Automorphisms of Generic 2 by 2 Matrices

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Abstract.
The automorphismgroup of the ring of two 2×2 matrices and related algebras is studied. Connections with the tame automorphism - and Jacobian conjecture are given.

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1. Introduction.

The purpose of this paper is to investigate the automorphisms of the ring of 2 generic 2 by 2 matrices G. Along the way, we will have to analyze the automorphisms of related algebras as well : the trace ring T, its center C, the ring of 2 generic 2 by 2 trace zero matrices T^0 and some Clifford algebras.

In [Be], G. Bergman has constructed wild automorphisms of rings of generic matrices in connection with the commutative tame automorphism problem that we recall in section 1. Concentrating on the ring of 2 generic 2 by 2 matrices, one can get more specific information on the automorphisms. According to [FHL] and [Pro], the center of the corresponding trace ring is a polynomial algebra in 5 variables and one can then investigate the induced automorphisms. The main result is that all automorphisms of G that we can construct induce tame automorphisms of C.

On the other hand, the study of T^0 leads to automorphisms that can reasonably be considered as wild as is defined in section 4 and such that the induced automorphisms on the 3-dimensional polynomial center is Nagata's automorphism up to a linear change of coordinates. This last automorphism is commonly considered as a good candidate for a wild automorphism on 3-dimensional affine space; and it appears as the restriction to the center of a wild automorphism of a 3-dimensional, yet 2-generator non-commutative algebra T^0. Perhaps this adds evidence to its conjectural wildness.

2. Tameness and Nagata's automorphism.

(2.1) : Let k be a reduced commutative ring and let $GA_n(k)$ be the group of k-algebra automorphisms of $k[X_1, ..., X_n]$. $GA_n(k)$ has two natural subgroups ; $Af_n(k)$ the affine automorphisms and $BA_n(k)$, the triangular (or Jonquière) automorphisms. They are defined by

$$Af_n(k) : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \to \sigma . \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$BA_n(k) : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} \alpha_1 x_1 + P_1(x_2, ..., x_n) \\ \alpha_2 x_2 + P_2(x_3, ..., x_n) \\ \vdots \\ \alpha_n x_n \end{pmatrix}$$

where $\sigma \in GL_n(k)$, $a_i \in k$, $\alpha_i \in k^* = G_m(k)$ and $P_i(x_{i+1}, ..., x_n) \in k[x_{i+1}, ..., x_n]$ for all $1 \leq i \leq n-1$.

The subgroup of $GA_n(k)$ generated by $Af_n(k)$ and $BA_n(k)$ is called the group of tame automorphisms. The main open problem in this context is

Problem 1 : If k is a field, are all automorphisms in $GA_n(k)$ tame ?

For $n = 2$, this is a classical result which describes $GA_2(k)$ as an amalgamated product of $Af_2(k)$ and $BA_2(k)$ along their intersection ([Ju],[Na],[Re],[Va]). If k is not a field, Nagata considers in [Na] the following automorphism of $K[x, y, z]$ where K is a field :

$$\sigma : \begin{array}{ccc} x & \to & x - 2y(y^2 + xz) - z(y^2 + xz)^2 \\ y & \to & y + z(y^2 + xz) \\ z & \to & z \end{array}$$

Theorem (Nagata) : σ considered as an element in $GA_2(K[z])$ is not tame, but it is tame considered in $GA_2(K[z, z^{-1}])$

To the best of our knowledge, the following problem is still open

Problem 2 : Is σ tame considered as an element in $GA_3(K)$?

