Etale Local Structure of Matrix Invariants and Concomitants.

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0 : INTRODUCTION

One of the basic open problems in linear algebra is to determine the equivalence classes of m-tuples of n by n matrices under simultaneous conjugation. This problem is readily seen to be equivalent to that of studying n-dimensional representations of the free algebra $\mathbb{C} < X_1, ..., X_m >$ upto equivalence.

Geometrically, one has to sudy the orbit structure of mn^2-dimensional affine space

$$X_{m,n} = M_n(\mathbb{C}) \bigoplus ... \bigoplus M_n(\mathbb{C})$$

under action by componentswise conjugation of the general linear group $GL_n(\mathbb{C})$. For this situation, there does not exist an orbit-space since there are non-closed orbits. The best algebraic approximation of this (nonexistant) orbit space is the quotient variety

$$V_{m,n} = X_{m,n}/GL_n(\mathbb{C})$$

which is determined by its coordinate ring

$$\mathbb{C}[V_{m,n}] = \mathbb{C}[X_{m,n}]^{GL_n(\mathbb{C})}$$

It is well known that $V_{m,n}$ parametrizes the closed orbits which , by Artins fundamental paper [Ar] , are in one-to-one correspondence with the isomorphism classes of semi-simple n-dimensional representations of the free algebra $\mathbb{C} < X_1, ..., X_m >$.

From [Pr2] we get that the coordinate ring of $V_{m,n}$ is the center of the so called trace ring of m generic n by n matrices $\mathbb{T}_{m,n}$. By this we mean the following : consider the coordinate ring

$$\mathbb{C}[X_{m,n}] = \mathbb{C}[x_{ij}(l) : 1 \leq i,j \leq n; 1 \leq l \leq m]$$

then we can consider in $M_n(\mathbb{C}[X_{m,n}])$ the so called generic matrices

$$X_l = (x_{ij}(l))_{i,j} \in M_n(\mathbb{C}[X_{m,n}])$$

The $\mathbb{C}$-algebra generated by these elements is called the ring of m generic n by n matrices , $\mathbb{G}_{m,n}$. Then $\mathbb{T}_{m,n}$ is the subalgebra of $M_n(\mathbb{C}[X_{m,n}])$ generated by $\mathbb{G}_{m,n}$ and $Tr(\mathbb{G}_{m,n})$.

It is known that $V_{m,n}$ is a unirational variety and that the Formanek center of $\mathbb{T}_{m,n}$ (see [Pr3]) determines an open smooth subvariety of $V_{m,n}$.

In this paper we aim to initiate the geometric study of the varieties $V_{mn,n}$ both globally and locally based on some powerfull results of D. Luna [Lu]. We will now briefly describe the main results :

We say that a point $\xi \in V_{m,n}$ is of representation type $\tau = (e_1, k_1; ...; e_r, k_r)$ if the corresponding isomorphism class of semi-simple representations is build from r distinct simple components of dimensions k_i occuring with multiplicities e_i.If τ is such a representation type we call $V_{m,n}(\tau)$ the subset of $V_{m,n}$ consisting of all points of representation type τ. We will prove that the sets $V_{m,n}(\tau)$ form a finite stratification into locally closed smooth subvarieties. Furthermore, we prove that $V_{m,n}(\tau)$ lies in the closure of $V_{m,n}(\tau')$ if and only if τ is a degeneration (or refinement) of τ' (theorem II.1.1).

