2, u_3, u_4 >, U \cong \mathbb{Z}[G]$ as G-lattices. From (***) we then obtain

$$1 \longrightarrow \mathrm{Ker}\beta \longrightarrow \mathbb{Z}[G] \xrightarrow{\beta} \mathbb{Z} \longrightarrow 1 \qquad (2)$$

exact as sequence of G-modules, whence

$$\mathbb{Z}[G]^G \xrightarrow{\beta} \mathbb{Z}^G = \mathbb{Z} \longrightarrow H^1(G, \mathrm{Ker}\beta) \longrightarrow 0 \qquad (3)$$

Since $\mathbb{Z}[G]^G = \mathbb{Z}[u_1+u_2+u_3+u_4]$ we have $H^1(G, \mathrm{Ker}\beta) \cong \mathbb{Z}/4\mathbb{Z}$. Further, $B_0 =< y_{11}, \ldots, y_{44} >$ can be shown to be a free $\mathbb{Z}[G]$-module, so from the exact sequence

$$1 \longrightarrow A_0 \longrightarrow B_0 \longrightarrow \mathrm{Ker}\beta \longrightarrow 1$$

one obtains $H^2(G, A_0) \cong H^1(G, \mathrm{Ker}\beta) \cong \mathbb{Z}/4\mathbb{Z}$. For any G-permutation module one has $H'(G, P) = 0$, so from (1) we obtain that $H^2(G, A_0) \hookrightarrow H^2(G, P_1)$. Since P_1 is a G-permutation module

$$P_1 \cong \oplus_{i=1}^{k} \mathbb{Z}[G/H_i]$$

with H_i subgroup of G. But

$$H^2(G, \mathbb{Z}[G/H_i]) \cong H^2(H_i, \mathbb{Z}) \cong \mathrm{Hom}(H_i, \mathbb{Q}/\mathbb{Z})$$

and hence $H^2(G, P_i)$ cannot contain an element of order 4 as it must because of $\mathbb{Z}/4\mathbb{Z} \cong H^2(G, A_0) \hookrightarrow H^2(G, P_1)$. Done. □

Of course, this does not imply that $Z_{2,4}$ is not rational over k. In fact, in an extremely skilful but laborious paper Ed Formanek has shown.

2.7.2. Proposition 5 (Formanek, 1980) :

$Z_{2,4}$ (and hence $Z_{m,4}$) is rational over k.

For details of the proof we refer to the original paper [20]. One of the main ideas of the proof is to use theorem 7(3) in order to reduce the problem of studying the S_4-action on A which is of rank 17 to that of an action on 3 variables.

2.8 : $PGL_n(k)$-invariants.

Although the description of $Z_{z,n}$ as the field of invariants of an S_n-lattice action is very usefull for small values of n, we will now need another interpretation of $Z_{z,n}$ in order to link it later with problems in algebraic geometry and representation theory.

Denote

$$X = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \\ x_{n1} & \dots & x_{nn} \end{pmatrix}; \quad Y = \begin{pmatrix} y_{11} & \dots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \dots & y_{nn} \end{pmatrix}$$

and

$$k[V] = k[x_{11}, \dots, x_{nn}; y_{11}, \dots, y_{nn}]$$

where v is the vectorspace of dimension $2n^2$ with basisvectors $\{x_{ij}, y_{ij} : 1 \leq i, j \leq n\}$. There is a natural $GL_n(k)$-action of V defined by :

$$\begin{cases} \alpha.x_{ij} = (\alpha.X.\alpha^{-1})(ij) \\ \alpha.y_{ij} = (\alpha.Y.\alpha^{-1})(ij) \end{cases}$$

for any $\alpha \in GL_n(k)$ (here $A(ij)$ denotes the (ij)-entry of the matrix A). Through this action $GL_n(k)$ acts as a group of k-automorphisms on the polynomial ring $k[V]$ and on its functionfield $k(V)$.

2.8.1. Theorem (Artin, 1969) :
For any n, $Z_{2,n} \cong k(V)^{GL_n(k)}$.

For a proof see [P]. In fact, more is true : let $\mathbb{T}_{m,n}$ be the subalgebra of $\Delta_{m,n}$ generated ny $\mathbb{G}_{m,n}$ and all its traces, then $\mathbb{T}_{m,n}$ is called the trace ring of m generic n by n matrices. The advantage of $\mathbb{T}_{m,n}$ to $\mathbb{G}_{m,n}$ is that it is a nicer ring e.g. it is Noetherian, affine and even a maximal order. Moreover, there are some indications that $\mathbb{T}_{m,n}$ also behaves nicely w.r.t. homology.

