Counterexamples to the Kac-conjecture on Schur roots.

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0. Introduction

One of the main achievements in the theory of representations of quivers is V. Kac's description of the dimension types of the indecomposables as a positive root system,[K1]. In the subsequent papers [K2] and [K3] Kac improved and completed his results. In section 1 we recall the main result and introduce the necessary definitions in order to make this paper selfcontained. For proofs and more information the reader is referred to the paper by Kraft and Riedtmann [KR] or to the original papers by Kac. Our notation is compatible with that of [KR].

This paper deals with some problems posed in [K2,4-6] and repeated in [K3,1.18-1.21]. They are concerned with describing for an arbitrary dimension type α the so called generic decomposition $\alpha = \beta_1 + ... + \beta_s$. That is, the unique decomposition of α such that the open sheet for the space of representations of type α under action of $GL(\alpha)$ contains an open set of representations $V = \bigoplus_{i=1}^{s} V_i$ where the V_i are indecomposable of type β_i, see [KR,Prop.2.7]. Then, each V_i has endomorphism ring $\mathbb{C}$ and, following Roiter, such representations are called Schur representations; the corresponding dimensions are called Schur roots.

As soon as the generic decomposition of α is known one can describe the endomorphism ring of a generic representation and show that the minimal codimension of an orbit of $GL(\alpha)$ in $R(Q,\alpha)$ is equal to $\sum_{i=1}^{s}(1 - T(\beta_i, \beta_i)$, where T is the Tits bilinear form of Q. This, in particular, gives the classification of linear algebraic groups, among those for which the restriction to any irreducible component is of the form $GL_m(\mathbb{C}) \times GL_r(\mathbb{C})^*$, which admit a dense orbit. Further, one can describe the semi-invariants and rational invariants of the action of $GL(\alpha)$ on $R(Q,\alpha)$ in terms of those for the Schur roots β_i.

In [K2,p155-p157] Kac conjectured a purely combinatorial description of the set of Schur roots and of the generic decomposition and verified it for quivers of finite or tame type and quivers of rank two. We will show, however, that in general there are counterexamples to the conjectures.

As is often the case with counterexamples, the first approach is not always the most direct or elegant one. In this case, I tried to find counterexamples by studying a similar concept to Schur representations in the theory of vector bundles over projective n-space $\mathbb{P}_n$; namely those bundles for which the only endomorphisms are the homotheties : the simple bundles [OSS,4.1]. Most known examples of such bundles (e.g. many stable ones) turn out to be the cohomology bundles of monads [OSS,II.3] of the form

$$(*) : K \bigotimes \mathcal{O}_{\mathbb{P}_n}(-u) \longrightarrow L \bigotimes \mathcal{B} \longrightarrow M \bigotimes \mathcal{O}_{\mathbb{P}_n}(v)$$

where K, L, M are finite dimensional vectorspaces of dimensions say k, l, m and $\mathcal{B}$ is a manageable simple bundle (for example $\mathcal{O}_{\mathbb{P}_n}(w), \Omega^1_{\mathbb{P}_n}(w)$ etc.). To a monad of the form $(*)$ one can associate a representation of the quiver $Q(x,y)$:

$$\circ \quad \overset{\phi_{1,1}}{\longrightarrow} \underset{\phi_{1,x}}{\longrightarrow} \quad \circ \quad \overset{\phi_{2,1}}{\longrightarrow} \underset{\phi_{2,y}}{\longrightarrow} \quad \circ$$

of dimension type (k,l,m) where $x = h^o(\mathcal{B}(u))$ and $y = h^o(\mathcal{B}^*(v))$. If a monad defines the bundle $\mathcal{E}$ and the representation $V_{\mathcal{E}}$ of $Q(x,y)$ then there exists a natural morphism $End(V_{\mathcal{E}}) \to End(\mathcal{E})$. The next result can be proved easily from [BH,Prop 4] and may be of some interest, either for constructing moduli spaces for simple bundles determined by the same monad-type or constructing new simple bundles using knowledge of the Schur roots for the quivers :

Lemma : Let $(*)$ be a monad as before such that $h^0(\mathcal{B}^*(-u)) = 0 = h^0(\mathcal{B}(-v))$; $h^1(\mathcal{B}(i)) = 0$ for all $i \geq -v$ and $h^{n-1}(\mathcal{B}(i)) = 0$ for $i < u + n$. (If $n = 2$ we assume moreover that $-u - v \geq -2$). Then, $End(\mathcal{E}) \cong End(V_{\mathcal{E}})$.

