THE REGULARITY PROBLEM FOR TRACE RINGS OF GENERIC MATRICES

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0. Introduction

Throughout, k will be an algebraically closed field of characteristic zero.Consider the polynomial ring

$$\mathcal{R} = k[x_{ij}(l); 1 \leq i,j \leq n; 1 \leq l \leq m]$$

and define the so called n by n generic matrices to be

$$X_l = (x_{ij}(l))_{i,j} \in M_n(\mathcal{R})$$

The ring of m generic n by n matrices, $\mathcal{G}_{m,n}$ is the subalgebra of $M_n(\mathcal{R})$ generated by the set

$$\{X_i; 1 \leq i \leq m\}$$

With $\mathcal{R}_{m,n}$ we will denote the k-subalgebra of $\mathcal{R}$ generated by the traces of elements from $\mathcal{G}_{m,n}$. The composite of $\mathcal{G}_{m,n}$ and $\mathcal{R}_{m,n}$ in $\mathcal{R}$ is called the trace ring of m generic n by n matrices and will be denoted by $\mathcal{T}_{m,n}$.

If $m \geq 2$ it is fairly easy to verify that $\mathcal{R}_{m,n}$ is the center of $\mathcal{T}_{m,n}$.One of the main motivations for studying the rings $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ is that they are respectively the ring of matrixinvariants and matrixconcomitants and of basic importance in the study of the classical problem of classifying m tuples of n by n matrices under simultaneous conjugation.

Of course, if $m = 1$ or $n = 1$ then $\mathcal{R}_{m,n} = \mathcal{T}_{m,n}$ and are commutative polynomial rings, so in particular they have finite global dimension (or equivalently, they are regular rings). Therefore it is natural to ask the following question :

Determine all couples (m,n) for which $\mathcal{T}_{m,n}$ (resp. $\mathcal{R}_{m,n}$) has finite global dimension

This is the regularity problem for trace rings of generic matrices. Small and Stafford proved that $\mathcal{T}_{2,2}$ and $\mathcal{R}_{2,2}$ have global dimension 5. Afterwards, I settled the regularity problem for 2 by 2 matrices in [LEBRUYN 1] using some important results of C.Procesi using the invariant theory of the (special) orthogonal group of dimension 3. Later, jointly with M. Van den Bergh we have shown in [LEBR-VDBERGH] that $\mathcal{T}_{2,3}$ has global dimension 10 and that for $n \leq 4$ there are no other trace rings which have finite global dimension.

However, the general case remained open and seemed to be extremely difficult. The reason for this is that the usual test for regularity of Noetherian rings depends on the rational form of its Poincaré series. In [FORMANEK] E. Formanek has described a method to compute the power series expansion of the Poincaré series but this method is very time-consuming. For $n = 3$ or $n = 4$ and $m = 2$ we were able to compute the first terms of this series with the aid of a computer, but already for $n = 5$ this is a virtually impossible task.

Finally, the general case was solved jointly with C. Procesi in [LEBR-PROCESI] using some powerfull invariant-theoretic results due to D. Luna (the so called étale slice results) and the description of invariants of tori mainly due to R. Stanley. The complete solution to the regularity problem is :

THEOREM : The trace ring of m generic n by n matrices has finite global dimension if and only if m or n is equal to one or $(m,n) = (2,2),(2,3)$ or $(3,2)$.

Its center $\mathcal{R}_{m,n}$ has finite global dimension if and only if m or n is equal to one or $(m,n) = (2,2)$.

The proof of the solution to the regularity problem of 2 by 2 matrices is well documented in [LEBRUYN1] and [LEBRUYN2], whereas the general case is proved in [LEBR-PROCESI]. To illustrate the basic methods as well as the computational difficulties, we restrict ourselves in this paper to the special case of 3 by 3 matrices.

1. A general strategy .

In this section we will outline an algorithm to find (at least in principle) the rational expression of the Poincaré series for the trace ring of m generic $n \times n$-matrices, $\mathcal{T}_{m,n}$, and for its center $\mathcal{R}_{m,n}$. This method also enables us to test trace rings of generic matrices for regularity (i.e. having finite global dimension).

1.1. Formanek's description of the Poincaré series.

It is known that $\mathcal{R}_{m,n}$ (resp. $\mathcal{T}_{m,n}$) are fixed rings of an action of $GL_n(k)$ on $\mathcal{R}$ (resp. on $M_n(\mathcal{R})$), where $\mathcal{R} = k[t_{ij}(l); 1 \leq i,j \leq n; 1 \leq l \leq m]$. Using this fact, Formanek applied the general theory developed by H. Weyl and I. Schur to obtain formulas for the Poincaré series

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) \text{ and } \mathcal{P}(\mathcal{T}_{m,n}; y_1, ..., y_m)$$

of the center (resp. trace ring) in a multigradation, i.e. by giving each indeterminate $t_{ij}(l) \in \mathcal{R}$ the degree $(0, ..., 0, 1, 0, ..., 0)$ with 1 on spot l. To describe the results of [FORMANEK] we must recall first some basic definitions and results on the ring of symmetric functions.

A degree sequence of lenght n is a sequence

$$\alpha = (\alpha_1, ..., \alpha_n)$$

of non-negative integers. The total degree of α is

$$| \alpha |= \alpha_1 + ... + \alpha_n$$

A partition of lenght $\leq n$ is a degree sequence $\lambda = (\lambda_1, ..., \lambda_n)$ satisfying

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$$

For any partition $\lambda = (\lambda_1, ..., \lambda_n)$ of lenght $\leq n$ we define the element in $\mathbb{Z}[x_1, ..., x_n]$

$$a_\lambda = a_\lambda(x_1, ..., x_n) = \sum_{\pi \in S_n} (sign(\pi)) x_{\pi(1)}^{\lambda_1} ... x_{\pi(n)}^{\lambda_n}$$

where S_n is the group of all permutations on n letters.

