Centers of Generic Division Algebras and Zeta-Functions.

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September 1986 86-24

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this paper is dedicated to the memory of Jose Carrasco Tapia, journalist,victim of a vicious regime,Santiago de Chile,september 7th 1986

1. Introduction

Let us fix an algebraically closed field of characteristic zero which we will denote by $\mathbb{C}$. Consider the polynomial ring :

$$P_{m,n} = \mathbb{C}\,[x_{ij}(l) : 1 \leq i,j \leq n, 1 \leq l \leq m]$$

and form the so called generic matrices

$$X_l = (x_{ij}(l))_{i,j} \in M_n(P_{m,n})$$

The ring of m generic n by n matrices , $\mathbb{G}_{m,n}$, is defined to be the sub $\mathbb{C}$-algebra of $M_n(P_{m,n})$ generated by the generic matrices $X_1, ..., X_m$. It is well known that this ring is a left and right Ore-domain, so we can form its classical ring of fractions, $\Delta_{m,n}$, which is a division algebra of dimension n^2 over its center $K_{m,n}$. One of the main open problems in the theory of rings satisfying polynomial identities is :

Problem 1 : Is the center $K_{m,n}$ of the generic division algebra $\Delta_{m,n}$ rational , i.e. is it purely trancendental over $\mathbb{C}$?

It is well known that the trancendence degree of $K_{m,n}$ over $\mathbb{C}$ is equal to $(m-1)n^2 + 1$.

algebra in m variables. Therefore, $V_{m,n}$ would be birational to $(m-1)n^2+1$ affine space giving an affirmative answer to problem 1.The main aim of this paper is to collect some evidence for this stronger conjecture.

In section two we recall some terminology which will be needed throughout the paper.We have tried to minimalize the representation lingo since we are mainly concerned with one specific quiver,but some results like the étale local structure of matrixinvariants are expressed most naturally in quiver terms. These results arising from joint work with C. Procesi are explained in section three.In the next section we explain the Kac conjecture in the special case of the bouquet quivers. For more details and a readable account of Kac's main result on the representations of quivers, we refer the interested reader to the excellent paper by H.P. Kraft and C. Riedtmann [KR].At the end of section four we give a generalization of problem 1 based on some joint work with C.M. Ringel on semi-simple representations of quivers.In section five we recall the results on the cellular decomposition of the orbitspace of representations of S_1 which we will need to simplify the computations.In the next section we have tried to describe the cellular decomposition of the orbitspace in the easiest noncommutative case, i.e. representations of S_2 with dimension vector 2. In section seven we outline the idea of Kac to check his conjecture by computing the zèta functions of the orbitspaces defined over finite fields.The next two section contain the details for these computations in the special case of interest to us. Since we know the decomposition for S_1 the calculations can be simplified a bit.It is somewhat surprising to realize that the computation of the zèta functions for the wild quivers S_m ($m \geq 2$) would follow from a thorough understanding of the tame quiver S_1. Probably the computations can be simplified even more by a specialist in the combinatorial aspects of $GL_n(\mathbb{F}_q)$ (which i am not). The next four sections are concerned with the explicit calculations of the zèta functions for $n \leq 5$. Unfortunately (for me but not for mathematics) the obtained results are consistent with the Kac conjecture. Therefore, our calculations can be viewed as an extra indication for the rationality of $K_{2,5}$.In the final section we prove a mysterious fact namely that there is no cell in the (conjectural) cellular decomposition of codimension one. I am willing to treat the one who can explain me this fact to a ridiculous amount of pints of Belgian beer. Finally, some connections with other fields (such as the rationality problem

Further, Procesi has shown in [Pr1] that $K_{m,n}$ is rational if $K_{2,n}$ is rational which reduces the problem to two matrices. He also showed that $K_{2,2}$ is rational with trancendence basis $Tr(X_1), Tr(X_2), Tr(X_1X_2), Det(X_1), Det(X_2)$. Later, E. Formanek proved the rationality of $K_{2,3}$ and $K_{2,4}$ and gave a trancendence basis for them [Fo1],[Fo2]. However, the technical and computational difficulties in the 4 by 4 case are so tremendous that it seems rather unlikely to prove the rationality of $K_{2,5}$ 'by hand'.

D. Saltman has proved some weaker versions of the rationality problem. In [Sa1] he showed that $K_{m,p}$ is 'retract rational' for prime numbers p and in [Sa2] he proved that the 'unramified Brauer group' of $K_{m,n}$ is trivial. Saltman seems to make a habit of re-inventing definitions and terminology , so let us translate his second result in more established lingo. The variety of matrixinvariants $V_{m,n}$ has as its field of functions $K_{m,n}$. Its coordinate ring $\mathbb{C}[V_{m,n}]$ is the center of the trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$ which is the subring of $M_n(P_{m,n})$ generated by $\mathbb{G}_{m,n}$ and the traces of its elements. Its points correspond to isoclasses of semi-simple n-dimensional representations of the free algebra in m variables. As we will see below, the geometry of this variety is reasonably well understood [LP]. Saltman's result states that the Brauer group of the desingularization of $V_{m,n}$ is trivial. If $V_{m,n}$ is rational (i.e. $K_{m,n}$ rational) then this desingularization is birational to $(m-1)n^2+1$ dimensional projective space. Now, Grothendieck has shown that the Brauer group of projective space is trivial and that the Brauer group is a birational invariant among smooth projective varieties. So, Saltmans result can be viewed as a fairly strong indication for the rationality of $K_{m,n}$.

In this paper we would like to close in from the other direction, i.e. we would like to test some stronger conjectures. In the representation theory of wild quivers there is a very powerfull conjecture of Kac [Ka1] which would immediatly imply the rationality of $K_{m,n}$. He conjectures that the variety of isoclasses of indecomposable representations of a quiver of a fixed dimension vector admits a cellular decomposition in a finite union of locally closed subvarieties each isomorphic to an affine space (we will define all these terms lateron). In the special case of the bouquet- quiver S_m , i.e. the quiver with one vertex and m edge-loops , and dimension-vector n we would get that the unique cell of maximal dimension has an open set consisting of isoclasses of simple n-dimensional representations of the free for the moduli space of stable vectorbundles with Chern-numbers $(0, n)$ over $\mathbb{P}^2$) are mentioned with the promise to come back to them in a future publication.

2. Some quiver lingo

Although we have tried to minimalize the quiver terminology in this paper, some results (e.g. the étale local structure of the varieties $V_{m,n}$) must be expressed in this language. This short section contains the necessary definitions.

Throughout this section, $\mathbb{C}$ will be an arbitrary field. A quiver Q consists of a finite set $Q_0 = \{1, ..., n\}$ of vertices, a set Q_1 of arrows and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow ϕ its tail $t\phi$ and its head $h\phi$. One does not exclude loops nor multiple arrows.

A representation V of Q is a family $\{V(i) : i \in Q_0\}$ of finite dimensional $\mathbb{C}$-vectorspaces together with a family of linear maps $\{V(\phi) : V(t\phi) \to V(h\phi) \mid \phi \in Q_1\}$. The vector $dim(V) = (dim(V(i)))_i \in \mathbb{N}^n$ is called the dimension vector of V. A morphism $f : V \to W$ is a family of linear maps $\{f(i) : V(i) \to W(i) \mid i \in Q_0\}$ such that $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$ for all arrows $\phi \in Q_1$.

The representation space $R(Q, \alpha)$ of Q of the dimension vector $\alpha = (\alpha(1), ..., \alpha(n)) \in \mathbb{N}^n$ is the set of all representations

$$R(Q, \alpha) = \{V : V(i) = \mathbb{C}^{\alpha(i)}; 1 \leq i \leq n\}$$

Since $V \in R(Q, \alpha)$ is determined by the maps $V(\phi)$

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_1} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ is the set of all $\alpha(t\phi)$ by $\alpha(h\phi)$ matrices with entries in $\mathbb{C}$. We consider $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and functionfield $\mathbb{C}(Q, \alpha)$.

