GLOBAL STUDY OF SEMI-SIMPLE REPRESENTATIONS OF QUIVERS

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Abstract

Isomorphism classes of semi-simple representations with dimension vector α of a quiver Q are parametrized by the quotient variety $R(Q, \alpha)/GL(\alpha)$. In this paper we will describe the coordinate ring of this variety thereby determining the polynomial invariants. Further, we will show that this variety admits a finite stratification into locally closed smooth subvarieties corresponding to the different types of semi-simple representations. Finally, we show how one can determine all these types as well as the generic semi- simple representation type.

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1. Introduction.

After the work of P. Gabriel [3], it became clear that several problems from linear algebra could be formulated and studied in a uniform way in the context of representations of quivers. Let us recall the setting. Throughout, we work over an algebraically closed field of characterisric zero and call it $\mathbb{C}$. A quiver Q is a quadruple (Q_0, Q_1, t, h) consisting of a finite $Q_0 = \{1, \ldots, n\}$ of vertices, a set Q_1 of arrows between these vertices and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow φ its tail $t(\varphi)$, and its head $h(\varphi)$, respectively. Note that we do not exclude loops nor multiple arrows. However, we will always assume that the underlying graph of the quiver is connected.

A representations V of the quiver Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional $\mathbb{C}$-vectorspaces together with a family of linear maps $\{V(\varphi) : V(t_\varphi) \to V(h\varphi); \varphi \in Q_1\}$. The n-tuple $\dim(V) = (\dim V(i))_i \in \mathbb{N}^n$ is called the dimension type of V. A morphism $f : V \to W$ between two representations is a family of linear maps $\{f_i : V(i) \to W(i); i \in Q_0\}$ such that for all arrows $\varphi \in Q_1$ we have :
$W(\varphi) \circ f(t\varphi) = f(h\varphi) \circ V(\varphi)$.

Given a dimension type $\alpha = (\alpha(1), \ldots, \alpha(n)) \in \mathbb{N}^n$ we define the representation space of $Q, R(Q, \alpha)$, to be the set of all representations of Q s.t. $V(i) = \mathbb{C}^{\alpha(i)}$ for all $i \in Q_0$. Because $V \in R(Q, \alpha)$ is completely determined by the maps $V(\varphi)$ we have that

$$R(Q, \alpha) = \bigoplus_{\varphi \in Q_1} \mathrm{Hom}_{\mathbb{C}} (\mathbb{C}^{\alpha(t\varphi)}, \mathbb{C}^{\alpha(h\varphi)}) = \bigoplus_{\varphi \in Q_1} M_\varphi(\mathbb{C})$$

where $M_\varphi(\mathbb{C})$ denotes the $\mathbb{C}$-vectorspace of all $\alpha(h\varphi)$ by $\alpha(t\varphi)$ matrices with entries in $\mathbb{C}$.

We will consider the representation space $R(Q, \alpha)$ as an affine variety with coordinating ring $\mathbb{C}[Q, \alpha]$ and function field $\mathbb{C}(Q, \alpha)$. We have a canonical action of the linear reductive group $GL(\alpha) = \prod_{i=1}^{n} GL_{\alpha(i)}(\mathbb{C})$ on $R(Q, \alpha)$ by

$$(g.V)(\varphi) = g_{h\varphi} V(\varphi) g_{t_\varphi}^{-1}$$

for all $g = (g_1, \ldots, g_n) \in GL(\alpha)$. By definition, the $GL(\alpha)$-orbits in $R(Q, \alpha)$ are just the isomorphism classes of representations.

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In general, if G is a linear reductive group acting linearly on a vector space X, an element $x \in X$ is said to be semisimple (resp. nilpotent) with respect to G iff the orbit $G.x$ is closed (resp. $0 \in \overline{G.x}$, the Zariski-closure of the orbit). We say that $X = X_s + X_n$ is a Jordan-decomposition of $x \in V$ if x_s is a semisimple element with respect to G, x_n is a nilpotent element with respect to G_{x_s} (the stabilizer subgroup of x_s in G, which is again a reductive linear group by a result of Matsuchima [11]) and $G_x = G_{x_s} \cap G_{x_n}$ where G_y is the stabilizer group of y.

Using results of Luna [9], V. Kac [5, prop. p. 161] has shown that for any $x \in X$ there exists a Jordan decomposition. Returning to the representation therory of quivers, this result says that the classification of all isomorphism classes can be divided up in two subproblems : (I) the study of the semisimple representations w.r.t. $G(\alpha)$ and (II) : the study of nilpotent representations w.r.t. reductive subgroups of $GL(\alpha)$.

We will primarily concentrate our attention to problem (I). However, concerning problem (II) we will determine the finitely many reductive subgroups of $GL(\alpha)$ which occur as stabilizer subgroups of semisimple elements.

