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Quiver concomitants are often reflexive Azumaya.

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RATIONAL INVARIANTS OF QUIVERS AND THE RING OF MATRIXINVARIANTS

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Abstract :

Let V be a finite dimensional vectorspace over $\mathbb{C}$ and let $\alpha = (\alpha(1), ..., \alpha(m)) \in \mathbb{N}^m$.An action of $GL(\alpha) = \prod_{i=1}^m GL(\alpha(i))$ on V is said to be Schurian if the stabilizer of a generic point is $\mathbb{C}^*$ embedded diagonally in $GL(\alpha)$. In this paper we show that the field of rational invariants for such an action is stably equivalent to the field of rational n by n matrixinvariants where $n = gcd(\alpha(i) : 1 \leq i \leq m)$.

Key words :

invariant theory,representations of quivers,generic matrices

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1. The problem

Throughout this paper, we consider an algebraically closed field of characteristic zero and call it $\mathbb{C}$. Let G be an affine linear reductive group acting allmost freely on a finite dimensional vectorspace V, that is, the stabilizer of a generic point is trivial. One of the main open problems in invariant theory is to determine for which groups G the field of rational invariants $\mathbb{C}(V)^G$ is (stably) rational.

A first approach might be the following : if G acts almost freely on V then by Luna's results [Lu] we know that there exists an affine G-invariant open subvariety U of V consisting of points with trivial stabilizer and we can form the quotient variety U/G. The canonical map $\pi : U \to U/G$ is then a principal G-bundle in the étale topology and such objects are classified by the cohomology group $H^1_{et}(U/G, G)$. Now, if the G-bundle is trivialisable in the Zariski topology then U is birational to $U/G \times G$ and since reductive groups are rational we see that U/G is stably rational. But of course, being Zariski trivialisable is a rather strong condition.

Another well known result concerning this problem is the following lemma which gives us some freedom on the particular choice of the vectorspace as long as we are interested in stable rationality :

no-name lemma : Let G be a reductive group acting on two vectorspaces V and W such that the action on V is almost freely. Then, the field of rational invariants of G acting on $V \oplus W$ is rational over the field of rational invariants on V.

Proof : Since $\mathbb{C}[V]$ is a unique factorization domain having only trivial units we can apply [Lu,p 103,lemme 2] in order to obtain a non-empty affine G-invariant open subvariety V' of V such that generic orbits in V' are closed. So, take a generic point $v \in V'$, then by the étale slice theorem [Lu,p.97] there exists an affine (!) $\pi_{V'}$-saturated subvariety $V"$ of V' containing v such that the G- action on V' induces an étale G-morphism

$$\psi : G \times V" \to V'$$

such that the image U of ψ is an open affine (!) $\pi_{V'}$- saturated subvariety of V' and the canonical morphism

$$\psi/G : (G \times V")/G \cong V" \to U/G$$

is étale and gives rise to a G-isomorphism

$$G \times V" \cong U \times_{U/G} V"$$

Therefore, $\pi_U : U \to U/G$ is an affine (!) principal G-bundle in the étale topology.

But now we can apply an old result of J.P. Serre [Se] or [Lu,p.86] : if U is a principal G-bundle in the étale topology and W a variety with G-action. Define a G-action on $U \times W$ by $g.(u, w) = (ug^{-1}, gw)$ and denote by $U \times_G W = (U \times W)/G$, then $U \times_G W$ is the total space of a fibration of type W with basis U/G. In the special case when V is a vectorspace this implies that $\mathbb{C}(U \times_G W) = \mathbb{C}(V \oplus W)^G$ is a rational extension of $\mathbb{C}(U/G) = \mathbb{C}(V)^G$ done.

As an immediate consequence we get that if G acts almost freely on both V and W then the rational invariants on V are stably equivalent to the rational invariants on W.