The linear reductive group $GL_\alpha(\mathbb{C}) = \prod_{i=1}^n GL_{\alpha(i)}(\mathbb{C})$ acts linearly on $R(Q, \alpha)$ by

$$(g.V)(\phi) = g(h\phi)^{-1} \circ V(\phi) \circ g(t\phi)$$

for $g \in GL_\alpha(\mathbb{C})$. By definition, the $GL_\alpha(\mathbb{C})$-orbits in $R(Q, \alpha)$ are just the isomorphism classes of representations.

The main aim of the representation theory of quivers is to describe the isoclasses of representations which by the Krull-Schmidt theorem reduces to the study of indecomposable representations (i.e.

representations which cannot be written as the direct sum of proper subrepresentations). Kac [Ka1] or [Ka2] has proved that the dimension vectors of indecomposable representations form a root system. Since we will not need this in the sequel, we do not go into this here. Finally, let us recall the definition of the Ringel bilinear form.Let r_{ij} be the number of arrows ϕ such that $t\phi = i$ and $h\phi = j$ and let $\alpha_i = (\delta_{1i}, ..., \delta_{ni})$. The Ringel bilinear form $R(-,-)$ on $\mathbb{Z}^n$ is then defined by

$$R(\alpha_i, \alpha_j) = \delta_{ij} - r_{ij}$$

The quiver Q is said to be wild if the symmetric bilinear form associated to R is indefinite.

In the sequel, we will be primarely interested in the bouquet-quivers S_m i.e. the quiver with one vertex and m edge loops.For a given dimension vector $n \in \mathbb{N}$ we get

$$R(S_m, n) = M_n(\mathbb{C}) \oplus ... \oplus M_n(\mathbb{C})$$

the vectorspace of m tuples of n by n matrices. The group acting on $R(S_m, n)$ is $GL_n(\mathbb{C})$ which acts by componentswise conjugation.Therefore, the classification problem is to study m tuples of n by n matrices under simultaneous conjugation.

A first aproximation to the description of all the orbits is to restrict attention to the closed orbits. In the next section we will recall some results in this direction.

3. The geometry of matrixinvariants

The results in this section are obtained jointly with C. Procesi. Proofs and more details can be found in [LP].

By Mumfords geometric invariant theory [Mu] we know that the closed orbits are in one-to-one correspondence with the points of the so called quotient variety $R(S_m, n)/GL_n(\mathbb{C})$ whose coordinate ring is by definition the fixed ring

$$\mathbb{C}[S_m, n]^{GL_n(\mathbb{C})} = \mathbb{C}[V_{m,n}]$$

by Procesi's result in [Pr2]. Moreover, Artin has shown that the closed orbits correspond to isoclasses of semi-simple representations (of course, a simple representation is one which has no proper subrepresentation).So, we can associate to each point $\xi \in V_{m,n}$ its representation type $\tau = (e_1, k_1; ...; e_r, k_r)$ if the corresponding semi-simple representation is built from r distinct simple representations of dimension vector k_i and occuring with multiplicity e_i. If τ is such a representation type we call $V_{m,n}(\tau)$ the subset of $V_{m,n}$ consisting of all points of representation type τ.

Usually, the varieties $V_{m,n}$ have lots of singular points. The next result shows that it is possible to cover $V_{m,n}$ with manageable smooth subvarieties :

Theorem 1 : (Le Bruyn-Procesi) The sets $V_{m,n}(\tau)$ form a finite stratification of $V_{m,n}$ into locally closed irreducible smooth algebraic subvarieties.

This result describes the global character of $V_{m,n}$. Next, we would like to know how $V_{m,n}$ looks like in a neighborhood of a point ξ. This can be done in the following way : let ξ be of representation type $\tau = (e_1, k_1; ..; e_r, k_r)$, then we will form a quiver Q^τ such that $Q_0^\tau = \{1, ..., r\}$ and $r_{ij} = \delta_{ij} + (m-1)k_i k_j$. Then, look at the dimension vector $\alpha^\tau = (e_1, ..., e_r)$ and consider the representation space $R(Q^\tau, \alpha^\tau)$. Denote by $V(Q^\tau, \alpha^\tau)$ the corresponding quotient variety $R(Q^\tau, \alpha^\tau)/GL_{\alpha^\tau}(\mathbb{C})$, i.e. the variety parametrizing isoclasses of semi-simple representations of Q^τ of dimension vector α^τ. As in the case of $V_{m,n}$, one can show that the coordinate ring of $V(Q^\tau, \alpha^\tau)$ is generated by traces of oriented cycles in the quiver Q^τ. We obtain

Theorem 2 : (Le Bruyn-Procesi) A neighborhood of ξ in $V_{m,n}$ is analytically isomorphic to a neighborhood of the origin in $V(Q^\tau, \alpha^\tau)$.

If the field $\mathbb{C}$ is not the field of complex numbers one can replace this analytic isomorphism by an étale mapping. This result reduces the local structure of $V_{m,n}$ to easier problems except for points of representation type $(n, 1)$.

4. The Kac conjecture

In this section we will describe the Kac conjecture on the structure of indecomposable representations. As promised, we will restrict to the case of m tuples of n by n matrices. A decomposable representation is an m-tuple $V = (V_1, ..., V_m)$ such that there exists an element $A \in GL_n(\mathbb{C})$ such that each $A^{-1}.V_i.A$ has the form

$$\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}$$

for some proper diagonal blocks $\star$ of fixed dimension for all i. For any $V \in R(S_m, n)$ we will denote by $M(V)$ the subalgebra of $M_n(\mathbb{C})$ generated by the m matrices $V_1, ..., V_m$. It is clear that the endomorphism ring of the representation V coincides with the centralizer of $M(V)$ in $M_n(\mathbb{C})$. It is fairly easy to verify that the representation V is indecomposable if and only if $End(V)$ is local, that is, the nilpotent endomorphisms form an ideal of codimension one.Equivalently, $\mathbb{C}^*$ is a maximal torus in $Aut(V)$ or every semi-simple element of $Aut(V)$ lies in $\mathbb{C}^*$.

We will now show that the set $I(S_m, n)$ consisting of all indecomposable representations in $R(S_m, n)$ is a constructible set, i.e. a finite union of locally closed sets , see [KR] for arbitrary quivers. By Chevalley's theorem [Kr,II.2.6] the function from $R(S_m, n)$ to $\mathbb{N}$ assigning to a representation V the dimension of $End(V)$ is upper semicontinuous so each of the sets

$$R(S_m, n)(d) = \{V \in R(S_m, n) \mid dim_{\mathbb{C}}(End(V)) = d\}$$

is locally closed (intersection of an open and a closed) in $R(S_m, n)$. Now, consider the closed subvariety N of $R(S_m, n) \oplus M_n(\mathbb{C})$ consisting of the couples (V, f) where f is a nilpotent endomorphism of V.If $\pi : N \rightarrow R(S_m, n)$ is projection onto the first factor then, again by Chevalley's theorem, the function $V \rightarrow dim(\pi^{-1}(V))$ is upper semicontinuous so the set

$$\{V \in R(S_m, n)(d) \mid dim(\pi^{-1}(V)) \geq d - 1\}$$

is closed in $R(S_m, n)(d)$, but this is also precisely the set of indecomposable representations in $R(S_m, n)(d)$ which we can call $I(S_m, n)(d)$. Finally, since

$$I(S_m, n) = \bigsqcup_{d \leq n^2} I(S_m, n)(d)$$

we get that $I(S_m, n)$ is constructible.

Next,we want to construct a variety whose points parametrize the isoclasses of indecomposable representations. For this we need a general remark about the action of a connected algebraic group G on an irreducible algebraic variety X. By a result of Rosenlicht, there exists a dense open subset U of X such that U is G-stable and all orbits in U are closed. Therefore, the quotient variety U/G parametrizes the G-orbits in U. To classify the remaining orbits one can repeat this procedure on the irreducible components of $X - U$ and so on.Finally, we get varieties Z_i such that the points of $\bigsqcup Z_i$ parametrize the G-orbits in X. Clearly, one can also perform this procedure for a constructible variety X. Therefore, we know that there must be a theoretical variety parametrizing isoclasses of indecomposable representations , which we will denote by $I^{iso}(S_m, n)$. Usually, such orbit-spaces are rather wild animals but Kac conjectures that they have a fairly nice structure in quiver-situations.

Kac conjecture for S_m : $I^{iso}(S_m, n)$ admits a cellular decomposition by locally closed subvarieties isomorphic to affine spaces.

