SIMULTANEOUS EQUIVALENCE OF SQUARE MATRICES

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Abstract :

Recent results of M. Maruyama on vector bundles over the projective plane give new information on the problem of classifying m-tuples of n by n matrices upto simultaneous equivalence.
As an application we also give a linearization procedure for partial differential equations of the form $\sum_{i+j+k=n} a_{ijk} \frac{\partial^n \psi}{\partial x^i \partial y^j \partial z^k} = c^n \psi$.

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1. Introduction.

In this paper we aim to show how certain recent results of M. Maruyama [14], [15] can be applied to obtain some grip on the following "hopeless" problem.

Question 1 : Parametrize m-tuples of n by n matrices under simultaneous equivalence.

That is, we aim to study the orbits of the group $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $M_n(\mathbb{C})^{\oplus m}$ by

$$(\alpha, \beta).(A_1, \ldots, A_m) = (\alpha A_1 \beta, \ldots, \alpha A_m \beta)$$

If $m = 1$, different orbits correspond to different ranks. If $m = 2$ and A_1 is invertible, then the orbit of (A_1, A_2) is completely determined by the Jordan normal form of $A_1^{-1}.A_2$.
If $m \geq 3$, however, this problem is known to be wild since it corresponds to classifying representations of the wild quiver

$$0 \quad \begin{matrix} \xrightarrow{\varphi_1} \\ \vdots \\ \xrightarrow[\varphi_m]{} \end{matrix} \quad 0$$

upto equivalence. In this paper we will restrict attention to representations which are generic, i.e. in sufficiently general position. To be more precise, we want to find an open subvariety $U_{m,n}$ of $M_n(\mathbb{C})^{\oplus m}$, and a morphism $\pi : U_{m,n} \longrightarrow V_{m,n}$ having the property that for each $\xi \in V_{m,n}$ the fiber $\pi^{-1}(\xi)$ consists of precisely one orbit. The main open problem concerning these parametrizing varieties $V_{m,n}$ is :

Question 2 :

Is $V_{m,n}$ (stably) rational ?

In the next two sections we will see that a positive solution to this question would be important also for the study of Brauer groups of functionfields and for the study of vectorbundles over the projective plane. We mention that question 2 would follow immediately from a positive solution to a rather daring conjecture of V. Kač [11]. He conjectured that, for any quiver Q and any dimension vector α, the variety parametrizing isoclasses of indecomposable α-representations would allow a cellular decomposition into locally closed subvarieties each isomorphic to some affine space. Since generic equidimensional representations of the two point quivers are indecomposable thin would imply rationality of $V_{m,n}$.

2. Connection with Brauer groups of functionfields.

For a long time, one of the main open problems on Brauer groups has been.

Question 3 : If X is a variety over $\mathbb{C}$, is the Brauer group $Br\mathbb{C}(X)$ of the functionfield $\mathbb{C}(X)$ generated by cyclic algebras ?

Recall that a cyclic algebra is an n^2-dimensional central simple $\mathbb{C}(X)$-algebra generated by two elements x and y satisfying the relations :

$$x^n = a, y^n = b, xy = \omega yx$$

where $a, b \in \mathbb{C}(X)^*$ and ω is a primitive n-th root of unity.
Question 3 was recently solved (in a more general setting) by Merkurjev and Suslin using heavy tools from algebraic K-theory, see for example [24].
Ringtheorists have tried to solve this problem by using generic methods. Let us briefly scetch their approach. For any $m, n \in \mathbb{N}$, consider the polynomial ring

$$P_{m,n} = \mathbb{C}[x_{ij}(k) : 1 \leq i, j \leq n; 1 \leq k \leq m]$$

The ring of m generic n by n matrices, $\mathbb{G}_{m,n}$, is the subring of $M_n(P_{m,n})$ generated by the generic matrices

$$X_k = (X_{ij}(k))_{i,j} \in M_n(P_{m,n})$$

It is well-known, see for example [17], that $\mathbb{G}_{m,n}$ is a left and right Öre-domain so we can consider its ring of fractions $\Delta_{m,n}$ which is a division algebra of dimension n^2 over its center $K_{m,n}$.
$\Delta_{m,n}$ is called the generic division algebra. Procesi [19] has shown that if all $\Delta_{m,n}$ are Brauer equivalent to a product of cyclic algebras, then question 3 has a positive solution.
Moreover, S. Bloch [4] has shown that if L is a field containing $\mathbb{C}$ then $Br(L)$ is generated by cyclic algebras if and only if $Br(L(x_1, \ldots, x_r))$ is generated by cyclic algebras.