(2.2) : Non commutative analogues of this problem have been studied extensively. The tameness notion is adapted to the considered non commutativity, the central idea being to be generated by the most natural automorphisms. In the sequel we take $k = \mathbb{C}$ in order to simplify the notation

(a) : The free algebra of rank 2 : In this case, tameness is defined as in the commutative case and one has the following theorem ([Co],[Cz],[Di],[Ma1])

Theorem : There is a natural isomorphism between $Aut_{\mathbb{C}}\, \mathbb{C} < X, Y >$ and $Aut_{\mathbb{C}}\, \mathbb{C}\,[x, y]$

(b) : The Weyl algebra $A_1(\mathbb{C})$: $A_1(\mathbb{C}) = \mathbb{C}\,[p, q]$ where $[p, q] = 1$. In this case, tameness is defined by considering the subgroups S and B defined as follows

$$S : \begin{array}{ccc} p & \rightarrow & \alpha p + \beta q + \gamma \\ q & \rightarrow & \alpha' p + \beta' q + \gamma' \end{array}$$

where $\alpha\beta' - \beta\alpha' = 1$ and $\gamma, \gamma' \in \mathbb{C}$ and

$$B : \begin{array}{ccc} p & \rightarrow & \alpha p + P(q) \\ q & \rightarrow & \alpha^{-1} q \end{array}$$

where $P(q) \in \mathbb{C}\,[q]$ and $\alpha \in \mathbb{C}^*$.Then, one has the following theorem ([A],[Dix],[Ma2])

Theorem : $Aut_{\mathbb{C}}\,(A_1(\mathbb{C}))$ is the amalgamated product of S and B along their intersection

(c) : Low dimensional enveloping algebras :

(i) : g soluble , $dim(g) = 2$ or 3 and g not nilpotent. In this case tameness is defined replacing $Af_n(\mathbb{C})$ by the group generated by $Aut_{\mathbb{C}}\, g$ and the translations, and $BA_n(\mathbb{C})$ by automorphisms triangular with respect to a basis of g adapted to the derived series of

g. In [Sm1], the automorphism groups of $U(g)$ are determined for different g's and simple inspection reveals tameness in all cases.

(ii) : $g = sl(2, \mathbb{C})$. In this case, tame automorphisms are defined to be the ones which are generated by automorphisms of $U(sl(2, \mathbb{C}))$ fixing an element $X \in sl(2, \mathbb{C})$. In [Jo], it is shown that there exists wild automorphisms of $U(sl(2, \mathbb{C}))$

(iii) : $g = g_3$, the nilpotent 3-dimensional Heisenberg algebra. In this case, tameness is defined as in (i) , the derived series being replaced by the central descending series. In [A], it is shown that a modified version of Nagata's automorphism which accounts for the non commutativity gives an example of a wild automorphism of $U(g_3)$

The case of generic matrix rings was considered by Bergman [Be] and we will briefly recall some of his results in the next section.

3. Bergman's wild automorphisms of generic matrices.

In [Be], G. Bergman constructs different types of wild automorphisms of generic matrix rings.In the special case of 2 generic 2 by 2 matrices, we will compute the extension of certain automorphisms to the trace ring and their restrictions to the 5-dimensional polynomial center. The wild automorphisms at the non commutative level induce then Nagata like automorphisms which become tame as we will see in the last section.

(3.1) : Let G be the ring of 2 generic 2 by 2 matrices , that is , the subring of $R = M_2(\mathbb{C}[x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4])$ generated by the generic matrices

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}; Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

With Z we will denote the center of G and T will be the trace ring of G , that is , the subalgebra of R generated by G and the traces of its elements. With C we denote the center of T. Finally we introduce the ring T^0 of 2 generic 2 by 2 trace zero matrices, that is, the subalgebra of R generated by $X^0 = X - \frac{1}{2}T(X)$ and $Y^0 = Y - \frac{1}{2}T(Y)$ where $T(-)$ is the trace map. The following result describes the relationship between these algebras [FHL]

Theorem (Formanek,Halpin,Li) :

(i) : The commutator ideal of G is equal to $T.[X, Y]$

(ii) : $C = \mathbb{C}[T(X), T(Y), D(X), D(Y), T(XY)]$ and T is a free C-module with basis $1, X, Y, XY$