Using the monads given in [OSS,p249-p268] it is then easy to determine a large set of Schur roots for quivers $Q(x,y)$ and after reflecting them one obtains counterexamples to the Kac conjectures.

Afterwards , it became clear that there is a more elegant and general approach using only results from representation theory. This approach is given in section 2. It will turn out that the main reason for the failure of the Kac-conjectures is that, unlike Schur roots, indecomposable roots are not preserved under reflections.

Acknowledgement

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1. The Kac conjectures on Schur roots.

(1.1) : Throughout, our basefield will be $\mathbb{C}$, but any algebraically closed field will do. A quiver Q consists of a finite set $Q_0 = \{1, ..., n\}$ of vertices, a set Q_1 of arrows and two maps $t, h : Q_1 \to Q_0$ assigning to an arrow ϕ its tail $t\phi$ and its head $h\phi$, respectively. We do not exclude loops nor multiple arrows.

A representation V of Q is a family $\{V(i) : i \in Q_0\}$ of finite-dimensional $\mathbb{C}$-vectorspaces together with a family of linear maps $\{V(\phi) : V(t\phi) \to V(h\phi) \mid \phi \in Q_1\}$. The vector $dim(V) = (dim(V(1)), ..., dim(V(n))$ is called the dimension type of V. A morphism $f : V \to W$ is a family of linear maps $\{f_i : V(i) \to W(i) \mid i \in Q_0\}$ such that $W(\phi) \circ f(t\phi) = f(h\phi) \circ V(\phi)$ for all arrows $\phi \in Q_1$.

The representation space $R(Q, \alpha)$ of Q of dimension type $\alpha = (\alpha(1), ..., \alpha(n)) \in \mathbb{N}^n$ is the set of representations

$$R(Q, \alpha) = \{V : V(i) = \mathbb{C}^{\alpha(i)}; 1 \leq i \leq n\}$$

Since $V \in R(Q, \alpha)$ is determined by the maps $V(\phi)$

$$R(Q, \alpha) = \bigoplus_{\phi \in Q_1} Hom_{\mathbb{C}}(\mathbb{C}^{\alpha(t\phi)}, \mathbb{C}^{\alpha(h\phi)}) = \bigoplus_{\phi \in Q_1} M_\phi(\mathbb{C})$$

where $M_\phi(\mathbb{C})$ is the set of all $\alpha(h\phi)$ by $\alpha(t\phi)$ matrices with entries in $\mathbb{C}$. We consider $R(Q, \alpha)$ as an affine variety with coordinate ring $\mathbb{C}[Q, \alpha]$ and functionfield $\mathbb{C}(Q, \alpha)$.

The linear reductive group $GL(\alpha) = \prod_{i=1}^n GL_{\alpha(i)}(\mathbb{C})$ acts linearly (and regularly) on $R(Q, \alpha)$ by

$$(g.V)(\phi) = g_{h\phi} \circ V(\phi) \circ q_{t\phi}^{-1}$$

for $g = (g_1, ..., g_n) \in GL(\alpha)$. By definition, the $GL(\alpha)$- orbits in $R(Q, \alpha)$ are just the isomorphism classes of representations.

(1.2) : Any representation of Q can be decomposed uniquely in a direct sum of indecomposable ones. The dimension types of indecomposable representations were described by Kac [K1,Th 1',p89], [K3,Th.1.10,p85] in the following way. Let $\alpha_i = (\delta_{1i}, ..., \delta_{ni})$, $1 \leq i \leq n$ be the standard basis of $\mathbb{Z}^n$ and let $r_{ij} = \sharp\{\phi \in Q_1 : t\phi = i, h\phi = j\}$. Define the Tits bilinear form $T(-,-)$ on $\mathbb{Z}^n$ by

$$T(\alpha_i, \alpha_j) = \delta_{ij} - \frac{1}{2}(r_{ij} + r_{ji})$$

let $\Phi = \{\alpha_i : r_{ii} = 0\}$ be the set of fundamental roots. For $\alpha \in \Phi$ define the fundamental reflexion $r_\alpha \in Aut(\mathbb{Z})$ by