Next, we determine explicitly the étale local structure in a point $\xi \in V_{m,n}$ of representation type $(1, k_1; ...; 1, k_r)$ (theorem II.2.1). A generalization of this result to arbitrary representation type can be phrased in the setting of representations of quivers and their invariants in the following way. Let $\Delta = (\Delta_0, \Delta_1)$ be a finite quiver where Δ_0 is the set of vertices and Δ_1 the set of directed arrows between these vertices (we allow loops and multiple edges). A representation of Δ is a couple $V = (V_x, V_\alpha)$ where each V_x for $x \in \Delta_0$ is a finite dimensional vector space and the V_α for $\alpha \in \Delta_1$ are linear maps between the corresponding vectorspace. $d = (dim(V_x))_x$ is called the dimension vector of the representation. Conversely, for any dimension vector d we can look at the space of all representations of Δ with this dimension vector. This is an affine space admitting a natural action of the reductive group

$$GL(\Delta, d) = \prod GL_{d_x}(\mathbb{C})$$

and we denote the corresponding quotient-variety by $V(\Delta, d)$. Now, let $\xi \in V_{m,n}$ be of representation type $(e_1, k_1; ...; e_r, k_r)$ then we can form a quiver Δ_ξ consisting of r vertices $\{x_1, ..., x_r\}$ and $(m-1)k_i^2+1$ loops in vertex x_i and $(m-1)k_ik_j$ directed edges from vertex x_i to x_j. Let d_ξ be the dimension vector $(e_1, ..., e_r)$. Then, one can show as in II.2 that a neighborhood of the origin in $V(\Delta_\xi, d_\xi)$ is analytically isomorphic to a neighborhood of ξ in $V_{m,n}$. Furthermore, we claim that the coordinate ring of these quotient varieties $V(\Delta, d)$ are generated by traces of oriented cycles in the quiver. In this paper we prove this claim for the special case Δ_ξ, d_ξ if ξ is of representation type $(1, k_1; ...; 1, k_r)$ using the results on tori-invariants.

In section II.3 we combine these results to prove that the singular locus of $V_{m,n}$ is determined by the Formanek center of $\mathbb{T}_{m,n}$.

In the final chapter we give an explicit description of the étale local structure of the trace ring of m generic n by n matrices $\mathbb{T}_{m,n}$ in a point $\xi \in V_{m,n}$ of representation type $(1, k_1; ...; 1, k_r)$. We prove that this 'noncommutative slice' is a Cohen-Macaulay module and that its Poincaré series satisfies a specific functional equation. In the final section, these results are applied in order to solve the regularity problem for trace rings of generic matrices : $gldim(\mathbb{T}_{m,n}) < \infty$ if and only if m or n is equal to one or $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$.

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I : PRELIMINARIES

For the reader's convenience, we have collected in this first chapter the basic results which will be used throughout the paper. Further,we fix notation and terminology which we will use freely in the next two chapters.

I.1 : THE ETALE SLICE THEOREM

Throughout this paper, the basefield will be $\mathbb{C}$. Let G be a linear algebraic group. A representation of G is a finite dimensional complex vectorspace V (the representation space) together with a homomorphism of algebraic groups

$$\rho : G \to GL(V)$$

We denote this representation by ρ or (V, G). We will always assume that G is reductive, that is, every representation of G is completely reducible.

An action of G on a complex algebraic variety X is said to be rational if the canonical map

$$G \times X \to X$$

is a morphism of varieties. If the action is rational, G acts on the coordinate ring $\mathbb{C}[X]$ of X and one can define an algebraic variety X/G by $\mathbb{C}[X/G] = \mathbb{C}[X]^G$, the invariant ring under this action which happens to be affine. The natural inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ gives rise to a morphism of varieties

$$\pi_X : X \to X/G$$

which is onto and separates disjoint G-stable closed subsets of X, [Mu,Ch.1,2].

Let $\xi \in X/G$, then it follows from these facts that the fiber $\pi_X^{-1}(\xi)$ contains exactly one closed orbit (that of minimal dimension) which we will denote by $T(\xi)$. Therefore, the points of the quotient variety X/G parametrize the closed G-orbits in X.

If x is a point of X , we will denote by $G(x)$ the orbit of x under G and by G_x the isotropy group of x, that is, the stabilizer of the point x. The theorem of Matsuchima [Ma] asserts that whenever the orbit $G(x)$ is closed, the isotropy group G_x is itself a reductive group.