(iii) : $G/[G, G] = \mathbb{C}[x, y]$ a polynomial ring in 2 variables

(3.2) : For our computations, it is more convenient to consider T^0 and its center $C^0 = \mathbb{C}[D(X^0), D(Y^0), T(X^0Y^0)]$. One has the following relations

(i) : $X^{0^2} = -D(X^0)$, $Y^{0^2} = -D(Y^0)$, $X^0Y^0 + Y^0X^0 = T(X^0Y^0)$

(ii) : $D(X) = D(X^0) + \frac{1}{4}T(X)^2$, $D(Y) = D(Y^0) + \frac{1}{4}T(Y)^2$, $T(XY) = T(X^0Y^0) + \frac{1}{2}T(X)T(Y)$

(iii) : $[X,Y] = [X^0,Y^0] = T(X^0Y^0) - 2Y^0X^0 = 2X^0Y^0 - T(X^0Y^0)$, $[X^0,Y^0]^2 = T(X^0Y^0)^2 - 4D(X^0Y^0)$

Hence in particular, $C = \mathbb{C}\,[T(X),T(Y),D(X^0),D(Y^0),T(X^0Y^0)]$ and T is a free C-module with basis $1, X^0, Y^0, X^0Y^0$

(3.3) : Following Bergman, consider the diagram

$$\begin{array}{ccc} \mathbb{C} < X,Y > & \stackrel{\varphi}{\rightarrow} & G \\ & \searrow^{\psi} & \downarrow \\ & & \mathbb{C}\,[x,y] \end{array}$$

which induces the diagram

$$\begin{array}{ccc} Aut_{\mathbb{C}}\,\mathbb{C} < X,Y > & \stackrel{\varphi}{\rightarrow} & Aut_{\mathbb{C}}\,G \\ & \searrow & \downarrow \pi \\ \psi & & Aut_{\mathbb{C}}\,\mathbb{C}\,[x,y] \end{array}$$

According to 2.2 (a) , ψ is an isomorphism so ϕ is mono and π is epi. Now, let us define the tame automorphisms of G to be the ones which are induced by automorphisms of $\mathbb{C} < X,Y >$ (which are all tame); they are generated by the following types

$$\eta_f : \begin{array}{ccc} X & \rightarrow & X + P(Y) \\ Y & \rightarrow & Y \end{array}$$

where $P(Y) \in \mathbb{C}\,[Y]$

$$\epsilon_c : \begin{array}{ccc} X & \rightarrow & cX \\ Y & \rightarrow & Y \end{array}$$

where $c \in \mathbb{C}^*$

$$\theta : \begin{array}{ccc} X & \rightarrow & Y \\ Y & \rightarrow & X \end{array}$$

Therefore, any non-trivial automorphism of G in the kernel of π will be wild. Let us give a few examples :

(1) : Bergman gives the following wild automorphism of G

$$\sigma_1 : \begin{array}{ccc} X & \rightarrow & X + [X,Y]^2 \\ Y & \rightarrow & Y \end{array}$$

One easily checks that σ_1 fixes T^0 and that the induced automorphism on C is the triangular one fixing all variables except $T(X)$ which is mapped to $T(X) + 2[X,Y]^2 = T(X) + 2T(X^0Y^0)^2 - 8D(X^0)D(Y^0)$.

(2) : A slightly more complicated example which gives rise to Nagata like automorphisms is

$$\sigma_2 : \begin{array}{ccc} X & \to & X + Y[X,Y]^2 \\ Y & \to & Y \end{array}$$

Then, one verifies that σ_2 induces an automorphism on T^0. In the next section we will give a more consistent procedure to produce automorphisms of the generic trace zero matrices. Moreover, the induced automorphism by σ_2 on the center of the trace algebra C is given by

$$\begin{array}{ccc} T(X) & \to & T(X) + T(Y)[X,Y]^2 \\ T(Y) & \to & T(Y) \\ D(X^0) & \to & D(X^0) - T(X^0Y^0)[X,Y]^2 + D(Y^0)[X,Y]^2 \\ D(Y^0) & \to & D(Y^0) \\ T(X^0Y^0) & \to & T(X^0Y^0) - 2D(Y^0)[X,Y]^2 \end{array}$$

The restriction of σ_2 to C^0 is then a Nagata like automorphism of C^0. In the last section we will see that $\sigma_2 \mid C$ is a tame automorphism.