Consider the centre of $\mathbb{T}_{m,n}$, $R_{m,n}$, then this is a normal, Cohen-Macaulay and even factorial hence Gorenstein domain. Moreover, $R_{m,n}$ can be described as being of the k-subalgebra of $K_{m,n}$ generated by elements of the form $Tr(X_{i_1}, \ldots, X_{i_q})$ with $q \leq n^2$ (Razmyslov-Formanek). Clearly, the field of fractions of $R_{m,n}$ is $Z_{m,n}$. Most of these results on $R_{m,n}$ follow from :

2.8.2. Theorem (Procesi, 1976).
For any n, m : $R_{m,n} = \mathcal{P}_{m,n}^{GL_n(k)}$ where the action of $GL_n(k)$ on $\mathcal{P}_{m,n}$ is similar to the one on k [V] described above. In terms of Mumford's geometric invariant theory this implies that $R_{2,n}$ is the coordinate ring of the quotient variety V/GL_n. The geometry of this variety is now reasonably understood (Le Bruyn, Procesi [29]). Finally, we note that the center $k^* \subset GL_n(k)$ acts trivially on V so it is really a $PGL_n(k)$-action.

2.9. Vectorbundles over the projective plane.

In this and the next section we will show that question 1 can be rephrased in geometrical terms.

Recall that a vectorbundle on $\mathbb{P}_2$ is a locally free sheaf. For an extensive exposition on vectorbundles over projective spaces and their invariants we refer to the monograph [39].

A very coarse classification of all vector bundles over $\mathbb{P}_2$ is given by their topological invariants such as the rank and the Chern numbers. Given such parameters r, c_1 and c_2 one wants to study sufficiently general bundles having these invariants. If turns out that they are stable which means that for all coherent subsheaves $\mathcal{F}$ or our vectorbundle $\mathcal{E}$ we have $\frac{c_1(\mathcal{F})}{rh\mathcal{F}} < \frac{c_1}{r}$. One wants to construct and study a variety $M(r, c_1, c_2)$ whose points correspond to isomorphism classes of stable vectorbundles $\mathcal{E}$ of rank r with Chern numbers $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$. This can do done, again see [39]. The variety $M(r, c_1, c_2)$ is then called the moduli space of stable $rk(r)$ bundles having Chern-numbers c_1 and c_2. Again, a major open problem is :

Question 2 : For which numbers r, c_1 and c_2 are the moduli spaces $M(r, c_1, c_2)$ rational (or stably rational), i.e. have a (stably) rational function field.

The motivation clearly is that whenever they are rational one can find additional algebraic invariants of bundles such that they classify freely and completely sufficietly general bundles with the given topological invariants r, c_1 and c_2.

Not much is known about question 2. Barth proved rationality of $M(2,0,2)$, Le Potier of $M(2,0,4)$ and rumour goes that Maeda proved rationality of $M(2,0,n)$. Apart from this, nothing seems to be known about the rationality of $M(r,0,n)$. What is the connection with the generic division algebras ?

2.9.1 Theorem (Le Bruyn, 1986)

$Z_{2,n}$ is the function field of the moduli space $M(n,0,n)$.

Modulo the rationality results of Formanek mentioned before, this theorem implies that $M(3,0,3)$ and $M(4,0,4)$ are rational. At this moment there does not seem to be an alternative proof for the rationality of $M(4,0,4)$.

The proof of theorem 10 is based on the following description of $M(n,0,n)$ due to K. Hulek [23] cfr. also the work of Maruyama [33].

Let $\mathcal{A} = (A_0, A_1, A_2) \in M_n(k) \oplus M_n(k) M_n \mathbb{Q}(k)$. $\mathcal{A}$ is said to be prestable if for any $0 \neq \theta \in k^n$ we have that

$$\dim(A_0 v, A_1 v, A_2 v) \geq 2; \quad \dim(A_0^\tau v, A_1^\tau v, A_2^\tau v) \geq 2$$

Note that this is clearly an open condition on $N_n(h) \oplus M_n(k) \oplus M_n(k)$. Let $V = \Gamma(\Theta_{\mathbb{P}_2}(1))^*$ with basis u, v, w dual to the usual x, y, z basis of $V^* = \Gamma(\Theta_{\mathbb{P}_2}(1))$ and define a linear map

$$\varphi_{\mathcal{A}} : k^n \otimes V \longrightarrow k^n \otimes V^*$$

given by the matrix

$$\begin{bmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{bmatrix} = \psi_{\mathcal{A}}$$

If we let $U = Im\varphi_{\mathcal{A}}$ and $s.\Gamma(\Theta_{\mathbb{P}_2}(1)) \otimes \Theta_{\mathbb{P}_2} \longrightarrow \Theta_{\mathbb{P}_2}(1)$ be the natural multiplication map then we have a complex of bundles

$$\begin{array}{ccccc} k^n \otimes \theta_{\mathbb{P}_2}(-1) & a & U \otimes \theta_{\mathbb{P}_2} & b & k^n \otimes \theta_{\mathbb{P}_2}(1) \\ 1 \otimes s^* \Big\downarrow & \varphi_{\mathcal{A}} \otimes 1 & \Big\downarrow & & 1 \otimes s \\ k^n \otimes V \otimes \theta_{\mathbb{P}_2} & & k^n \otimes V^* \otimes \theta_{\mathbb{P}_2} & & \end{array}$$

where a is monomorphism of bundles (i.e. locally split mono) and b is an epimorphism. The cohomology bundle of this complex $\mathcal{E}_{\mathcal{A}}$ has rank $\dim U - 2n$ and Chern numbers $c_1 = 0$, $c_2 = n$. Moreover one can show that $\mathcal{E}_{\mathcal{A}} \cong \mathcal{E}_{\mathcal{A}}$ if and only if there are invertible matrices $\alpha, \beta \in GL_n(k)$ s.t. $A'_i = \alpha A_i \beta$ for $0 \leq i \leq 2$.