Before stating the étale slice theorem, we need to recall some facts about fibers. Let H be a reductive group acting rationally on an affine variety Y and let G be a reductive group containing H as a subgroup. The group H acts on G by translations on the right which makes G into the total space of a principal fibration over H (locally trivial in the étale topology). That is, there exists an affine variety Z, an étale and onto morphism $Z \to G/H$ and an H-isomorphism

$$H \times Z \simeq G \times_{G/H} Z$$

Recall that a map between smooth complex algebraic varieties is étale if its differential is everywhere an isomorphism.

Now, H acts on $G \times Y$ by $h.(g,y) = (gh^{-1}, hy)$ and we denote the quotient under this action by $G \times^H Y$. The action of G on itself by translations on the left passes through this quotient in such a way that the projection

$$G \times^H Y \to G/H$$

commutes with the action of G. The action of G on $G \times^H Y$ is totally determined by the action of H on Y and $(G \times^H Y)/G$ is identical to Y/H.

Again, let X be a rational affine G-variety and let X' be a G-invariant subset of X. We say that X' is a G-saturated subset of X if $\pi_X^{-1}(\pi_X(X')) = X'$. If X' is affine, we also have that the natural map $X'/G \to X/G$ is an open embedding.

We can now state a version of Luna's slice theorem [Lu,p.97] or [Sc,Th.5.3,p.55]

Theorem I.1.1 : Let X be a representation space of G. Let $x \in X$ be such that the orbit $G(x)$ is closed. Choose a G_x-splitting of $X \simeq T_x(X)$ as $T_x(G(x)) \oplus N_x$ and let ϕ denote the canonical equivariant map

$$\phi : G \times^{G_x} N_x \to X$$

$$[g,n] \to g.(x+n)$$

Then there exists an affine open G-saturated subset Y of X and an affine open G_x-saturated neighborhood B_x of 0 in N_x such that the maps

$$\phi : G \times^{G_x} B_x \to Y$$

$$\overline{\phi} : (G \times^{G_x} B_x)/G \to Y/G$$

are both étale, where $\overline{\phi}$ denotes the map induced by ϕ. Also, ϕ and the natural map $G \times^{G_x} B_x \to B_x/G_x$ give rise to a fiber product diagram

$$\begin{array}{ccc} G \times^{G_x} B_x & \longrightarrow & B_x/G_x \simeq (G \times^{G_x} B_x)/G \\ \downarrow \phi & & \downarrow \overline{\phi} \\ Y & \longrightarrow & Y/G \end{array}$$

with a G-isomorphism

$$G \times^{G_x} B_x \cong Y \times_{Y/G} B_x/G_x$$

Whereas ϕ is a morphism between smooth varieties, $\overline{\phi}$ usually is not. We recall that a morphism between two affine complex algebraic varieties is étale in a point iff

it is a local isomorphism of analytic spaces in that point,i.e. a homeomorphism which induces an isomorphism of the structure sheaves (in the analytic topology,obtained after embedding both varieties in a suitable affine space).

Therefore, it is sometimes profitable to look at the following realization of X/G. Let $y_1, ..., y_d$ be generators of $\mathbb{C}[X]^G$ and let

$$y = (y_1, ..., y_d) : X \to \mathbb{C}^d$$

The image of y ,which we will call Z, is closed in $\mathbb{C}^d$ and naturally isomorphic to X/G , see [Sc,Prop.5.2.1]. If X, G, y, Z and d are as above, we call y and the quintuple (X, G, y, Z, d) orbit maps.

Now, let H be a reductive algebraic subgroup of G. Then $Z_{(H)}$ will denote the set of points in Z whose corresponding closed orbit has an isotropy group which is conjugate (in G) to H. If $Z_{(H)} \neq \emptyset$, then we say that (H) is an isotropy class of (X, G). If H and H' are subgroups of G, we will write $(H) \leq (H')$ iff H is conjugate to a subgroup of H'. As a consequence of the étale slice theorem, one obtains the next result of G. Schwartz,[Sc,lemma 5.5] :

Theorem I.1.2 : Let (X, G, y, Z, d) be an orbit map.