In the special case of rank two quivers and dimension vectors $(n, 1)$ the orbit space of indecomposable representations is a Grassmann variety which has such a cellular decompositions. This example must have been in the back of Kac's mind when he dared to make this conjecture.

In a forthcoming publication with C.M. Ringel we will study the structure of semi-simple representations of arbitrary quivers and generalize the results of section three. One of the key results is the determination of all dimension vectors α for a given quiver Q such that there exists a simple representation of dimension α. This condition is that the support of α must have the property that any two vertices in it belong to an oriented cycle in it and that $R(\alpha, \alpha_i)$ and $R(\alpha_i, \alpha)$ are ≤ 0 for all i in the support (with the possible exception that the support is one oriented cycle, then, of course, the only possibility for α is $(1, 1, ..., 1)$).

An immediate consequence of the conjecture of Kac would be that the quotient variety of $R(Q, \alpha)$ would be rational for such a dimension vector α.

5. Cellular decomposition for S_1 and any n .

In this section we recall a result due to Dixmier-Kraft or Peterson showing that the orbitspace of $R(S_1, n)$ under $GL_n(\mathbb{C})$ admits a cellular decomposition in affine spaces. Note that the Kac-conjecture implies the existence of a cellular decomposition by affine spaces of the total orbitspace (by the Krull-Schmidt theorem).

The classification problem we have to solve is the description of conjugacy classes of n by n matrices. By the Jordan-normalform theorem we know that the conjugacy class of $A \in M_n(\mathbb{C})$ contains a matrix of the form

$$\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ & & \\ 0 & 0 & J_r \end{pmatrix}$$

where each of the diagonal blocks J_i has the form

$$\begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}$$

Moreover, this matrix in uniquely determined upto interchanging these blocks. The numbers λ_i are the roots of the characteristic polynomial of A and J_i is an m_i by m_i matrix if the multiplicity of the root λ_i is m_i.

Let us first consider the special case of 2 by 2 matrices. Then,

$$\{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{C}\} \bigcup \{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}\}$$

describe all conjugacy classes. Therefore, the orbit space $R^{iso}(S_1, 2)$ has a cellular decomposition as $\mathbb{C}^2 \bigsqcup \mathbb{C}^1$.

We will now derive this result in another manner which will be more helpfull to understand the next section. The quotient variety $R(S_1, 2)/GL_2(\mathbb{C})$ (i.e. the variety parametrizing the closed orbits) is $\mathbb{C}^2$ and the morphism $\pi : R(S_1, 2) \to \mathbb{C}^2$ is given by assigning to a matrix A the couple $(Tr(A), Det(A))$. Now, we have to count the number of orbits lying over a point $(x, y) \in \mathbb{C}^2$. The statification result of section 3 can be made more concrete in this situation as follows. Consider the parabola $P : X^2 - 4Y = 0$, points in $\mathbb{C}^2 - P$ correspond to matrices with distinct eigenvalues and hence are of representation correspond to matrices with distinct eigenvalues and hence are of representation type $(1,1;1,1)$ whereas those on P are of representation type $(2,1)$. Points in $\mathbb{C}^2 - P$ have as their π-fiber precisely one conjugacy class whereas those on P have a fiber with two conjugacy classes.So, the orbitspace has a decomposition into

$$((\mathbb{C}^2 - P) \bigcup P) \bigsqcup P$$

and because $P = \mathbb{C}^1$ we obtain the same result.

Let us go back to the general case. The multiplicities of the roots $(m_1, m_2, ..., m_r)$ (we may assume $m_1 \geq m_2 \geq ... \geq m_r$) form a partition of n , i.e. $\sum m_i = n$.Consider the conjugate of this partition $(n_1, ..., n_s)$ where n_i is the number of m_j such that $m_j \geq i$.

Conversely , for a partition λ of n we can consider $R(S_1, n)(\lambda)$ which is the set of all $A \in R(S_1, n)$ s.t. the conjugate partition of the rootmultiplicity-partition is equal to λ. It can be shown that $dim_{\mathbb{C}}(End(A))$ is constant on $R(S_1, n)(\lambda)$, so all orbits are closed in $R(S_1, n)(\lambda)$ (if B lies in the closure of the orbit of A , then $dim_{\mathbb{C}}(End(B)) > dim_{\mathbb{C}}(End(A))$ if B does not belong to the orbit itself). This dimension is equal to $d(\lambda) = \sum m_i^2$ where $\lambda' = (m_1, ..., m_r)$ is the root multiplicity partition (i.e. λ' is the conjugate of λ).Hence, the quotient variety of $R(S_1, n)(\lambda)$ is equal to its orbitspace. The main result can now be stated as follows, see [Kr2],[Pe] :

Theorem 3 : (Dixmier-Kraft-Peterson)

(1) : The connected components of $R(S_1, n)(d)$ are the subsets $R(S_1, n)(\lambda)$ such that $d(\lambda) = d$

(2) : Each $R(S_1, n)(\lambda)$ is a smooth algebraic subvariety of $R(S_1, n)$

(3) : The orbitspace $R(S_1, n)(\lambda)/GL_n(\mathbb{C})$ is in a natural way an affine space of dimension equal to the number of distinct roots

As an application, let us compute the number of cells in $R^{iso}(S_1, 7)$. We get the following table of partitions and cells :

(7)	$\mathbb{C}^7$
$(6,1)$	$\mathbb{C}^6$
$(5,2)$	$\mathbb{C}^5$
$(5,1,1)$	$\mathbb{C}^5$
$(4,3)$	$\mathbb{C}^4$
$(4,2,1)$	$\mathbb{C}^4$
$(4,1,1,1)$	$\mathbb{C}^4$
$(3,3,1)$	$\mathbb{C}^3$
$(3,2,2)$	$\mathbb{C}^3$
$(3,2,1,1)$	$\mathbb{C}^3$
$(3,1,1,1,1)$	$\mathbb{C}^3$
$(2,2,2,1)$	$\mathbb{C}^2$
$(2,2,1,1,1)$	$\mathbb{C}^2$
$(2,1,1,1,1,1)$	$\mathbb{C}^2$
$(1,1,1,1,1,1,1)$	$\mathbb{C}^1$

Therefore, $R^{iso}(S_1, 7)$ admits a cellular decomposition into

$$\mathbb{C}^7 \bigsqcup \mathbb{C}^6 \bigsqcup 2.\mathbb{C}^5 \bigsqcup 3.\mathbb{C}^4 \bigsqcup 4.\mathbb{C}^3 \bigsqcup 3.\mathbb{C}^2 \bigsqcup \mathbb{C}^1$$

6. Cellular decomposition for S_2 and $n = 2$

In this section we will show that the conjugacy classes of couples of 2 by 2 matrices admit a cellular decomposition. First note that the quotient variety $V_{2,2}$ is $\mathbb{C}^5$ and the quotient map $\pi : R(S_2, 2) \to \mathbb{C}^5$ is given by

$$(A, B) \to (Tr(A), Tr(B), Tr(AB), Det(A), Det(B))$$

First, we like to have an explicit description of the stratification result (theorem 1) in this situation. We have three different representation types $(1,2)$ (the simple representations) , $(1,1;1,1)$ and $(2,1)$.Couples op matrices (A, B) mapping to points in $\mathbb{C}^5$ of the last two representation types satisfy $(AB - BA)^2 = 0$. Translating this equation in the coordinates of $\mathbb{C}^5$ we obtain that the simple representations are precisely those points of $\mathbb{C}^5$ not lying on the hypersurface

$$H : x_3^2 - x_1x_2x_3 + x_1^2x_5 + x_2^2x_4 - 4x_4x_5 = 0$$

In this hypersurface we still have to distinguish between points of representation type $(1,1;1,1)$ and those of type $(2,1)$.The last ones have a representant-couple of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

So, they lie on the surface S in $\mathbb{C}^5$ of all points of the form $(2a, 2b, 2ab, a^2, b^2)$.