4. Constructing weird automorphisms.

In this section we will present a method to construct automorphisms of T^0, the generic trace zero ring for 2 2×2 matrices. This method rests on the description of T^0 as a generic Clifford algebra, see [LB]. Our construction can be used also to construct weird automorphisms on commutative polynomial ring and on the ring of m generic 2 by 2 trace zero matrices T^0_m.

(4.1) : Let us first recall some basic facts on quadratic forms and their Clifford algebras, see for example [Ba] for more details.Let R be a commutative $\mathbb{C}$-algebra. Then any quadratic form

$$q = \sum_{i,j=1}^{m} \alpha_{ij} X_i X_j$$

with $\alpha_{ij} = \alpha_{ji} \in R$ induces a symmetric bilinear form on a free R-module of rank m

$$P = Re_1 \oplus ... \oplus Re_m$$

by defining $B(e_i, e_j) = \alpha_{ij}$. The Clifford algebra of P associated to the quadratic form q is defined to be the quotient of the tensor R-algebra $T(P)$ of P by the ideal generated by the elements of the form

$$p \otimes p - B(p,p)$$

for all $p \in P$. If we give the tensor-algebra the usual gradation, it follows that the Clifford algebra $Cl(P,q)$ has an induced $\mathbb{Z}/2\mathbb{Z}$-gradation, i.e. $Cl(P,q) = C_0 \oplus C_1$. There is a canonical R-algebra automorphism on $Cl(P,q)$ sending $c_0 \oplus c_1$ to $c_0 \oplus (-c_1)$ which is called the main automorphism.

In [LB] the so called generic Clifford algebras Cl_m were introduced. Cl_m is the Clifford algebra over the polynomial algebra

$$S_m = \mathbb{C}\,[a_{ij} : 1 \leq i \leq j \leq m]$$

that is, the homogeneous coordinate ring of the variety of all symmetric m by m matrices, corresponding to the generic quadratic form

$$q_m = \sum_{i,j=1}^{m} a_{ij} X_i X_j$$

Genericity here means that any Clifford algebra of an m-ary quadratic form over $\mathbb{C}$ can be obtained as a specialization of Cl_m. One can show, [LB] , that Cl_m is isomorphic to the iterated Öre extension

$$\mathbb{C}\,[a_{ij} : 1 \leq i < j \leq m][a_1][a_2, \sigma_2, \delta_2]...[a_m, \sigma_m, \delta_m]$$

where $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = 2a_{ij}$ for all $i < j$ and trivial actions on the other variables and $a_{ii} = a_i^2$. In particular, Cl_m has finite global dimension equal to $\frac{m(m+1)}{2}$, is a maximal order ans has p.i.-degree equal to 2^α where α is the largest natural number smaller or equal to $\frac{m}{2}$.