Since semisimple representations are, by definition, those for which the corresponding orbit $GL(\alpha).V$ is closed we see by Mumford's theory [12] that the isomorphism classes of semisimple representation are classifified by the points of the quotient variety $R(Q, \alpha)/GL(\alpha) = V(Q, \alpha)$, or if no ambiguity is possible we set $V(Q, \alpha) = V(\alpha)$. The coordinate ring of this variety $\mathbb{C}[V(\alpha)]$ is the ring of polynomial invariants under $GL(\alpha)$ in $\mathbb{C}[Q, \alpha]$. In section two we will show, using results of Procesi [13], [14], that this invariant ring $\mathbb{C}[V(\alpha)]$ is generalized by traces of oriented cycles in the quiver Q. A result of Razmyslow gives us an upper bound on the length of these cycles needed to generate $\mathbb{C}[V(\alpha)]$.

In section three we will give a nice stratification of $V(\alpha)$. In general, if a reductive group G acts on a vector-space X then the quotient variety X/G can be covered by finitely many locally, closed smooth algebraic subvarieties $(X/G)_H$ consisting of those points $\xi \in X/G$ s.t. the fiber $\pi^{-1}(\xi)$ contains a semisimple element x with stabilizer subgroup conjugated to H; there H is a reductive subgroup of G

[10]. In our situation, these different strata coorespond to the different types of semi-simple decompositions of representations of dimension type α. Moreover, one strata lies in the closure of another if the corresponding representations are deformations of each other.

Using these results we will determine in section four all dimension vectors which occur as the dimension type of a simple representation. This problem can be viewed analogous to (but much easier than) the corresponding problem for indecomposable representations, which has been solved by V. Kač [4]. Our description is expressed in terms of the so called Ringel bilinear form, that is $R = (r_{ij})_{i,j} \in M_n(\mathbb{Z})$ s.t. $r_{ij} = -\#\{\varphi \in Q_1 : t\varphi = i, h\varphi = j\} + \delta_{ij}$ and

$$R : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}^n : \mathrm{R}(\alpha, \beta) = \alpha^r . \mathrm{R}. \beta$$

Therefore, we obtain a purely a purely combatorial method to determine all type of semi-simple representations of dimensional type α and hence of all the reductive subgroup of $GL(\alpha)$ which occur in subproblem (II) above.

From the stratification result it is clear that exactly one of the strata is an open subvariety of the quotient variety $V(Q, \alpha)$. The semi-simple representation type corresponding to it is called the generic semi-simple representation type. We will present a method to determine this generic type. Note that the corresponding problem for arbitrary representations is still open, although A. Schofield [15] has recently obtained some encouraging results.

In a subsequent paper we will concentrate on the local study of the quotient variety $V(Q, \alpha)$. For each type τ of semi-simple representations we will construct a new quiver Q_τ depending only on the dimension vectors of the simple components and a new dimension type α_τ depending only on the multiplicities with which these simple components occur s.t. the étale local structure of $V(\alpha)$ near a point of type τ is the same as that of the origin in the quotient variety $R(Q_\tau, \alpha_\tau)/GL(\alpha_\tau)$. To be more precise, if $\tau = (e_1, \beta_1; ...; e_l, \beta_l)$ is a semi-simple representation type then Q_τ will be a quiver with vertices $\{1, \ldots, l\}$ and there are $\delta_{ij} - R(\beta_i, \beta_j)$ arrows pointing from i to j ; the new dimension vector α_τ will be $(e_1, \ldots, e_l)$. This procedure simplifies our study in all points except for an m-dimensional subspace

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where m is the number of loops in the quiver Q. Moreover, it will enable us to calculate the dimension of the quotient variety as well as to determine its singular locus. If $\tau = (e_1, \alpha_1; ...; e_k, \alpha_k)$ is the generic semi-simple representation type, then this dimension is $\sum_{i=1}^{k}(1 - R(\alpha_i, \alpha_i))$ and the nonsingular locus coincides with the generic stratum except for low dimensional anomalities.

2. The coordinate ring.

By Mumford's theory, the coordinate ring of the quotient variety $V(Q, \alpha)$ is the fixed ring $\mathbb{C}[Q, \alpha]^{GL(\alpha)}$, i.e. the ring of polynomial invariants of the action of $GL(\alpha)$ on $R(Q, \alpha)$. The main result of this section will be the following description of this invariant ring :

Theorem 1.

The ring of polynomial invariants of $GL(\alpha)$ acting on $\mathbb{R}(Q, \alpha)$ is generated by traces of oriented cycles in Q of length at most N^2 where $N = \alpha(1) + \ldots + \alpha(n)$.

By this we mean the following : let $C = (\varphi_1, \ldots, \varphi_k)$ be an oriented cycle in Q, then one can consider the d by d matrix with entries in $\mathbb{C}[Q, \alpha]$:

$$X_{\varphi_1} X_{\varphi_2} \ldots X_{\varphi_k} = X_c$$

where $d = \alpha(t\varphi_1) = \alpha(h\varphi_k)$, then the trace of this d by d matrix $Tr(X_c)$ is clearly an invariant polynomial function. We will now show that these are essentially the only invariants.

we begin by recalling the definition of the path algebra $\mathbb{C}\langle Q \rangle$ of the quiver Q. A path of length $l \geq 1$ from vertex i to vertex j in a quiver Q is if form

$$(i|\varphi_1, \ldots, \varphi_l|j)$$

with arrows φ_j satisfying $h(\varphi_j) = t(\varphi_{j+1})$ for $1 \leq j \leq l$ such that $i = t(\varphi_1)$ and $j = h(\varphi_l)$. In addition, we define for any vertex $i \in Q_0$ a path of length 0 denoted by $(i|i)$. A path of length $\geq i$ from i to i is called an oriented cycle.