So, by applying Bloch's result (twice), (stable) rationality of $K_{m,n}$ would imply the Merkurjev-Suslin result for fields containing $\mathbb{C}$.
What is known about the rationality of $K_{m,n}$? Procesi [17,] has proved that $K_{m,n}, m \geq 3$, is rational over $K_{2,n}$ thereby reducing the problem to two generic matrices. Moreover, he proved that $K_{2,2} = \mathbb{C}(Tr(X_1), Tr(X_2), \mathrm{Det}(X_1), \mathrm{Det}(X_2), Tr(X_1, X_2))$ setling the problem for 2 by 2 matrices.
Formanek, [8] and [9], has proved rationality of $K_{m,3}$ and $K_{m,4}$.

Apart from these results, D. Saltman [20], [21], has obtained some partial results : he proved that the unramified Brauer group of $K_{m,n}$ is trivial and that $K_{m,n}$ is retract rational for n prime.
Further, we note that the obvious approach (i.e. trying to prove that $K_{2,n}$ is rational over $K_{1,n}$) fails for $n = 4$. This was shown by R. Snider and mentioned in [9] and [19]. A proof of this fact is contained in the paper [7] by Colliot-Thélène and Sansuc. D. Saltman has communicated to us that $K_{2,n}$ cannot be stably rational over $K_{1,n}$ for any non-squarefree n.

What does all this have to do with our problem ? The connection is given by the (easy) proof of the wildness of the quivers for $m \geq 3$. Consider the open subvariety

$$U_{m,n} = GL_n(\mathbb{C}) \oplus M_n(\mathbb{C})^{\oplus m-1}$$

of $M_n(\mathbb{C})^{\oplus m}$, then a representant for the orbit under action of the first component of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ is of the form

$$(I_n; A_2; \ldots; A_m)$$

Now, we have to calculate the action of the second component on these representants

$$\begin{aligned}(I_n, \beta).(I_n, A_2, \ldots, A_m) \\ &= (\beta, A_2\beta, \ldots, A_m\beta) \\ &= (I_n.\beta^{-1}A_2\beta, \ldots, \beta^{-1}A_m\beta)\end{aligned}$$

That is, the orbit structure of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $U_{m,n}$ is the same as that of $GL_n(\mathbb{C})$ acting on $(m-1)$-tuples of n by n matrices by componentswise conjugation.
Procesi [18] has shown that the quotient variety $M_n(\mathbb{C})^{\oplus m-1}/GL_n(\mathbb{C})$ has as its function field the field $K_{m-1,n}$. Therefore,

Proposition 1. : $V_{m,n}(m \geq 3)$ is birational to the quotient variety

$$M_n(\mathbb{C})^{\oplus m-1}/GL_n(\mathbb{C})$$

Therefore, rationality of $V_{m,n}$ would imply a positive answer to question 3. Moreover, from Procesi's observation that $K_{m,n}$ is rational over $K_{2,n}$ we also obtain the following (perhaps surprising) result :

Proposition 2 : $V_{m,n}(m \geq 3)$ is birational to $V_{3,n} \times \mathbb{A}^{(m-3)n^2}$ which basically reduces question 1 to the special case of triples of n by n matrices. That is, from now on we will restrict attention to the isomorphism problem of equidimensional generic representations of

$$0 \;\begin{array}{c}\longrightarrow \\ \longrightarrow \\ \longrightarrow\end{array}\; 0$$

3. Connection with vectorbundles over $\mathbb{P}_2$

A very coarse classification of all vectorbundles over the projective plane $\mathbb{P}_2$ is given by topological invariants such as the rank and the chern classes, which can be interpreted as integers in the case of projective spaces.
Given the numbers r, c_1, c_2 one wants to study how sufficiently general bundles with these invariants look like. Such bundles will turn out to be stable, i.e. for all coherent subsheaves $\mathcal{F} \in \mathcal{E}$ we have $\frac{c_1 \mathcal{F}}{rk(\mathcal{F})} < \frac{c_1}{r}$.