Restriction of automorphisms of Cl_m to S_m gives an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Aut_{\mathbb{C}}\, Cl_m \to Aut_{\mathbb{C}}\, S_m = GA_{\frac{m(m+1)}{2}}(\mathbb{C}\,)$$

the kernel being generated by the main automorphism. Therefore, in order to describe the automorphism group of the generic Clifford algebra we aim to compute $Aut(S_m : Cl_m)$ which is the subgroup of $Aut_{\mathbb{C}}\, S_m$ consisting of those automorphisms which can be extended to Cl_m. A large subgroup of them can be described in the following way :

Let σ be an automorphism of S_m , then σ extends to Cl_m if and only if $Cl_m \cong_\sigma Cl_m$ as S_m-algebras where $_\sigma Cl_m$ is equal to Cl_m as an abelian group and with S_m-action given via $s * c = \sigma^{-1}(s).c$ for all $s \in S_m$ and $c \in Cl_m$. Now, it follows from the definition that $_\sigma Cl_m$ is the Clifford algebra over S_m associated to the m-ary quadratic form

$$\sigma^{-1}(q) = \sum_{i,j=1}^{m} \sigma^{-1}(a_{ij}) X_i X_j$$

Now, it is well known that two Clifford algebras are isomorphic if their corresponding matrices are congruent.That is, if there exists an invertible matrix $A \in GL_m(S_m)$ such that

$$A^\tau.(a_{ij})_{i,j}.A = (\sigma^{-1}(a_{ij}))_{i,j}$$

These observations make it possible to determine lots of elements of $Aut(S_m : Cl_m)$ by determining which of the endomorphisms of S_m determined by sending a_{ij} to the (i,j)-entry of the matrix $A^\tau(a_{ij})_{i,j}A$ for $A \in GL_m(S_m)$ are automorphisms (which can be tested by calculating the Jacobian).

(4.2) : We will now see what the above general procedure gives us in the special case that $m = 2$. For notational simplicity we will let $x = a_{11}, y = a_{12}$ and $z = a_{22}$.Let us consider the easiest case of an elementary matrix

$$A = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$$

for some $f \in \mathbb{C}[x,y,z] = S_2$. As we have seen above, this matrix induces an endomorphism on $\mathbb{C}[x,y,z]$ which is given by

$$\begin{array}{ccc} x & \rightarrow & x + 2fy + f^2z \\ y & \rightarrow & y + fz \\ z & \rightarrow & z \end{array}$$

To check when this is an automorphism we can calculate its Jacobian

$$\begin{pmatrix} 1 + 2y\partial_x f + 2fz\partial_x f & 2f + 2y\partial_y f + 2fz\partial_y f & * \\ z\partial_x f & 1 + z\partial_y f & * \\ 0 & 0 & 1 \end{pmatrix}$$

If we set the determinant equal to 1 we get the equation

$$2y\partial_x f + z\partial_y f = 0$$

which implies that $f \in \mathbb{C}[z, y^2 - xz]$ as can be readily verified. Moreover, any such f clearly induces an automorphism on $\mathbb{C}[x,y,z]$.

(4.3) : Note that for any m there is a natural epimorphism

$$\pi : \mathbb{C} < X_1, ..., X_m > \to Cl_m$$

which is determined by $\pi(X_i) = a_i$. Therefore, the natural notion of tame automorphisms of Cl_m is that they are the automorphisms which can be lifted to tame automorphisms of the free algebra $\mathbb{C} < X_1, ..., X_m >$. Using this convention we can now prove

Theorem : If $f \in \mathbb{C}[y^2 - xz]_+$ then the induced automorphism on Cl_2 is wild

Proof : In general,if an endomorphism of S_m determined by an element $A \in GL_m(S_m)$ is an automorphism, the extension of it to Cl_m is given by sending a_i to the i-th entry of the vector

$$A^{\tau}.\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

Now, if $f \in \mathbb{C}[y^2 - xz]_+$ the induced automorphism on Cl_2 is given by

$$\begin{array}{ccc} a_1 & \to & a_1 + f a_2 \\ a_2 & \to & a_2 \end{array}$$

It is east to verify that this automorphism fixes the normalizing element $[a_1, a_2]$ and hence induces an automorphism on the quotient $Cl_2/Cl_2[a_1, a_2]$ which is a polynomial ring in the images of the a_i say $\mathbb{C}[u, v]$. Since $[a_1, a_2]^2 = y^2 - xz$ it is clear that this induced automorphism on $\mathbb{C}[u, v]$ is the identity. But we have seen before that the natural morphism

$$Aut_{\mathbb{C}}\,\mathbb{C} < X_1, X_2 > \to Aut_{\mathbb{C}}\,\mathbb{C}[u, v]$$

is an isomorphism, so the automorphism on Cl_2 being non-trivial cannot be lifted to an automorphism of $\mathbb{C} < X_1, X_2 >$ and hence is not tame.