In the special case that

$$\delta = (n-1, n-2, ..., 1, 0)$$

we get

$$a_\delta = \prod_{i<j} (x_i - x_j)$$

In $\mathbb{Z}[x_1, ..., x_n]$, $a_{\delta+\lambda}$ is divisible by a_δ and the quotient $s_\lambda = a_{\delta+\lambda}/a_\delta$ is invariant under the natural action of S_n on $\mathbb{Z}[x_1, ..., x_n]$, i.e. by permuting the indeterminates. $s_\lambda = s_\lambda(x_1, ..., x_n)$ is said to be the Schur function associated to the partition of lenght $\leq n$, λ.

The set

$$\{s_\lambda \mid \text{ a partition of lenght } \leq n\}$$

forms a $\mathbb{Z}$-basis for Λ_n , the ring of symmetric functions in n variables i.e. the ring of invariants of $\mathbb{Z}[x_1, ..., x_n]$ under action of S_n.

One can define an inner product $<,>$ on Λ_n such that the s_λ form an ortonormal basis and it can be extended to

$$\Gamma_n = \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]^{S_n}$$

Formanek [FORMANEK] defines this inproduct intrinsically in the following way. Let

$$(-)^* : \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \to \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$$

be the involution defined by $(x_i)^* = x_i^{-1}$. Now, define the linear functional

$$\int : \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \to \mathbb{Z}$$

by $\int 1 = 1$ and $\int x_1^{\alpha_1}...x_n^{\alpha_n} = 0$ if $\alpha_1, ..., \alpha_n$ are not all zero. For any $a, b \in \Gamma_n$ the inproduct is defined to be

$$< a, b > = \frac{1}{n!} \int a.(b)^* .a_\delta.(a_\delta)^*$$

Now consider the finite dimensional $GL_n(k) \times GL_m(k)$- module $M_n(k) \otimes U_m$ where U_m is the standard $GL_m(k)$-module of dimension m and the action of $GL_n(k)$ on $M_n(k)$ is given by conjugation. Then it is clear that $M_n(k) \otimes U_m$ is rational as a $GL_n(k)$-module and polynomial as a $GL_m(k)$- module.

This action of $GL_n(k) \times GL_m(k)$ extends in the natural way to the symmetric algebra of $M_n(k) \otimes U_m$ which is just

$$\mathcal{R} = k[t_{ij}(l); 1 \leq i, j \leq n; 1 \leq l \leq m]$$

By giving each of the generators $t_{ij}(l)$ degree one, $\mathcal{R}$ is a positively graded k-algebra

$$\mathcal{R} = k \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus ...$$

where each homogeneous part $\mathcal{R}_i$ is a finite dimensional $GL_n(k) \times GL_m(k)$-module , rational in the first factor and polynomial in the second.Therefore the Poincaré series of $\mathcal{R}$ as a $GL_n(k) \times GL_m(k)$-module is

$$\mathcal{P}(\mathcal{R}; x_i, x_i^{-1}, y_j) = 1 + \chi(\mathcal{R}_1) + \chi(\mathcal{R}_2) + ...$$

where χ is the isomorphism between the Grothendieck ring $Mod(n, m)$ of all finite dimensional $GL_n(k) \times GL_m(k)$-modules which are rational in the first factor and polynomial in the second, and

$$\mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}, y_1, ..., y_m]^{S_n \times S_m}$$

see for example [FORMANEK,lemma 11].

Therefore, $\mathcal{P}(\mathcal{R})$ is a formal power series over $\Gamma_n \otimes \Lambda_m$. It is fairly easy to see that $\chi(\mathcal{R}_i)$ is the i-th complete symmetric function of

$$\{x_i.x_j^{-1}.y_k \mid 1 \leq i, j \leq n; 1 \leq k \leq m\}$$

i.e. the coefficient of t^i in the power series expansion of

$$\prod_{i,j,k}(1 - x_i.x_j^{-1}.y_k.t)^{-1}$$

Further, we have

$$\prod(1 - x_i.x_j^{-1}.y_k)^{-1} = \sum_\lambda s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

where the sum is taken over all partitions λ of lenght $\leq min(n,m)$. Therefore

$$\mathcal{P}(\mathcal{R}) = \sum_\lambda s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

$$\chi(\mathcal{R}_i) = \sum_{|\lambda|=1} s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

Here, $s_\lambda(x_i.x_j^{-1})$ is the image of $s_\lambda(z_{ij}; 1 \leq i,j \leq n)$ under the homomorphism

$$\mathbb{Z}[z_{ij}; 1 \leq i,j \leq n] \to \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$$

sending z_{ij} to $x_i.x_j^{-1}$.

Now, for any $GL_n(k) \times GL_m(k)$-module of the form $N \otimes M$ where N is a rational $GL_n(k)$-module and M is a polynomial $GL_m(k)$-module we have :

$$\chi((N \otimes M)^{GL_n(k)}) = \chi(N^{GL_n(k)} \otimes M) = < \chi_n(N), 1 > .\chi_m(M)$$

where χ_n is the isomorphism between the Grothendieck ring of all finite dimensional rational $GL_n(k)$-modules and Γ_n and χ_m the natural isomorphisms between the Grothendieck ring of all finite dimensional polynomial $GL_m(k)$-modules and Λ_m and the inproduct $<,>$ is taken in $\mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]^{S_n}$.

Theorem 1 : [FORMANEK,Theorem 12]

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) = \sum_\lambda < s_\lambda(x_i.x_j^{-1}), 1 > .s_\lambda(y_l)$$

where λ varies over all partitions of lenght $\leq min(n,m)$.

Similarly, Formanek computes the Poincaré series of the trace ring $\mathcal{T}_{m,n}$ as fixed ring of $M_n(\mathcal{R})$ under action of $GL_n(k)$.

Theorem 2 : [FORMANEK,Theorem 12]

$$\mathcal{P}(\mathcal{T}_{m,n}; y_1, ..., y_m) = \sum_{\lambda} < s_\lambda(x_i.x_j^{-1}), s_{(1)}(x_i.x_j^{-1}) > .s_\lambda(y_k)$$

where λ varies over all partitions of lenght $\leq min(n,m)$ and $(1) = (1, ..., 1)$, i.e. $s_{(1)}(x_i.x_j^{-1}) = \sum_{i,j} x_i.x_j^{-1}$.