2. Representations of quivers

Our main motivation comes from the study of representations of quivers. In this section we will briefly recall the setting. A quiver Q is a quadruple (Q_0, Q_1, t, h) consisting of a finite set $Q_0 = \{1, ..., m\}$ of vertices, a set Q_1 of arrows between these vertices and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow ϕ its tail $t\phi$ and its head $h\phi$ respectively.

A representation V of a quiver Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional vectorspaces over $\mathbb{C}$ together with a family of linear maps $\{V(\phi) : V(t\phi) \to V(h\phi); \phi \in Q_1\}$. The m-tuple $dim(V) = (dim(V(i)))_i \in \mathbb{N}^m$ is called the dimension vector of V. A morphism $f : V \to W$ between two representations is a family of linear maps $\{f(i) : V(i) \to W(i); i \in Q_0\}$ such that for all arrows $\phi \in Q_1$ we have $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$.

Given a dimension vector $\alpha = (\alpha(1), ..., \alpha(m)) \in \mathbb{N}^m$ we define the representation space $R(Q, \alpha)$ to be the vectorspace of all representations V of Q such that $V(i) = \mathbb{C}^{\alpha(i)}$ for all $i \in Q_0$. Because $V \in R(Q, \alpha)$ is completely determined by the maps $V(\phi)$ we have that

$$R(Q,\alpha) = \oplus_{\phi \in Q_1} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ denotes the vectorspace of all $\alpha(h\phi)$ by $\alpha(t\phi)$ matrices with entries in $\mathbb{C}$.

We will consider the representation space $R(Q,\alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q,\alpha]$ and functionfield $\mathbb{C}(Q,\alpha)$. We have a canonical action of the linear reductive group $GL(\alpha) = \prod_{i=1}^m GL(\alpha(i))$ on $R(Q,\alpha)$ by

$$(g.V)(\phi) = g_{h\phi} V(\phi) g_{t\phi}^{-1}$$

for all $g = (g_1, ..., g_m) \in GL(\alpha)$. The $GL(\alpha)$-orbits in $R(Q,\alpha)$ are precisely the isomorphism classes of representations.

Ultimately, one is interested in the description of this orbit structure. It suffices clearly to restrict attention to indecomposable representations. V. Kac [Ka] conjectured that the variety parametrizing isoclasses of indecomposable α-dimensional representations admits a finite cellular decomposition into locally closed subvarieties each isomorphic to some affine space. Unfortunately there is, at this moment, not much evidence to support this conjecture. An immediate consequence of it would be that the field of rational invariants $\mathbb{C}(Q,\alpha)^{GL(\alpha)}$ is rational whenever α is a Schur root. Recall that α is said to be a Schur root if α-representations in general position are indecomposable, or equivalently, if there exists an α-dimensional representation with endomorphism ring reduced to $\mathbb{C}$.

Therefore, if we denote $PGL(\alpha) = GL(\alpha)/\mathbb{C}^*$ where $\mathbb{C}^*$ is embedded diagonally in $GL(\alpha)$ then Schur roots are precisely those dimensionvectors α such that $PGL(\alpha)$ acts almost freely on the representation space $R(Q,\alpha)$. In the special case of the two loop quiver (the classification of couples of n by n matrices under simultanous conjugation) such a rationality result would immediatly imply the Merkurjev-Suslin result, the lifting problem for crossed products over local algebras and the rationality of the moduli space of stable rank n bundles over the projective plane with Chern-numbers $c_1 = 0$ and $c_2 = n$.

In this paper we will show that this special case is really the heart of the problem. More precisely we will prove that it α is a Schur root for the quiver Q such that $gcd(\alpha(i); i \in Q_0) = n$, then the rational invariants are stably rational to the field of rational n by n matrixinvariants. The Schur root assumption is no real restriction since Kac [Ka] has indicated how the rational invariants of an arbitrary dimension vector can be computed in terms of the rational invariants for the Schur roots occuring in the generic decomposistion. Finally, we mention that C.M. Ringel [Ri] has proved rationality of the rational invariants in case Q is a tame quiver.