(1) : $\{Z_{(H)} : (H)$ an isotropy class of $(X, G)\}$ is a finite stratification of Z into locally closed irreducible smooth algebraic subvarieties.

(2) : If (H) is an isotropy class of (X, G) , then $cl(Z_{(H)}) = \bigcup_{(H) \leq (H')} Z_{(H')}$.

Here, $cl(-)$ denotes the Zariski closure in Z, but since $Z_{(H)}$ is constructible this closure coincides with the closure in the analytic topology.

If (H) is an isotropy class of (X, G) and if $y : X \to Z$ is an orbit map, we denote by $X_{(H)}$ the inverse image $y^{-1}(Z_{(H)})$. $X_{(H)}$ is equipped with the structure of a locally closed subvariety of X, but is not necessarely reduced. Moreover, we have [Lu,Cor.5,p.101] :

Theorem I.1.3 : With the reduced structure, $X_{(H)}$ is smooth and the map $y : X_{(H)} \to Z_{(H)}$ is a smooth map and a locally trivial fibration in the étale topology with fibers G/H ,i.e. there exists an affine variety Z', an étale and onto morphism $Z' \to Z_{(H)}$ and a G-isomorphism

$$(G/H) \times Z' \cong X_{(H)} X_{Z_{(H)}} Z'$$

I.2 : INVARIANTS OF TORI

Let Φ be an r by s matrix with integer coefficients such that $rk(\Phi) = r \leq s$. Let α be an r-dimensional (column) vector over the integers and define the following sets

$$E_\Phi = \{\beta \in \mathbb{N}^s : \Phi.\beta = 0\}$$

$$E_{\Phi,\alpha} = \{\beta \in \mathbb{N}^s : \Phi.\beta = \alpha\}$$

It is clear that E_Φ is a submonoid of $\mathbb{N}^s$ and that $E_{\Phi,\alpha}$ is an E_Φ-module in the sense that $E_\Phi + E_{\Phi,\alpha} \subset E_{\Phi,\alpha}$.

$\mathbb{C}E_\Phi$ will denote the monoid algebra of E_Φ over $\mathbb{C}$, that is the subalgebra of $\mathbb{C}[x_1, ..., x_s]$ generated by the monomials x^β for all $\beta \in E_\Phi$ where we denote $x_1^{\beta_1}...x_s^{\beta_s} = x^\beta$ for any $\beta \in \mathbb{N}^s$.

Similarly, we define $\mathbb{C}E_{\Phi,\alpha}$ to be the $\mathbb{C}$-subvectorspace of $\mathbb{C}[x_1, ..., x_s]$ generated by the monomials x^β where $\beta \in E_{\Phi,\alpha}$. It is clear that $\mathbb{C}E_{\Phi,\alpha}$ is a $\mathbb{C}E_\Phi$-module in the obvious way.

Both the ring $\mathbb{C}E_\Phi$ and the module $\mathbb{C}E_{\Phi,\alpha}$ have an interpretation in terms of invariant theory. Suppose that

$$\Phi = [\gamma_1, ..., \gamma_s]$$

where each γ_i is an r-dimensional vector, then we define

$$T_r = \{diag(u^{\gamma_1}, ..., u^{\gamma_s}) : u \in (\mathbb{C}^*)^r\}$$

where for $u = (u_1, ..., u_r)^\tau$ and $\gamma_i = (\gamma_{i_1}, ..., \gamma_{i_r})^\tau$ we denote $u^{\gamma_i} = u_1^{\gamma_{i_1}}...u_r^{\gamma_{i_r}}$.