1.2. Towards a rational expression.

It is known [PROCESI] that $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ are both affine algebras over k and that $\mathcal{T}_{m,n}$ is a finite module over $\mathcal{R}_{m,n}$.

Therefore, it follows from the Hilbert-Serre theorem that the Poincaré series of $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ are rational, i.e. there exist polynomials $f, g, h, j \in \mathbb{Z}[y_1, ..., y_m]$ such that

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) = \frac{f(y_1, ..., y_m)}{g(y_1, ..., y_m)}$$

$$\mathcal{P}(\mathcal{T}_{m,n}; y_1, ..., y_m) = \frac{h(y_1, ..., y_m)}{j(y_1, ..., y_m)}$$

The main problem in determining the rational expressions is to find out which polynomials can occur in the numerator.

In the special case that $n = 2$, this is easy because there exists an epimorphism

$$\pi_m : \Gamma_m \to \mathcal{T}_{m,2}$$

where Γ_m is the iterated Öre-extension

$$\Gamma_m = k[a_{ij}; 1 \leq i, j \leq m][a_1][a_2, \sigma_2, \delta_2]...[a_m, \sigma_m, \delta_m][b_1, ..., b_m]$$

where $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = a_{ij}$ for all $i < j$ and on all other variables both σ_j and δ_j act trivially.

The epimorphism π_m is defined by sending a_i to $X_i - Tr(X_i)$, b_i to $Tr(X_i)$ and a_{ij} to $Tr((X_i - Tr(X_i))(X_j - Tr(X_j))$, cfr. [LE BRUYN] for more details.

The Poincaré series of Γ_m is easy to determine

$$\mathcal{P}(\Gamma_m; y_1, ..., y_m) = \frac{1}{\prod_{i<j}(1 - y_i.y_j).\prod_i(1 - y_i)^2}$$

and because $\mathcal{T}_{m,2}$ has a finite resolution as Γ_m-module one can write

$$\mathcal{P}(\mathcal{T}_{m,2}; y_1, ..., y_m) = \frac{f(y_1, ..., y_m)}{\prod_{i<j}(1 - y_i.y_j).\prod_i(1 - y_i)^2}$$

Comparing the power series expansion of $\mathcal{P}(\Gamma; y_1, ..., y_m)$ with that of $\mathcal{P}(\mathcal{T}_{m,2}; y_1, ..., y_m)$ as can be calculated using (1.1) it is fairly easy to calculate the functions $f(y_1, ..., y_m)$.

In the general case, however, one has to find another approach. Our starting point will be the following theorem of Procesi

Theorem 3 [PROCESI,Theorem 3.4]

$\mathcal{R}_{m,n}$ is generated as a k-algebra by the elements $Tr(X_{i_1}...X_{i_j})$ with $j \leq 2^n - 1$.

Therefore, $\mathcal{R}_{m,n}$ has a finite resolution over the ring

$$\mathcal{S}_{m,n} = k[a_{i_1...i_j}; j \leq 2^n - 1, i_k \in \{1, ..., m\}]$$

whose Poincaré series is

$$\frac{1}{\prod_i(1 - y_i).\prod_{i_1,i_2}(1 - y_{i_1}.y_{i_2})...\prod_{i_1,..,i_{2^n-1}}(1 - y_{i_1}...y_{i_{2^n-1}})}$$

Therefore, $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ being finite modules over $\mathcal{S}_{m,n}$ we get

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) = f(y_1, ..., y_m).\mathcal{P}(\mathcal{S}_{m,n}; y_1, ..., y_m)$$

$$P(\mathcal{T}_{m,n}; y_1, ..., y_m) = h(y_1, ..., y_m).P(\mathcal{S}_{m,n}; y_1, ..., y_m)$$

and , again, comparing the power series expansion of $P(\mathcal{R}_{m,n}; y_1, ..., y_m)$ (resp. $\mathcal{T}_{m,n}; y_1, ..., y_m)$) and that of $P(\mathcal{S}_{m,n}; y_1, ..., y_m)$ gives us an algorithm to compute the functions $f(y_1, ..., y_m)$ and $h(y_1, ..., y_m)$.

Of course, this is a very laborous method and usually we will contend ourselves with computing the rational expression of the Poincaré series in one variable. These are obtained from the multi-graded ones by setting

$$y_1 = y_2 = ... = y_m = t$$

and we have :

Theorem 4 :

There exist polynomials $f(t), h(t) \in \mathbb{Z}[t]$ such that

$$P(\mathcal{R}_{m,n}; t) = \frac{f(t)}{(1-t)^m.(1-t^2)^{m^2}...(1-t^{2^n-1})^{m^{2^n-1}}}$$

$$P(\mathcal{T}_{m,n}; t) = \frac{g(t)}{(1-t)^m.(1-t^2)^{m^2}...(1-t^{2^n-1})^{m^{2^n-1}}}$$

A direct consequence of this result is the determination of all possible rational expressions of the Poincaré series of $\mathcal{R}_{m,n}$ (resp. of $\mathcal{T}_{m,n}$) providing it has finite global dimension.

For, in that case, the Poincaré series has to have the form $\frac{1}{g(t)}$ and comparing this with the foregoing theorem we get

Corollary 5 :

If $\mathcal{R}_{m,n}$ or $\mathcal{T}_{m,n}$ has finite global dimension, then its Poincaré series has the form

$$\frac{1}{F_1^{\alpha_1}...F_k^{\alpha_k}}$$

where the F_j are irreducible factors (in $\mathbb{Z}[t]$) of $1-t^l$ for some $1 \leq l \leq 2^n - 1$ and $\sum_1^k \alpha_i$ is clearly bounded by

$$m + 2.m^2 + ... + (2^n - 1).m^{2^n - 1}$$

2. Computation of the Poincaré series of $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$.

In this section we will explicitate the foregoing general results in the special case of 2 generic 3×3-matrices. The computation of the first terms in the power series expansion of the Poincaré series will enable us in the next section to prove that $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$ both have infinite global dimension.

