Some Remarks on Rational Matrix Invariants.

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Abstract.

Wildness of the rank two quiver P_3 provides a link between the study of rational matrix invariants and that of stable vectorbundles over the projective plane. Using this dictionary, results of Formanek imply the rationality of the moduli spaces of rank three and rank four vectorbundles. Conversely, the stable rationality result of Maruyama implies the Merkurjev-Suslin theorem and gives a positive answer to the lifting problem for crossed products to local algebras. Finally, we recover a recent result of Van den Bergh showing that the field of rational n by n matrixinvariants is the functionfield of the generic Jacobian variety for smooth plane curves of degree n.

1. Introduction.

(1.1) : Throughout this paper,we consider an algebraically closed field of characteristic zero and call it $\mathbb{C}$. Let $GL_n(\mathbb{C})$ act on m-tuples of n by n matrices $X_{m,n} = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$ by componentswise conjugation. The topic of this paper is the field $K_{m,n}$ of rational invariants for this situation. That is, consider the rational field $L_{m,n} = \mathbb{C}(x_{ij}(l) : 1 \leq i,j \leq n; 1 \leq l \leq m)$ and $\gamma \in GL_n(\mathbb{C})$ acts on it by sending the variable $x_{ij}(l)$ to the (i,j)-entry of the matrix $\gamma^{-1}X_l\gamma$ where $X_l = (x_{ij}(l))_{i,j} \in M_n(L_{m,n})$. Then, $K_{m,n}$ is the fixed field under this action.

$K_{m,n}$ is easily seen to be the field of functions of the variety of matrixinvariants $V_{m,n} = X_{m,n}/GL_n(\mathbb{C})$. That is, the variety parametrizing simultaneous conjugacy classes of m tuples of n by n matrices which generate a semi-simple subalgebra of $M_n(\mathbb{C})$. See for example [6],[14].

(1.2) : It is still an open question whether $K_{m,n}$ is always a rational functionfield. For ringtheorists this question is important because it would imply the Merkurjev-Suslin result for fields containing $\mathbb{C}$ (the Brauer group is generated by cyclic algebras). Let us scetch the argument : consider the ring of m generic n by n matrices, that is the subring of $M_n(L_{m,n})$ generated by the matrices X_l. This ring is known to be a left and right Öre domain, so we can form its classical ring of quotients $\Delta_{m,n}$ which is a division algebra of dimension n^2 over its center $K_{m,n}$. Rationality of $K_{m,n}$ and a result of Bloch [3,Th.1.1] would imply that $\Delta_{m,n}$ is Brauer equivalent to a product of cyclic algebras. Then by the generic property of the $\Delta_{m,n}$ for all $m \geq 2$ every central simple algebra of dimension n^2 over a field L containing $\mathbb{C}$ would be a product of cyclic algebras in $Br(L)$, see for example [15]. For more details we refer to [6],[15] and [17].

Procesi [13] proved that $K_{m,n}$ is rational whenever $K_{2,n}$ is , thereby reducing the problem to two matrices. He also solved the rationality problem for $n = 2$. Later, E. Formanek [4],[5] proved the rationality for $n = 3$ and $n = 4$. He used the following elegant description of $K_{2,n}$: let $\{x_i, y_{ij} \mid 1 \leq i, j \leq n\}$ be independent commuting indeterminates and let L be the subfield of $\mathbb{C}\,(x_i, y_{ij} : 1 \leq i, j \leq n)$ generated by $\{x_i, y_{ii}, y_{ij}y_{ji}, y_{ij}y_{jk}y_{ki} \mid 1 \leq i, j, k \leq n\}$. Then L is a rational functionfield of trancendence degree $n^2 + 1$ and the permutation group S_n acts on it by $\sigma(x_i) = x_{\sigma(i)}, \sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. Then, $K_{2,n}$ is the fixed field under this action.

(1.3) : In this paper we aim to show that this rationality problem may also be of interest to geometers. Using the results of K. Hulek [9] we will show that $K_{2,n}$ is the functionfield of the moduli space $M(n, 0, n)$ of stable (rank n) vectorbundles over the projective plane with Chern-numbers $(0, n)$. Therefore, Formaneks results imply the rationality of $M(3, 0, 3)$ and $M(4, 0, 4)$ which was (perhaps) not known. Rationality of $M(2, 0, 2)$ was proved by Barth [2].

Conversely, one can give ringtheoretical interpretations of geometrical results. For example, M. Maruyama [11] has shown that the moduli spaces $M(n, 0, n)$ are stably rational.