Because $rk(\Phi) = r$, T_r is a subgroup of $GL_s(\mathbb{C})$ isomorphic to $(\mathbb{C}^*)^r$ and hence by definition an r-dimensional torus. T_r acts in a natural way on the polynomial ring $\mathbb{C}[x_1, ..., x_s]$, viz. if $\tau_u = diag(u^{\gamma_1}, ..., u^{\gamma_s}) \in T_r$, then

$$\tau_u.f(x_1, ..., x_s) = f(u^{\gamma_1}x_1, ..., u^{\gamma_s}x_s)$$

One verifies easily that $\mathbb{C}E_\Phi$ is equal to the ring of invariants $\mathbb{C}[x_1, ..., x_s]^{T_r}$. Since T_r is linearly reductive, it follows from this that $\mathbb{C}E_\Phi$ is an affine Cohen-Macaulay algebra, [HR].

We also wish to interpret the module $\mathbb{C}E_{\Phi,\alpha}$ in terms of invariant theory. Suppose that the equation $\Phi.\beta = \alpha$ has at least one integral solution $\beta \in \mathbb{Z}^s$, then the map

$$\chi_\alpha : T_r \to \mathbb{C}^*$$

defined by $\chi_\alpha(\tau_u) = u^\alpha$ is a one- dimensional representation (or character) of T_r. Then, one can verify that $\mathbb{C}E_{\Phi,\alpha}$ is the module of semi-invariants or relative invariants of T_r with respect to the character χ_α, i.e.

$$\mathbb{C}E_{\Phi,\alpha} = \{f(x_1, ..., x_s) \in \mathbb{C}[x_1, ..., x_s] : \forall \tau_u \in T_r : \tau_u.f(x_1, ..., x_s) = \chi_\alpha(\tau_u)f(x_1, ..., x_s)\}$$

From this one can deduce that $\mathbb{C}E_{\Phi,\alpha}$ is a finitely generated $\mathbb{C}E_\Phi$-module. However, it is not true in general that $\mathbb{C}E_{\Phi,\alpha}$ is a Cohen-Macaulay module. Stanley [St,Th.3.2] has proved the following sufficient condition :

Proposition I.2.1 : Suppose there exists a rational solution $\beta = (\beta_1, ..., \beta_s)^\tau \in \mathbb{Q}^s$ to $\Phi.\beta = \alpha$ satisfying $-1 < \beta_i \leq 0$ for all i , then $\mathbb{C}E_{\Phi,\alpha}$ is a Cohen-Macaulay module.

In [St] and [Ho] it was shown that ring- and moduletheoretic properties of $\mathbb{C}E_\Phi$ and $\mathbb{C}E_{\Phi,\alpha}$ are closely linked to topological properties of the $(s-r)$-dimensional convex polyhedral cone C_Φ in $\mathbb{R}^s$ consisting of the set of all solutions $\beta \in \mathbb{R}^s$ to $\Phi.\beta = 0$.

Let $\mathcal{P}_\Phi$ be the non-degenerate cross-section $C_\Phi \cap \{(\beta_1, ..., \beta_s) \in \mathbb{R}^s : \sum \beta_i = 1\}$ of C_Φ , then $\mathcal{P}_\Phi$ is an $(s-r-1)$-dimensional convex polytope.