Next, we have to compute the number of distinct orbits in the fiber $\pi^{-1}(\xi)$. If ξ is a simple representation, the fiber is a single conjugacy class, i.e. $\pi^{-1}(\xi) \cong PGL_2(\mathbb{C})$. If ξ has representation type $(1,1;1,1)$ then $\pi^{-1}(\xi)$ consists of two conjugacy classes with representants say V_1 and V_2 such that

$$M(V_1) = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}; M(V_2) = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$$

Finally, if ξ is of type $(2,1)$ it is not difficult to verify that $\pi^{-1}(\xi)$ consists in the representant-couple given above and a one-parameter family of conjugacy classes parametrized by $\mathbb{P}^1$. If $y = (y_0, y_1) \in \mathbb{P}^1$ a representant has the form

$$\begin{pmatrix} a & y_0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & y_1 \\ 0 & b \end{pmatrix}$$

We can simplify the computations by the process of 'separating traces', i.e. we can replace $R(S_2, 2)$ by R' consisting of couples of trace zero matrices and $\mathbb{C}^5$ by $\mathbb{C}^3$ where the projection map $\pi' : R' \to \mathbb{C}^3$ is given by sending a couple (A^0, B^0) to $(Tr(A^0B^0), Det(A^0), Det(B^0))$. Then it is easy to see that $H = \mathbb{C}^2 \times Y$ where Y is the surface in $\mathbb{C}^3$ determined by the equation $y_1^2 - y_2y_3 = 0$ (the affine cone) and $S = \mathbb{C}^2 \times o$ where o is the origin of the cone.So it suffices to determine the cell decomposition of this simpler problem and after crossing with $\mathbb{C}^2$ we get our wanted decomposition.

Over a point of Y there are two conjugacyclasses of trace zero couples. The first one has as a representant

$$\begin{pmatrix} a & 1 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & 1 \\ 0 & -b \end{pmatrix}$$

i.e. an indecomposable representation and the second one is represented by

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$$

i.e. a decomposable one.Gluing the first type with the uniquely determined orbit over points in $\mathbb{C}^3 - Y$ gives a cell $\mathbb{C}^3$ of indecomposable representations. The second type gives a cell $\mathbb{C}^2$ of decomposable representations. The remaining orbits lie over the origin , they are parametrized by

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1-a \\ 0 & 0 \end{pmatrix}$$

thereby giving a cell $\mathbb{C}^1$.

Crossing with $\mathbb{C}^2$ we obtain the following result

Theorem 4 : The orbit space $R^{iso}(S_2, 2)$ admits a cellular decomposition into $\mathbb{C}^5 \bigsqcup \mathbb{C}^4 \bigsqcup \mathbb{C}^3$
The orbit space of the indecomposables $I^{iso}(S_2, 2)$ admits a cellular decomposition into $\mathbb{C}^5 \bigsqcup \mathbb{C}^3$

7. Implications on the zéta functions

In 1949, André Weil stated his famous conjectures concerning the number of solutions of polynomial equations over finite fields. These conjectures suggested a deep connection between the arithmetic of algebraic varieties defined over finite fields and the topology of algebraic varieties defined over the complex numbers.

In this section we will show what implications the Kac-conjecture has on the rational form of the ζ-function of certain varieties. We will not give too much details about the reduction steps from $\mathbb{C}$ to finite fields since they follow from some general yoga and the fact that affine space is a representable functor.

So, let us start with our algebraically closed field of characteristic zero $\mathbb{C}$ and suppose the Kac conjecture is true over it , then it is also true over the field $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$.But then it has to be true also over allmost all $\overline{\mathbb{F}_p}$ where $\mathbb{F}_p$ is the finite field on p elements.

In the preceeding sections we have seen that $I^{iso}(S_m, n)$ is a constructible variety over any algebraically closed basefield. Since there are only a finite number of equations involved in its definition, $I^{iso}(S_m, n)(\overline{\mathbb{F}_p})$ is defined over some finite field $\mathbb{F}_q$ where $q = p^s$, that is, there exists a variety V over $\mathbb{F}_q$ such that $V \otimes \overline{\mathbb{F}_p} \cong I^{iso}(S_m, n)(\overline{\mathbb{F}_p})$. Now, what is this variety V ? At first sight one might think that V is the orbit-space for the indecomposable representations over $\mathbb{F}_q$. This is allmost true, except that we have to replace 'indecomposable' by 'absolutely indecomposable'. Recall that an m tuple of n by n matrices is called absoltely indecomposable over $\mathbb{F}_q$ if it remains indecomposable over its algebraic closure $\overline{\mathbb{F}_p}$.We will denote this variety by

$$AI^{iso}(S_m, n)(\mathbb{F}_q)$$

Now, if the Kac conjecture is true then $AI^{iso}(S_m, n)(\mathbb{F}_q)$ needs to have a cellular decomposition into affine spaces. But then its number of points must satisfy the equation

$$\sharp(AI^{iso}(S_m, n)(\mathbb{F}_q)) = a_N q^N + ... + a_1 q + a_0$$

if there are a_i cells of dimension i. Of course, in this case $N = (m-1)n^2 + 1$ and one can show that

$a_N = 1$, [Ka2]. If one is interested in counting points on varieties over finite fields , tradition wants us to associate to this problem a zèta function. Let us abreviate $\sharp(AI^{iso}(S_m, n)(\mathbb{F}_q) = ai_{m,n}(q)$ and form the power series

$$\zeta_{m,n,q}(z) = exp(\sum_{k\geq 1} \frac{1}{k} a_{m,n}(q^k) z^k$$

which is an element in $\mathbb{Q}[[z]]$.If the Kac conjecture is true and if $ai_{m,n}(q)$ is given by the above equation, then this zèta-function has the following rational expression

$$\zeta_{m,n,q}(z) = \frac{1}{(1-q^N z)^{a_N} \ldots (1-qz)^{a_1}(1-z)^{a_0}}$$

This follows from the fact that if a variety is a disjoint union of locally closed subvarieties, then its zèta-function is the product of the zèta-functions of these subvarieties ([Ha,Ex 5.1,p 457]) and clearly the zèta-function of affine r-space is $(1-q^r z)^{-1}$.

In the rest of this paper we aim to show how one can compute these zèta-functions and that the obtained results (for small values of n) are consistent with the above form and hence can be viewed as extra evidence for the Kac-conjecture.

8. Conjugacy classes in $GL_n(\mathbb{F}_q)$

In order to compute the zèta function, we have to recall some results on the conjugacy classes in $GL_n(\mathbb{F}_q)$. More details can be found in [Mc].

Each $g \in GL_n(\mathbb{F}_q)$ acts on the vectorspace $\mathbb{F}_q^n$ and hence defines a $\mathbb{F}_q[t]$-module structure on $\mathbb{F}_q^n$ with the property that $t.v = gv$ for all $v \in \mathbb{F}_q^n$. We shall denote this module by V_g. Two such modules V_g and V_h are isomorphic if and only if g is conjugated to h. The conjugacy classes of $GL_n(\mathbb{F}_q)$ are thus in one-to-one correspondence with the isoclasses of $\mathbb{F}_q[t]$-modules V of dimension n such that $t.v = 0$ implies $v = 0$.

Because $\mathbb{F}_q[t]$ is a p.i.d. any finite dimensional module is of the form

$$V \cong \oplus \mathbb{F}_q[t]/(f_i)^{m_i}$$

for some $m_i \geq 1$ and irreducible polynomials f_i. Therefore, V determines a partition valued function

$$\mu : \Phi \rightarrow Par$$

from the set of all irreducible polynomials over $\mathbb{F}_q$ (with the exception of t) ,Φ, to the set of all partitions. If we denote $\mu(f) = (\mu_1(f), \mu_2(f), ...)$ then

$$V \cong \oplus_{f,i} \mathbb{F}_q[t]/(f)^{\mu_i(f)}$$

Of course, if $dim(V) = n$ then μ must satisfy

$$* :\mid \mu \mid = \sum_{f \in \Phi} deg(f)(\sum_i \mu_i(f)) = n$$

In this way we find that there is a one-to-one correspondence between conjugacy classes in $GL_n(\mathbb{F}_q)$ and functions μ satisfying $*$. We can make this a bit more explicit in the following way : for each $f = t^d - \sum_{i=1}^d a_i t^{i-1} \in \Phi$ we can form its companion matrix $J(f)$

$$\begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & ... \\ 0 & 0 & 0 & ... & 1 \\ a_1 & a_2 & a_3 & ... & a_d \end{pmatrix}$$

and for each integer $m \geq 1$ let us denote by $J_m(f)$

$$\begin{pmatrix} J(f) & I_d & 0 & \dots & 0 \\ 0 & J(f) & I_d & \dots & 0 \\ \dots & & & & \dots \\ 0 & 0 & 0 & \dots & J(f) \end{pmatrix}$$

with m diagonal blocks $J(f)$. Then the Jordan canonical form for elements of the conjugacy class associated to the function μ is the diagonal sum of the matrices $J_{\mu_i(f)}(f)$ for all $i \geq 1$ and $f \in \Phi$.