By a theorem of S. Bloch and the argument given in (1.2) this implies the Merkurjev-Suslin result . Moreover, using some ideas of D. Saltman [16] it also implies that every central simple algebra over a residue field can be lifted to the corresponding local algebra. In particular , the canonical morphism between the Brauer groups is surjective. This fact also follows from the Merkurjev-Suslin result.

Another consequence of our result is a recent theorem of M. Van den Bergh [18] who showed that $K_{2,n}$ is the functionfield of a Picard scheme of a bundle of nonsingular curves over a rational variety. This result will now follow from the fact [9,1.7] that a sufficiently general stable vectorbundle over $\mathbb{P}_2$ having Chern-numbers $(0, n)$ is classified by a smooth plane curve of degree n and an invertible sheaf over it (generalizing the curve of jumping lines and the θ-characteristic in the rank two case , see [2]). Added in proof : it is clear from [11,p.88] that this result was also proved by M. Maruyama.

2. Vector bundles over $\mathbb{P}_2$.

(2.1) : In this section we aim to prove the following result :

Theorem 1 : $K_{2,n}$ is the functionfield of the moduli space $M(n,0,n)$ of stable rank n vectorbundles over the projective plane with Chern-numbers $(0,n)$.

(2.2) : Let us recall the connection between so called s-stable vectorbundles over $\mathbb{P}_2$ and certain triples of n by n matrices $A = (A_0, A_1, A_2)$. One calls A prestable if for any $v \in \mathbb{C}^n$ we have $dim_{\mathbb{C}} (A_0v + A_1v + A_2v) \geq 2$ and $dim_{\mathbb{C}} (A_0^\tau v + A_1^\tau v + A_2^\tau v) \geq 2$. Hulek associates to a prestable triple A a vectorbundle $\mathcal{E}_A$ in the following way :

Let $\mathcal{O}_{\mathbb{P}}$ be the structure sheaf of $\mathbb{P}_2$ and let X_0, X_1, X_2 be the usual basis for $\Gamma(\mathcal{O}_{\mathbb{P}}(1))$. Let $V = \Gamma(\mathcal{O}_{\mathbb{P}}(1))^*$ and let Y_0, Y_1, Y_2 be a basis of V dual to X_0, X_1, X_2. Define a linear map $\phi_A : \mathbb{C}^n \otimes V \to \mathbb{C}^n \otimes V^*$ by sending $v \otimes Y_i$ to $A_{i+1}v \otimes X_{i-1} - A_{i-1}v \otimes X_{i+1}$ for all $v \in \mathbb{C}^n$ and $i = 0, 1, 2$ mod 3.For a canonical choice of bases for $\mathbb{C}^n$ and the bases defined before for V and V^* the matrix of ϕ_A is given by

$$\begin{pmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{pmatrix}$$

If U denotes the image of ϕ_A , we obtain a complex of vectorbundles

$$M_A : \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{a} U \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{c} \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}}(1)$$

where a denotes the composite morphism

$$\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{1\otimes s^*} \mathbb{C}^n \otimes V \otimes \mathcal{O}_{\mathbb{P}} \xrightarrow{\phi_A \otimes 1} U \otimes \mathcal{O}_{\mathbb{P}}$$

and c is the restriction to $U \otimes \mathcal{O}_{\mathbb{P}}$ of the morphism $1 \otimes s : \mathbb{C}^n \otimes V^* \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}}(1)$ where $s : \Gamma(\mathcal{O}_{\mathbb{P}}(1)) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$ is the multiplication map and s^* its dual. The complex M_A is a monad in the sense of Horrocks. Its cohomology turns out to be a vector bundle $\mathcal{E}_A$ which is s-stable in the sense that $H^0(\mathcal{E}_A) = H^0(\mathcal{E}_A^*) = 0$. The bundle $\mathcal{E}_A$ has rank $rk(\phi_A) - 2n$, has Chern-numbers $(0,n)$ and the map $A \to \mathcal{E}_A$ induces a bijection between the set of isoclasses of s-stable vector bundles over $\mathbb{P}_2$ with Chern-numbers

$(0, n)$ and isomorphism classes of prestable triples when considered as representations of dimension vector (n, n) of the wild quiver P_3

$$\bigcirc \;\; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \;\; \bigcirc$$

that is, orbits of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $X_n = M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ by $(\gamma_1, \gamma_2).(A_0, A_1, A_2) = (\gamma_2^{-1} A_0 \gamma_1, \gamma_2^{-1} A_1 \gamma_1, \gamma_2^{-1} A_2 \gamma_1)$.