Precisely as parametrizing problems in representation theory one therefore wants to study a variety $M(r, c_1, c_2)$ whose points correspond to isomorphism classes of stable bundles $\mathcal{E}$ of rank r and with Chern numbers $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$. Again, a major open problem is

Question 4 : which of these moduli spaces $M(r, c_1, c_2)$ are (stably) rational ?

The motivation is that in case they are rational we can find additional algebraic invariants of bundles such that they classify freely and completely sufficiently general bundles with the invariants r, c, and c_2. What is known about this problem ? Barth has shown in [1] that $M(2,0,2)$ is rational. Moreover, $M(2,0,2)$ is just the variety of nonsingular plane conics. Unfortunately, his proof of the rationality of $M(2,0,n)$ contains a gap. The only other moduli space $M(2,0,n)$ which is known to be rational is $n = 4$ and will by published by Le Potier. Apart from this, nothing seems to be known about the rationality of $M(r,0,n)$. What has this to do with our problem ?
Hulek [10] gave the following elegant description of $M(r,0,n)$ which can be rephrased in terms of represention. He calls an equidimensional representation

$\mathcal{A} = (A_0, A_1, A_2)$ of $0 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} 0$ prestable if and only if for every $v \in \mathbb{C}^n$:

$$\dim\,(A_0v, A_1v, A_2v) \geq 2$$
$$\dim\,(A_0^\tau v, A_1^\tau, v, A_2^\tau v) \geq 2$$

Remark that this is an open condition on $M_n(\mathbb{C})^{\oplus 3}$. With such a representation one can construct a bundle over $\mathbb{P}_2$ in the following way.
Let $V = \Gamma(\theta_{\mathbf{P}_2}(1))^*$ with basis u, v, w dual to the usual x, y, z basis of $\Gamma(\theta_{\mathbf{P}_2}(1))$, then we can define a linear map

$$\varphi_{\mathcal{A}} : \mathbb{C}^n \otimes V \longrightarrow \mathbb{C}^n \otimes V^*$$

given by the matrix

$$\begin{bmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{bmatrix} = \psi_{\mathcal{A}}$$

and denote $U = Im\varphi_{\mathcal{A}}$. Further, $s : \Gamma(\theta_{\mathbf{P}2}(1)) \otimes (\theta_{\mathbf{P}_2} \to \theta_{\mathbf{P}_2}(1)$ is the natural multiplication map and s^* is its dual. Then we have a complex of vectorbundles :

$$\begin{array}{ccccc} \mathbb{C}^n \otimes \theta_{\mathbf{P}_2}(-1) & \xrightarrow{a} & U \otimes \theta_{\mathbf{P}_2} & \xrightarrow{b} & \mathbb{C}^n \otimes \theta_{\mathbf{P}_2}(1) \\ {\scriptstyle 1 \otimes s^*} \downarrow & \nearrow {\scriptstyle \varphi_{\mathcal{A}} \otimes 1} & \downarrow & \nearrow {\scriptstyle 1 \otimes s} & \\ \mathbb{C}^n \otimes V \otimes \theta_{\mathbf{P}_2} & & \mathbb{C}^n \otimes V^* \otimes \theta_{\mathbf{P}_2} & & \end{array}$$

where a is a mono and b is epi. The cohomology of this complex is a bundle $\mathcal{E}_A$ of rank dim $U - 2n$ and with Chern-numbers $c_1(\mathcal{E}_A) = 0, c_2(\mathcal{E}_A) = n$. Moreover. Hulek has shown that isoclasses of prestable representations and isoclasses of the corresponding bundles coincide. Clearly, for sufficiently general representations, the corresponding bundle will be of rank n and stable. Therefore, we obtain :

Proposition 3 : $V_{3,n}$ is birational to the moduli space of stable rank n bundles over $\mathbb{P}_2$ with Chern-numbers $c_1 = 0, c_2 = n$; $M(n,0,n)$

In particular, rationality of $V_{3,n}$ would imply rationality of the moduli spaces $M(n,0,n)$.