Our first job is to calculate the Schur functions in 9 variables , $z_1, ..., z_9$, associated to partitions λ of lenght ≤ 2, i.e. λ has the form

$$\lambda = (a, b, 0, 0, 0, 0, 0, 0, 0)$$

where $a, b \in \mathbb{N}$ such that $a \geq b$.

By definition the Schur function s_λ is

$$s_\lambda(z_1, ..., z_9) = \frac{det(A)}{\prod_{i<j}(z_i - z_j)}$$

where A is the following 9×9-matrix

$$\begin{pmatrix} z_1^{8+a} & z_1^{7+b} & z_1^6 & \cdots\cdots & z_1 & 1 \\ z_2^{8+a} & z_2^{7+b} & z_2^6 & \cdots\cdots & z_2 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ z_9^{8+a} & z_9^{7+b} & z_9^6 & \cdots\cdots & z_9 & 1 \end{pmatrix}$$

by elementary row alterations on A it is easy to schow that the Schur functions becomes

$$s_\lambda(z_1, ..., z_9) = \det \begin{pmatrix} F & G \\ H & I \end{pmatrix}$$

where

$$F = \sum_{|i|=a+1} z_1^{i_1} ... z_8^{i_8}$$

$$G = \sum_{|j|=b} z_1^{j_1} ... z_8^{j_8}$$

$$H = \sum_{|k|=a} z_1^{k_1} ... z_9^{k_9}$$

$$I = \sum_{|l|=b-1} z_1^{l_1} ... z_9^{l_9}$$

where i, j are 8-tuples and k, l are 9-tuples of nonnegative integers. If $b \leq 1$ then the under right corner becomes 0 by definition.

Next, we calculate the image of $s_\lambda(z_1, ..., z_9)$ under the map

$$\mathbb{Z}[z_1, ..., z_9]^{S_9} \to \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]^{S_3}$$

defined by

$$\begin{array}{ccc} z_1 \to 1 & z_4 \to x_2 x_1^{-1} & z_7 \to x_3 x_1^{-1} \\ z_2 \to x_1 x_2^{-1} & z_5 \to 1 & z_8 \to x_3 x_2^{-1} \\ z_3 \to x_1 x_3^{-1} & z_6 \to x_2 x_3^{-1} & z_9 \to 1 \end{array}$$

Therefore, $s_\lambda(x_i . x_j^{-1})$ is the determinant of the following 2×2-matrix

$$\begin{pmatrix} F_1 & G_1 \\ H_1 & I_1 \end{pmatrix}$$

where :

$$F_1 = \sum_{|i|=a+1} 1^{i_1+i_5} . x_1^{i_2+i_3-i_4-i_7} . x_2^{i_4+i_6-i_2-i_8} . x_3^{i_7+i_8-i_3-i_6}$$

$$G_1 = \sum_{|j|=b} 1^{j_1+j_5} . x_1^{j_2+j_3-j_4-j_7} . x_2^{j_4+j_6-j_2-j_8} . x_3^{j_7+j_8-j_3-j_6}$$

$$H_1 = \sum_{|k|=a} 1^{k_1+k_5+k_9} . x_1^{k_2+k_3-k_4-k_7} . x_2^{k_4+k_6-k_2-k_8} . x_3^{k_7+k_8-k_3-k_6}$$

$$I_1 = \sum_{|l|=b-1} 1^{l_1+l_5+l_9} . x_1^{l_2+l_3-l_4-l_7} . x_2^{l_4+l_6-l_2-l_8} . x_3^{l_7+l_8-l_3-l_6}$$

Our next job is to compute the inproducts in $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]^{S_3}$

$$< s_\lambda(y_i . y_j^{-1}), 1 > = \frac{1}{6} \int s_\lambda(y_i . y_j^{-1}) . a_\delta . a_\delta^*$$

$$< s_\lambda(y_i.y_j^{-1}), s_{(1)}(y_i.y_j^{-1}) >= \frac{1}{6}\int s_\lambda(y_i.y_j^{-1}).\sum_{i,j} y_i.y_j^{-1}.a_\delta.a_\delta^*$$

where we have :

$$a_\delta = (y_1 - y_2).(y_2 - y_3).(y_1 - y_3)$$

$$a_\delta^* = (y_1^{-1} - y_2^{-1}).(y_2^{-1} - y_3^{-1}).(y_1^{-1} - y_3^{-1})$$

If we denote $y_1^{\alpha_1}.y_2^{\alpha_2}.y_3^{\alpha_3}$ by $(\alpha_1, \alpha_2, \alpha_3)$, we find that $a_\delta.a_\delta^*$ is equal to

$$6.(0,0,0)$$

$$-2.[(1,-1,0) + (-1,1,0) + (1,0,-1) + (-1,0,1) + (0,1,-1) + (0,-1,1)]$$

$$+2.[(2,-1,-1) + (-2,1,1) + (-1,2,-1) + (1,-2,1) + (-1,-1,2) + (1,1,-2)]$$

$$-1.[(2,-2,0) + (-2,2,0) + (2,0,-2) + (-2,0,2) + (0,2,-2) + (0,-2,2)]$$

Therefore, we have all the necessary ingredients to compute the inproduct $< s_\lambda(y_i.y_j^{-1}), 1 >$ for partitions of the form $\lambda = (a,b,0,0,0,0,0,0,0)$. In appendix 1, a listing is given of a Pascal program which computes this inproduct as well as the obtained values for $a + b \leq 10$.