On the other hand one can study this $GL_n \times GL_n$-action on $M_n(k)^{\oplus 3}$. There is open subvariety $GL_n(k) \oplus M_n(k) \oplus M_n(k)$ on which representants of the orbits under $GL_n \times I_n$ can clearly to be chosen of the form (I_n, A, B). We have to investigate the action of $I_n \times GL_n$ on these representants. $(I_n, \beta)(I_n, A, B) = (\beta, A\beta, B\beta) = (\beta, I_n).(I_n, \beta^{-1}A\beta, \beta^{-1}B\beta)$. That is the action of $GL_n \times GL_n$ on $GL_n \oplus M_n(h) \oplus M_n(k)$ is the same as the action of $GL_n(h)$ on $V = M_n(h) \oplus M_n(h)$ described in section H.

Finally, the set of all prestable triples $\mathcal{A}$ s.t. rank $\mathcal{E}_{\mathcal{A}} = n$ forms an open subvariety so it has a nonempty open intersection with $GL_n \oplus M_n(h) \oplus M_n(k)$. So, the module space $M(n, 0, n)$ is birational to the quotient variety $GL_n(h) \oplus M_n(h) \oplus M_n(h)/GL_n \times GL_n \cong V/GL_n(k)$ which has $K_{2,n}$ as its functionfield, done. □

Maruyama [34] claims to have proved stable rationality of $M(n, 0, n)$. In view of theorem 10 this would settle our question 1. However, a closer investigation of Maruyama's result (modulo the above sketched translation from vectorbundles to conjugacy classes of couples of matrices) shows that he proves a stronger result namely that $K_{2,n}$ is stably rational over $K_{1,n}$ which we have seen to be impossible if $n = 4$. The error of Maruyama is where he claims $PGL_n(k)$-invariance of a map which cannot be the case.

2.10. Halfcanonical divisors on plane curves.

In this section we will give a very precise parametrization of an open piece of the quotient variety $V/GL_n(k)$ by plane projective curves and points on their Jacobians (or more precisely on an homogeneous space over the Jacobian).

Recall that the projective space $\mathbb{P}^{\frac{1}{2}n(n+3)}$ parametrizes plane projective curves of degree n. Let U be an open subvariety consisting of nonsingular curves. Consider the flagvariety $W \subset \mathbb{P}^2 \times U$ consisting of all couples (P, C) with P a point on C. The natural projection $W \to U$ is then a flat bundle of smooth curves.

We can then investigate the relative Picard scheme introduced by Grothen dieck [21] and studied by Artin [13] and Mumford [35]. This functor $\mathrm{Pic}_{W/U}$

associates to an U-scheme S the following group

$$\mathrm{PIC}_{W/U} = \frac{\{gp \text{ of invertible sheaves on } WX_U S\}}{\{\text{subgroup of sheafs of form } p*_2(K) \text{ for } K \text{ on } S\}}$$

where p_2, the projection on the second component. Since $W \to U$ is a bundle of smooth curves we can associate to invertible sheaves a discrete invariant, the degree. With $\mathrm{Pic}^d_{W/U}$ we denote the subfunctor consisting of invertible sheaves of degree d. The sheafification of this functor with a spect to the flat topology is represented by the variety $\mathrm{Pic}^d_{W/U}$ consisting of couples $(C, \mathcal{L})$ where C is a nonsingular plane curve f degree n and $\mathcal{L}$ is a divisor on C of degree d. With this terminology one has the following beautifull result.

2.10.1. Theorem (Van den Bergh, 1986)
$Z_{z,n}$ is the function field of the relative Picard scheme $\mathrm{Pic}^d_{W/U}$ for $d = \frac{1}{2}n(n-1)$.

Note that since the degree of the canonical divisor on a smooth curve of degree n is $n(n-1)$, this variety can be viewed as the generic variety of halfcanonical divisors on plane curves. For the original proof of Theorem 11 we refer the reader to [48]. Here, we will outline how his result can be deduced from theorem 10. So, we have to associate to a sufficiently general vectorbundle $\mathcal{E}$ over $\mathbb{P}^2$ of rank n and with Chern-classes $c_1 = 0, c_2 = n$ a smooth plane curve C of degree n and a halfcanonical division $\mathcal{L}$ on it. Again, Hulek [23, 17] has indicated how this can be done by a suitable generalization of Barth's characterization of rank two bundles by their curve of jumping lines and Θ-characteristic [14].
Let $\mathcal{E}_{\mathcal{A}}$ be a stable vectorbundle associated to the triple $\mathcal{A} = (A_0, A_1, A_2)$ if n by n matrices as in section I and define

$$\Delta_{\mathcal{A}} = \det(A_0 u + A_1 v + A_2 w) \in \Gamma(\theta_{\mathbb{P}_2^*}(n))$$