$$r_\alpha(\lambda) = \lambda - 2.T(\lambda, \alpha)\alpha$$

for $\lambda \in \mathbb{Z}^n$. The group $W(Q) \subset Aut(\mathbb{Z}^n)$ generated by all fundamental reflexions is called the Weyl group of the quiver Q. The fundamental set $F \subset \mathbb{Z}^n$ is defined by

$$F = \{\alpha \in \mathbb{N}^n - 0 : T(\alpha, \alpha_i) \leq 0; 1 \leq i \leq n \text{ and } supp(\alpha) \text{ is connected } \}$$

where the support of α is the full subquiver of Q on the i's such that $\alpha(i) \neq 0$ and connected means as a graph forgetting the orientation. The real roots are defined by $\Delta^{re}(Q) = \bigcup_{w \in W(Q)} w(\Phi)$, the imaginary roots by $\Delta^{im} = \bigcup_{w \in W(Q)} w(F \bigcup -F)$ and the root system of Q is the set $\Delta(Q) = \Delta^{re}(Q) \bigcup \Delta^{im}(Q)$.The dimension types of indecomposable representations turns out to be the set of all positive roots $\Delta_+(Q) = \Delta(Q) \bigcap \mathbb{N}^n$.

(1.3) : The description of the dimension types of the indecomposable representations is only the first step in the classification of all representations. However, the later problem seems to be hopeless in general. According to general principles of invariant theory, it is natural to try to solve a simple problem : classifying the "generic" representations of a given dimension type α. By [KR,Prop.2.7] there exists a unique decomposition $\alpha = \beta_1 + ... + \beta_s$ such that the set

$$\{V \in R(Q, \alpha) : V = \bigoplus_{i=1}^{s} V_i; dim(V_i) = \beta_i; V_i \text{ indecomposable } \}$$

contains an open and dense subset of $R(Q, \alpha)$. This is called the generic decomposition of α. If this decomposition is known, one can describe the endomorphism ring of a generic representation [K2,Prop.4] and show that the minimal codimension of an orbit of $GL(\alpha)$ in $R(Q, \alpha)$ is equal to $\sum_{i=1}^{s}(1 - T(\beta_i, \beta_i)) = trdeg\mathbb{C}(Q, \alpha)^{GL(\alpha)}$ [K3,p99].Further, $trdeg\mathbb{C}(Q, \alpha)^{(GL(\alpha), GL(\alpha))} = \sum_{i=1}^{s}(1 - T(\beta_i, \beta_i) + \sharp supp(\alpha) - s - r$ where s and r are the number of distinct real roots and the dimension of the $\mathbb{Q}$-span of all imaginary roots in the generic decomposition of α, respectively. Further, one can describe the semi-invariants and rational invariants of the action of $GL(\alpha)$ on $R(Q, \alpha)$ in terms of those of $GL(\beta_i)$ on $R(Q, \beta_i)$.

(1.4) : In view of the foregoing it would be important to find a purely combinatorial description of the generic decomposition. Each of the $R(Q, \beta_i)$ has an open set of indecomposable representations entailing that their endomorphismring must be reduced to $\mathbb{C}$, [K2,Prop.1],[KR,Th.2.6]. Following Roiter one calls the dimension type of such representations Schur roots. In [K3,1.20] Kac calls an element $\alpha \in \mathbb{N}^n - 0$ an indecomposable root (quasi-indecomposable in [K2,p154]) if α cannot be decomposed into a sum $\alpha = \beta + \gamma$ where $\beta, \gamma \in \mathbb{N}^n - 0$ and $R(\beta, \gamma) \geq 0$, $R(\gamma, \beta) \geq 0$. Here $R(-,-)$ is the Ringel bilinear form om $\mathbb{Z}^n$ defined by $R(\alpha_i, \alpha_j) = \delta_{ij} - r_{ij}$. He then conjectured

Kac conjecture 1 : ([K2,Conjecture 1],[K3,Conjecture 9])
If α is a Schur root, then α is an indecomposable root.

The converse implication is true [K3,Prop.p98]. For an arbitrary dimension vector $\alpha \in \mathbb{N}^n$, Kac calls the decomposition $\alpha = \gamma_1 + ... + \gamma_t$ where $\gamma_i \in \mathbb{N}^n - 0$ natural if it satisfies the following properties

(1) : $R(\gamma_i, \gamma_j) \geq 0$ if $i \neq j$

(2) : γ_i is an indecomposable root for every i

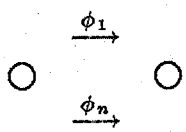
(3) : the number $\sum_{i=1}^{t}(1 - R(\beta_i, \beta_i))$ is maximal among the decompositions satisfying properties (1) and (2).