The Schur functions in 2 variables are easy to compute

$$s_{(k,0)}(y_1, y_2) = \sum_{i=0}^{k} y_1^i.y_2^{k-i}$$

$$s_{(k,l)}(y_1, y_2) = (y_1.y_2)^{k-l}.s_{(k-l,0)}(y_1, y_2)$$

Therefore, we have all the necessary information to calculate the first terms of the Poincaré series of $\mathcal{R}_{2,3}$.

$$1 +$$

$$y_1 + y_2 +$$

$$2y_1^2 + 2y_1.y_2 + 2y_2^2 +$$

$$3y_1^3 + 4y_1^2.y_2 + 4y_1.y_2^2 + 3y_2^3 +$$

$$4y_1^4 + 6y_1^3.y_2 + 9y_1^2.y_2^2 + 6y_1.y_2^3 + 4y_2^4 +$$

$$5y_1^5 + 9y_1^4.y_2 + 14y_1^3.y_2^2 + 14y_1^2.y_2^3 + 9y_1.y_2^4 + 5y_2^5$$

$$7y_1^6 + 12y_1^5.y_2 + 22y_1^4.y_2^2 + 25y_1^3.y_2^3 + 22y_1^2.y_2^4 + 12y_1.y_2^5 + 7y_2^6 +$$

.....

From this we deduce the Poincaré series in one variable

$$\mathcal{P}(\mathcal{R}_{2,3};t) = 1 + 2t + 6t^2 + 14t^3 + 29t^4 + 56t^5 + 107t^6 + ...$$

For the trace ring, we have to compute

$$\sum_{i,j} y_i.y_j^{-1}.a_\delta.a_\delta^*$$

which is equal to

$$6.(0,0,0)$$

$$-1.[(1,-1,0)+(-1,1,0)+(1,0,-1)+(-1,0,1)+(0,1,-1)+(0,-1,1)]$$

$$-1.[(2,-2,0)+(-2,2,0)+(2,0,-2)+(-2,0,2)+(0,2,-2)+(0,-2,2)]$$

$$-1.[(3,-3,0)+(-3,3,0)+(3,0,-3)+(-3,0,3)+(0,3,-3)+(0,-3,3)]$$

$$+1.[(3,-2,-1)+(-3,2,1)+(3,-1,-2)+(-3,1,2)+(-1,3,-2)+(1,-3,2)+$$

$$(-2,3,-1)+(2,-3,1)+(-1,-2,3)+(1,2,-3)+(-2,-1,3)+(2,1,-3)]$$

In appendix 2 we give a listing of a Pascal program which computes the inproduct

$$< s_\lambda(y_i.y_j^{-1}), s_{(1)}(y_i.y_j^{-1}) >$$

for partitions $\lambda = (a, b, 0, 0, 0, 0, 0, 0, 0)$. Also contained in this appendix are the values for $a + b \leq 10$.

They enable us to compute the first terms in $P(\mathcal{T}_{2,3}; y_1, y_2)$

$$1+$$

$$2y_1 + 2y_2 +$$

$$4y_1^2 + 6y_1.y_2 + 4y_2^2 +$$

$$6y_1^3 + 13y_1^2.y_2 + 13y_1.y_2^2 + 6y_2^3 +$$

$$9y_1^4 + 22y_1^3.y_2 + 31y_1^2.y_2^2 + 22y_1.y_2^3 + 9y_2^4 +$$

$$12y_1^5 + 34y_1^4.y_2 + 56y_1^3.y_2^2 + 56y_1^2.y_2^3 + 34y_1.y_2^4 + 12y_2^5 +$$

$$16y_1^6 + 48y_1^5.y_2 + 91y_1^4.y_2^2 + 109y_1^3.y_2^3 + 91y_1^2.y_2^4 + 48y_1.y_2^5 + 16y_2^6 +$$

$$.....$$

Hence, the Poincaré series in one variable is

$$P(\mathcal{T}_{2,3}; t) = 1 + 4t + 14t^2 + 38t^3 + 93t^4 + 204t^5 + 419t^6 + 806t^7 + 1480t^8 ...$$

3. The regularity problem for 3 by 3 matrices.

The calculations carried out in the foregoing section enable us to prove the following result

Theorem 6 : The center $\mathcal{R}_{2,3}$ of the trace ring of 2 generic 3 by 3 matrices has infinite global dimension

Proof : It follows from section 1 that if $gldim(\mathcal{R}_{2,3}) < \infty$ then its Poincaré series should have the following rational form

$$\frac{1}{(1-t)^a(1-t^2)^b(1-t^3)^c(1-t^4)^d(1-t^5)^e(1-t^6)^f(1-t^7)^g}$$

with $a+b+c+d+e+f+g=10$. A comparison of the coefficient of t in the power series expansion of this form with the actual Poincaré series learns us that $a=2$. The coefficient of t^2 then becomes $3+b$ whence $b=3$. The coefficient of t^3 finally, becomes $4+c$ whence $c=10$ but $2+3+10>10$ and we obtain a contradiction.

From the calculation of the first terms of the Poincaré series of the trace ring of 2 generic 3 by 3 matrices one deduces

Theorem 7 : If the trace ring of 2 generic 3 by 3 matrices has finite global dimension, its Poincaré series must have the following rational form

$$\frac{1}{(1-t)^4(1-t^2)^4(1-t^3)^2}$$

Proof : If the trace ring $\mathcal{R}_{2,3}$ has finite global dimension, its Poincaré series must have the rational form

$$\frac{1}{F_1^{a_1}...F_k^{a_k}}$$

where F_i is an irreducible factor in the factorial domain $\mathbb{Z}[t]$ of terms $1-t^l$ where $l \leq 7$. Consequently, we must find natural numbers a, b, c, d, e, f, g, h such that

$$\mathcal{P}(\mathcal{T}_{2,3};t) = \frac{1}{(1-t)^a A_1^b A_2^c A_3^d A_4^e A_5^f (1+t^2)^g (1+t^3)^h}$$

where

$A_1 = 1+t$

$A_2 = 1+t+t^2$

$A_3 = 1-t+t^2$

$A_4 = 1+t+t^2+t^3+t^4$

$A_5 = 1+t+t^2+t^3+t^4+t^5+t^6$

Because $A_1.A_3 = 1+t^3$ we will separate two cases :

case i : $b < d$, then the numerator has the form

$$(1-t)^a (1-t+t^2)^v A_2^c A_4^e A_5^f (1+t^2)^g (1+t^3)^h$$

we will investigate the two possibilities : (i.a) $a \geq c+e+f$, then the numerator becomes

$$(1-t)^u (1-t+t^2)^v (1-t^3)^w (1-t^5)^x (1-t^7)^y (1+t^2)^g (1+t^3)^h$$

and calculating the first terms in the powe series expansion of this formula we get

$$1 + (u+v)t + (\frac{u(u+1)}{2} + \frac{v(v+1)}{2} - v + uv - g)t^2 + ...$$

and therefore the set of equations

$$u+v = 4; u-g = 8$$

which has no solutions in $\mathbb{N}$.