The discriminant $\Delta_{\mathcal{A}}$ is a homogenous polynomial of degree n and we can consider the curve $C_{\mathcal{A}} \subset \mathbb{P}_2^*$ it defines. The interpretation of $C_{\mathcal{A}}$ is that it contains those lines $L \subset \mathbb{P}_2$ such that $\mathcal{E}|L \neq \theta_L^{\oplus n}$ so it generalizes the curve of jumping lines in the rank two bundle case.
One can define a map

$$\psi_{\mathcal{A}} = (A \otimes 1) \circ (1 \otimes s) : k^n \otimes \theta_{\mathbb{P}_2^*}(-1) \longrightarrow k^n \otimes V^* \otimes \theta_{\mathbb{P}_2^*} \to k^n \otimes \theta_{\mathbb{P}_2^*}$$

Over a point $L \in \mathbb{P}_2^*$ with coordinates $(y_0 : y_1 : y_2)$ this map is just $\mathcal{A}(y) = A_0 y_c + A_1 y_1 + A_2 y_2$. One can define a sheaf $\mathcal{L}_{\mathcal{A}}$ by the sequence

$$0 \longrightarrow k^n \otimes \theta_{\mathbb{P}_2^*}(-1) \xrightarrow{\psi_{\mathcal{A}}} k^n \otimes \theta_{\mathbb{P}_2^*} \longrightarrow \mathcal{L}_{\mathcal{A}} \longrightarrow 0 \qquad (+)$$

which has its support on $C_{\mathcal{A}}$. Moreover by [23, 1.7.3. iv] the couple $(C_{\mathcal{A}}, \mathcal{L}_{\mathcal{A}})$ determines $\mathcal{E}_{\mathcal{A}}$ uniquely.

If we restrict the sequence (+) to $\mathcal{C}_{\mathcal{A}}$ we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^n \otimes \theta_{\mathbb{P}_2^*}(-1) & \xrightarrow{\psi_A} & k^n \otimes \theta_{\mathbb{P}_2^*} & \longrightarrow & \mathcal{L}_{\mathcal{A}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & k^n \otimes \theta_{\mathcal{C}_{\mathcal{A}}}(-1) & \xrightarrow{\psi_A|} & k^n \otimes \theta_{\mathcal{C}_{\mathcal{A}}} & \longrightarrow & \mathcal{L}' & \longrightarrow & 0 \end{array}$$

For sufficiently general $\mathcal{A}$ we have that $rk(\psi_{\mathcal{A}}|\mathcal{C}_{\mathcal{A}}) = n-1$ whence $\mathcal{L}'$ is an invertible sheaf over $\mathcal{C}_{\mathcal{A}}$. The induced map $\mathcal{L}_{\mathcal{A}} \longrightarrow \mathcal{L}'$ is surjective and can be shown to be injective, too. So, for generic $\mathcal{A}$ we have that $\mathcal{L}_{\mathcal{A}} \in \mathrm{Pic}(\mathcal{C}_{\mathcal{A}})$ of degree $\frac{1}{2}n(n-1)$ [23, 1.7.3 iii]. Conversely, starting from a plane curve G and $\mathcal{L} \in \mathrm{Pic}(\mathcal{C})$ of degree $\frac{1}{2}n(n-1)$ one can reconstruct a trip;e $\mathcal{A}$ s.t. is prestable for sufficiently general G and $\mathcal{L}$ [23, 1.7.], done. □

Theorem 11 gives another proof of rationality of $Z_{2,2}$. For, in the case of degree 2 (i.e. rational) plane curves, $\mathrm{Pic}^d = 0$ hence $\mathrm{Pic}^d_{W/U} \cong U \hookrightarrow \mathbb{P}^5$.

More surprisingly is that one can give a geometrical proof of the rationality of $Z_{2,3}$ due to Van den Bergh [48, §7] : Let $(\mathcal{C}, \mathcal{L})$ represent a point of $\mathrm{Pic}^d_{W,U}$ for the case $n = 3$, i.e. C is an elliptic curve $\mathcal{L} \in \mathrm{Pic}^0(C) = \mathrm{Jac}(C) \cong C$ i.e. $\mathcal{L}$ determined by a point D on C.
Fix a line $L \subset \mathbb{P}^2$ and denote by $\{P_1, P_2, P_3\} = L \cap C$. Note that this is only canonical upto an element of S_3. There are uniquely determined points Q_i on C s.t. $P_i + D \sim Q_i$ where $\sim$ denotes linear equivalence. Moreover, $Q_i + P_i \sim Q_j + P_i$ and hence $R_{ij} = \mathrm{line}(P_i, Q_j) \cap \mathrm{line}(P_j, Q_i)$ also lie on C. Conversely, if we are given points P_i, Q_i and a curve C of degree 3 passing through the points $\{P_i, Q_i, R_{ij} : i, j\}$ then D is uniquely determined as the divisor class containing $Q_1 - P_1$.