The path algebra $\mathbb{C}\langle Q \rangle$ of the quiver Q is defined to be the $\mathbb{C}$-vectorspace with basis the set of all paths in Q. The multiplication of two composable paths is defined to be the corresponding composition, the product if two noncomposable paths is, by definition, zero. We obtain an associative algebra with unit element $(1|1) + \ldots + (n|n)$.

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For a quiver Q, let Q^{op} be the opposite quiver, i.e. $Q_0 = Q_0^{\mathrm{op}}$ and with a bijection between Q_1 and Q_1^{op} such that $\varphi^{\mathrm{op}} : i \longrightarrow j$ iff $\varphi : j \longrightarrow i$ (i.e. reversing the orientation). Then the category of representations of Q can be identified with the category $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$-mod of left $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$-modules in the following way : given a representation V of Q, the corresponding $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$-module is given by $\overset{n}{\underset{i=1}{\oplus}} V(i)$ with module structure if $t \in Q_0 = Q_0^{\mathrm{op}}$ then $(t|t)v_i = v_t$ and if $\varphi^{\mathrm{op}} \in Q_1^{\mathrm{op}}$ s.t. $\varphi^{\mathrm{op}} : i \longrightarrow j$, the $\varphi^{\mathrm{op}} \Sigma v_i = V_\varphi(v_j)$. Conversely, given a $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$-module M. Let $M(i) = (i|i)M$ for all $i \in Q_0 = Q_0^{\mathrm{op}}$ and for $\varphi : j \longrightarrow i$ let $M(\varphi) : M(j) \to M(i)$ be the multiplication with $\varphi^{\mathrm{op}} \in \mathbb{C} \langle Q^{\mathrm{op}} \rangle$.

Let $N \in \mathbb{N}$, then we denote by $\mathbb{C} \langle Q^{\mathrm{op}} \rangle_{(N)}$ the quotient of $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$ by the ideal of evaluations of $N \times N$ matrix identities. Thus, $\mathbb{C} \langle Q^{\mathrm{op}} \rangle_{(N)}$ is the universal quotient of $\mathbb{C} \langle Q^{\mathrm{op}} \rangle$ of Pi-degree N and hence has a presentation

$$\mathbb{C} \langle Q^{\mathrm{op}} \rangle_{(N)} \cong \mathbb{G}_{m,N} / I$$

as a quotient of a ring of m generic n by n matrices $\mathbb{G}_{m,N}$, i.e. the subring of $M_N = M_N(\mathbb{C}[X_{ij}(l) : 1 \le i,j \le N, i \le l \le m])$ generated by the so called generic matrices $X_l = (X_{ij}(l))_{i,j}$. In our case, $m = \#Q_0 + \#Q_1$ and the natural epimorphism

$$\chi : \mathbb{G}_{m,n} \longrightarrow \quad \mathbb{C} \langle Q^{\mathrm{op}} \rangle_{(N)}$$

is given by sending the first $n = \#Q_0$ generic matrices to the idempotents $(i|i)$, $i \in Q_0$, and the last $\#Q_1$ generic matrices to the generators φ^{op} where $\varphi \in Q_1$. Then,

$$M_N / M_N I M_N \cong M_N(\Lambda)$$

where $\Lambda = \mathbb{C}[X_{ij}(l) : i,j,l]/J$ and J is the ideal generated by the entries of the matrices in I. Thus, we have the following situation :

$$
\begin{array}{ccc}
\mathbb{G}_{m,n} & \hookrightarrow & M_N(\mathbb{C}[X_{ij}(l)]) \\
\downarrow & & \downarrow {\scriptstyle M_N(p)} \\
\mathbb{C} \langle Q^{\mathrm{opp}} \rangle \longrightarrow \mathbb{C} \langle Q^{\mathrm{opp}} \rangle_{(N)} & \overset{\sigma}{\longrightarrow} & M_N(\Lambda)
\end{array}
$$

Let $\mathbb{C}[T_N(Q^{\mathrm{op}})]$ be the subring of Λ generated by the traces of $\sigma(\mathbb{C} \langle Q^{\mathrm{opp}} \rangle_{(N)})$.

Artin and Schelter proved in [1, Prop. 3.10., Th. 3.20] that $\mathbb{C}\,[T_n(Q^{\mathrm{op}})]$ is an affine $\mathbb{C}$-algebra and the corresponding variety parametrizes the isomorphism classes of N-dimensional semi-simple representation of $\mathbb{C}\,\langle Q^{\mathrm{op}}\rangle$.