Kac conjecture 2 : ([K2,Conjecture 2],[K3,Conjecture 10])
A natural decomposition of $\alpha \in \mathbb{N}^n - 0$ coincides with the generic decomposition.

Actually, conjecture 10 in [K3] is slightly different in the sense that one assumes that Q has no oriented cycles but one removes condition (3) in the definition of a natural decomposition.

2. The counterexamples

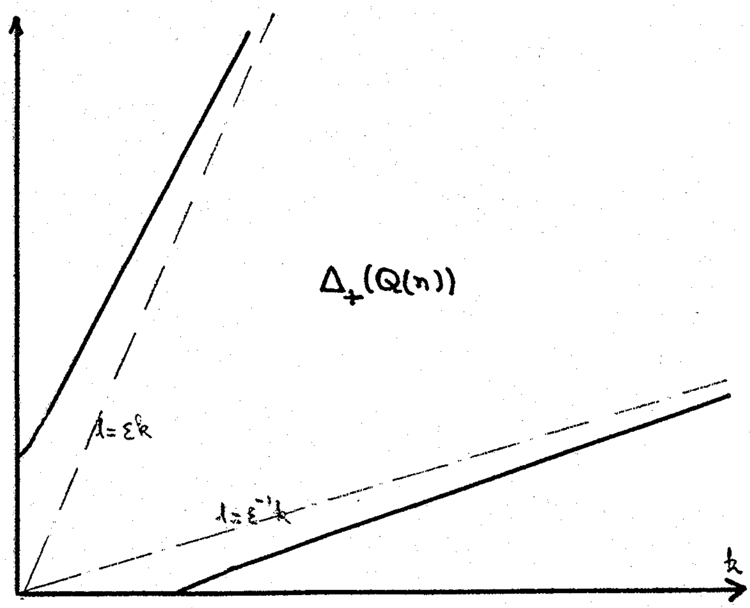
(2.1) : Kac verified his conjecture when Q is a quiver of finite type [K2,Ex(a),p158], tame type [K2,Ex(b),p158] or of rank two [K2,Ex(c),p159] or [Ri]. Let us concentrate on the special case of quivers $Q(n)$ for $n \geq 3$



It is shown in [K1,2.6] that every positive root is a Schur root (and,of course,also conversely) and

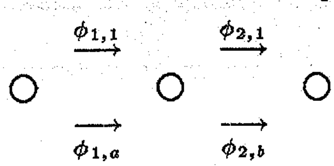
$$\Delta_+(Q(n)) = \{(k,l) \in \mathbb{N}^2 : \frac{1}{2}(n - \sqrt{n^2-4})l \leq k \leq \frac{1}{2}(n + \sqrt{n^2-4})l\}$$

So, we get the following picture :



where $\epsilon = (\sqrt{\frac{n+2}{n-2}} + 1).(\sqrt{\frac{n+2}{n-2}} - 1)^{-1}$ and all Schur roots (or positive roots) lie in the region between the two blades of the hyperbola $(H) : k^2 + l^2 - nkl = 1$. The real roots are the integer points on (H).

(2.2) : Let us now consider the easiest rank three quivers $Q(a,b)$ for $a, b \geq 3$



Whereas the determination of all Schur roots may turn out to be difficult, there is at least a large subclass which is easy to describe. For, take a dimension-type (k, l, m) where (k, l) is a Schur root of $Q(a)$ and (l, m) is a Schur root of $Q(b)$, then clearly (k, l, m) is a Schur root of $Q(a, b)$. Combining this observation with Kac's result mentioned above we get

Proposition 1 : If $\alpha = (k, l, m) \in \mathbb{N}^3$ satisfies $k^2 + l^2 - akl \leq 1$ and $l^2 + m^2 - blm \leq 1$, then α is a Schur root of $Q(a, b)$.

We will see later that there may be other Schur roots of $Q(a, b)$ than the ones described by proposition 1.

(2.3) : Again, let us consider the easiest situation. We take $a = b = n$ and we want to investigate Schur roots of $Q(n, n)$ of the form (k, l, k).