In order to compute the number of conjugacy classes we have to know how many irreducible polynomials there are over $\mathbb{F}_q$ of given degree d. For $d = 1$ this number is $q - 1$ since we excluded t from Φ. For $d > 1$ we have

$$\sharp\{f \in \Phi : deg(f) = d\} = \frac{1}{d} \sum_{j|d} M(j) q^{\frac{d}{j}}$$

where M is the classical Möbius-function i.e. $M(1) = 1$, $M(d) = (-1)^k$ if d is the product of k distinct prime numbers and $M(d) = 0$ if d has a multiple prime factor.

From the Jordan normal form given before it is not that difficult to deduce that the centralizer of an element in the conjugacy class associated to μ has order

$$a_\mu = \prod_{f \in \Phi} a_{\mu(f)}(q^{deg(f)})$$

where

$$a_\lambda(q) = q^{\sum (\lambda')_i^2} . \prod_i (1 - q^{-1}) \dots (1 - q^{-m_i(\lambda)})$$

if λ' is the conjugate partition of λ and $m_i(\lambda)$ is the multiplicity of the number i occuring in the partition λ. Finally, we recall that the order of the group $GL_n(\mathbb{F}_q)$ is equal to

$$q^{\frac{n(n-1)}{2}} (q - 1)(q^2 - 1) \dots (q^n - 1)$$

9. How to compute the zèta function ?

In this section we will outline the method to compute the rational form of the zèta-function for arbitrary values of m and n. In the next few sections we will give the details of the computations for $n \leq 5$.

If we denote by X/G the number of orbits of a group G acting on a variety X we will define

$$o_{m,n}(q) = \sharp(R(S_m, n)(\mathbb{F}_q)/GL_n(\mathbb{F}_q))$$

$$i_{m,n}(q) = \sharp(I(S_m, n)(\mathbb{F}_q)/GL_n(\mathbb{F}_q)) = \sharp(I^{iso}(S_m, n)(\mathbb{F}_q))$$

$$ai_{m,n}(q) = \sharp(AI(S_m, n)(\mathbb{F}_q)/GL_n(\mathbb{F}_q)) = \sharp(AI^{iso}(S_m, n)(\mathbb{F}_q))$$

Our first task will be to compute the orbit-number $o_{m,n}(q)$. A general result which is attributed to Burnside states that the number of orbits of a finite group G acting on a finite set X is equal to

$$\sharp(X/G) = \frac{1}{|G|} \sum_{g \in G} | X^g |$$

where X^g is the set of elements of X fixed by g. This expression can be rewritten as follows

$$\sharp(X/G) = \sum_{C(g)} \frac{| X^g |}{| C_G(g) |}$$

where the summation is taken over all conjugacy classes $C(g)$ in G and where $C_G(g)$ denotes the centralizer of the element g in G.

In this formula we can substitute the information on conjugacy classes in $GL_n(\mathbb{F}_q)$ of the foregoing section. If μ is a function from Φ to Par satisfying $*$ and if g_μ is a representant of the corresponding conjugacy-class (e.g. the Jordan normalform described before) then one can show that

$$dim_{\mathbb{F}_q}(M_n(\mathbb{F}_q)^{g_\mu}) = \sum_{f \in \Phi} deg(f).(\sum_i (\mu(f)')_i^2)$$

Therefore, we have all the necessary material at our disposal to compute the orbit number. We get

$$o_{m,n}(q) = \sum_{\mu} \frac{q^{(m-1)\sum_{f\in\Phi} deg(f)(\sum_i (\lambda')_i^2)}}{\prod_{f\in\Phi}\prod_i (1-q^{deg(f)})\ldots(1-q^{m_i(\lambda)deg(f)})}$$

The difficulty in computing this number is of course the vast number of possible functions μ. We will now indicate another method which reduces somewhat the number of calculations (and errors !).

Again, our starting point is the Burnside result, that is

$$o_{m,n}(q) = \frac{1}{\mid GL_n(\mathbb{F}_q) \mid} \sum_{g \in GL_n(\mathbb{F}_q)} \mid M_n(\mathbb{F}_q)^g \mid^m$$

For any function μ satisfying $*$ we will define

$$d(\mu) = \sum_{f \in \Phi} deg(f) (\sum_i (\mu(f)')_i^2)$$

It is clear that $d(\mu)$ has to be one of the numbers $\sum_i \lambda_i^2$ where $\lambda = (\lambda_1, \lambda_2, ...)$ is a partition of n.Then we can rewrite

$$o_{m,n} = \frac{1}{\mid GL_n(\mathbb{F}_q) \mid} (\alpha_{n^2}(q) q^{n^2 m} + ... + \alpha_n(q) q^{nm})$$

where $\alpha_i(q)$ is the number of elements of $GL_n(\mathbb{F}_q)$ whose conjugacy class is determined by a function μ satisfying $d(\mu) = i$.Notice that this number depends only on n. We can reformulate it as follows : $\alpha_i(q)$ is the number of elements of $GL_n(\mathbb{F}_q)$ such that its commutator ring in $M_n(\mathbb{F}_q)$ has dimension i. Of course, we can compute this number as follows

$$\alpha_i(q) = \sum_{\mu : d(\mu) = i} \frac{\mid GL_n(\mathbb{F}_q) \mid}{a_\mu}$$

But maybe there is a closed expression of $\alpha_i(q)$ known to people more familiar with the combinatorial aspects of $GL_n(\mathbb{F}_q)$ than i am. At any rate, the advantedge of this approach is that most functions μ have a low value for $d(\mu)$ and it is not necessary to compute $\alpha_n(q)$ and $\alpha_{n+2}(q)$ because they can be deduced from the following two equations

$$\mid GL_n(\mathbb{F}_q) \mid = \sum_n^{n^2} \alpha_i(q)$$

$$o_{1,n}(q) = \frac{1}{\mid GL_n(\mathbb{F}_q) \mid} \sum_n^{n^2} \alpha_i(q) q^i$$

and $o_{1,n}(q)$ can be readily computed from the results of section 5. That is

$$o_{1,n}(q) = \sum_{i=1}^{n} \sharp \{ \lambda : \sum \lambda_j = i \} q^i$$

where $\lambda = (\lambda_1, ...)$ runs trough all partitions of n.As we will see in the next sections, this reduces the number of computations drastically for small values of n.

Once we know the values of $o_{m,k}(q)$ for all $k \leq n$ we can calculate the number $i_{m,n}(q)$ by using the Krull-Schmidt theorem. Indeed, as in section 3 we can associate to a point in the orbitspace $R(S_m, n)/GL_n(\mathbb{F}_q)$ its decomposition-type $\tau = (e_1, k_1; ...; e_r, k_r)$ if it corresponds to a representation which is the direct sum of r distinct indecomposable representations of dimension vector k_i and occuring with multiplicity e_i. The number of points of decomposition-type τ can then be computed from the numbers $i_{m,k_i}(q)$ which we know by induction since $i_{m,1}(q) = o_{m,1}(q)$.