There is a natural action of $GL_N(\mathbb{C})$ on $\mathbb{C}\,[x_{ij}(l)]$ such that the image of $x_{ij}(l)$ under $\alpha \in GL_n(\mathbb{C})$ is the entry (i,j) of the matrix $\alpha^{-1}.X_l.\alpha$. Procesi proved [14] that this action induces an action of $GL_N(\mathbb{C})$ in Λ such that

$$\Lambda^{GL_N(\mathbb{C})} = \mathbb{C}\,[T_N(Q^{\mathrm{op}})]$$

This fact will enable us in a moment to describe $\mathbb{C}\,[T_N(Q^{\mathrm{op}})]$. First, let us describe Λ. Let $k = \#Q_0, l = \#Q_1$ and denote

$$\mathbb{G}_{m,n} = \mathbb{C}\,\{X_1,\ldots,X_k; Y_1,\ldots,Y_l\} \subseteq M_n(\mathbb{C}\,[X_{ij}(p)])$$

the the ideal I is generated by the relations

$$
\begin{array}{llll}
(I): & X_i^2 = X_i & & \\
(II): & X_iY_j = Y_j & \text{if} & (\varphi_j) = i \\
& Y_jX_i = Y_j & \text{if} & h(\varphi_j) = i \\
& X_iY_j = Y_jX_i = 0 & & \text{otherwise} \\
(III): & Y_iY_j = 0 & \text{if} & h(\varphi_i) \neq t(\varphi_j)
\end{array}
$$

These matrix relations give rise to relations among the $x_{ij}(p)$ in every entry. These relations renerate the ideal J which is clearly $GL_n(\mathbb{C})$-invariant.

Having described the kernel of : $\varphi : \mathbb{C}\,[X_{ij}(p)] \longrightarrow \Lambda$, we will now concentrate on $\mathbb{C}\,[T_n(Q^{\mathrm{op}})]$. Since $GL_n(\mathbb{C})$ is a reductive group, the epi φ induces an epi

$$\varphi : \mathbb{C}\,[X_{ij}(p)]^{GL_N(\mathbb{C})} \longrightarrow \Lambda^{GL_N(\mathbb{C})} = \mathbb{C}\,[T_n(Q^{\mathrm{op}})]$$

By [13], $\mathbb{C}\,[X_{ij}(p)]^{GL_N(\mathbb{C})}$ is generated by traces of $\mathbb{G}_{m,N}$ and Rasmylov and Formanek [2] showed that one may restrict to traces of elements of degree $\leq N^2$. So we get :

Proposition 1 : With notations as before, the ring $\mathbb{C}\,[T_N(Q^{\mathrm{op}})]$ is generated by

(a): traces of oriented cycles of length $\leq N^2$ in the quiver Q^*, i.e. $Tr(Y_{i_1}\ldots Y_{i_p})$ if $(i|\varphi_{i_1},\ldots,\varphi_{i_p}|i)$ is a cycle.

(b): coefficients of the characteristic polynomial of the X_i (the generic matrices corresponding to the vertices).

In general, the variety corresponding to $\mathbb{C}\left[T_N(Q^{op})\right]$ is not irreducible. The irreducible components are determined by the dimension types $\alpha = (\alpha(1), \ldots, \alpha(n)) \in \mathbb{N}^n$ s.t. $\sum_{i=1}^{n} \alpha(i) = N$.

If α is such a dimension type, then the corresponding component is determined by the ring :

$$\mathbb{C}\left[T_N(Q^{op})\right]/D_\alpha$$

where D_α is generated by the $\alpha(i) + 1$ minors of X_i for all $i \leq i \leq n$.

Moreover, associate to each vertex i and N by N diagonal matrix $A_i = \mathrm{diag}(0,\ldots,0,1,\ldots,0,\ldots,0)$ with 1's from place $\sum\limits_{j=1}^{i-1} \alpha(j)+1$ to $\sum\limits_{j=1}^{i} \alpha(j)$. To every arrow $\varphi^{op} : j \rightarrow i$ in Q^{op} we associate an N by N matrix of indeterminates $B_{\varphi^{op}}$ with zeroes everywhere except in place (p,q) where $\sum\limits^{j-1} \alpha(k) < p \leq \sum\limits^{j} \alpha(k)$ and $\sum^{i-1} \alpha(k) < q \leq \sum^{i} \alpha(k)$.

Note that the polynomial ring in all the entries of the $B_{\varphi^{op}}$ is just $\mathbb{C}\left[Q, \alpha\right]$ and that there exists a natural algebra morphism

$$\mathbb{C}\langle Q^{op}\rangle_{(N)} \longrightarrow M_n(\mathbb{C}\left[Q, \alpha\right])$$

by sending $X_i \vdash A_i$ and $Y_{\varphi^{op}} \vdash B_{\varphi^{op}}$. This induces epimorphism

$$
\begin{array}{ccc}
\Lambda & \longrightarrow & \mathbb{C}\left[Q, \alpha\right] \\
\downarrow & \nearrow & \\
\Lambda/D_\alpha & &
\end{array}
$$