Remark that the special case when $f = y^2 - xz$ gives us the Nagata automorphism so the foregoing result may add some evidence to the conjectural wildness of this automorphism.

Of course, one can repeat the same argumentation for more general elements of $GL_2(\mathbb{C}[x,y,z])$. Note that there is a fairly precise description of this group as an amalgamated product with $GL_2(\mathbb{C})$ as one of the components. This prompts the following question

Problem 3 : Is the subgroup $Aut(S_2 : Cl_2)$ of $GA_3(\mathbb{C})$ an amalgamated product with $GL_2(\mathbb{C})$ as one of the components ?

At a time, we had the following fairly optimistic procedure to find a counterexample to the Jacobian conjecture in three variables : consider the elements $A \in GL_2(\mathbb{C}[x,y,z])$ such that the Jacobian of the associated endomorphism of S_2 is invertible. Then (modulo the Jacobian conjecture) the endomorphism of Cl_2 given by sending a_i to the i-th entry of $A.(a_i)_i$ should be an automorphism fixing the normalizing element $[a_1,a_2]$ and hence should induce an automorphism on the quotient $Cl_2/Cl_2[a_1,a_2] = \mathbb{C}[u,v]$ for which there exists a test by computing the Jacobian. At first sight there is not much relation between these two Jacobians but , due to lacking technical support , we were not able to construct interesting A's to verify whether this aproach has any chance.

(4.4) : Of course, the general method can be used to provide weird automorphisms on arbitrary polynomial algebras. Let us consider, as an example, the case $m = 3$ and denote $a_{12} = u, a_{13} = v, a_{23} = w, a_{11} = x, a_{22} = y, a_{33} = z$ and consider an elementary matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f & 0 & 1 \end{pmatrix}$$

where $f \in \mathbb{C}[x,y,z,u,v,w]$. Then the induced endomorphism is given by

$$\begin{array}{ccc} x & \rightarrow & x + 2fv + f^2 z \\ y & \rightarrow & y \\ z & \rightarrow & z \\ u & \rightarrow & u + fw \\ v & \rightarrow & v + fz \\ w & \rightarrow & w \end{array}$$

Setting the Jacobian equal to one gives this time the condition

$$w\partial_u f + z\partial_v f + 2v\partial_x f = 0$$

which gives as possible solutions $f \in \mathbb{C}[uz - vw, v^2 - xz, y, z, w]$.For example, is the special case when $f = uz - vw$ a tame automorphism ?

(4.5) : It is about time to clarify what the above has to do with our original topic. If T_m^0 denotes the ring of m generic 2 by 2 trace zero matrices, then there is a natural epimorphism

$$\pi_m : Cl_m \to T_m^0$$

sending a_i to X_i^0 and a_{ij} to $\frac{1}{2}T(X_i^0 X_j^0)$.This epimorphism is an isomorphism in case $m = 2$ or 3 and in general the kernel is the unique graded ideal of Cl_m lying over the ideal of S_m generated by the four by four minors of the generic symmetric matrix $(a_{ij})_{i,j}$. Note that this ideal is invariant for all automorphisms of the form described above, therefore any automorphism of Cl_m corresponding to a matrix $A \in GL_m(S_m)$ induces an automorphism on T_m^0, thus providing a large class of interesting examples. In particular, the automorphism on $Cl_2 = T^0$ induced by the triangular matrix with $f = y^2 - xz$ is the restriction of σ_2 determined in the previous section to T^0.