(2.3) : We can study the following projective quotient variety as a first approximation to the orbitstructure problem in X_n. Consider the open subvariety $X' = \{(A_0, A_1, A_2) \in X_n \mid rg(A_0, A_1, A_2) = n\}$. We can eliminate the action of the first component of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ on this subvariety and get the Grassmann variety $Grass(n, 3n)$ as a representing space. The second $GL_n(\mathbb{C})$ component acts on this space via its diagonal embedding in $GL_{3n}(\mathbb{C})$. The projective variety of interest to us is $Y_n = Grass(n, 3n)^{ss}/GL_n(\mathbb{C})$ where $Grass(n, 3n)^{ss}$ is the set of semi-stable points under this action. These points come from representations in X_n having no subrepresentation of dimension vector (k, l) where $0 < l \leq k < n$. The stable points come from representations having no subrepresentation of type (k, k) where $0 < k < n$.

We will show that the variety of matrixinvariants $V_{2,n} = X_{2,n}/GL_n(\mathbb{C})$ is birational to Y_n. On the open subvariety $X"$ of X' determined by those triples (A_0, A_1, A_2) s.t. $det(A_0) \neq 0$ we can eliminate the action of the first component of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ by multiplying on the right by A_0^{-1} and get representants of the form (I_n, B_1, B_2). The action of the second component on these representants is $\gamma.(I_n, B_1, B_2) = (\gamma^{-1}, \gamma^{-1}B_1, \gamma^{-1}B_2) = (I_n, \gamma^{-1}B_1\gamma, \gamma^{-1}B_2\gamma)$. That is, the orbits of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $X"$ correspond to orbits of $GL_n(\mathbb{C})$ acting on couples of n by n matrices by simultaneous conjugation. Clearly, the map $X_{2,n} \to X"$ given by sending a couple (B_1, B_2) to the representation (I_n, B_1, B_2) induces an open immersion of $V_{2,n}^{simple}$ in Y_n. Here, $V_{2,n}^{simple}$ is the open set of $V_{2,n}$ corresponding to couples which generate $M_n(\mathbb{C})$ and therefore the corresponding representations give rise to stable points in $Grass(n, 3n)$.

(2.4) : We now have all the relevant information to prove theorem 1 :

Consider the open subvariety $X'_{2,n}$ of $X_{2,n}$ consisting of couples (B_1, B_2) which generate $M_n(\mathbb{C})$ and such that $[B_1, B_2] \in GL_n(\mathbb{C})$. The corresponding representation $B = (I_n, B_1, B_2) \in X"$ is prestable. For otherwise, B_1 and B_2 would have a common eigenvector v , but then $[B_1, B_2]v = 0$ whence $v = 0$. Therefore, we can associate to B an s-stable vectorbundle $\mathcal{E}_B$ of rank n. This follows from (2.2) and

$$\begin{pmatrix} I_n & -B_1 & -B_2 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & B_2 & -B_1 \\ -B_2 & 0 & I_n \\ B_1 & -I_n & 0 \end{pmatrix} = \begin{pmatrix} [B_1, B_2] & 0 & 0 \\ * & 0 & I_n \\ * & -I_n & 0 \end{pmatrix}$$

By a result of Maruyama [9,Th.2.8] stability is an open property, hence there exist open subvarieties $X^s_{2,n} \subset X'_{2,n}$ and $X^s \subset X"$ whose points give rise to stable rank n vectorbundles over $\mathbb{P}_2$ with Chern-numbers $(0, n)$. Using the observations from (2.2) and (2.4) it is clear that $V^s_{2,n} = X^s_{2,n}/GL_n(\mathbb{C})$ embeds in $M(n, 0, n)$ finishing the proof.

(2.5) : Using the results of E. Formanek, we get as an immediate consequence :

Corollary : The moduli spaces $M(n, 0, n)$ are unirational and they are rational for $n \leq 4$

To the best of our knowledge, rationality of $M(3, 0, 3)$ and $M(4, 0, 4)$ has not been noted before. Rationality of $M(2, 0, 2)$ is due to W. Barth [2].

3. Lifting crossed products.

(3.1) : As was pointed out to me by K. Hulek, M. Maruyama has recently proved stable rationality of the moduli spaces $M(n,0,n)$, [11]. Recall that a field K is said to be stably rational over $\mathbb{C}$ if $K(x_1,...,x_r)$ is rational over $\mathbb{C}$ for some r. Therefore, using theorem 1 we have the following (partial) positive answer to problem 11 from Formanek [7,p.34]

Theorem 2 : $K_{2,n}$ is stably rational for all n.

