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SOME REMARKS ON RATIONAL MATRIX INVARIANTS

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Abstract.

Wildness of the rank two quiver P_3 provides a link between the study of rational matrix invariants and that of stable vectorbundles over the projective plane. Using this dictionary, results of Formanek imply the rationality of the moduli spaces of rank three and rank four vectorbundles. Further,we recover a recent result of Van den Bergh showing that the field of rational n by n matrixinvariants is the functionfield of the generic Jacobian variety for smooth plane curves of degree n.

1. Introduction.

(1.1) : Throughout this paper,we consider an algebraically closed field of characteristic zero and call it $\mathcal{C}$. Let $GL_n(\mathcal{C})$ act on m-tuples of n by n matrices $X_{m,n} = M_n(\mathcal{C}) \oplus ... \oplus M_n(\mathcal{C})$ by componentswise conjugation. The topic of this paper is the field $K_{m,n}$ of rational invariants for this situation. That is, consider the rational field $L_{m,n} = \mathcal{C}(x_{ij}(l) : 1 \leq i,j \leq n; 1 \leq l \leq m)$ and $\gamma \in GL_n(\mathcal{C})$ acts on it by sending the variable $x_{ij}(l)$ to the (i,j)-entry of the matrix $\gamma^{-1}X_l\gamma$ where $X_l = (x_{ij}(l))_{i,j} \in M_n(L_{m,n})$. Then, $K_{m,n}$ is the fixed field under this action.

$K_{m,n}$ is easily seen to be the field of functions of the variety of matrixinvariants $V_{m,n} = X_{m,n}/GL_n(\mathcal{C})$. That is, the variety parametrizing simultaneous conjugacy classes of m tuples of n by n matrices which generate a semi-simple subalgebra of $M_n(\mathcal{C})$. See for example [6],[14].

(1.2) : It is still an open question whether $K_{m,n}$ is always a rational functionfield. For ringtheorists this question is important because it would imply the Merkurjev-Suslin result for fields containing $\mathcal{C}$ (the Brauer group is generated by cyclic algebras). Let us scetch the argument : consider the ring of m generic n by n matrices, that is the subring of $M_n(L_{m,n})$ generated by the matrices X_l. This ring is known to be a left and right Öre domain, so we can form its classical ring of quotients $\Delta_{m,n}$ which is a division algebra of dimension n^2 over its center $K_{m,n}$. Rationality of $K_{m,n}$ and a result of Bloch [3,Th.1.1] would imply that $\Delta_{m,n}$ is Brauer equivalent to a product of cyclic algebras. Then by the generic property of the $\Delta_{m,n}$ for all $m \geq 2$ every central simple algebra of dimension n^2 over a field L containing $\mathcal{C}$ would be a product of cyclic algebras in $Br(L)$, see for example [15]. For more details we refer to [6],[15] and [17].

Procesi [13] proved that $K_{m,n}$ is rational whenever $K_{2,n}$ is , thereby reducing the problem to two matrices. He also solved the rationality problem for $n = 2$. Later, E. Formanek [4],[5] proved the rationality for $n = 3$ and $n = 4$. He used the following elegant description due to C. Procesi of $K_{2,n}$: let $\{x_i, y_{ij} \mid 1 \leq i,j \leq n\}$ be independent commuting indeterminates and let L be the subfield of $\mathcal{C}(x_i, y_{ij} : 1 \leq i,j \leq n)$ generated by $\{x_i, y_{ii}, y_{ij}y_{ji}, y_{ij}y_{jk}y_{ki} \mid 1 \leq i,j,k \leq n\}$. Then L is a rational functionfield of trancendence degree n^2+1 and the permutation group S_n acts on it by $\sigma(x_i) = x_{\sigma(i)}, \sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. Then, $K_{2,n}$ is the fixed field under this action.

(1.3) : In this paper we aim to show that this rationality problem may also be of interest to geometers. Using the results of K. Hulek [9] we will show that $K_{2,n}$ is the functionfield of the moduli space $M(n,0,n)$

of stable (rank n) vectorbundles over the projective plane with Chern-numbers $(0, n)$. Therefore, Formaneks results imply the rationality of $M(3, 0, 3)$ and $M(4, 0, 4)$ which was (perhaps) not known. Rationality of $M(2, 0, 2)$ was proved by Barth [2].