Now, define W_1 as the variety parametrizing the tuples $((P_i)i = 1,2,3; (Q_i)_{i=1,2,3})$ with P_1, P_2, P_3 on L. Let W_i' be the variety parametrizing tuples $((P_i)_{i=1,2,3}; (Q_i)_{i=1,2,3}; C)$ where P_1, P_2, P_3 lie on L and C is a curve of degree 3 passing through the points $\{P_i, Q_i, R_{ij} : 1 \leq i, j \leq 3\}$. Then, there is a natural S_3-action on W' via $\sigma.(P_i, Q_i, C) = (P_{\sigma(i)}; Q_{\sigma(i)}; C)$ and W_i'/S_3 is birational to $\mathrm{Pic}^d_{W/U}$.

One can then verify that $k(W_1')$ is rational over $k(W_1)$ of trancendence degree one. Hence, $k(W_1')^{S_3}$ is the functionfield of a conic over $k(W_1)^{S_3}$ which itself is a rational field over k. Finally, this conic has a rational point since $W_1'/S_3 \to W_1/S_3$ has generically a section by fixing a point $S \in \mathbb{P}^2$ and associating to a tuple (P_i, Q_i) the unique curve of degree 3 passing through $\{P_i, Q_i, R_{ij}, S\}$, so $k(W_1')^{S_3}$ is rational over k, done. □

2.11. Representation theory of hereditary algebras.

Form now on we assume that the basefield k is algebraically closed. If Λ is a finite dimensional, basic, connected hereditary algebra it is well known that Λ is the path algebra of some quiver Q. The study of finite dimensional Λ-modules then coincides with the study of representations of Q. Let us recall some of the basics :

A quiver Q is a quadruple (Q_0, Q_1, t, h) consisting of a finite set $Q_0 = \{1, \ldots, m\}$ of vertices, a set Q_1 of arrows between these vertices and two maps $t, h : Q_1 \to Q_\circ$ assigning to an arrow φ its tail t_φ and its head $h\varphi$, respectively.

A representation V of the quiver Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional k-vectorspace together with a family of linear maps $\{V(\varphi) : V(t_\varphi) \to V(h\varphi); \varphi \in Q_1\}$. The m-tuple $\dim(V) = (\dim V(i)); \in \mathbb{N}^m$ is called the dimension type of V. A morphism $f : V \to W$ between two representations of the quiver Q is a family $\{f(i); V(i) \to W(i); i \in Q_0\}$ such that for all arrows $\varphi \in Q$, one has $W(\varphi) \circ f(t\varphi) = f(h\varphi) \circ V(\varphi)$.

Given a dimension vector $\alpha = (\alpha(1), \ldots, \alpha(m)) \in \mathbb{N}^m$ we define the representationspace $R(Q, \alpha)$ to be the set of representations V of Q such that $V(i) = k^{\alpha(i)}$ for all $i \in Q_0$. Because $V \in R(Q, \alpha)$ is completely determined by the maps $V(\varphi)$ we have that

$$R(Q, \alpha) = \oplus_{\varphi \in Q_1} M_\varphi(t)$$

where $M_\varphi(k)$ denotes the k-vectorspace of all $\alpha(h_\varphi)$ by $\alpha(t\varphi)$ matrices with entries in k.

We will consider the representationspace $R(Q, \alpha)$ as an affine variety with coordinatering $k[Q, \alpha]$ and functionfield $k(Q, \alpha)$. We have a canonical action of the linear reductive group $GL(\alpha) = \prod_{i=1}^m GL_{\alpha(i)}(k)$ on $R(Q, \alpha)$ by the rule

$$(g.V)(\varphi) = g_{h\varphi}.V.g_{t\varphi}^{-1}$$

for all $g = (g_1, \ldots, g_m) \in GL(\alpha)$ and $V \in R(Q,\alpha)$. Note that the orbits under this action correspond precisely to isomorphism classes of representations of Q and hence to isoclasses of Λ-modules if Λ is the path algebra of Q.

This action induces an action of $GL(\alpha)$ as a group of k-automorphisms on $k[Q,\alpha]$ and $k(Q,\alpha)$. The geometrical interpretation of the invariant field $k(Q,\alpha)^{Gi(\alpha)}$ is that it is the functionfield of the variety parametrizing the orbits of representations in sufficiently general position. So, proving rationality or stable rationality of these invariantfields would be a decisive step in understanding the representation theory of wild hereditary algebras. Note that C.M. Ringel [42] proved rationality of $k(Q,\alpha)^{GL(\alpha)}$ in case Q (or equivalently Λ) is tame.