5. Stable tameness and central tameness.

(5.1) : The following stable tameness result is due to M. Smith.

Definition : (i) Let D be a derivation of $\mathbb{C}[x_1, ..., x_n]$. D is said to be triangular if $D(x_i) = \phi_i(x_{i+1}, ..., x_n)$ for all $1 \leq i \leq n-1$ and $D(x_n) = 0$

(ii) D is said to be locally nilpotent if for every $P \in \mathbb{C}[x_1, ..., x_n]$ we can find $n(P) \in \mathbb{N}^*$ such that $D^n(P) = 0$

Theorem (M. Smith,[Sm2]) : Let D be a triangular,locally nilpotent derivation of $\mathbb{C}[x_1, ..., x_n]$ and $u \in Ker(D)$. Then, uD is locally nilpotent and the automorphism $exp(uD)$ becomes tame when extended to $\mathbb{C}[x_1, ..., x_n, t]$ by fixing t.

(5.2) : Let σ be an automorphism of G. Then, theorem 3.1 implies that σ induces an automorphism of $G/[G, G]$ which has to be tame by the Jung-Van der Kulk result (2.1). Up to a tame automorphism of G, σ can then be brought in the following form

$$\begin{array}{lcl} X & \rightarrow & X + t_1[X, Y] \\ Y & \rightarrow & Y + t_2[X, Y] \end{array}$$

where $t_1, t_2 \in T$. Unfortunately, we cannot continue the analysis in this generality. Restricting to automorphisms of the form

$$\begin{array}{lcl} X & \rightarrow & X + t[X, Y] \\ Y & \rightarrow & Y \end{array}$$

for $t \in T$, one can show the following lemma

Lemma : t has to be of the form $(P_1 + P_2Y)[X, Y]$ with $P_i \in C$

Proof : First, remark that $\sigma([X, Y]) = \lambda[X, Y]$ for some $\lambda \in \mathbb{C}^*$.This implies the following equivalences

$$[X,Y] + [t[X,Y],Y] = \lambda[X,Y]$$
$$:: [X^0,Y^0] + [t[X^0,Y^0],Y^0] = \lambda[X^0,Y^0]$$
$$:: [t[X^0,Y^0],Y^0] = (\lambda-1)[X^0,Y^0]$$
$$:: [t,Y^0][X^0,Y^0] + t[[X^0,Y^0],Y^0] = (\lambda-1)[X^0,Y^0]$$
$$:: [t,Y^0][X^0,Y^0] - 2tY^0[X^0,Y^0] = (\lambda-1)[X^0,Y^0] \text{ (since } [[X^0,Y^0]Y^0] = -2Y^0[X^0,Y^0])$$
$$:: [t,Y^0] - 2tY^0 = \lambda - 1 \text{ (since } T \text{ is a domain)}$$

Now, if we let $t = a + bX^0 + cY^0 + dX^0Y^0$ with $a,b,c,d \in C$ then this is equivalent to

$$:: [bX^0,Y^0] + d[X^0Y^0,Y^0] - 2tY^0 = \lambda - 1$$

$:: b(2X^0Y^0 - T(X^0Y^0)) + d(2X^0Y^0 - T(X^0Y^0))Y^0 - 2aY^0 - 2bX^0Y^0 + 2cD(Y^0) + 2dD(Y^0)X^0 = \lambda - 1$

which entails then that

$$2cD(Y^0) = bT(X^0Y^0)$$
$$2a = -dT(X^0Y^0)$$

By factoriality of C, there exists an element $\mu \in C$ such that

$$b = 2\mu D(Y^0) \text{ , } c = \mu T(X^0Y^0)$$

We then have that t is equal to

$$a + dX^0Y^0 + 2\mu D(Y^0)X^0 + \mu T(X^0Y^0)Y^0$$
$$= \tfrac{d}{2}(-T(X^0Y^0) + 2X^0Y^0) + \mu(-2{Y^0}^2X^0 + T(X^0Y^0)Y^0)$$
$$= \tfrac{d}{2}[X^0,Y^0] + \mu Y^0[X^0,Y^0] = (\tfrac{d}{2} + \mu Y^0)[X^0,Y^0]$$