4. Some consequences.

In the foregoring two sections we have reduced our original problem to that of two existing varieties :

$$\begin{array}{ccccc} & & V_{3,n} & & \\ & \sim & & \sim & \\ M_n(\mathbb{C})^{\oplus z}/GL_n(\mathbb{C}) & & & & M(n,0,n) \end{array}$$

where $\sim$ denotes birationality. Note that we did not have to use anything but the wildness of the quiver $0 \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} 0$ to prove these results. Nevertheless, we get using Formanek's result on the rationality of the quotient varieties $M_n(\mathbb{C})^{\oplus 2}/GL_n(\mathbb{C})$ the following rather surprising result :

Proposition 4 : The moduli spaces $M(3,0,3)$ and $M(4,0,4)$ are rational.

As far as I know these results were not known. In [16] M. Maruyama claims stable rationality of $M(n,0,n)$ for all $n \in \mathbb{N}$. As we have mentioned before such a result would immediately imply the Merkurjev-Suslin result for function fields of varieties. Unfortunately, in view of Sniders's remark mentioned before the method of proof of [16] cannot be correct. In fact, D. Saltman, K. Hulek and Le Potier have communicated to us a gap in [16]. For more details we refer the reader to [13].

5. Plane Curves and their Jacobians.

Although we have reduced reduced our original problem on classifying simultaneous equivalence classes of m-tuples of square matrices to some existing varieties it is by no means clear that these varieties are easier to handle than $V_{3,n}$.

In this section, we will outline an elegant approach originally due to Maruyama [15] and rediscovered in [13] and [23].

Let us start by considering the open subvariety

$$U_{3,n} = \{\mathcal{A} = (A_0, A_1, A_2) : \det(A_0x + A_1y + A_2z) \neq 0\}$$

of $M_n(\mathbb{C})^{\oplus 3}$. With such an $\mathcal{A}$ we can associate a monomorphism of vectorbundles

$$\lambda_{\mathcal{A}} = A_0x + A_1y + A_2z : \mathcal{O}_{\mathbf{P}_2}(-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbf{P}_2}^{\oplus n}$$

and we can consider its cokernel :

$$0 \to \mathcal{O}_{\mathbf{P}_2}(-1)^{\oplus n} \xrightarrow{\lambda} \mathcal{O}_{\mathbf{P}_2}^{\oplus n} \longrightarrow \mathcal{L}(1) \longrightarrow 0$$

then it follows that $\mathcal{L}$ is a torsion sheaf satisfying $H^0(\mathbb{P}_2, \mathcal{L}) = H^1(\mathbb{P}_2, \mathcal{L}) = 0$ and $c_1(\mathcal{L}) = n$.
Conversely, suppose we start off with a torsion coherent sheaf $\mathcal{L}$ on $\mathbb{P}_2$ with $H^0(\mathbb{P}_2, \mathcal{L}) = H^1(\mathbb{P}_2, \mathcal{L}) = 0$ and $c_1(\mathcal{L}) = n$, then it follows that $\mathcal{L}$ is 1-regular in Mumford's terminology [12]. That is, $H^q(\mathcal{L}(1-q))$ has to vanish for all $q \geq 1$. A pleasant consequence of this is that $\mathcal{L}(n)$ is generated by its sections $H^0(\mathcal{L}(n))$ for all $n \geq 1$ [12]. That is, we get an exact sequence :

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{P}_2}^{\oplus n} \longrightarrow \mathcal{L}(1) \longrightarrow 0$$

Because $\mathcal{L}$ is torsion, $\mathcal{F}$ is a vector bundle and using Horrock's classification of vector bundles which are sums of line bundles, see for example [3], one can show that $\mathcal{F}$ is $\bigoplus_{i=1}^{n} \mathcal{O}(a_i)$. Finally, using $c_1(\mathcal{L}) = n$ one gets $\mathcal{F} = \mathcal{O}_{\mathbf{P}_2}(-1)^{\oplus n}$. That is, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}_2}(-1)^{\oplus n} \xrightarrow{\lambda} \mathcal{O}_{\mathbf{P}_2}^{\oplus n} \longrightarrow \mathcal{L}(1) \longrightarrow 0$$

and hence a representation $\mathcal{A}_1 = (A_0, A_1, A_2)$ of $0 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ 0$. Moreover, isoclasses of representations correspond to isoclasses of torsions sheaves. Therefore we have :

Proposition 5 : Orbits in $U_{3,n}$ correspond bijectively to isomorphism classes of torsion coherent sheaves $\mathcal{L}$ on $\mathcal{L}$ on $\mathbb{P}_2$ satisfying $H^0(\mathbb{P}_2, \mathcal{L}) = H^1(\mathbb{P}_2, \mathcal{L}) = 0$ and $c_1(\mathcal{L}) = n$.