The following reduction is due to V. Kač [25] : $\alpha = \beta_1 + \ldots + \beta_k$ is said to be the generic decomposition of the dimension vector α provided representations V of Q in sufficiently general position decompose as

$$V = W_1 \oplus \ldots \oplus W_k$$

where the W_i are indecomposable representations of dimension type β_i. Moreover, Kač shows that the β_i are so called Schur roots. That is, there is an open subvariety U of $R(Q,\beta_i)$ s.t. for $W \in U$ one has $\text{End}(W) \cong k$. One can use the generic decomposition in order to reduce the problem of studying the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ to the study of rational invariants for Schur roots in the following way : let $\beta_i = (\beta_i(1), \ldots, \beta_i(m))$ and fix decompositions

$$k^{\alpha(i)} = \oplus_{j=1}^{k} k^{\beta_j(i)}$$

then this given inclusions $\oplus_{j=1}^{k} R(Q,\beta_j) \hookrightarrow R(Q,\alpha)$ and $X_{i=1}^{j} GL(\beta_j) \subset GL(\alpha)$. Note that if $\beta_j(i) = \beta_j(i)$ for some $j \neq j'$ we have an involution $\sigma_i(j,j')$ on the space $\oplus_{j=1}^{k} R(Q,\beta_j)$ which permutes $k^{\beta_j(i)}$ and $k^{\beta_j(i)}$. Denote by Σ_α the group generated by all these involutions. One then has :

2.11.1. Theorem. If $\alpha = \beta_1 + \ldots + \beta_k$ is the generic decomposition of α and if Σ_α is defined as above. Then,

$$k(Q,\alpha)^{GL(\alpha)} \cong \left[\otimes_{j=1}^{k} k(Q,\beta_j)^{GL(\beta_j)} \right]^{\Sigma_\alpha}$$

For a proof we refer to [26, prop. 5 p. 153]. This result urges us to investigate the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ for a Schur root α.

Note that rationality of $k(Q,\alpha)^{GL(\alpha)}$ would follow immediately from a positive solution to a rather daring conjecture of Kač : he conjectured in [26] that the variety of indecomposable representations of Q of dimension vector α admits a finite cellular decomposition into locally closed subvarieties each isomorphic to some affine space k. Since for a Schur root α there is an open subvariety $U \subset R(Q,\alpha)$ consisting of indecomposables, this would immediately imply that $k(Q,\alpha)^{GL\alpha}$ is rational. Unfortunately, at this moment there is not much evidence to support this conjecture.

In the next sections we will show how stable rationality of the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ for a Schur root α and any quiver Q would follow from a positive solution to problem 1. The proof of this result rests on two major results : the so called "no-name-lemma" in invariant theory on almost free actions and the Bernstein-Gelfand-Ponomarev theory of reflexion functors. We will first recall these results.

2.12. The no-name lemma.

The following result seems to be common knowledge to people working in invariant theory. However, there does not seem to exist an establashed reference for it and therefore it is called the no-name lemma (some even say : no-proof lemma).

Recall that two k-fields K and L are said to be stably equivalent if $K(X_1,\ldots,X_l) \cong L(y_1,\ldots,y_m)$ for some l and m. Further, a linear reductive group G is said to act almost freely on a finite dimensional vectorspace V if the stabilizer of a sufficiently general point is trivial. Of course, G then acts as a group of automorphisms on the functionfield $k(V)$ and one wants to study the rational invariants $k(V)^G$. The no-name lemma gives us some freedom in the particular choice of V as long as we are interested in stable equivalence :

2.12.1. Theorem (No-name lemma)

Let a reductive linear group G act almost freely on vectorspaces V and W. Then the fields of rational invariants $k(V)^G$ and $k(W)^G$ are stably equivalent.

Since there is no fixed reference, we will outline the proof of this result due to Le Bruyn and Schofield [30] which is based on some results of D. Luna [32].

Because the coordinate ring of V, $k[V]$, is a unique factorization domain having only trivial units we can apply [32, p. 103 lemma 2], in order to obtain a non-empty affine G-invariant subvariety V' of V such that $k(V)^G$ is the field of fractions of $k[V'/G] = k[V']^G$, or equivalently, generic orbits in V' are closed. So, take a generic point $v \in V'$, then by the étale slice theorem [32, p. 97] there exists an affine (!) subvariety V'' of V' containing v such that the G-action on V' induces an étale G-morphism :

$$\psi : G \times V'' \longrightarrow V'$$

such that the image U of ψ is an open affine (!) $\Pi_{V'}$-saturated subvariety of $V'(\pi_V : V' \longrightarrow /G$ is the canonical quotient map) and the canonical morphism :

$$\psi/G : (G \times V'')/G \cong V'' \longrightarrow U/G$$

is étale and gives rise to a G-isomorphism

$$G \times V'' \cong U x_{U/G} V''$$

Therefore, $\pi_U : U \longrightarrow U/G$ is an affine (!) principal G-bundle in the étale topology.
But now we can apply an old result of J.P. Serre [44] : let $\pi_U : U \rightarrow U/G$ be an affine principal G-bundle in the étale topology and W a variety with G-action. Define a G-action on $U \times W$ by $g.(u,w) = (ug^{-1}, gw)$ and denote by $U \times_G W = (U \times W)/G$, then $U \times_G W$ is the total space of a fibration of type W with basis U/G.
But this implies that (in case W is a vector space) $k(U \times_G W) = k(U \oplus W)^G$ is a rational extension of $k(U/G) = k(U)^G = k(V)^G$. We can repeat the same argument with W instead of V and obtain that $k(V \oplus W)^G$ is rational over both $k(V)^G$ and $k(W)^G$ finishing the proof. □