Let V be the closed subvariety subvariety of the variety of Λ determined by D_α, then the $GL_N(\mathbb{C})$-orbits in V are the same as the $GL_{\alpha(1)}(\mathbb{C}) \times \ldots \times GL_{\alpha(n)}(\mathbb{C}) = GL(\alpha)$-orbits. By the universal property of quotient varieties this gives us

$$V/GL_N(\mathbb{C}) \cong V/GL(\alpha)$$

whence $(\Lambda/D_\alpha)^{GL_N(\mathbb{C})} = (\Lambda/D_\alpha)^{GL(\alpha)}$. We have epimorphisms

$$\Lambda \longrightarrow \Lambda/D_\alpha \longrightarrow \mathbb{C}\left[Q, \alpha\right]$$

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and since both $GL_N(\mathbb{C})$ and $GL(\alpha)$ are reductive, we have epis :

$$\mathbb{C}\left[T_N(Q^{\mathrm{opp}})\right] \longrightarrow (\Lambda/D_\alpha)^{GL_N(\mathbb{C})}$$
$$\parallel$$
$$(\Lambda/D_\alpha)^{GL(\alpha)} \longrightarrow \mathbb{C}\left[Q,\alpha\right]^{GL(\alpha)}$$

Therefore, proposition 1 finishes the proof of Theorem 1. ∎

3. Luna's stratification for $V(\alpha)$.

In this section we aim to give a concrete description of the Luna stratification of quotient varieties [10,16]. Let us recall his construction.

Let a linear reductive group G act on a finite dimensional vectorspace X and let $\pi : X \longrightarrow V = X/G$ be the quotient map. If $\xi \in V$. its fiber $\pi^{-1}(\xi)$ contains precisely one closed orbit, say $G.x$. By a result of Matsuchima [11] the stabilizer subgroup G_x of x in G is a reductive linear group.

Conversely, if M is is a reductive subgroup of G we can look at the set $V_{(H)}$ consisting of all those points $\xi \in V$ s.t. for corresponding $x \in X, G_x$ is conjugated to H in G. We then have [16, lemma 5.5.] :

(a): $\{V_{(H)} : H$ reductive subgroup of $G\}$ is a finite stratification of V into locally closed irreducible smooth algebraic subvarieties. In particular, for only finitely many reductive subgroup H of G, $V_{(H)}$ is nonempty.

(b): $V_{(H')}$ lies in the closure of $V_{(H)}$ if anly if H is congugated to a subgroup of H', i.e. the smaller the stabilizer subgroup the larger the stratum.

We will now try to make this general result more concrete in the special case of the quotient varieties $V(\alpha)$. The points $\xi \in V(\alpha)$ correspond one-to-one to isomorphism classes of semi-simple representations of Q of dimension type α. Let V be a semi-simple representation in the fiber $\pi^{-1}(\xi)$ where $\pi : R(Q,\alpha) \longrightarrow V(\alpha) = R(Q,\alpha)/GL(\alpha)$. Then we can decompose V into its simple components

$$V = W_1^{\oplus e_1} \oplus \ldots \oplus W_k^{\oplus e_k}$$

where W_i is a simple representation of dimension type α_i and occuring in V with multiplicity e_i. We will then say that ξ is a point of representation of dimension type $\tau = (e_1, \alpha_1; \ldots, e_k, \alpha_k)$. In the next section we will give a combinatorial method to describe all possible representation types.

With $V(\alpha)_\tau$ we will denote the set of all points ξ of $V(\alpha)$ of representation type $t(\xi) = \tau$.

Theorem : $\{V(\alpha)_\tau : \tau$ representation type$\}$ is a finite stratification of $V(\alpha)$

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into locally closed irreducible smooth subvarieties.

Proof. In view of the results mentioned above we have to verify that the representation type determines the stabilizer group upto conjugation.

So, let ξ be a point of representation type $\tau = (e_1, \beta_1; \ldots; e_k, \beta_k)$ where $\beta_i = (b_{i1}, \ldots, b_{in}) \in \mathbb{N}^n$ and we define $b_i = \sum\limits_{j=1}^{n} b_{ij}$. We can choose of basis of $\bigoplus\limits_{i \in Q_o} \mathbb{C}^{\alpha(i)}$ in the following way : the first $e_1 b_1$ vectors give a basis of the simple component of type W_1 with dimension vector β_1 where $V = \oplus W_i^{\oplus e_i}$ is a semisimple representation lying in the fiber $\pi^{-1}(\xi)$, and so on.