(3.2) : So we can find r and s such that $K_{2,n}(x_1,...,x_r) \cong \mathbb{C}(x_1,...,x_s)$. S. Bloch proved that the cokernels of the natural morphisms

$$K_2(L)_n \longrightarrow Br(L)_n$$

$$K_2(L(x_1,...,x_k))_n \longrightarrow Br(L(x_1,...,x_k))_n$$

are isomorphic for all k and all fields L containing a primitive n-th root of unity. Therefore, this map is surjective for $\mathbb{C}(x_1,...,x_s) \cong K_{2,n}(x_1,...,x_r)$ and hence also for $K_{2,n}$. This implies that $\Delta_{2,n}$ is Brauer equivalent to a product of cyclic algebras. Now, one can proceed as in (1.2) to obtain

Theorem 3 : (Merkurjev-Suslin) If L is a field containing $\mathbb{C}$, then the Brauer group of L is generated by cyclic algebras.

(3.3) : Using some ideas of D. Saltman [16] we will give another consequence of Maruyama's result for Azumaya algebras. Let R be a local $\mathbb{C}$-algebra with maximal ideal m and residue field $k = R/m$. A central R-algebra A which is a free module of rank n^2 is called an Azumaya R-algebra iff A/mA is central simple of dimension n^2 over its center k. Two R-Azumaya algebras A and B are called equivalent if $M_r(A) \cong M_s(B)$ for some r, s. The tensorproduct $\otimes_R$ induces a groupstructure on the set of equivalence classes which is called the Brauer group of R, Br(R).

For a long time it has been an open question whether the natural morphism π :

$Br(R) \rightarrow Br(k)$ is surjective. We will show that this is indeed the case. Actually, a much stronger result holds

Theorem 4 : If R is a local $\mathbb{C}$-algebra with residue field k and if Δ is a central simple k-algebra of dimension n^2, then there exists an R-Azumaya algebra A such that $A \otimes_R k \cong \Delta$

(3.4) : Since $V_{2,n} \times A^r$ is birational to some A^s, we can find an open set U of $V_{2,n}$ and V of A^s and a split morphism $\mathbb{C}[U] \rightarrow \mathbb{C}[V] \rightarrow \mathbb{C}[U]$. Since the ideal of central polynomials defines an open piece of $V_{2,n}$ we may assume that U is determined by some central polynomial f. Let $g \in \mathbb{C}[x_1, ..., x_s]$ be the corresponding element defining V. Let Δ be a central simple k-algebra of dimension n^2. As a k-algebra Δ is generated by two elements, so we have a morphism $\phi : \Lambda \rightarrow \Delta$ from the ring of two generic n by n matrices Λ to Δ such that $\Lambda \otimes_\phi k = \Delta$. ϕ extends to Λ_f which is an Azumaya algebra over $\mathbb{C}[U]$ (f is a central polynomial for n by n matrices).We have the split morphisms $\eta : \mathbb{C}[U] \rightarrow \mathbb{C}[V]$ and $\delta : \mathbb{C}[V] \rightarrow \mathbb{C}[U]$. Take for any $1 \leq i \leq s$ an element $b_i \in R$ s.t. $\pi(b) = \phi(\delta(x_i))$ then this defines a morphism $\mu : \mathbb{C}[V] \rightarrow R$ because $\mu(g) \in R^*$ since $\pi(\mu(g)) = \phi(\delta(g)) \neq 0$. Then $\phi' = \mu o \eta$ extends ϕ and $\Lambda_f \otimes_{\phi'} R$ is the desired Azumaya algebra

(3.5) : As an immediate consequence we obtain

Corollary : If R is a local $\mathbb{C}$-algebra with residue field k, then the natural morphism $Br(R) \rightarrow Br(k)$ is surjective

4. The generic Jacobian variety.

(4.1) : In [15,§6], M. Van den Bergh showed that $V_{2,n}$ is birational to a Picard scheme of a bundle of nonsingular curves over a rational variety. The projective space $\mathbb{P}^{\frac{1}{2}n(n+3)}$ parametrizes plane curves of degree n. Let U be the open subvariety corresponding to nonsingular curves. Consider the flagvariety $W \subset \mathbb{P}^2 \times U$ consisting of all couples (P, Y) s.t. $P \in Y$. The projection $W \to U$ is a flat bundle of smooth curves. Let $PIC_{W/U}$ be the functor which associates to an U-scheme S the group

$$PIC_{W/U}(S) = \frac{\{\text{group of invertible sheaves on } W \times_U S\}}{\{\text{subgroup of sheaves of the form } p_2^*(K) \text{ for } K \text{ on } S\}}$$

Since $W \to U$ is a bundle of smooth curves we can associate to invertible sheaves a discrete invariant, the degree. $PIC^d_{W/U}$ is the subfunctor consisting of invertible sheaves of degree d. The sheafification of this functor with respect to the flat topology is represented by the variety $Pic^d_{W/U}$ consisting of couples $(Y, \mathcal{L})$ where Y is a nonsingular curve of degree n in $\mathbb{P}_2$ and $\mathcal{L}$ is a divisor on Y of degree d (which I like to call the generic Jacobian variety for smooth plane curves of degree n).For more details we refer the reader to [1],[7],[10] or the preliminary sections of [15].