What has the no-name lemma to do with our investigation of rational invariants of quivers ?
Well, α being a Schur root of a quiver Q is equivalent to $PGL(\alpha) = GL(\alpha)/k^*$ acting almost freely on $R(Q,\alpha)$. Hence, we get as an immediate consequence of theorem 13 :

2.12.2. Proposition (Le Bruyn-Schofield)
Let α be a Schur root of the quivers Q and Q', then the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ and $k(Q',\alpha)^{GL(\alpha)}$ are stably equivalent.

2.13. Reflection functors.
Theorem 13 allows us to correlate the rational invariants for a Schur root α with respect to different quivers. Now, we will see that one can also vary the dimension vector α by applying the theory of reflexion functors due to Bernstein, Gelfand and Ponomarev [16].
Let $i \in Q_0$ be a sink, that is for no $\varphi \in Q_1$ we have $t\varphi = i$ and let α be a dimension vector. We form a new quiver Q' by reversing the direction of the arrows connected to i and we define a new dimension vector β by $\beta(j) = \alpha(j)$ if $i \neq j$ and $\beta(i) = \sum_{h\varphi=i} \alpha(t\varphi) - \alpha(i)$.

Consider the set

$$R'(Q,\alpha) = \{V \in R(Q,\alpha); \oplus V(\varphi) : \oplus_{h\varphi=i} V(t\varphi) \to V(i) \text{ is surjective}\}$$

and observe that all indecomposable representations belong to $R'(Q,\alpha)$ so in case α is a Schur root for Q then $R'(Q,\alpha)$ is an open subvariety of $R(Q,\alpha)$. Similarly we consider the set

$$R'(Q',\beta) = \{V \in R(Q',\beta) : \oplus V(\varphi) : V(i) \longrightarrow \oplus_{t\varphi=i} V(h\varphi) \text{ is injective}\}$$

then, if $R'(Q,\alpha)$ is open in $R(Q,\alpha)$ so is $R'(Q',\beta)$ in $R(Q,\beta)$.

2.13.1. Theorem (Bernstein, Gelfand, Ponomarev)
With notation as above, there exists an homeomorphism between $R'(Q,\alpha)/GL(\alpha)$ and $R'(Q',\beta')/GL(\beta)$ such that corresponding reservations have isomorphic endomorphism rings.

For a nice proof of this result we refer the reader to the excellent survey paper by Kraft and Riedtmann [28]. Here we content ourselves by indicating the map between $R'(Q,\alpha)$ and $R'(Q',\beta)$. Consider a $V \in R'(Q,\alpha)$, then we have the exact sequence

$$0 \longrightarrow \operatorname{Ker} \oplus V(\varphi) \stackrel{i}{\longrightarrow} \oplus_{h\varphi=i} V(t\varphi) \stackrel{\oplus V\varphi}{\longrightarrow} V(i) \longrightarrow 0$$

Now, the corresponding representation $W \in R'(Q',\beta)$ consists of $W(j) = V(j)$ if $i \neq j$ and $W(i) = \operatorname{Ker} \oplus V(\varphi)$. Moreover, the maps corresponding to the arrows in Q' beginning in i are $\mathrm{pr}_j \circ i$. Of ourse, for all other $\varphi \in Q'_1$:

$W(\varphi) = V(\varphi)$. From theorem 14 it follows that β is a Schur root for Q' and $k(Q,\alpha)^{GL(\alpha)} \cong k(Q',\beta)^{GL(\beta)}$.

2.14. Connection with matrixinvariants.

We are now in a position to link our study of the representation theory of (wild) hereditary algebras to that of matrix-invariants.

2.14.1. Theorem. [**Le Bruyn, Schofield 1987**]
Let α be a Schur root for the quiver Q. Let $n =$ g.c.d.$(\alpha(i) : 1 \leq i \leq m)$. Then the field of rational invariants $k(Q,\alpha)^{GL(\alpha)}$ is stably equivalent to $Z_{2,n}$.

Proof. Let $i \in Q_\circ$ be such that $\alpha(i) = kn$ is minimal and let $j \in Q_\circ$ be such that $\alpha(j) = ln$ and k does not divide l, say $l = ak - b$ with $0 \neq b < k$.
We can now form a new quiver Q' on the same vertex set with precisely a arrows pointing from i to j. We demand that all other arrows live on $Q'_0 - \{j\}$ in such a way that α is a Schur root for Q'. Note that this can always be done for example by throwing in lots of loops.