(i.b) : $a < c + e + f$, then the rational form becomes

$$\frac{(1-t)^u}{(1-t+t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t^2)^g(1+t^3)^h}$$

and its power series expansion begins with

$$1 + (v-u)t + (\frac{u(u-1)}{2} + \frac{v(v+1)}{2} - v - uv - g)t^2 + ...$$

Therefore, we obtain the equations

$$u - v = 4; u + g = -8$$

which has no solutions in $\mathbb{N}$.

This reduces our study to case ii : $b \geq d$, i.e. the numerator becomes

$$(1-t)^a A_1^b A_2^c A_4^e A_5^f (1+t^2)^g (1+t^3)^h$$

It is easy to check that $a \geq b + c + e + f$ for else the coefficient of t in the power series expansion would be negative.Hence we have the rational form

$$\frac{1}{(1-t)^u(1-t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t)^g(1+t^3)^h}$$

Comparing the coefficient of t in its power series expansion with the actual coefficient in the Poincaré series we get $u = 4$. But then, the first terms become

$$1 + 4t + (10 - v - g)t^2 + (20 + 4(v-g) + w - h)t^3 +$$

$$(35 + 10(v-g) + 4(w-h) + \frac{v(v+1)}{2} + \frac{g(g+1)}{2} - vg)t^4 + ...$$

which gives us the equations

$$v - g = 4; w - h = 2$$

Substituting this information in the coefficient of t^4 we get

$$v^2 + v + g^2 + g - 2vg = 20$$

whence $v = 4$ and $g = 0$. The coefficient of t^5 then becomes

$$176 + 14(w - h) + x$$

and comparing this with the Poincaré series we get $x = 0$. Finally, comparing the coefficients of t^6 gives us the equation

$$w^2 + w + h^2 + h - 2wh = 6$$

whence $w = 2$ and $h = 0$. Since $u + v + w = 10 = Kdim$ we must have $y = 0$, finishing the proof of the theorem.

In a similar manner it is possible to compute the rational form in a multi-gradation. One obtains

$$\frac{1}{(1-t_1)^2(1-t_2)^2(1-t_1^2)(1-t_2^2)(1-t_1t_2)^2(1-t_1^2t_2)(1-t_1t_2^2)}$$

In [LEBR-VDBERGH] it is shown that $\mathcal{T}_{2,3}$ does indeed have finite global dimension and has the prescribed rational form for its Poincaré series.

Of course, one can try to apply the method outlined above to calculate the Poincaré series of the trace ring of 3 generic 3 by 3 matrices and show that its rational form cannot be a pure inverse. It should be clear however that the computations become extremely complicated. Nevertheless, we can follow a more elegant approach to solve the regularity problem for 3 by 3 matrices

Theorem 8 : The trace ring of m generic 3 by 3 matrices $\mathcal{T}_{m,3}$ has finite global dimension if and only if $m \leq 2$.

Proof :

As we mentioned before, regularity of $\mathcal{T}_{2,3}$ was proved in [LEBR-VDBERGH]. Now, let us consider the special case of 3 generic 3 by 3 matrices. In order to prove that the global dimension is infinite it suffices to show that $\mathcal{P}(\mathcal{T}_{3,3}; t_1, t_2, t_3)$ is not a pure inverse in $\mathbb{Z}(t_1, t_2, t_3)$.Suppose that the Poincare series has the rational expression

$$\frac{1}{\prod_{i=1}^{\alpha} g_i(t_1, t_2, t_3)}$$

where each of the components $g_i(t_1, t_2, t_3)$ is an irreducible factor in the factorial domain $\mathbb{Z}[t_1, t_2, t_3]$ of a term $1 - t_1^k t_2^l t_3^m$ and we may assume by the Procesi-results that $k + l + m \leq 2^3 - 1 = 7$.

Lert us consider the subproduct consisting of those terms involving only two variables t_i and t_j : $G(t_i, t_j)$.We obtain

$$\frac{1}{G(t_i, t_j)} = \mathcal{P}(\mathcal{T}_{3,3}; t_1, t_2, t_3) \mid_{t_k = 0}$$

Therefore, from our knowledge of the rational form in a multigradation of the Poincaré series of the trace ring of 2 generic 3 by 3 matrices we obtain that

$$G(t_i, t_j) = (1 - t_i)^2 (1 - t_j)^2 (1 - t_i^2)(1 - t_j^2)(1 - t_i t_j)^2 (1 - t_i^2 t_j)(1 - t_i t_j^2)$$

Since this holds for each couple (i, j) from $\{1, 2, 3\}$ we obtain a subproduct equal to

$$(1 - t_1)^2 (1 - t_2)^2 (1 - t_3)^2 (1 - t_1^2)(1 - t_2^2)(1 - t_3^2)$$

$$(1 - t_1 t_2)^2 (1 - t_2 t_3)^2 (1 - t_1 t_3)^2$$

$$(1 - t_1^2 t_2)(1 - t_2^2 t_1)(1 - t_2^2 t_3)(1 - t_3^2 t_2)(1 - t_1^2 t_3)(1 - t_3^2 t_1)$$

If we set $t_1 = t_2 = t_3 = t$ we obtain that the residue of the pole in $t = 1$ of the Poincaré series of the trace ring of 3 generic 3 by 3 matrices is at least 21. This is in contradiction with the fact that this residue must be equal to its Krull dimension which is 19, so we obtain a contradiction.