If $\beta \in \mathbb{R}^s_+$, we define its support $supp(\beta) = \{i : \beta_i > 0\}$ and if $\beta \in \mathbb{R}^s$ we define its negative support $supp_-(\beta) = \{i : \beta_i < 0\}$. If $\mathcal{F}$ is a face of $\mathcal{P}_\Phi$, then all elements of the relative interior of $\mathcal{F}$ have the same support, $supp(\mathcal{F})$. It follows that the faces of $\mathcal{P}_\Phi$ are in one-to-one correspondence with the supports of elements $\beta \in E_\Phi$ and that two faces $\mathcal{F}, \mathcal{G}$ satisfy $\mathcal{F} \subset \mathcal{G}$ iff $supp(\mathcal{F}) \subset supp(\mathcal{G})$. If v is a vertex of $\mathcal{P}_\Phi$, then those elements $\beta \in E_\Phi$ satisfying $supp(\beta) = supp(v)$ are $\mathbb{N}$-multiples of a unique element $\beta_v \in E_\Phi$, the so called completely fundamental solution corresponding to the vertex v. Recall that $\beta \in E_\Phi$ is said to be completely fundamental if whenever $m\beta = \gamma + \delta$ where $m \geq 1$ and $\gamma, \delta \in E_\Phi$, then $\gamma = i\beta$ for some $0 \leq i \leq m$. With $CF(E_\Phi)$ one denotes the set of completely fundamental solutions. The Krull dimension of $\mathbb{C}E_\Phi$ is equal to the dimension of the $\mathbb{Q}$- vectorspace spanned by $CF(E_\Phi)$, i.e. is equal to $s-r$.

Let $< E_\Phi >$ denote the group generated by E_Φ in $\mathbb{Z}^s$ and let $< E_{\Phi,\alpha} >= < E_\Phi > + E_{\Phi,\alpha}$, the coset of $< E_\Phi >$ in $\mathbb{Z}^s$ containing $E_{\Phi,\alpha}$. Now, let $\mathcal{P}^*_\Phi$ be the dual polytope of $\mathcal{P}_\Phi$ and define

$$\Gamma_\beta = \bigcup \{\mathcal{F}^* : \mathcal{F} \text{ face of } \mathcal{P}_\Phi \text{ s.t. } supp_-(\beta) \subset supp(\mathcal{F})\}$$

From [St2,p.47] we retain that if $\mathbb{C}E_{\Phi,\alpha}$ is a Cohen-Macaulay module of dimension d , its d-th local cohomology module with respect to the irrelevant ideal of $\mathbb{C}E_\Phi$ is given by

$$H^d(\mathbb{C}E_{\Phi,\alpha}) = \mathbb{C}\{x^\beta : \beta \in < E_{\Phi,\alpha} >, \Gamma_\beta = \emptyset\}$$

The $\mathbb{C}E_\Phi$-module structure on it is given by

$$x^\gamma . x^\beta = x^{\beta+\gamma} \text{ if } \Gamma_{\gamma+\beta} = \emptyset$$

$$= 0 \text{ if } \Gamma_{\gamma+\beta} \neq \emptyset$$

for all $\gamma \in E_\Phi$ and $x^\beta \in H^d(\mathbb{C}E_{\Phi,\alpha})$.

Suppose there exists $\gamma = (\gamma_1, ..., \gamma_s) \in \mathbb{Q}^s$ with $-1 < \gamma_i \leq 0$ such that $\Phi.\beta = \alpha$, then by Prop.I.2.1, $\mathbb{C}E_{\Phi,\alpha}$ is a Cohen-Macaulay module. In this case, $H^d(\mathbb{C}E_{\Phi,\alpha})$ can be described as follows : choose $m \in \mathbb{N}$ such that $m\gamma$ is integral and let $\beta \in < E_{\Phi,\alpha} >$, then $\Phi.m(\beta - \gamma) = 0$ and $supp_-(\beta) = supp_-(\beta - \gamma) = supp_- m(\beta - \gamma)$. Since Γ_β depends only upon $supp_-(\beta)$ we have $\Gamma_\beta = \Gamma_{m(\beta-\gamma)}$. If we assume,further,that there exists a solution $\delta = (\delta_1, ..., \delta_s) \in E_\Phi$ such that $\delta_i > 0$ for all i , then since $m(\beta - \gamma) \in < E_\Phi >$ we have [St2,p.47] : $\Gamma_{m(\beta-\gamma)} = \emptyset$ iff $m(\beta - \gamma) < 0$. Therefore, we have

$$H^d(\mathbb{C}E_{\Phi,\alpha}) = \mathbb{C}\{x^\beta : \beta \in < E_{\Phi,\alpha} >: \beta < 0\}$$