Finally, we have to pass from $i_{m,n}(q)$ to $ai_{m,n}(q)$.Now, every absolutely indecomposable representation has a minimal field of definition. Let us denote by $mai_{m,n}(q)$ the number of isoclasses of absolutely indecomposable representations in $R(S_m, n)$ with minimal field of definition $\mathbb{F}_q$. Then, clearly

$$(1) : ai_{m,n}(q) = \sum_{\mathbb{F}_{q'} \subseteq \mathbb{F}_q} mai_{m,n}(q')$$

Now, suppose we have an indecomposable representation V over $\mathbb{F}_q$ which is not absolutely indecomposable. Assume that $\mathbb{F}_{q^r}$ is its minimal splitting field, i.e. the minimal field extension such that $V \otimes \mathbb{F}_{q^r}$ is the direct sum of absolutely indecomposables. Let $G = Gal(\mathbb{F}_{q^r}, \mathbb{F}_q)$ which is a cyclic group, then G acts on all the representation spaces $R(S_m, k)(\mathbb{F}_{q^r})$ by letting G act on all the entries of the matrices. Then an easy Galois-descent argument shows that there exists an absolutely indecomposable representation $I \in R(S_m, \frac{n}{r})(\mathbb{F}_{q^r})$ such that

$$V \otimes \mathbb{F}_{q^r} \cong \oplus_{\sigma \in G} \sigma.I$$

Therefore, we have the following equality

$$i_{m,n}(q) = ai_{m,n}(q) + \sum_{r|n; r \neq 1} \frac{1}{r} mai_{m,\frac{n}{r}}(q^r)$$

Now, we can apply Möbius-inversion to (1) and substitute this in the above equation in order to get

$$i_{m,n}(q) = \sum_{d|n} \frac{1}{d} \sum_{e|d} M(e) ai_{m,\frac{n}{d}}(q^{\frac{d}{e}})$$

Therefore, we are able to compute $ai_{m,n}(q)$ (and hence the rational form of the zèta function) from $i_{m,n}(q)$ and $ai_{m,k}(q^l)$ for $k < n$.

10. The zèta function for 2 by 2 matrices

Let us first consider the trivial case of 1 by 1 matrices, then $GL_1(\mathbb{F}_q)$ acts trivially on $R(S_m,1)(\mathbb{F}_q)$ and therefore we get

$$o_{m,1}(q) = i_{m,1}(q) = ai_{m,1}(q) = q^m$$

ans so the zèta-function is just

$$\zeta_{m,1,q}(z) = \frac{1}{1 - q^m z}$$

Now, consider the case of 2 by 2 matrices. There are only two possible values for $d(\mu)$ namely 2 and 4. So,

$$o_{m,2}(q) = \frac{1}{q(q-1)(q^2-1)}(\alpha_4(q)q^{4m} + \alpha_2(q)q^{2m})$$

where the functions $\alpha_i(q)$ satisfy the equations

$$q(q-1)(q^2-1) = \alpha_2(q) + \alpha_4(q)$$

$$q(q-1)(q^2-1)o_{1,2}(q) = \alpha_4(q)q^4 + \alpha_2(q)q^2$$

Since $o_{1,2}(q) = q^2 + q$ we obtain that

$$\alpha_4(q) = q - 1$$

$$\alpha_2(q) = q^4 - q^3 - q^2 + 1$$

In this case it is still possible to rewrite the obtained formula for $o_{m,2}(q)$ in a polynomial form for any m

$$o_{m,2}(q) = (q^{4m-3} + q^{4m-5} + \dots + q^{2m+1} + q^{2m-1}) + q^{2m}$$

In order to compute $i_{m,2}(q)$ we note that there are three decomposition-types $(1,2)$, $(1,1;1,1)$ and $(2,1)$. Therefore,

$$i_{m,2}(q) = o_{m,2}(q) - \binom{i_{m,1}(q)}{2} - i_{m,1}(q)$$

and substituting the information obtained before we get

$$i_{m,2}(q) = (q^{4m-3} + q^{4m-5} + \dots + q^{2m+1} + q^{2m-1}) + \frac{1}{2}q^{2m} - \frac{1}{2}q^m$$

Finally, to compute $ai_{m,2}(q)$ we have to use the formula

$$ai_{m,2}(q) = i_{m,2}(q) - \frac{1}{2}ai_{m,1}(q^2) + \frac{1}{2}ai_{m,1}(q)$$

whence we obtain

$$ai_{m,2}(q) = q^{4m-3} + q^{4m-5} + \ldots + q^{2m+1} + q^{2m-1}$$

or, for the rational form of the zèta-function

$$\zeta_{m,2,q}(z) = \frac{1}{(1 - q^{4m-3}z)(1 - q^{4m-5}z)\ldots(1 - q^{2m-1}z)}$$

11. The zèta function for 3 by 3 matrices

In this case there are three possible values for $d(\mu)$ namely 9,5 and 3. There is only one type of function μ with $d(\mu) = 9$ namely sending precisely one irreducible polynomial of degree one to the partition $(1,1,1)$. There are $q-1$ such functions. Therefore

$$\alpha_9(q) = (q-1).\frac{q^3(q-1)(q^2-1)(q^3-1)}{q^9(1-q^{-1})(1-q^{-2})(1-q^{-3})} = q-1$$

$\alpha_5(q)$ and $\alpha_3(q)$ are then solutions to the following two equations

$$(q-1) + \alpha_5(q) + \alpha_3(q) = q^3(q-1)(q^2-1)(q^3-1)$$

$$(q-1)q^9 + \alpha_5(q)q^5 + \alpha_3 q^3 = q^3(q-1)(q^2-1)(q^3-1)o_{1,3}(q)$$

where $o_{1,3}(q) = q^3 + q^2 + q$. Solving these equations gives us

$$o_{m,3}(q) = \frac{1}{q^3(q-1)(q^2-1)(q^3-1)}(\alpha_9(q)t^{9m} + \alpha_5(q)t^{5m} + \alpha_3(q)t^{3q})$$

where we have

$$\alpha_9(q) = q-1$$

$$\alpha_5(q) = q^6 - q^5 - 2q^3 + q^2 + 1$$

$$\alpha_3(q) = q^9 - q^8 - q^7 - q^6 + 2q^5 + q^4 + q^3 - q^2 - q$$

Let us give a few concrete examples

$$o_{2,3}(q) = q^{10} + q^8 + 2q^7 + 2q^6 + 2q^5 + q^4$$

$$o_{3,3}(q) = q^{19} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 3q^{10} + 2q^9 + 2q^8 + q^7$$

In this case the different representation-types are

$(3,1)$
$(2,1;1,1)$
$(1,1;1,1;1,1)$
$(1,2;1,1)$
$(1,3)$

where the degenerations are given from bottom to top.Therefore

$$i_{m,3}(q) = o_{m,3}(q) - i_{m,2}(q)i_{m,1}(q) - \binom{i_{m,1}(q)}{3} - i_{m,1}(q)(i_{m,1}(q) - 1) - i_{m,1}(q)$$

Again, we will compute the first two examples

$$i_{2,3}(q) = q^{10} + q^8 + q^7 + \frac{4}{3}q^6 + q^5 + q^4 - \frac{1}{3}q^2$$

$$i_{3,3}(q) = q^{19} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + q^{12} + 2q^{11} + 2q^{10} + \frac{4}{3}q^9 + q^8 + q^7 - \frac{1}{3}q^3$$

Finally, in order to compute the number of absolutely indecomposable orbits we have to use the formula

$$ai_{m,3}(q) = i_{m,3}(q) - \frac{1}{3}ai_{m,1}(q^3) + \frac{1}{3}ai_{m,1}(q)$$

and this gives us in our examples

$$ai_{2,3}(q) = q^{10} + q^8 + q^7 + q^6 + q^5 + q^4$$

$$ai_{3,3}(q) = q^{19} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + q^{12} + 2q^{11} + 2q^{10} + q^9 + q^8 + q^7$$

consistent with the Kac-conjecture.