Another consequence of our result is a recent theorem of M. Van den Bergh [18] who showed that $K_{2,n}$ is the functionfield of a Picard scheme of a bundle of nonsingular curves over a rational variety. This result will now follow from the fact [9,1.7] that a sufficiently general stable vectorbundle over $\mathbb{P}_2$ having Chern-numbers $(0, n)$ is classified by a smooth plane curve of degree n and an invertible sheaf over it (generalizing the curve of jumping lines and the θ-characteristic in the rank two case , see [2]).

2. Vector bundles over $\mathbb{P}_2$.

(2.1) : In this section we aim to prove the following result :

Theorem 1 : $K_{2,n}$ is the functionfield of the moduli space $M(n,0,n)$ of stable rank n vectorbundles over the projective plane with Chern-numbers $(0,n)$.

(2.2) : Let us recall the connection between so called s-stable vectorbundles over $\mathbb{P}_2$ and certain triples of n by n matrices $A=(A_0,A_1,A_2)$. One calls A prestable if for any $v\in\mathbb{C}^n$ we have $dim_{\mathbb{C}}(A_0v+A_1v+A_2v)\geq 2$ and $dim_{\mathbb{C}}(A_0^tv+A_1^tv+A_2^tv)\geq 2$ where $(-)^t$ denotes transposition. Hulek associates to a prestable triple A a vectorbundle $\mathcal{E}_A$ in the following way :

Let $\mathcal{O}_{\mathbb{P}}$ be the structure sheaf of $\mathbb{P}_2$ and let X_0,X_1,X_2 be the usual basis for $\Gamma(\mathcal{O}_{\mathbb{P}}(1))$. Let $V=\Gamma(\mathcal{O}_{\mathbb{P}}(1))^*$ and let Y_0,Y_1,Y_2 be a basis of V dual to X_0,X_1,X_2. Define a linear map $\phi_A:\mathbb{C}^n\otimes V\to\mathbb{C}^n\otimes V^*$ by sending $v\otimes Y_i$ to $A_{i+1}v\otimes X_{i-1}-A_{i-1}v\otimes X_{i+1}$ for all $v\in\mathbb{C}^n$ and $i=0,1,2$ mod 3.For a canonical choice of bases for $\mathbb{C}^n$ and the bases defined before for V and V^* the matrix of ϕ_A is given by

$$\begin{pmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{pmatrix}$$

If U denotes the image of ϕ_A , we obtain a complex of vectorbundles

$$M_A:\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}}(-1)\xrightarrow{a}U\otimes\mathcal{O}_{\mathbb{P}}\xrightarrow{c}\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}}(1)$$

where a denotes the composite morphism

$$\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}}(-1)\xrightarrow{1\otimes s^*}\mathbb{C}^n\otimes V\otimes\mathcal{O}_{\mathbb{P}}\xrightarrow{\phi_A\otimes 1}U\otimes\mathcal{O}_{\mathbb{P}}$$

and c is the restriction to $U\otimes\mathcal{O}_{\mathbb{P}}$ of the morphism $1\otimes s:\mathbb{C}^n\otimes V^*\otimes\mathcal{O}_{\mathbb{P}}\longrightarrow\mathbb{C}^n\otimes\mathcal{O}_{\mathbb{P}}(1)$ where $s:\Gamma(\mathcal{O}_{\mathbb{P}}(1))\otimes\mathcal{O}_{\mathbb{P}}\longrightarrow\mathcal{O}_{\mathbb{P}}(1)$ is the multiplication map and s^* its dual. The complex M_A is a monad in the sense of Horrocks. Its cohomology turns out to be a vector bundle $\mathcal{E}_A$ which is s-stable in the sense that $H^0(\mathcal{E}_A)=H^0(\mathcal{E}_A^*)=0$. The bundle $\mathcal{E}_A$ has rank $rk(\phi_A)-2n$, has Chern-numbers $(0,n)$ and the map $A\to\mathcal{E}_A$ induces a bijection between the set of isoclasses of s-stable vector bundles over $\mathbb{P}_2$ with Chern-numbers $(0,n)$ and isomorphism classes of prestable triples when considered as representations of dimension vector (n,n) of the wild quiver P_3

$$\begin{array}{ccc} & \longrightarrow & \\ \bigcirc & \longrightarrow & \bigcirc \\ & \longrightarrow & \end{array}$$

that is, orbits of $GL_n(\mathbb{C})\times GL_n(\mathbb{C})$ acting on $X_n=M_n(\mathbb{C})\oplus M_n(\mathbb{C})\oplus M_n(\mathbb{C})$ by $(\gamma_1,\gamma_2).(A_0,A_1,A_2)=(\gamma_2^{-1}A_0\gamma_1,\gamma_2^{-1}A_1\gamma_1,\gamma_2^{-1}A_2\gamma_1)$.