3. Rational invariants and Azumaya algebras

Although we are primarely interested in the rational invariants of $GL(\alpha)$ acting on a representation space $R(Q,\alpha)$ where α is a Schur root for Q, our first result can be stated in a more general setting.

Let V be an affine variety with a Schurian action of $GL(\alpha)$. Suppose there exist an open subvariety V' of V in which generic orbits are closed, then as in the proof of the no-name lemma we can find an affine open subvariety U of V such that the natural morphism $\pi : U \rightarrow X = U/GL(\alpha)$ is a principal $PGL(\alpha)$-bundle in the étale topology determining an element of the cohomology group $H^1_{et}(X, PGL(\alpha))$.

In this case,one can obtain this cohomology class in a more concrete way in terms of an Azumaya algebra whose triviality is equivalent to the existence of a Zariski cover splitting the cohomology class.

Recall that an R-ring S is a ring with a specified homomorphism from R to S. If $K = \times_{i \in Q_0} \mathbb{C}$ then the centre of $\times_{i \in Q_0} M_{\alpha(i)}(\mathbb{C})$is K so we can regard it as a K-ring; in turn, if $m = \sum_{i \in Q_0} \alpha(i)$, there is an embedding of $\times_{i \in Q_0} M_{\alpha(i)}(\mathbb{C})$ in $M_m(\mathbb{C})$ along the diagonal and we regard $M_m(\mathbb{C})$ as a K-ring via this embedding.

The group of automorphisms of $M_m(\mathbb{C})$ that fix K is isomorphic to $PGL(\alpha)$ since all automorphisms of $M_m(\mathbb{C})$ that fix the center are inner. Therefore, $H^1_{et}(X, PGL(\alpha))$ classifies twisted forms of $M_m(\mathbb{C})$ over K, that is, Azumaya algebras over X with a distinguished embedding of K that are split by an étale cover so that on the étale cover the embedding of K in matrices is conjugate to the original embedding defined by α.

So, let $\delta \in H^1_{et}(X, PGL(\alpha))$ and let $U(\delta) \rightarrow X$ be the corresponding principal $PGL(\alpha)$-bundle in the étale topology and let $A(\delta)$ be the sheaf of Azumaya algebras over X determined by δ. Then one can identify $A(\delta)$ with the ring of $GL(\alpha)$-concomitants from $U(\delta)$ to $M_m(\mathbb{C})$. This allows us at once to deduce the following result :

Theorem 1 : Let $GL(\alpha)$ act Schurian on an affine variety V such that generic orbits are closed in an open affine subvariety. Let $U \to X = U/GL(\alpha)$ be the corresponding affine principal $PGL(\alpha)$- bundle. Then, if $gcd(\alpha(i) : 1 \leq i \leq m) = 1$ then $\mathbb{C}(V)$ is rational over $\mathbb{C}(X) = \mathbb{C}(V)^{GL(\alpha)}$.

Proof : In this case the corresponding Azumaya algebra must be split on a Zariski cover since the natural map from $K_0(K)$ to $K_0(M_m(\mathbb{C}))$ is surjective and this forces the same to be true for the Azumaya algebra. Since the Azumaya algebra is split on a Zariski cover the same thing is true for the principal $PGL(\alpha)$-bundle $U \to X$, which implies that V is birational to $X \times PGL(\alpha)$ and $PGL(\alpha)$ being rational finishes the proof.

Corollary 2 :

(1) : Let α be a Schur root for the quiver Q such that $gcd(\alpha(i); i \in Q_0) = 1$. Then, the corresponding field of rational invariants is stably rational.

(2) : Let α be a Schur root for the quiver Q and let Q' be a larger quiver on the same vertexset. Then, the field of rational invariants for α on Q' is rational over the field of rational invariants on Q.