12. The zèta function for 4 by 4 matrices

Here $\delta(\mu)$ can take the values 16, 10, 8, 6 and 4. As in the previous section it is easy to show that $\alpha_{16}(q) = q-1$ so we have to calculate $\alpha_{10}(q)$ and α_8.There are precisely two types of functions μ with $d(\mu) = 10$ namely

$$\begin{array}{lccc} (I): & P_1 & \rightarrow & (2,1,1) \\ (II): & P_1 & \rightarrow & (1,1,1) \\ & Q_1 & \rightarrow & (1) \end{array}$$

where the subscripts give the degrees of the irreducible polynomials.There are $q-1$ functions of type (I) and $(q-1)(q-2)$ of type (II).Therefore,$\alpha_{10}(q)$ is equal to

$$(q-1).\frac{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}{q^{10}(1-q^{-1})^2(1-q^{-2})} + (q-1)(q-2).\frac{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}{q^{10}(1-q^{-1})^2(1-q^{-2})(1-q^{-3})}$$

There are three types of functions μ with $d(\mu) = 8$:

$$\begin{array}{lccc} (I): & P_1 & \rightarrow & (2,2) \\ (II): & P_2 & \rightarrow & (1,1) \\ (III): & P_1 & \rightarrow & (1,1) \\ & Q_1 & \rightarrow & (1,1) \end{array}$$

where subscripts indicate the degree of the polynomials. Therefore, there are $q-1$ functions of type (I) , $\frac{q(q-1)}{2}$ functions of type (II) and $\frac{(q-1)(q-2)}{2}$ of type (III). This enables us to compute $\alpha_8(q)$ which is equal to

$$(q-1).\frac{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}{q^8(1-q^{-1})(1-q^{-2})} + \frac{q(q-1)}{2}.\frac{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}{q^8(1-q^{-2})(1-q^{-4})}$$
$$+\frac{(q-1)(q-2)}{2}.\frac{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}{q^8(1-q^{-1})^2(1-q^{-2})^2}$$

As before, the functions $\alpha_6(q)$ and $\alpha_4(q)$ are then the solutions to the following set of equations

$$\sum \alpha_i(q) = q^6(q-1)(q^2-1)(q^3-1)(q^4-1)$$

$$\sum \alpha_i(q)q^i = q^6(q-1)(q^2-1)(q^3-1)(q^4-1)o_{1,4}(q)$$

where $o_{1,4}(q) = q^4+q^3+2q^2+q$. Solving these equations we get that $o_{m,4}(q)$ is eqaul to

$$\frac{1}{q^6(q-1)(q^2-1)(q^3-1)(q^4-1)}(\alpha_{16}(q)q^{16m} + \alpha_{10}(q)q^{10m} + \alpha_8(q)q^{8m} + \alpha_6(q)q^{6m} + \alpha_4(q)q^{4m})$$

where

$$\alpha_{16}(q) = q - 1$$

$$\alpha_{10}(q) = q^8 - q^7 - 2q^4 + q^3 + 1$$

$$\alpha_8(q) = q^{10} - q^9 - 2q^7 + q^6 - q^5 + 2q^4 + q^2 - q$$

$$\alpha_6(q) = q^{13} - q^{12} - 3q^{10} + q^9 + q^8 + 4q^7 + 2q^6 - 2q^4 - 2q^3 - q^2$$

$$\alpha_4(q) = q^{16} - q^{15} - q^{14} - q^{13} + q^{12} + 2q^{11} + 2q^{10} - 3q^8 - 2q^7 - 2q^6 + q^5 + 2q^4 + q^3$$

Let us give a few examples :

$$o_{2,4}(q) = q^{17} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 6q^9$$

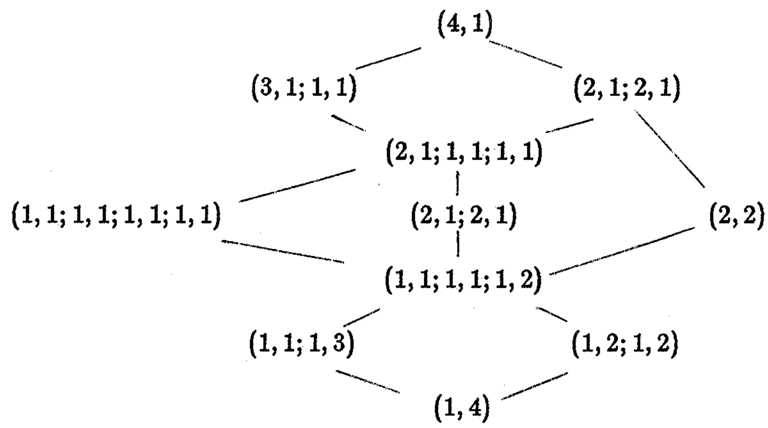
$$+5q^8 + 5q^7 + 3q^6 + q^5$$

$$o_{3,4}(q) = q^{33} + q^{31} + q^{30} + 2q^{29} + q^{28} + 3q^{27} + 2q^{26} + 4q^{25}$$

$$+3q^{24} + 5q^{23} + 5q^{22} + 7q^{21} + 6q^{20} + 9q^{19} + 8q^{18} + 10q^{17}$$

$$+9q^{16} + 10q^{15} + 9q^{14} + 9q^{13} + 6q^{12} + 5q^{11} + 3q^{10} + q^9$$

In this case the different representation-types are



where we have indicated the possible degenerations.The number of indecomposable orbits is therefore given by the formula

$$i_{m,4}(q) = o_{m,4}(q) - i_{m,1}(q)i_{m,3}(q) - \binom{i_{m,2}(q)}{2} - i_{m,2}(q)\binom{i_{m,1}(q)}{2}$$

$$-\binom{i_{m,1}(q)}{4} - \binom{i_{m,1}(q)}{2} - i_{m,2}(q) - 3\binom{i_{m,1}(q)}{3}$$

$$-i_{m,1}(q)(i_{m,1}(q)-1) - \binom{i_{m,1}(q)}{2} - i_{m,1}(q)$$

Let us compute the easiest example

$$i_{2,4} = q^{17} + q^{15} + q^{14} + 2q^{13} + q^{12} + 3q^{11} + \frac{5}{2}q^{10} + 4q^9$$

$$+\frac{27}{12}q^8 + 3q^7 + \frac{3}{2}q^6 + \frac{1}{2}q^5 - \frac{1}{4}q^4 - \frac{1}{2}q^3$$

Finally, the number of absolutely indecomposable orbits can be computed using the formula

$$ai_{m,4}(q) = i_{m,4}(q) - \frac{1}{2}ai_{m,2}(q^2) + \frac{1}{2}ai_{m,2}(q) - \frac{1}{4}ai_{m,1}(q^4) + \frac{1}{4}ai_{m,1}(q^2)$$

and for the easiest example we get

$$ai_{2,4}(q) = q^{17} + q^{15} + q^{14} + 3q^{13} + q^{12} + 3q^{11} + 2q^{10} + 4q^9$$

$$+2q^8 + 3q^7 + q^6 + q^5$$

consistent with the Kac conjecture.

13. The zèta function for 5 by 5 matrices

In this case $\delta(\mu)$ can take the values 25,17,13, 11,9,7 and 5. Of course, $\alpha_{25}(q) = q - 1$. There are two types of functions with $d(\mu) = 17$

$$\begin{array}{rcll} (I): & 1 & \rightarrow & (2,1,1,1) \\ (II): & 1 & \rightarrow & (1,1,1,1) \\ & 1 & \rightarrow & (1) \end{array}$$

which enables us to compute that

$$\alpha_{17}(q) = q^{10} - q^9 - 2q^5 + q^4 + 1$$

Likewise, there are two types of functions μ with $d(\mu) = 13$

$$\begin{array}{rcll} (I): & 1 & \rightarrow & (2,2,1) \\ (II): & 1 & \rightarrow & (1,1,1) \\ & 1 & \rightarrow & (1,1) \end{array}$$

and one can compute that

$$\alpha_{13}(q) = q^{14} - q^{13} + q^{12} - 2q^{11} - q^{10} - 2q^9 + q^8 - q^7 + 3q^6 + q^5 + q^4 - q$$

There are five types of functions with $d(\mu) = 11$,namely

$$\begin{array}{rcll} (I): & 1 & \rightarrow & (3,1,1) \\ (II): & 1 & \rightarrow & (1,1,1) \\ & 1 & \rightarrow & (2) \\ (III): & 1 & \rightarrow & (1,1,1) \\ & 1 & \rightarrow & (1) \\ & 1 & \rightarrow & (1) \\ (IV): & 1 & \rightarrow & (1,1,1) \\ & 2 & \rightarrow & (1) \\ (V): & 1 & \rightarrow & (2,1,1) \\ & 1 & \rightarrow & (1) \end{array}$$

which allows us, at the cost of a headache, to compute

$$\alpha_{11}(q) = q^{18} - q^{17} - q^{15} - q^{14} - q^{13} + 3q^{12} + q^{11} + 3q^{10} + 2q^9 - 2q^7 - q^6 - 2q^5 - q^4$$