Finally, let us consider the general case. Suppose that $\mathcal{T}_{m,3}$ has finite global dimension, then the rational form of its Poincaré series in a multigradation must be a pure inverse. However, if we set $t_4 = ... = t_m = 0$ we obtain the rational form of the Poincaré series of the trace ring of 3 generic 3 by 3 which cannot be a pure inverse, fininshing the proof.

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APPENDIX 1

```
program centertr(input,output);

{this program computes the coefficient in the Poincare-series}
{of the center of T of 2 generic 3x3-matrices for the Schur-function}
{associated to the partition (x,y,0,0,0,0,0,0,0)}

var x,y,z,t,v,result :integer;

function hulp(a,b : integer):integer;

var i1,i2,i3,i4,i5,i6,i7,i8 : integer;
    j1,j2,j3,j4,j5,j6,j7,j8,j9 : integer;
    c,d,e,k  : integer;

begin{hulp}
k:=0;
for i1:=0 to a do begin
  for i2:=0 to (a-i1) do begin
    for i3:=0 to (a-i1-i2) do begin
      for i4:=0 to (a-i1-i2-i3) do begin
        for i5:=0 to (a-i1-i2-i3-i4) do begin
          for i6:=0 to (a-i1-i2-i3-i4-i5) do begin
            for i7:=0 to (a-i1-i2-i3-i4-i5-i6) do
   begin
      i8:=(a-i1-i2-i3-i4-i5-i6-i7);
      for j1:=0 to b do begin
        for j2:=0 to (b-j1) do begin
          for j3:=0 to (b-j1-j2) do begin
            for j4:=0 to (b-j1-j2-j3) do begin
              for j5:=0 to (b-j1-j2-j3-j4) do begin
                for j6:=0 to (b-j1-j2-j3-j4-j5) do begin
                  for j7:=0 to (b-j1-j2-j3-j4-j5-j6) do begin
                    for j8:=0 to (b-j1-j2-j3-j4-j5-j6-j7) do
      begin
         j9:=(b-j1-j2-j3-j4-j5-j6-j7-j8);
         c:=(i2+j2+i3+j3-i4-j4-i7-j7);
         d:=(i4+j4+i6+j6-i2-j2-i8-j8);
         e:=(i7+j7+i8+j8-i3-j3-i6-j6);
         if (c*c+d*d+e*e)=0 then k:=k+6
                            else begin
                              if (c*d*e)=0 then begin
                                 if (c*c+d*d+e*e)=2 then k:=k-2
                                          else begin
                                 if (c*c+d*d+e*e)=8 then k:=k-1;
                                               end;
                                               end
                                           else begin
                                 if (c*c+d*d+e*e)=6 then k:=k+2;
                                               end;
                                 end;
        end;
                                                          end;
                                                      end;
                                                  end;
                                              end;
                                          end;
                                      end;
                                  end;
                              end;
                                                 end;
                                             end;
                                         end;
                                     end;
                                 end;
                             end;
 hulp:=trunc(k/6);
 end{hulp};

 begin{main program}
 read(x);read(y);
 z:=hulp(y,x);
 if (y-1)<0 then result:=z
            else begin
                    t:=hulp(x+1,y-1);
                    result:=(z-t);
                 end;
 writeln('coefficient of s(',x:3,',',y:3,',0,0,0,0,0,0,0) = ',result:3);
 end.
```

```
coefficient of s(  1,  0,0,0,0,0,0,0,0) =  1
coefficient of s(  2,  0,0,0,0,0,0,0,0) =  2
coefficient of s(  1,  1,0,0,0,0,0,0,0) =  0
coefficient of s(  3,  0,0,0,0,0,0,0,0) =  3
coefficient of s(  2,  1,0,0,0,0,0,0,0) =  1
coefficient of s(  4,  0,0,0,0,0,0,0,0) =  4
coefficient of s(  3,  1,0,0,0,0,0,0,0) =  2
coefficient of s(  2,  2,0,0,0,0,0,0,0) =  3
coefficient of s(  5,  0,0,0,0,0,0,0,0) =  5
coefficient of s(  4,  1,0,0,0,0,0,0,0) =  4
coefficient of s(  3,  2,0,0,0,0,0,0,0) =  5
coefficient of s(  6,  0,0,0,0,0,0,0,0) =  7
coefficient of s(  5,  1,0,0,0,0,0,0,0) =  5
coefficient of s(  4,  2,0,0,0,0,0,0,0) = 10
coefficient of s(  3,  3,0,0,0,0,0,0,0) =  3
coefficient of s(  7,  0,0,0,0,0,0,0,0) =  8
coefficient of s(  6,  1,0,0,0,0,0,0,0) =  8
coefficient of s(  5,  2,0,0,0,0,0,0,0) = 14
coefficient of s(  4,  3,0,0,0,0,0,0,0) =  9
coefficient of s(  8,  0,0,0,0,0,0,0,0) = 10
coefficient of s(  7,  1,0,0,0,0,0,0,0) = 10
coefficient of s(  6,  2,0,0,0,0,0,0,0) = 21
coefficient of s(  5,  3,0,0,0,0,0,0,0) = 15
coefficient of s(  4,  4,0,0,0,0,0,0,0) = 10
coefficient of s(  9,  0,0,0,0,0,0,0,0) = 12
coefficient of s(  8,  1,0,0,0,0,0,0,0) = 13
coefficient of s(  7,  2,0,0,0,0,0,0,0) = 27
coefficient of s(  6,  3,0,0,0,0,0,0,0) = 27
coefficient of s(  5,  4,0,0,0,0,0,0,0) = 18
coefficient of s( 10,  0,0,0,0,0,0,0,0) = 14
coefficient of s(  9,  1,0,0,0,0,0,0,0) = 16
coefficient of s(  8,  2,0,0,0,0,0,0,0) = 36
coefficient of s(  7,  3,0,0,0,0,0,0,0) = 36
coefficient of s(  6,  4,0,0,0,0,0,0,0) = 37
coefficient of s(  5,  5,0,0,0,0,0,0,0) = 10
```