Theorem 2 : (Van den Bergh,[15,Th.6.1.3])
If $d = \frac{1}{2}n(n-1)$, then $K_{2,n}$ is the functionfield of the variety $Pic^d_{W/U}$

(4.2) : In view of theorem 1 we have to associate to a sufficiently general vectorbundle $\mathcal{E}$ of rank n over $\mathbb{P}_2$ with Chern-numbers $(0, n)$ a nonsingular curve Y of degree n and an invertible sheaf $\mathcal{L}$ which determine $\mathcal{E}$ upto isomorphism. Hulek [8,1.7] has indicated how this can be done by a suitable generalization of Barth's characterization of rank two bundles by their curve of jumping lines and θ- characteristic , [2].

Let $\mathcal{E}_A$ be an s-stable vectorbundle associated to the prestable triple $A = (A_0, A_1, A_2)$ and define $\Delta_A = \det(A_0Y_0 + A_1Y_1 + A_2Y_2) \in \Gamma(\mathcal{O}_{\mathbb{P}_2^*}(n))$ and let $Y_A = \{\Delta_A = 0\}$. The discriminant Δ_A is a homogeneous polynomial of degree n and $Y_A \subset \mathbb{P}_2^*$ will be a curve of degree n or the whole plane. The interpretation of Y_A is that it contains those lines L

in $\mathbb{P}_2$ such that $\mathcal{E} \mid L \neq \mathcal{O}_L^{\oplus r}$, so it generalizes the curve of jumping lines in the rank two case.

In case Y_A is a curve (which is the generic case) one defines a map $\psi_A = (A\otimes 1)o(1\otimes s)$: $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \to \mathbb{C}^n \otimes V^* \otimes \mathcal{O}_{\mathbb{P}^*} \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}_*}$. Over a point $L \in \mathbb{P}_2^*$ with coordinate vector $y = (y_0, y_1, y_2)$ the map ψ_A is just $A(y) = A_0 y_0 + A_1 y_1 + A_2 y_2$. We can define a sheaf $\mathcal{L}_A$ by the sequence

$$(*) : 0 \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \xrightarrow{\psi_A} \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*} \to \mathcal{L}_A \to 0$$

which has its support in Y_A. By [8,1.7.3.iv] the pair $(Y_A, \mathcal{L}_A)$ determines $\mathcal{E}_A$ uniquely. Restricting the sequence $(*)$ to Y_A we obtain

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) & \xrightarrow{\psi_A} & \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*} & \to & \mathcal{L}_A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{C}^n \otimes \mathcal{O}_{Y_A}(-1) & \xrightarrow{\psi_A \mid Y_A} & \mathbb{C}^n \otimes \mathcal{O}_{Y_A} & \to & \mathcal{L}' & \to & 0 \end{array}$$

For sufficiently general A we get that $rk(\psi_A \mid Y_A) = n - 1$ whence $\mathcal{L}'$ is an invertible sheaf over Y_A. The induced map $\mathcal{L}_A \to \mathcal{L}'$ is surjective and will be injective too if every section in $\mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}$ vanishing on Y_A comes by ψ_A. This is a consequence of $\psi_A^{adj} o \psi_A = det(A) 1_{\mathbb{C}^n}$ where $det(A) : \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-n) \xrightarrow{\psi_A^{adj}} \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}(-1) \to \mathbb{C}^n \otimes \mathcal{O}_{\mathbb{P}^*}$. So, for generic A we have that $\mathcal{L}_A \in Pic(Y_A)$ of degree $\frac{1}{2}n(n-1)$ by [8,1.7.3.iii].

Conversely, starting from a plane curve Y of degree n and $\mathcal{L} \in Pic(Y)$ of degree $\frac{1}{2}n(n-1)$ one can reconstruct a triple A which will be prestable (and hence determine a vectorbundle) for a sufficiently general choice of Y and $\mathcal{L}$, [8,1.7].

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