The subring of $M_N(\mathbb{C})$ where $N = \sum \alpha(i)$ generated by this representation V is then (in this basis) :

$$
\begin{bmatrix}
M_{b_1}(\mathbb{C}) \otimes 1_{e_1} & & 0 \\
& \ddots & \\
0 & & M_{b_k}(\mathbb{C}) \otimes 1_{e_k}
\end{bmatrix} = \Gamma(V)
$$

The stabilizer subgroup $GL(\alpha)_V$ is easily seen to be the group of units of the centralizer of $\Gamma(V)$. The centralizer of $\Gamma(V)$ is the ring

$$
\begin{bmatrix}
M_{e_1}(\mathbb{C} \otimes 1_{b_1}) & & 0 \\
& \ddots & \\
0 & & M_{e_k}(\mathbb{C} \otimes 1_{b_k})
\end{bmatrix}
$$

whence $GL(\alpha)_\gamma \cong GL_{e_1}(\mathbb{C}) \times \ldots \times GL_{e_k}(\mathbb{C})$ embedded in $GL(\alpha)$ (with respect to the new basis) as

$$
GL(\alpha)_V =
\begin{bmatrix}
GL_{e_1}(\mathbb{C} \otimes 1_{b_1}) & & 0 \\
& \ddots & \\
0 & & GL_{e_k}(\mathbb{C} \otimes 1_{b_k})
\end{bmatrix}
$$

Using this computation it is now easy to verify that the conjugacy class of $GL(\alpha)_V$ in $GL(\alpha)$ depends only upon the type τ, finishing the proof.

Further one can verify that $GL(\alpha)_{\tau'}$ is comjugated to a subgroup of $GL(\alpha)_\tau$ iff semisimple representations of type τ can be obtained as degenerations of semisimple representation of type τ'.

One can express this combinatorially in the following way. Two types $\tau = (e_1, \beta_1; \ldots; e_r, \beta_r)$ and $\tau' = (e'_1, \beta'_1, \ldots, e'_{r'}, \beta'_r)$ are said to be direct successors $\tau < \tau'$ iff

(1) : $r' = r + 1$ and for all but one $1 \leq i \leq r$ we have $(e_i, \beta_i) = (e'_j, \beta'_j)$ for precisely one j one for the remaining i we have $(e_i, \beta_i) = (e_i, \beta'_r; e_i, \beta'_e)$ where $\beta_i = \beta'_k + \beta'_l$. Or

(2) : $r' = r - 1$ and for all but one $1 \leq i \leq r'$ we have $(e'_i, \beta'_i) = (e_j, \beta_j)$ for precise one j and for the remaining j we have $(e'_i, \beta'_i) = (e_k, \beta_i; e_e, \beta'_i)$ where $e_k + e_l = e'_i$. Two types τ and τ' are said to be successors $\tau << \tau'$ iff there exist type $\tau_1 \ldots, \tau_l$ s.t. $\tau = \tau_1 < \ldots < \tau_l = \tau'$. Combining this with the result mentioned in the beginning of this section we get :

Theorem : $V(\alpha)_{\tau'}$ lies in the closure of $V(\alpha)_\tau$ iff $\tau << \tau'$.

So, we see that the Luna stratification of $V(\alpha)$ can be described completely by representation theoretic features. The remaining problem of which representation types can occur, which comes down to a description of the dimension types of simple representations, will be solved in the next section.

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4. Simple representations.

In this section we will characterize the dimension vectors of simple representations of a quiver Q.

A full subquiver Q' of Q is called a club iff every couple of its vertices belongs to an oriented cycle. It is clear that we can divide Q into maximal clubs, say $G_1, \ldots, G_k$. The direction of all arrows between elements of G_i and elements of G_j is the same and can be used to define an orientation between G_i and G_j.

The clubquiver, $\mathrm{Club}(Q)$, of Q has as its vertices the maximal clubs and there is one arrow from G_i to G_j iff there is an arrow in Q from an element of G_i to an element of G_j. Note that $\mathrm{Club}(Q)$ is a quiver without oriented cycles.

It is fairly easy to deduce necessary conditions on the dimension vectors of simple representations. Consider a simple representation V of Q then we claim that its support, $\mathrm{supp}(V)$, is a club. For otherwise, consider its club quiver, $\mathrm{Club}(\mathrm{supp}(V))$, and let $\mathcal{D}_1$ be a sink in it. Then one can find a proper subrepresentation W of V by

(1) : $W_X = V_X$ iff $x \in \mathcal{D}_1$; $W_X = 0$ otherwise

(2) : $W(\varphi) = V(\varphi)$ iff $\varphi \in (\mathcal{D}_1)_1$; $W(\varphi) = 0$, otherwise.

In order to state the second necessary condition, let us recall some facts about the bilinear form R on $\mathbb{Z}^n$ (see e.g. [6] for more details) which is defined by

$$R(\alpha_i, \alpha_j) = \delta_{ij} - r_{ij}$$

where r_{ij} is the number of arrows from vertex i to vertex j and $\alpha_i = (\delta_{ij})_j$ is the standard basis of $\mathbb{Z}^n$. If V_i is a representation of Q with $\dim(V_i) = \gamma_i, i = 1, 2$, then

$$R(\gamma_1, \gamma_2) = \dim_{\mathbb{C}} \mathrm{Hom}(V_1, V_2) - \dim_{\mathbb{C}} \mathrm{Ext}^1(V_1, V_2)$$