(2.3) : We can study the following projective quotient variety as a first approximation to the orbitstructure problem in X_n. Consider the open subvariety X' consisting of triples (A_0,A_1,A_2) s.t. the rank of the $n\times 3n$ matrix (A_0,A_1,A_2) is maximal. We can eliminate the action of the first component of $GL_n(\mathbb{C})\times GL_n(\mathbb{C})$ on this subvariety and get the Grassmann variety $Grass(n,3n)$ as a representing space. The second $GL_n(\mathbb{C})$ component acts on this space via its diagonal embedding in $GL_{3n}(\mathbb{C})$. The projective variety of interest to us is $Y_n=Grass(n,3n)^{ss}/GL_n(\mathbb{C})$ where $Grass(n,3n)^{ss}$ is the set of semi-stable points under this action. These points come from representations in X_n having no subrepresentation of dimension vector (k,l) where $0<l\leq k<n$. The stable points come from representations having no subrepresentation of type (k,k) where $0<k<n$,see [9].

We will show that the variety of matrixinvariants $V_{2,n} = X_{2,n}/GL_n(\mathbb{C})$ is birational to Y_n. On the open subvariety $X^{"}$ of X' determined by those triples (A_0, A_1, A_2) s.t. $det(A_0) \neq 0$ we can eliminate the action of the first component of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ by multiplying on the right by A_0^{-1} and get representants of the form (I_n, B_1, B_2). The action of the second component on these representants is $\gamma.(I_n, B_1, B_2) = (\gamma^{-1}, \gamma^{-1}B_1, \gamma^{-1}B_2) = (I_n, \gamma^{-1}B_1\gamma, \gamma^{-1}B_2\gamma)$. That is, the orbits of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $X^{"}$ correspond to orbits of $GL_n(\mathbb{C})$ acting on couples of n by n matrices by simultaneous conjugation. Clearly, the map $X_{2,n} \rightarrow X^{"}$ given by sending a couple (B_1, B_2) to the representation (I_n, B_1, B_2) induces an open immersion of $V_{2,n}^{simple}$ in Y_n. Here, $V_{2,n}^{simple}$ is the open set of $V_{2,n}$ corresponding to couples which generate $M_n(\mathbb{C})$ and therefore the corresponding representations give rise to stable points in $Grass(n, 3n)$.

(2.4) : We now have all the relevant information to prove theorem 1 :
Consider the open subvariety $X'_{2,n}$ of $X_{2,n}$ consisting of couples (B_1, B_2) which generate $M_n(\mathbb{C})$ and such that $[B_1, B_2] \in GL_n(\mathbb{C})$. The corresponding representation $B = (I_n, B_1, B_2) \in X^{"}$ is prestable. For otherwise, B_1 and B_2 would have a common eigenvector v , but then $[B_1, B_2]v = 0$ whence $v = 0$. Therefore, we can associate to B an s-stable vectorbundle $\mathcal{E}_B$ of rank n. This follows from (2.2) and

$$\begin{pmatrix} I_n & -B_1 & -B_2 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & B_2 & -B_1 \\ -B_2 & 0 & I_n \\ B_1 & -I_n & 0 \end{pmatrix} = \begin{pmatrix} [B_1, B_2] & 0 & 0 \\ * & 0 & I_n \\ * & -I_n & 0 \end{pmatrix}$$

By a result of Maruyama [10,Th.2.8] stability is an open property, hence there exist open subvarieties $X^s_{2,n} \subset X'_{2,n}$ and $X^s \subset X^{"}$ whose points give rise to stable rank n vectorbundles over $\mathbb{P}_2$ with Chern-numbers $(0, n)$. Using the observations from (2.2) and (2.4) it is clear that $V^s_{2,n} = X^s_{2,n}/GL_n(\mathbb{C})$ embeds in $M(n, 0, n)$ finishing the proof.

(2.5) : Using the results of E. Formanek, we get as an immediate consequence :

Corollary : The moduli spaces $M(n, 0, n)$ are rational for $n \leq 4$

To the best of my knowledge, rationality of $M(3, 0, 3)$ and $M(4, 0, 4)$ has not been noted before. Rationality of $M(2, 0, 2)$ is due to W. Barth [2].