At first sight we have not gained much. We reduced our problem again to some moduli problem.
However, it is fairly easy to show that the torsion coherent sheaf $\mathcal{L}$ associated to a representation $\mathcal{A} = (A_0, A_1, A_2)$ lives on the curve $C_{\mathcal{A}} \hookrightarrow \mathbb{P}_2$ of degree n determined by

$$\det\,(A_0x + Ay + A_2z) = 0$$

Moreover, Barth [2] has shown that if $C_{\mathcal{A}}$ is reduced and if x is a smooth point of $C_{\mathcal{A}}$, then $\mathcal{L}_x$ is invertible. Therefore, if $C_{\mathcal{A}}$ is a nonsingular curve of degree

n, and hence of genus $g = \frac{(n-1)(n-2)}{2}$, then $\mathcal{L}$ is a divisor over it. Because $H^0(c\mathcal{L}) = H^1(c\mathcal{L}) = 0$ we obtain from the Riemann-Roch theorem for C :

$$\chi(\mathcal{L}) = \deg\mathcal{L} + 1 - g$$

that the degree of $\mathcal{L}$ is $d = g - 1 = \frac{n(n-3)}{2}$.

Conversely, let $C \in \mathbb{P}_2$ be a smooth plane curve of degree n, then the degree determines an exact sequence

$$0 \longrightarrow \mathrm{Jac}(C) \longrightarrow \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

where Jac (C) is the Jacobian variety of C. Restricting to the set of all divisors of degree d we get an homogeneous space $\mathrm{Pic}_d(C)$ over $\mathrm{Jac}(C)$.
Moreover, there exists an open subvariety $\mathrm{Pic}'_{g-1}(C) \subset \mathrm{Pic}_{g-1}(C)$ consisting of those divisors $\mathcal{L}$ s.t. $H^0(C,\mathcal{L}) = H^1(C,\mathcal{L}) = 0$. One can show, see for example [6], that $\mathrm{Pic}'_{g-1}(C)$ is precisely the complement in $\mathrm{Pic}_{g-1}(C)$ of the image of the natural map

$$\pi_{g-1} : \underbrace{C \times \ldots \times C}_{g-1 \text{ copies}} \longrightarrow \mathrm{Pic}_{g-1}(C)$$

associating to a $(g-1)$-tuple of points of C the divisor $\theta(p_1 + \ldots + p_{g-1})$.
In view of the vast amount of theory on Jacobian varieties and the explicit description of π_{g-1} one can consider $\mathrm{Pic}'_{g-1}(C)$ as a tractable variety.

Now, let us restrict attention to the following open subvariety of $U_{3,n}$:

$$U'_{3,n} = \{\mathcal{A} = (A_0, A_1, A_2) | C_{\mathcal{A}} \text{ is smooth}\}$$

then we have the following answer to our problem 1 :

Theorem 1 : Orbits of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $U'_{3,n}$ by simultaneous equivalence can be parametrized by couples $(C, \mathcal{L})$ where

(1) C is a smooth plane curve of degree n

(2) $\mathcal{L} \in \mathrm{Pic}'_{g-1}(C)$

Let us check the dimension of this variety. Smooth plane curves of degree n form an open subvariety of $\mathbb{P}_{\frac{(n+1)(n+2)}{2}-1}$, i.e. is of dimension $\frac{(n+1)(n+2)}{2} - 1$ and the dimension of $\mathrm{Pic}'_{g-1}(C)$ is equal to that of $\mathrm{Jac}(C)$ which is known to be equal to the genus $= \frac{(n-1)(n-2)}{2}$. Therefore, the dimension of the parametrizing variety $V_{3,n}$ is

$$\frac{(n+1)(n+2)}{2} + \frac{(n-1)(n-2)}{2} - 1 = n^2 + 1$$

as expected.