Finally, there are five types of functions with $d(\mu) = 9$

$$\begin{array}{llcl}
(I): & 1 & \rightarrow & (3,2) \\
\\
(II): & 1 & \rightarrow & (1,1) \\
 & 1 & \rightarrow & (1,1) \\
 & 1 & \rightarrow & (1) \\
\\
(III): & 1 & \rightarrow & (2,1) \\
 & 1 & \rightarrow & (1,1) \\
\\
(IV): & 1 & \rightarrow & (2,2) \\
 & 1 & \rightarrow & (1) \\
\\
(V): & 2 & \rightarrow & (1,1) \\
 & 1 & \rightarrow & (1)
\end{array}$$

which gives us that $\alpha_9(q)$ is equal to

$$q^{19} - q^{18} - 3q^{16} + q^{15} - 2q^{14} + 6q^{13} + 2q^{12} + 7q^{11} - 2q^{10} + q^9 - 6q^8 - 2q^7 - 4q^6 + q^5 + q^3$$

Again, the remaining polynomials $\alpha_7(q)$ and $\alpha_5(q)$ can be obtained as the solutions to the standard set of two equations where we have to use the fact that $o_{1,5}(q) = q^5 + q^4 + 2q^3 + 2q^2 + q$. These calculations lead to the result that

$$\alpha_7(q) = q^{22} - q^{21} - 3q^{19} + q^{18} + 7q^{16} + q^{15} + 2q^{14} - 7q^{13} - 7q^{12}$$

$$-10q^{11} - 2q^{10} + q^9 + 6q^8 + 6q^7 + 4q^6 + q^5$$

and

$$\alpha_5(q) = q^{25} - q^{24} - q^{23} - q^{22} + q^{21} + q^{20} + 3q^{19} + q^{18} - 2q^{17} - 4q^{16} - 3q^{15}$$

$$+3q^{13} + 6q^{12} + 3q^{11} + 2q^{10} - 2q^9 - 3q^8 - 3q^7 - q^6$$

Combining these computations with the fact that

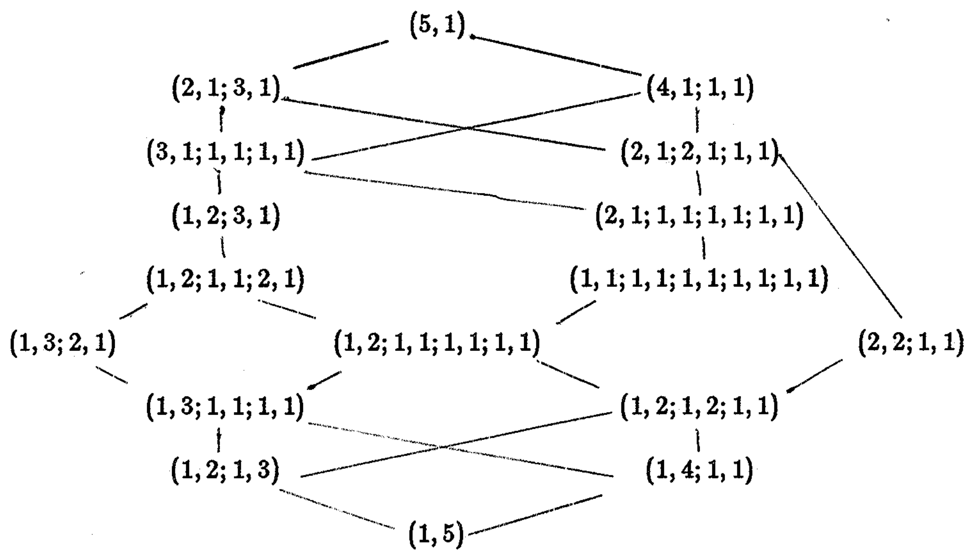
$$o_{m,5}(q) = \frac{1}{q^{10}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)} \cdot \sum_i \alpha_i(q) q^{mi}$$

we are able to calculate the number of orbits.For example

$$o_{2,5}(q) = q^{26} + q^{24} + q^{23} + 2q^{22} + 2q^{21} + 3q^{20} + 4q^{19} + 5q^{18} + 6q^{17} + 8q^{16} + 10q^{15}$$

$$+11q^{14}+14q^{13}+15q^{12}+17q^{11}+15q^{10}+13q^{9}+8q^{8}+4q^{7}+q^{6}$$

The different representation-types and their degenerations are given in the following diagram



which enables us to deduce the formula for $i_{m,5}(q)$ as in the foregoing sections. In the special case under consideration we get that

$$i_{2,5}(q)=q^{26}+q^{24}+q^{23}+2q^{22}+2q^{21}+3q^{20}+3q^{19}+5q^{18}+5q^{17}+7q^{16}+7q^{15}$$

$$+9q^{14}+9q^{13}+10q^{12}+9q^{11}+\frac{41}{5}q^{10}+6q^{9}+4q^{8}+2q^{7}+q^{6}-\frac{1}{5}q^{2}$$

Finally, the number of absolutely indecomposable orbits is

$$ai_{m,5}(q)=i_{m,5}(q)-\frac{1}{5}ai_{m,1}(q^{5})+\frac{1}{5}ai_{m,1}(q)$$

In our special case we get therefore

$$ai_{2,5}(q)=q^{26}+q^{24}+q^{23}+2q^{22}+2q^{21}+3q^{20}+3q^{19}+5q^{18}+5q^{17}+7q^{16}+7q^{15}$$

$$+9q^{14}+9q^{13}+10q^{12}+9q^{11}+8q^{10}+6q^{9}+4q^{8}+2q^{7}+q^{6}$$

consistent with the Kac-conjecture. These calculations can therefore be viewed as an extra indication for the rationality of $\mathcal{K}_{m,5}$.

14. The missing cell mystery and other problems

It can be seen from the calculations made in the foregoing sections that the coefficient of $q^{(m-1)n^2+1}$ in $ai_{m,n}(q)$ is always zero for $m \geq 2$. If the Kac- conjecture is true (i.e. if there is a cellular decomposition by affine spaces) this means that there is no cell in codimension one.

It is almost trivial to prove that this coefficient is zero for all $n \geq 2$. One uses

$$o_{1,n}(q). \mid GL_n(\mathbb{F}_q) \mid = \sum \alpha_i(q) q^i$$

to get a bound on the degrees of the polynomials $\alpha_i(q)$ and the fact that $\alpha_{n^2}(q) = q-1$ to obtain that for $m \geq 2$ the degree of $(q-1)q^{mn^2}$ is greater than the degrees of the remaining terms in $\sum_i \alpha_i(q) q^{mi}$. Then we divide first by $q-1$ and then by the remaining terms in $\mid GL_n(\mathbb{F}_q) \mid$ and obtain that the coefficient of $q^{(m-1)n^2+1}$ is always zero in $o_{m,n}(q)$ (with the exception $n = m = 2$).

Unlike its proof, it is fairly difficult to understand this fact. So, we ask

Problem 2 : What is the reason for the non-existence of a cell in codimension one ?

Of course, one could check the Kac conjecture for other quivers than the bouquet quiver S_m. Of particular interest to us may be the rank two quivers P_m , i.e. the quiver with two vertices and m edges between them (e.g. all with the same orientation). The orbit-space of the representation space with dimension vector (n,n) is birational to the center of the trace ring of m generic n by n matrices. The main advantedge of this other approach is that we are in the setting of projective varieties and hence we can for example use the Deligne theorems (former Weil conjectures) to compute the Betti numbers of the projective variety

$$I(P_m;(n,n))(\mathbb{C})^{ss}/(GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))$$

where the superscript ss denotes the set of semi-stable points in the sense of Mumfords G.I.T. This calculation will be carried out by the author in a future publication.

The special case when $m = 3$ is of interest to vector bundle adicts.For the orbitspace of indecompasable representations of P_3 of dimension vector (n,n) is birational to the moduli space of stable vectorbundles over the projective plane $\mathbb{P}^2$ with Chern numbers (o, n). So, problem 1 on the rationality of $K_{m,n}$ is equivalent to the rationality problem for these moduli spaces.

It would be interesting to make all these connections between trace rings of generic matrices, representations of P_m and vector bundles over $\mathbb{P}^2$ as explicit as possible.

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