(3) : Let α be a Schur root for the quivers Q and Q'. Then, the field of rational invariants for α on Q is stably equivalent to the field of rational invariants on Q'.

Proof : (1) : immediate from theorem 1 ; (2) : write $R(Q', \alpha) = R(Q, \alpha) \oplus W$ and apply the no-name lemma ; (3) : apply the no-name lemma twice.

One final construction is needed. Let α be a Schur root for the quiver Q and let us denote $n = gcd(\alpha(i); i \in Q_0)$.We form a new quiver Q^* by adjoining one vertex 0 and an arrow from 0 to some vertex $i \in Q_0$.Let α^* be the extended dimensionvector such that $\alpha^*(0) = n$ and $\alpha^*(i) = \alpha(i)$ for all $i \in Q_0$.Let U be an affine open subvariety of $R(Q, \alpha)$ such that there is an orbit map $U \to X = U/GL(\alpha)$ which is a principal $PGL(\alpha)$-bundle in the étale topology (note that this is always possible by the argument in the proof of the no-name lemma). Let $A(\alpha)$ be the corresponding Azumaya algebra. Because

$gcd(\alpha(i); i \in Q_0) = n$, we may assume by passing to a Zariski open subvariety that $A(\alpha)$ is isomorphic to $M_t(B(\alpha))$ where $m = tn$ and the embedding of K in $A(\alpha)$ may be refined to a set of matrixunits.

Let W be the open subvariety of $R(Q^*, \alpha^*)$ whose image in $R(Q, \alpha)$ is in U and where the new arrow is injective and let

$$W/PGL(\alpha) \rightarrow U/PGL(\alpha)$$

be the induced map. Then, $W/PGL(\alpha)$ represents rank one $B(\alpha)$- submodules of free rank s (where $sn = \alpha(i)$) $B(\alpha)$-modules, which is a rational projective variety. Then, by applying corrolary 2 we get :

Theorem 2 : Let α be a Schur root for the quiver Q and let $n = gcd(\alpha(i); i \in Q_0)$. Let Q' be a quiver with one extra vertex 0 and at least one more arrow. Let α' be the dimensionvector determined by $\alpha'(0) = n$ and $\alpha'(i) = \alpha(i)$ where defined. Then, the rational invariants for α' on Q' are rational over the rational invariants for α on Q.

Theorem 3 : Let α be a Schur root for the quiver Q and let $n = gcd(\alpha(i); i \in Q_0)$. Then, the field of rational invariants for α on Q is stably equivalent to the field of rational n by n matrixinvariants.

Proof : Consider the one point extension quiver Q' where the number of arrows from the extra point 0 to the point i is $\alpha(i)/n$. Consider the open subvariety of $R(Q', \alpha')$ where the $\alpha(i)/n$ maps from $V(0)$ to $V(i)$ define an isomorphism from $V(0)^{\alpha(i)/n}$ to $V(i)$. This reduces the classification problem of the quiver Q' to representations of the original quiver Q where each vertex space is in addition given a fixed representation as a vectorspace $V^{\alpha(i)/n}$ where V is a vectorspace of dimension n. But this is the same as the classification of $\sum_{\phi \in Q_1} \alpha(t\phi)\alpha(h\phi)/n^2$ square n by n matrices upto simultanous conjugation. This shows that the rational invariants for α' on Q' are stably equivalent to the rational invariants for n by n matrices and theorem 2 finshes the proof.