Example :

Q	Q'		Q''		
0	0	0	0	0	0
2_n	3_n	$2n$	$3n$	$2n$	n

From proposition 6 it follows that $k(Q,\alpha)^{GL(\alpha)}$ is stably equivalent to $k(Q',\alpha)^{GL(\alpha)}$. Now, we can apply the reflexion functor in the vertex j and we obtain a quiver Q'' and a dimension vector α'' s.t. $\alpha''(l) = \alpha(l)$ for $l \neq j$ and $\alpha''(j) = bn < kn$. Moreover, by the results mentioned in section m we have that $k(Q',\alpha)^{GL(\alpha)}$ is isomorphic to $k(Q'',\alpha)^{GL(\alpha)}$. Note, that Q'' has a vertex with smaller dimension than α.

Applying the same game to the quiver Q'' a finite number of times we end up with a quiver Q''' and a dimension vector α''' s.t. g.c.d. $(\alpha'''(i)) = n$ and for some $i \in Q_0 : \alpha'''(i) = n$.
Now, we form a quiver Q^+ with precisely k arrows pointing from i to j if $\alpha'''(j) = k, n$ and two loops in i.

In our example

Q'''		Q^+	
0	0	0	0
2_n	n	2_n	n

then α''' is easily seen to be a Schur root for Q^+. So, the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ are stably rational to $k(Q^+,\alpha''')^{GL(\alpha''')}$. Now, we can apply reflexion functors in all vertices $j \neq i$ and we end up with the fact that

$$k(Q',\alpha''')^{GL(\alpha''')} \cong k(M_n(h) \oplus M_n(k))^{GL_n} \cong Z_{2,n}$$

which finishes the proof. □

Again, applying Formanek's results on the rationality of $Z_{2,n}$ we obtain.

2.14.2 Proposition : Let α be a Schur root for a quiver Q with g.c.d. $(\alpha(i) : i \in Q_0) \leq 4$, then the rational invariants $k(Q,\alpha)^{GL(\alpha)}$ are stably rational.

As an example of the usefullness of theorem, 15 let us consider the special case of the n-subspace problem. This problem asks for the determinations of the possible positions of r subspaces in a vector space, or, equivalently, of the indecomposable representations of the quiver

$$\begin{matrix} 1_0 & & \\ 2_0 & & \\ \vdots & 0\ 0 & (Q_r) \\ r_0 & & \end{matrix}$$

on $r+1$-vertices, i.e. the study of modules over the finite dimensional k-algebra

$$\Lambda_r = \begin{pmatrix} k & k & k & \dots & k \\ 0 & k & 0 & \dots & 0 \\ & & & 0 & \\ & 0 & & \ddots & k \end{pmatrix}$$

For a given dimension vector $(a_0; a_1, \dots, a_r)$ the geometrical problem is that of studying $GL(a_0)$-orbits in the variety

$$\text{Grass}(a_1, a_0) \times \dots \times \text{Grass}(a_r, a_\circ)$$

which was one of the testingexamples for Mumford's Geometric Invariant Theory. It turns out that one can only have a nice quotient variety provided there are stable points. We will not go into the formal definition of stability w.t.z. a groupaction on a variety (see [37]) but we merely recall that Mumford gave a combinatorial description of stable points in the above variety. He proved that a point $(u_1, \dots, u_r)$ in $\text{Grass}(a_1, a_0) \times \dots \times \text{Grass}(a_r, a_0)$ is stable w.z.t. the natural $GL_{a_o}(k)$-action if and only if there does not exist a proper subspace $0 \neq V \not\subseteq k^{a_o}$ s.t.

$$\frac{\sum_{i=1}^{r} \dim_k(V \cap U_i)}{\dim_k(V)} \geq \frac{\sum \dim_k U_k}{a_0} = \frac{\sum_{i=1}^{r} a_i}{q_o}$$

When one has stable points one can construct the quotient variety

$$X_{i=1}^{r} \text{Grass}(a_i, a_0)/GL(a_0),$$

see e.g. [37], [38], [27]. Apart from some easy cases where one can explicitly describe this quotient variety nothing seems to be known about its rationality. A. Schofield has shown that $\alpha = (a_0; a_1, \ldots, a_r)$ is a Schur root for the quiver Q_r if and only if there are stable points for the groupaction. So, we get

2.14.3. Proposition : Suppose there are stable points for the natural action of $GL(a_0)$ on $\text{Grass}(a_1, a_\circ) \times \ldots \times \text{Grass}(a_r, a_0)$. If $n = \text{g.c.d.}(a_0, a_1, \ldots, a_r)$, then the functionfield of the quotient variety is stably equivalent to $Z_{2,n}$. In particular, if $n \leq 4$ the quotient variety is stably rational.

One again, there does not exist a geometrical proof of this result. We hope to have shown that a seemingly innocent problem in p.i. theory turns out to have deep connection with algebraic geometry. Surprisingly, ringtheoretical results such as Formanek's rationality give new geometrical results, such as rationality of $M(3,0,3)$ and $M(4,0,4)$ or stable rationality of many instances of the r-subspace problem, for which there is no geometrical proof at this moment. In short, good ringtheory cannot be so bad after all.

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