Purists who like to know the structure of this variety rather than a description of its points may consult [15], [13] or [23]. In short, the variety is a relative Picard scheme of a smooth family of curves.

6. Linearization of partial differential equations.

The Schrödinger equation of a free particle in relativistic free quantum mechanics is

$$i\,\hbar\frac{\partial\psi(\bar{r},t)}{\partial t} = H(\bar{r},\frac{\hbar}{i}\nabla)\psi(\bar{r},t) \tag{1}$$

where $|\psi(\bar{r},t)|^2$ is the probability that the particle is in place $\bar{r}$ at time t and H is the Hamiltonian, i.e.

$$H = \frac{p^2}{2m} + V(\bar{r},t)$$

One is primarily interested in solutions of the form

$$\psi(\bar{r},t) = \psi(\bar{r})e^{-iEt/\hbar}$$

where E denotes the energy. Substituting this form in equation (1) we obtain

$$E\psi(\bar{r},t) = H\psi(\bar{r},t)$$

where the energy E is an eigenvalue of the Hamiltonian operator. For a free particle ($V(\bar{r},t) = 0$) we obtain

$$H = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

and we therefore obtain the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 2mE\psi = (\sqrt{2}mc)^2\psi \tag{2}$$

In order to solve this equation (2) one tries to replace it by a system of first order partial differential equations. In this case, the Pauli matrices :

$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

do this so called linearization trick. For,

$$\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}\right)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) I_2$$

and therefore any solution to the system of first order partial differential equations :

$$A\frac{\partial\psi}{\partial x}+B\frac{\partial\psi}{\partial y}+C\frac{\partial\psi}{\partial z}=(\sqrt{2}mc)\psi.S$$

where $S\in M_2(\mathbb{C})$ s.t. $S^2=I_2$ is a solution to (2).

Of course, one can try to generalize this linearization procedure to more variables and/or higher order equations. The present knowledge of this problem is as follows, see for example [5] :
In two variables partial differential equations of any order can be linearized and for any number of variables second order equations can be linearized. The first open case, that of three variables and a third order partial differential equation was (under some extra assumptions) recently solved by M. Van den Bergh [22].

In this section we will use the results of the foregoing section to give a linearization procedure for partial differential equations of the form :

$$\sum_{i+j+k=n} a_{ijk}\frac{\partial^n\psi}{\partial x^i\partial y^i\partial z^k}=c^n\psi \tag{3}$$

Proposition 6 : Partial differential equations of type (3) can always be linearized.

Proof. Suppose that we can find n by n matrices (A_0,A_1,A_2) such that

$$\det\left(A_0\frac{\partial}{\partial x}+A_1\frac{\partial}{\partial y}+A_2\frac{\partial}{\partial z}\right)=\sum a_{ijk}\frac{\partial^n}{\partial x^i\partial y^j\partial z^k}$$

then one can show using the Cayley-Hamilton polynomial that any solution to the system of first order equations

$$A_0\frac{\partial\psi}{\partial x}+A_1\frac{\partial\psi}{\partial y}+A_2\frac{\partial\psi}{\partial z}=c\psi S$$

where $S\in SL_n(\mathbb{C})$ is a solution to (3).

For notational reasons let us write $X=\frac{\partial}{\partial x},Y=\frac{\partial}{\partial y},Z=\frac{\partial}{\partial z}$. If $\sum a_{ijk}X^iY^jZ^k$ determines a smooth curve such matrices exist by §5.
If $\sum a_{ijk}X^iY^jZ^k$ determines an irreducible curve C having singularities one can look at the desingularization

$$f:\tilde{C}\to C\hookrightarrow\mathbb{P}_2$$

and take an element $\mathcal{L}\in\mathrm{Pic}'_{g-1}(\tilde{C})$, then $f_*(\mathcal{L})$ is a torsion coherent sheaf on $\mathbb{P}_2$ having the required properties and hence it determines the wanted triple of matrices.

In case $\sum a_{ijk}X^iY^jZ^k = \prod_{l=1}^{s} f_l(X,Y,Z)$ is a factorization, one can construct torsion sheafs on each of the $f_l(X,Y,Z)$, say $\mathcal{L}_l$, and then $\oplus\mathcal{L}_l$ is a torsion sheaf of the required type. ∎

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