Proposition 2 : If $(k, l) \in \mathbb{N}^2_+$ satisfies $\frac{2}{n}k \leq l \leq nk$, then $\alpha = (k, l, k)$ is an indecomposable root for $Q(n, n)$. In particular, α is a Schur root.

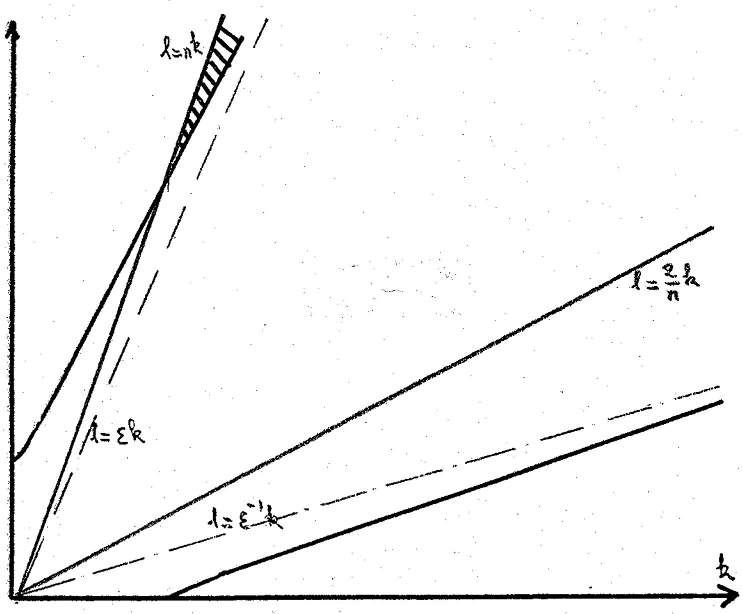
Proof :

The Tits bilinear form of the quiver $Q(n, n)$ is given by the matrix

$$\begin{pmatrix} 1 & -\frac{1}{2}n & 0 \\ -\frac{1}{2}n & 1 & -\frac{1}{2}n \\ 0 & -\frac{1}{2}n & 1 \end{pmatrix}$$

Therefore, (k, l, k) lies in the fundamental domain of $Q(n, n)$ iff $k - \frac{1}{2}nl \leq 0$ and $l - nk \leq 0$. Since the support of (k, l, k) is not a tame quiver and (k, l, k) lies by assumption in the fundamental domain, it is an indecomposable root by [K1,lemma 2.5,p72] or [KR,Th.3.3].

The fact that there are more Schur roots than those given by proposition 1 is now clear from the following picture



(2.4) : Let us recall the concept of reflexion functors (or castling transforms),[BGP],[SK],[K1,2.3]or [KR,4].Let Q be any quiver on n vertices and $\alpha \in \mathbb{N}^n - 0$. Fix a source i of Q , that is a vertex such that there is no $\phi \in Q_1$ with $h\phi = i$ and suppose $\sum_{t\phi=i} \alpha(h\phi) \geq \alpha(i)$. Consider $R'(Q,\alpha)$ which is the set

$$\{V \in R(Q,\alpha) \mid \bigoplus_{t\phi=i} V(\phi) : V(i) \to \bigoplus_{t\phi=i} V(h\phi) \text{ is injective } \}$$

Now, look at the quiver Q^* which is obtained from Q by reversing all arrows with tail i and the new dimension type α^* which is given by $\alpha^*(k) = \alpha(k)$ for all $k \neq i$ and $\alpha^*(i) = \sum_{t\phi=i} \alpha(h\phi) - \alpha(i)$.Consider the set $R'(Q^*.\alpha^*)$

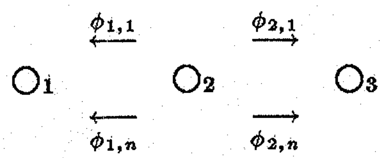
$$\{V \in R(Q^*,\alpha^*) \mid \bigoplus_{h\phi=i} V(\phi) : \bigoplus_{h\phi=i} V(t\phi) \to V(i) \text{ is surjective } \}$$

then there is an homeomorphism $R'(Q,\alpha) \cong R'(Q^*,\alpha^*)$, called the reflexion functor, such that corresponding representations have isomorphic endomorphism rings, see for example [KR,Prop.4.1]. Of course, the same process can be started from a sink (that is a vertex such that there is no ϕ with $t\phi = i$) instead of from a source. In particular we get that if α is a Schur root for the quiver Q, then α^* is a Schur root for the quiver Q^*.