(2.6) : We take this opportunity to warn the reader for possible misuse of theorem 1 in view of [11]. In this paper Maruyama claims stable rationality of the moduli spaces $M(n, 0, n)$ and hence , via theorem 1, of $K_{2,n}$. By applying Bloch's result twice this would immediatly imply the Merkurjev-Suslin result for fields containing $\mathbb{C}$. In fact, the method of proof of [11] would even give the stronger result that $K_{2,n}$ is stably rational over $K_{1,n}$ (which would imply the full Merkurjev-Suslin result in characteristic zero). For, if one takes $x = (0, 0, 1) \in \mathbb{P}_2$ and the description of stable bundles by triples (I_n, B_1, B_2) as in (2.4) then it is fairly easy to compute that the variety Y_n constructed in [11,p.83] has as its functionfield $K_{1,n}$. Maruyama constructs a vectorbundle V on Y_n birational to a bundle Z over $M(n, 0, n)$ s.t. the image of $\mathbb{C}(Y_n)$ in $\mathbb{C}(Z)$ coincides with $K_{1,n}$ as subfield of $\mathbb{C}(M(n, 0, n)) = K_{2,n}$. However, Snider has remarked that this is impossible for $n = 4$ see for example [5,p.319] or [15] and proved by Colliot-Thélène and Sansuc in "Principal homogeneous spaces under flasque tori with applications to various problems" which will appear in the Journal of Algebra.A possible gap in Maruyama's proof was communicated to me by D. Saltman ([11,p.86 l.-7] the trivialization of $\tilde{H}_{\tilde{M}_o}$ is not a trivialization as $GL(N)$-sheaf) and by Le Potier and Hulek which I hereby like to thank. D. Saltman also has an elegant extension of Sniders remark for any non-squarefree n.

3. The generic Jacobian variety.

(3.1) : In [18,§6], M. Van den Bergh showed that $V_{2,n}$ is birational to a Picard scheme of a bundle of nonsingular curves over a rational variety. The projective space $\mathbb{P}^{\frac{1}{2}n(n+3)}$ parametrizes plane curves of degree n. Let U be the open subvariety corresponding to nonsingular curves. Consider the flagvariety $W \subset \mathbb{P}^2 \times U$ consisting of all couples (P, Y) s.t. $P \in Y$. The projection $W \to U$ is a flat bundle of smooth curves. Let $PIC_{W/U}$ be the functor which associates to an U-scheme S the group

$$PIC_{W/U}(S) = \frac{\{\text{group of invertible sheaves on } W \times_U S\}}{\{\text{subgroup of sheaves of the form } p_2^*(K) \text{ for } K \text{ on } S\}}$$

Since $W \to U$ is a bundle of smooth curves we can associate to invertible sheaves a discrete invariant, the degree. $PIC^d_{W/U}$ is the subfunctor consisting of invertible sheaves of degree d. The sheafification of this functor with respect to the flat topology is represented by the variety $Pic^d_{W/U}$ consisting of couples $(Y, \mathcal{L})$ where Y is a nonsingular curve of degree n in $\mathbb{P}_2$ and $\mathcal{L}$ is a divisor on Y of degree d (which I like to call the generic Jacobian variety for smooth plane curves of degree n).For more details we refer the reader to [1],[8],[12,ch 6] or the preliminary sections of [18].

Theorem 2 : (Van den Bergh,[18,Th.6.1.3])
If $d = \frac{1}{2}n(n-1)$, then $K_{2,n}$ is the functionfield of the variety $Pic^d_{W/U}$

(3.2) : In view of theorem 1 we have to associate to a sufficiently general vectorbundle $\mathcal{E}$ of rank n over $\mathbb{P}_2$ with Chern-numbers $(0, n)$ a nonsingular curve Y of degree n and an invertible sheaf $\mathcal{L}$ which determine $\mathcal{E}$ upto isomorphism. Hulek [9,1.7] has indicated how this can be done by a suitable generalization of Barth's characterization of rank two bundles by their curve of jumping lines and θ- characteristic , [2]. For the reader's convenience we will briefly recall the main ideas of his proof.

Let $\mathcal{E}_A$ be an s-stable vectorbundle associated to the prestable triple $A = (A_0, A_1, A_2)$ and define $\Delta_A = \det(A_0Y_0 + A_1Y_1 + A_2Y_2) \in \Gamma(\mathcal{O}_{\mathbb{P}^*_2}(n))$ and let $Y_A = \{\Delta_A = 0\}$. The discriminant Δ_A is a homogeneous polynomial of degree n and $Y_A \subset \mathbb{P}^*_2$ will be a curve of degree n or the whole plane. The interpretation of Y_A is that it contains those lines L in $\mathbb{P}_2$ such that $\mathcal{E} \mid L \neq \mathcal{O}_L^{\oplus r}$, so it generalizes the curve of jumping lines in the rank two case.