Remark : Unfortunately, not every t of this form will give an automorphism as the following example shows : consider the endomorphism σ of G defined by

$$\begin{array}{ccc} X & \rightarrow & X + T(X)[X,Y]^2 \\ Y & \rightarrow & Y \end{array}$$

Then, $T(X)$ is send under the extension of σ to T to $T(X)(1 + 2[X,Y]^2)$ whereas $T(Y), D(X^0), D(Y^0)$ and $T(X^0Y^0)$ are fixed. But by computing the Jacobian on C we see that this is not an automorphism. So, additional restrictions on t are necessary. The following result summarizes the most general automorphisms we can construct and their behaviour on C

Theorem : (i) : All tame automorphisms of G induce tame automorphisms on C

(ii) : All wild automorphisms σ of G of the form

$$\begin{array}{rcl} X & \to & X + (P_1 + P_2 Y)[X,Y]^2 \\ Y & \to & Y \end{array}$$

where $P_1 \in \mathbb{C}[T(Y), D(X^0), D(Y^0), T(X^0Y^0)]$ and $P_2 \in \mathbb{C}[T(Y), D(Y^0)]$ induce tame automorphisms on C

Proof : (i) : This is an easy verification on the generators $\eta_f, \epsilon_c, \theta$ of the tame automorphisms given in 3.3

(ii) : Remark that $\sigma = \sigma_1 \circ \sigma_2$ where σ_i fixes Y and $\sigma_1(X) = X + P_1[X,Y]^2$ whereas $\sigma_2(X) = X + P_2Y[X,Y]^2$ and both σ_1 and σ_2 are automorphisms of G. We will first consider σ_1 :

σ_1 fixes $T(Y), D(X^0), D(Y^0)$ and $T(X^0Y^0)$ and it sends $T(X)$ to $T(X) + 2P_1[X,Y]^2$, hence $\sigma_1 \mid C$ is a triangular automorphism.

On the other hand, σ_2 has the following action on C

$$\begin{array}{rcl} T(X) & \to & T(X) + T(Y)P_2[X^0,Y^0]^2 \\ T(Y) & \to & T(Y) \\ D(X^0) & \to & D(X^0) - T(X^0Y^0)P_2[X^0,Y^0]^2 + D(Y^0)(P_2[X^0,Y^0]^2)^2 \\ D(Y^0) & \to & D(Y^0) \\ T(X^0Y^0) & \to & T(X^0Y^0) - 2D(Y^0)P_2[X^0,Y^0]^2 \end{array}$$

Now, we define

$$\Delta = -2D(Y^0)\frac{\partial}{\partial T(X^0Y^0)} - T(X^0Y^0)\frac{\partial}{\partial D(X^0)}$$

Then Δ is a triangular, locally nilpotent derivation of C. Moreover, $\Delta(P_2[X^0,Y^0]^2) = P_2\Delta(T(X^0Y^0)^2 - 4D(X^0)D(Y^0)) = 0$. Now, put $C_1 = \mathbb{C}[T(Y), D(X^0), D(Y^0), T(X^0Y^0)]$ then $\sigma_2 \mid C_1 = exp(P_2[X^0,Y^0]^2\Delta)$. By M. Smith's stable tameness result in 5.1 , $\sigma_2 \mid C_1$ becomes tame when extended to C by fixing $T(X)$. Composing this last automorphism of C with the triangular one which sends $T(X)$ to $T(X)+T(Y)P_2[X^0,Y^0]^2$ and fixes the other variables one gets $\sigma \mid C$, finishing the proof of the tameness of σ

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