We now claim that if V is a simple representation with $\dim V = \gamma$, then $R(\alpha_i, \gamma) \leq 0$ and $R(\gamma, \alpha_i) \leq 0$ for all $1 \leq i \leq n$. For let $\gamma = (\gamma(1), \ldots, \gamma(n))$ then $R(\alpha_i, \gamma) = \gamma(i) - \sum_{i \to j} r_{ij}\gamma(j)$. Suppose that $R(\alpha_i, \gamma) > 0$, then the natural map :

$$\underset{i \overset{\varphi}{\longrightarrow} j}{\oplus} V(\varphi) : V_i \longrightarrow \underset{i \overset{\varphi}{\longrightarrow} j}{\oplus} V_j$$

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has a kernel say K which determines a proper subrepresentation W of V by $W = K; W = 0, j \neq i$ and $W(\varphi) = 0$. Similarly, $R(\gamma, \alpha_i) = \gamma(i) - \sum\limits_{j \to i} r_{ji}\gamma(j)$ and if $R(\gamma, \alpha_i) > 0$, then the image of the natural map :

$$\bigoplus\limits_{j \xrightarrow{\varphi} i} V(\varphi) : \bigoplus\limits_{j \xrightarrow{\varphi} i} V_j \longrightarrow V_i$$

is not all of V_i. Therefore, we have a proper subrepresentation W of V by $W_i = \mathrm{Im}(\oplus V(\varphi)); W_j = V_j$ if $i \neq j$ and $W(\varphi) = V(\varphi)$ for all $\varphi \in Q_1$.

The two conditions (1) : $\mathrm{supp}(\gamma)$ is a club and (2) for all $1 \leq i \leq n$, $R(\alpha_i, \gamma) \leq 0$ and $R(\gamma, \alpha_i) \leq 0$; are not sufficient to ensure that γ is the dimension vector of a simple representation of Q. For take the extended Dynkin diagram $\widetilde{A}_n$ with the cyclic orientation
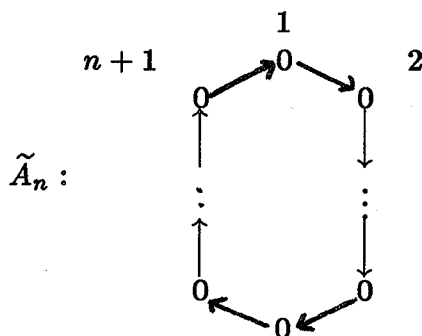
then $\gamma = (a, a, \ldots, a)$ for $a \in \mathbb{N}_+$ satisfies the conditions, however it is well known that the only nontrivial simple representation has dimension vector $(1, 1, \ldots, 1)$. Nevertheless, we will show that these are the only exceptions :

Theorem : $\gamma \in \mathbb{N}_+^n$ is the dimension vector of a simple representation of the quiver Q iff

(a) : either $\mathrm{supp}(\gamma)$ is an oriented cycle and all $\gamma(i)$ are 0 or 1;

(b) : or $\mathrm{supp}(\gamma)$ is a club and for all $i \leq i \leq n$ we have $R(\alpha_i, \gamma) \leq 0$ and $R(\gamma, \alpha_i) \leq 0$

The proof goes by induction on the number of vertices n of Q and on $|\gamma| = \sum \gamma(i)$.

We will need some extra terminology. We call a vertex i a focus (resp. prisma) iff $\exists!\varphi \in Q_1$ with $t\varphi = i$ (resp. $h\varphi = i$). A vertex i is said to be large iff $\gamma(i)$ is maximal among the $\gamma(j)$.

Finally, we call a vertex i good iff i is large and it has no large successor which is a prisma nor a large predecessor which is a focus. We begin with an easy observation.

Lemma : If Q is a club which is not an oriented cycle, then there is no cycle of prisma (resp. focus) vertices.

Proof : Suppose there is a cycle of prisma $(i_1, \ldots, i_k)$ then for each j the unique arrow coming into i_j belongs to the cycle. However, Q is not an oriented cycle so there is not atleast one more vertex, say i, But there can be no path from i to any of the i_j contrary to the club assumption.

Using this lemma, we can find either a good vertex or a large prisma i which has no large prisma successors. Suppose we are in the second case, then the unique predecessor of i has to be a large focus, so we have the following situation :

$$(*)$$
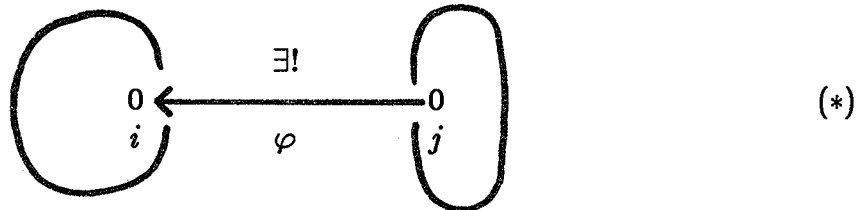
and we can apply a shrinking process :

Lemma : If we are in situation (*) with $\gamma(i) = \gamma(j)$ then γ is the dimension vector of a simple representation of Q iff $\gamma' = (\gamma(1), \ldots, \check{\gamma}(i), \ldots, \gamma(n))$ is the dimension vector of a simple representation of Q' which is the quiver obtained from Q by identifying i and j.