In case Y_A is a curve (which is the generic case) one defines a map $\psi_A = (A \otimes 1) o (1 \otimes s) : \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \to \mathbb{C}^n \otimes V^* \otimes \mathcal{O}_{\mathbb{P}^*} \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}$. Over a point $L \in \mathbb{P}^*_2$ with coordinate vector $y = (y_0, y_1, y_2)$ the map ψ_A is just $A(y) = A_0y_0 + A_1y_1 + A_2y_2$. We can define a sheaf $\mathcal{L}_A$ by the sequence

$$(*) : 0 \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \xrightarrow{\psi_A} \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*} \to \mathcal{L}_A \to 0$$

which has its support in Y_A. By [9,1.7.3.iv] the pair $(Y_A, \mathcal{L}_A)$ determines $\mathcal{E}_A$ uniquely. Restricting the sequence $(*)$ to Y_A we obtain

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) & \xrightarrow{\psi_A} & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*} & \to & \mathcal{L}_A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{C}^n \otimes \mathcal{O}_{Y_A}(-1) & \xrightarrow{\psi_A|Y_A} & \mathbb{C}^n \otimes \mathcal{O}_{Y_A} & \to & \mathcal{L}' & \to & 0 \end{array}$$

For sufficiently general A we get that $rk(\psi_A \mid Y_A) = n-1$ whence $\mathcal{L}'$ is an invertible sheaf over Y_A. The induced map $\mathcal{L}_A \to \mathcal{L}'$ is surjective and will be injective too if every section in $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}$ vanishing on Y_A comes by ψ_A. This is a consequence of $\psi_A^{adj} o \psi_A = det(A) 1_{\mathbb{C}^n}$ where $det(A) : \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-n) \xrightarrow{\psi_A^{adj}} \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}$. So, for generic A we have that $\mathcal{L}_A \in Pic(Y_A)$ of degree $\frac{1}{2}n(n-1)$ by [9,1.7.3.iii].

Conversely, starting from a plane curve Y of degree n and $\mathcal{L} \in Pic(Y)$ of degree $\frac{1}{2}n(n-1)$ one can reconstruct a triple A which will be prestable (and hence determine a vectorbundle) for a sufficiently general choice of Y and $\mathcal{L}$, [9,1.7].

References .

[1] : M. Artin ; Algebraization of formal moduli I , Global Analysis , Tokyo (1969) 21-71

[2] : W. Barth ; Moduli of vector bundles on the projective plane , Invent.Math. 42 (1977) 63-91

[3] : S. Bloch ; Torsion algebraic cycles,K_2 and the Brauer group of functionfields,Bull AMS 80 (1974) 941-945

[4] : E. Formanek : The center of the ring of 3 by 3 generic matrices, Lin. Mult. Alg. 7 (1979) 203-212

[5] : E. Formanek : The center of the ring of 4 by 4 generic matrices, J. Algebra 62 (1980) 304-319

[6] : E. Formanek : The polynomial identities of matrices, Contemp. Math. 13 (1982) 41-79

[7] : E. Formakek : Rational function fields,Noether's problem and related questions JPAA 31 (1984) 28-36

[8] : A. Grothendieck : Les schémas de Picard, Sém Bourbaki 232 et 236

[9] : K. Hulek : On the classification of stable rank r vector bundles over the projective plane, Birkhäuser PM 7 (1980) 113-144

[10] : M. Maruyama : Openness of a family of torsion free sheaves, J.Math.Kyoto Univ. 16 (3) (1976) 627-637

[11] : M. Maruyama : Stable rationality of some moduli spaces over vectorbundles [11] : M. Maruyama : Stable rationality of some moduli spaces of vectorbundles over $\mathbb{P}^2$, Springer LNM 1194 (1986) 80-89

[12] : D. Mumford : Geometric invariant theory , Ergebn. Math. 33 (1965) Springer Verlag

[13] : C. Procesi : Rings with polynomial identities, Dekker New York (1973)

[14] : C. Procesi : Finite dimensional representations of algebras, Israel J. Math. 19 (1974) 169-184

[15] : C. Procesi : Relazioni tra geometrica algebrica ed algebra non commutativa.Algebre cicliche e problema di Lüroth, Boll. Un. Math. Ital. (5) 18-A (1981) 1-10

[16] : D. Saltman : Retract rational fields and cyclic Galois extensions, Israel J. Math. 47 (1984) 165-215

[17] : R. Snider : Is the Brauer group generated by cyclic algebras ? LNM 734 (1979) 279-301

[18] : M. Van den Bergh : The center of generic division algebras,to appear