4. Rational invariants and reflection functors

In this section we will give a purely representation theoretic proof of theorem 3 using the Bernstein-Gelfand-Ponomarev theory of reflection functors. Let $i \in Q_0$ be a sink, that is for no $\phi \in Q_1$ we have $t\phi = i$, and let α be a dimensionvector. We form a new quiver Q' by reversing the direction of all arrows connected to i and define a new dimension vector α' by $\alpha'(j) = \alpha(j)$ whenever $i \neq j$ and $\alpha'(i) = \sum_{h\phi=i} \alpha(t\phi) - \alpha(i)$. Consider the open subvariety of $R(Q,\alpha)$

$$R^*(Q,\alpha) = \{V \in R(Q,\alpha) : \oplus V(\phi) : \oplus_{h\phi=i} V(t\phi) \rightarrow V(i) \text{ is surjective } \}$$

and similarly we consider the open subvariety of $R(Q',\alpha')$

$$R^*(Q',\alpha') = \{V \in R(Q',\alpha') : \oplus V(\phi) : V(i) \rightarrow \oplus_{t\phi=i} V(h\phi) \text{ is injective } \}$$

then there exists a homeomorphism between $R^*(Q,\alpha)/GL(\alpha)$ and $R^*(Q',\alpha')/GL(\alpha')$ such that corresponding representations have isomorphic endomorphism rings. In particular, if α is a Schur root for the quiver Q, then α' is a Schur root for the quiver Q' and their corresponding fields of rational invariants are isomorphic.

Proof of theorem 3 : let $i \in Q_0$ be such that $\alpha(i) = kn$ is minimal and let $j \in Q_0$ be such that $\alpha(j) = ln$ with k not dividing l say $l = ak - b$ with $0 \neq b < k$. Form a new quiver Q' on the same vertices with a arrows pointing from i to j and all other arrows living on $Q_0 - \{j\}$ in such a way that α is a Schur root for Q'. Note that this can always be done by trowing in lots of loops. By corollary 2(3) the rational invariants for α on Q are stably equivalent to the rational invariants for α on Q' which are in turn isomorphic to the rational invariants for α' on the reflected quiver in j where $\alpha'(j) = bn < kn$. So, by induction we may assume that the rational invariants for α on Q are stably equivalent to the rational invariants for γ on some quiver Q^* with $gcd(\gamma(i) : i \in Q_0) = n$ and $\gamma(i) = n$ for some $i \in Q_0$.

Now, we can proceed by induction on the number of vertices. Take $j \neq i$ such that $\gamma(j) = kn$. Form a quiver Q' with precisely k arrows from i to j and the arrows in $Q'_) - \{j\}$ such that γ is a Schur root for Q'. Then, after applying reflection with respect to the sink

j we get a quiver with one vertex less such that the rational invariants are still stably equivalent to the original.Continuing in this way we will end up with the two loop quiver in i finishing the proof.

By appllying the no-name lemma we have therefore proved

Theorem 4 : Let $GL(\alpha)$ have a Schurian action on a finite dimensional vectorspace V. Then, the field of rational invariants for this action is stably equivalent to the field of rational n by n matrixinvariants where $n = gcd(\alpha(i) : 1 \leq i \leq m)$.

We recall that Ed Formanek proved rationality of the field of matrixinvariants for $n \leq 4$ and that David Saltman proved retract rationality for all squarefree n. Recall that the unramified Brauer group of a field L is the Brauer group of a smooth projective model for L. D. Saltman proved that the unramified Brauer group of the field of rational n by n matrix- invariants is trivial. Hence we obtain :

Corollary 5 : Let α be a Schur root for the quiver Q and let $n = gcd(\alpha(i) : i \in Q_0)$, then

(1) : if $n \leq 4$ the rational invariants for α on Q are stably rational

(2) : if n is squarefree , the rational invariants for α on Q are retract rational

(3) : the Brauer group of a projective smooth model for the rational invariants for α on Q is trivial

References

[Ka] : V. Kac ; Root systems,representations of quivers and invariant theory,Springer LNM 996 74-108

[Lu] : D. Luna ; Slices étales , Bull.Soc.Math.France Mém 33 , 81-105 (1973)

[Ri] : C.M. Ringel ; The rational invariants of tame quivers, Invent Math 58 (1980)

217-239

[Se] : J.P. Serre ; Espaces fibrés algébriques , Sém C. Chevalley 21 avril 1958 E.N.S.