APPENDIX 2

```
program tracering(input,output);

{this program computes the coefficient in the Poincare-series}
{of the trace ring of 2 generic 3x3-matrices for the Schur-function}
{associated to the partition (x,y,0,0,0,0,0,0,0)}

var x,y,z,t,v,result :integer;

function hulp(a,b : integer):integer;

var i1,i2,i3,i4,i5,i6,i7,i8 : integer;
    j1,j2,j3,j4,j5,j6,j7,j8,j9 : integer;
    c,d,e,k  : integer;

begin{hulp}
k:=0;
for i1:=0 to a do begin
  for i2:=0 to (a-i1) do begin
    for i3:=0 to (a-i1-i2) do begin
      for i4:=0 to (a-i1-i2-i3) do begin
        for i5:=0 to (a-i1-i2-i3-i4) do begin
          for i6:=0 to (a-i1-i2-i3-i4-i5) do begin
            for i7:=0 to (a-i1-i2-i3-i4-i5-i6) do
   begin
      i8:=(a-i1-i2-i3-i4-i5-i6-i7);
      for j1:=0 to b do begin
        for j2:=0 to (b-j1) do begin
          for j3:=0 to (b-j1-j2) do begin
            for j4:=0 to (b-j1-j2-j3) do begin
              for j5:=0 to (b-j1-j2-j3-j4) do begin
                for j6:=0 to (b-j1-j2-j3-j4-j5) do begin
                  for j7:=0 to (b-j1-j2-j3-j4-j5-j6) do begin
                    for j8:=0 to (b-j1-j2-j3-j4-j5-j6-j7) do
      begin
         j9:=(b-j1-j2-j3-j4-j5-j6-j7-j8);
         c:=(i2+j2+i3+j3-i4-j4-i7-j7);
         d:=(i4+j4+i6+j6-i2-j2-i8-j8);
         e:=(i7+j7+i8+j8-i3-j3-i6-j6);
         if (c*c+d*d+e*e)=0 then k:=k+6
                             else begin
                               if (c*d*e)=0 then begin
                                  if (c*c+d*d+e*e)=2 then k:=k-1
                                            else begin
                                  if (c*c+d*d+e*e)=8 then k:=k-1
                                                  else begin
                                  if (c*c+d*d+e*e)=18 then k:=k-1;
                                                       end;
                                                end;
                                                end
                                            else begin
                                  if (c*c+d*d+e*e)=14 then k:=k+1;
                                                end;
                                  end;
      end;

                                                          end;
                                                      end;
                                                  end;
                                              end;
                                          end;
                                      end;
                                  end;
                              end;

                                                end;
                                             end;
                                         end;
                                     end;
                                 end;
                             end;
hulp:=trunc(k/6);
end{hulp};

begin{main program}
read(x);read(y);
z:=hulp(y,x);
if (y-1)<0 then result:=z
           else begin
                  t:=hulp(x+1,y-1);
                  result:=(z-t);
                end;
writeln('coefficient of s(',x:3,',',y:3,',0,0,0,0,0,0,0) = ',result:3);
end.
```

```
coefficient of s(  1,   0,0,0,0,0,0,0,0) =   2
coefficient of s(  2,   0,0,0,0,0,0,0,0) =   4
coefficient of s(  1,   1,0,0,0,0,0,0,0) =   2
coefficient of s(  3,   0,0,0,0,0,0,0,0) =   6
coefficient of s(  2,   1,0,0,0,0,0,0,0) =   7
coefficient of s(  4,   0,0,0,0,0,0,0,0) =   9
coefficient of s(  3,   1,0,0,0,0,0,0,0) =  13
coefficient of s(  2,   2,0,0,0,0,0,0,0) =   9
coefficient of s(  5,   0,0,0,0,0,0,0,0) =  12
coefficient of s(  4,   1,0,0,0,0,0,0,0) =  22
coefficient of s(  3,   2,0,0,0,0,0,0,0) =  22
coefficient of s(  6,   0,0,0,0,0,0,0,0) =  16
coefficient of s(  5,   1,0,0,0,0,0,0,0) =  32
coefficient of s(  4,   2,0,0,0,0,0,0,0) =  43
coefficient of s(  3,   3,0,0,0,0,0,0,0) =  18
coefficient of s(  7,   0,0,0,0,0,0,0,0) =  20
coefficient of s(  6,   1,0,0,0,0,0,0,0) =  45
coefficient of s(  5,   2,0,0,0,0,0,0,0) =  68
coefficient of s(  4,   3,0,0,0,0,0,0,0) =  52
coefficient of s(  8,   0,0,0,0,0,0,0,0) =  25
coefficient of s(  7,   1,0,0,0,0,0,0,0) =  59
coefficient of s(  6,   2,0,0,0,0,0,0,0) = 101
coefficient of s(  5,   3,0,0,0,0,0,0,0) =  97
coefficient of s(  4,   4,0,0,0,0,0,0,0) =  46
coefficient of s(  9,   0,0,0,0,0,0,0,0) =  30
coefficient of s(  8,   1,0,0,0,0,0,0,0) =  76
coefficient of s(  7,   2,0,0,0,0,0,0,0) = 138
coefficient of s(  6,   3,0,0,0,0,0,0,0) = 159
coefficient of s(  5,   4,0,0,0,0,0,0,0) = 114
coefficient of s( 10,   0,0,0,0,0,0,0,0) =  36
coefficient of s(  9,   1,0,0,0,0,0,0,0) =  94
coefficient of s(  8,   2,0,0,0,0,0,0,0) = 183
coefficient of s(  7,   3,0,0,0,0,0,0,0) = 232
coefficient of s(  6,   4,0,0,0,0,0,0,0) = 215
coefficient of s(  5,   5,0,0,0,0,0,0,0) =  83
```