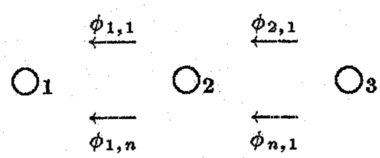
(2.5) : Proposition 3 : If $(k,l) \in \mathbb{N}^2_+$ satisfies $\frac{n}{n^2-1}k \leq l \leq \frac{2}{n}k$, then $\alpha = (k,l,k)$ is a Schur root for the quiver $Q(n,n)$.

Proof :

Start with $\alpha' = (k', l, k')$ where $(k', l) \in \mathbb{N}^2_+$ satisfying $\frac{2}{n}k' \leq l \leq nk'$, then α' is a Schur root for $Q(n,n)$ by proposition 2. Now, apply the reflexion functor with respect to the source 1. Then $\alpha" = (nl - k', l, k')$ is a Schur root for the quiver $Q(-n, n)$



Next, we can apply the reflexion functor to $Q(-n,n)$ with respect to the sink 3. Then, $\alpha = (nl - k', l, nl - k')$ is a Schur root for the quiver $Q(-n, -n)$



which is just the arabic way of writing $Q(n,n)$. Finally, since $\frac{2}{n}k' \leq l \leq nk'$ we get for $k = nl - k'$ that $\frac{n}{n^2-1}k \leq l \leq \frac{2}{n}k$.

(2.6) : Now, we are in a position to give a class of counterexamples to the Kac conjectures :

Proposition 4 : All Schur roots $\alpha = (k, l, k)$ of $Q(n,n)$ satisfying $l \leq \frac{2}{n}k - \frac{2}{n}$ are not indecomposable roots.

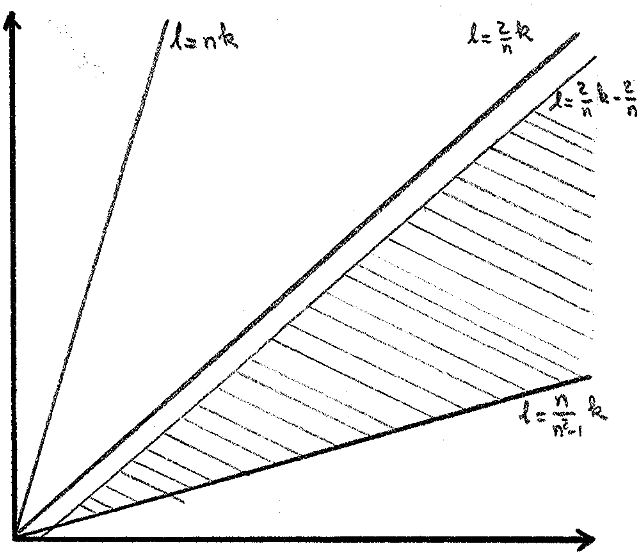
Proof :

The Ringel bilinear form associated to $Q(n,n)$ is determined by the matrix

$$\begin{pmatrix} 1 & -n & 0 \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{pmatrix}$$

It is now trivial to compute that $R((1,0,1),(k-1,l,k-1)) = R((k-1,l,k-1),(1,0,1)) = 2(k-1) - nl \geq 0$ by assumption.

Therefore, we obtain the following picture



By proposition 2 and 3 all $(k,l) \in \mathbb{N}_+^2$ such that $\frac{n}{n^2-1}k \leq l \leq nk$ give rise to Schur roots $\alpha = (k,l,k)$ for $Q(n,n)$. The line $l = \frac{2}{n}k$ divides this region in two and our reflexion process interchanges the two parts fixing the line. All Schur roots in the upper part are indecomposable roots,whereas most in the lower part (except those for which $\frac{2}{n}k - \frac{2}{n} < l \leq \frac{2}{n}k$ which seem to be indecomposable) are not indecomposable roots. Therefore, we see that the main reason for the failure of the Kac conjecture is

Theorem : In contrast to Schur roots, indecomposable roots need not be preserved under reflexion functors.

It would be interesting to know whether there exist counterexamples to Kac 1 where the components β and γ are both roots (resp. Schur roots).

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