Proof : Let V be a simple representation of Q with $\dim(V) = \gamma$, then $V(\varphi)$ is an isomorphism. For otherwise, either W determined by $W_k = 0$ if $k \neq j, W_j = \mathrm{Ker}V(\varphi)$ or W' determined by $W'_k = V_k$ if $k \neq i$, $W_i = \mathrm{Im}V(\varphi)$ is a proper subrepresentation. Therefore, we can identify V_i and V_j and obtain a simple

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representation of Q'.

Conversely, let V' be a simple representation of Q' and form a representation V of Q by $V_k = V'_k$ for $k \neq i$ and $V_i = V'_j$, $V(\varphi) = 1_{V'_j}$. We claim that V is a simple representation of Q. If not, there are subvectorspaces $W_k \subseteq V_k$ s.t. W is a proper subrepresentation. But then W', determined by $W'_k = W_k$ if $k \neq i$ would be a proper subrepresentation of V', a contradiction.

The foregoing lemma finishes the proof of the theorem in case (*) by induction on the number of vertices.

So, we are left to consider the case of a good vertex i. If $\gamma(i) = 1$, then all $\gamma(j) = 1$ for $j \in \text{supp}(\gamma)$ and defining for all $V_j = \mathbb{C}$ and $V(\varphi) = 1_{\mathbb{C}}$ we get by the club-assumption a simple representation V. If $\gamma(i) > 1$ we replace γ by γ' where $\gamma'(j) = \gamma(j)$ for $j \neq i$ and $\gamma'(i) = \gamma(i) - 1$. Clearly, the $\text{supp}(\gamma')$ is still a club and we claim that $R(\gamma', \alpha_j) \leq 0$ and $R(\alpha_j, \gamma') \leq 0$ for all $1 \leq j \leq n$. The only j's where it might go wrong are direct predecessors or direct successors of i. Suppose $R(\alpha_j, \gamma') > 0$, then $\gamma'(j) > \sum_{k \to j} r_{kj} \gamma'(k)$ whence $\gamma(j) = \gamma'(j)$ must be large and a focus with endpoint i, contradicting the goodness of i.

So, by inducting on $|\gamma|$ we may assume that there exist simple representations of Q with dimension vector γ'. Take such a representation V' then since $R(\alpha_i, \gamma') < 0$ and $R(\gamma', \alpha_i) < 0$ we know that $\text{Ext}^1(V', S_i) \neq 0 \neq \text{Ext}^1(S_i, V')$ where S_i is the trivial simple representation in i.

Now look at the space of all representations V in $R(Q, \gamma)$ s.t. $V|\gamma' = V'$ and $V|\alpha_i = S_i$. This is an affine space $X_{V'}$ of dimension $\sum_{j \to i} r_{ji} \gamma(j) + \sum_{i \to j} r_{ij} \gamma(j)$. Loosely speaking, $X_{V'}$ consists of representations worse than $V' \oplus S_i$. We can choose V' s.t. $X_{V'}$ contains representations with a trace of an oriented cycle different from the corresponding trace of $V' \oplus S_i$, for being simple in an open condition in $R(Q, \gamma')$. Therefore, the Jordan-Hölder factors of these representations cannot be V' and S_i, see §2 but still these representations degenerate to $V' \oplus S_i$ hence by §3 they must be simple; finishing the proof.

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Recall that a representation V is called a Schur representation iff its endomorphisms ring is $\mathbb{C}$; the corresponding dimension vector is called a Schur root. V. Kac has conjectured a purely combinatorial description of these Schur roots [6]. He calls an element $\alpha \in \mathbb{N}_+^n$ an indecomposable root if α cannot be decomposed into a sum $\alpha = \beta + \gamma$ where $\beta, \gamma \in \mathbb{N}_+^n$ and $R(\beta, \gamma) \geq 0$, $R(\gamma, \beta) \geq 0$. He then conjectured that Schur roots and indecomposable roots coincide. In general, this conjecture is wrong, see [8]. However, it is clear from [7,lemma 3.2] and our result that th Schur roots which are dimension vectors of simple representations are indecomposable .

In view of the stratification result there is precisely one semi-simple representation type τ_{gen} such that the corresponding stratum is an open subvariety of $V(Q, \alpha)$. We will now briefly describe how to determine this so called generic semi-simple representation type. Given our quiver Q and a dimension vector α we can consider as before the clubquiver $Club(supp(\alpha))$. A simple subrepresentation of a representation of dimension vector α must live on one of the sinks of this clubquiver. Therefore, by induction it will be sufficient to find the generic semi-simple representation type in case $supp(\alpha)$ is a club. In this case there is precisely one dimension vector $\beta \leq \alpha$ with $R(\beta, \alpha_i) \leq 0$ and $R(\alpha_i, \beta) \leq 0$ for all i. Then the generic semi-simple representation type is

$$(1, \beta; \alpha(1) - \beta(1), \alpha_i; \ldots; \alpha(n) - \beta(n), \alpha(n))$$

If we have determined the generic semi-simple representation type in one of the sinks of the clubquiver, we delete this sink from it and proceed similarly untill we reach the empty graph. The generic semi-simple representation type will then be the sum of the generic types on the maximal clubs.

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