I.3 : FINITE DIMENSIONAL REPRESENTATIONS

An n-dimensional representation of the free algebra in m variables is an algebra morphism

$$\phi : \mathbb{C} < X_1, ..., X_m > \rightarrow M_n(\mathbb{C})$$

Note that this is equivalent to giving m elements $\phi(X_1), ..., \phi(X_m)$ in $M_n(\mathbb{C})$, i.e. n-dimensional representations of $\mathbb{C} < X_1, ..., X_m >$ can be parametrized by an affine variety $X_{m,n}$ which is just mn^2-dimensional affine space.

Two n-dimensional representations are said to be equivalent if they differ upto a $\mathbb{C}$-automorphism of $M_n(\mathbb{C})$. So, the projective linear group $PGL_n(\mathbb{C})$ acts on $X_{m,n}$ and the orbits under this action are the equivalence classes of representations.

If ϕ is an n-dimensional representation, then $\mathbb{C}^{(n)}$ becomes a $\mathbb{C} < X_1, ..., X_m >$-module via ϕ. If $\mathbb{C}^{(n)}$ is completely reducible as such, then ϕ is said to be semi-simple. In general one has a composition series

$$0 = V_t \subset V_{t-1} \subset ... \subset V_1 \subset V_0 = \mathbb{C}^{(n)}$$

of $\mathbb{C} < X_1, ..., X_m >$-modules. Then, $W = \oplus(V_i/V_{i+1})$ is completely reducible and n-dimensional. For a suitable choice of basis of $\mathbb{C}^{(n)}$,ϕ can be expressed in the matrix-form

$$\phi = \begin{pmatrix} \phi_1 & & N \\ & \ddots & \\ 0 & & \phi_t \end{pmatrix}$$

where the $\phi_j : \mathbb{C} < X_1, ..., X_m > \rightarrow M_{k_j}(\mathbb{C})$ for $k_j = dim_{\mathbb{C}}(V_{j-1}/V_j)$ are the irreducible components. With ϕ we can associate the semi-simple representation

$$\phi^{ss} = \phi_1 \oplus ... \oplus \phi_t$$

isomorphic to W as module.

Artin [Ar] proved that ϕ^{ss} lies in the closure of the orbit $GL_n(\mathbb{C})(\phi)$.Moreover he proved that the closed orbits in $X_{m,n}$ under action of $GL_n(\mathbb{C})$ correspond precisely to the semi-simple representations. Therefore, the quotient variety

$$V_{m,n} = X_{m,n}/GL_n(\mathbb{C})$$

parametrizes the equivalence classes of semi-simple n- dimensional representations of $\mathbb{C} < X_1, ..., X_m >$.

A concrete description of the coordinate ring $\mathbb{C}[V_{m,n}]$ can be given as follows. Consider the polynomial ring

$$P_{m,n} = \mathbb{C}[x_{ij}(l) : 1 \leq i, j \leq n; 1 \leq l \leq m]$$

and let $\mathbb{G}_{m,n}$ be the ring of m generic n by n matrices, that is the subalgebra of $M_n(\mathcal{P}_{m,n})$ generated by the m elements

$$X_l = (x_{ij}(l))_{i,j} \in M_n(\mathcal{P}_{m,n})$$

The ring of matrixinvariants $\mathcal{R}_{m,n}$ is the subalgebra of $\mathcal{P}_{m,n}$ generated by the traces of elements of $\mathbb{G}_{m,n}$. See for example [Pr] for a proof that $\mathcal{R}_{m,n}$ is an affine algebra and $\mathbb{C}[V_{m,n}] = \mathcal{R}_{m,n}$.

Not much is known about the geometry of $V_{m,n}$. In [Pr] it was proved that the points corresponding to the equivalence classes of irreducible n-dimensional representations form an open smooth subvariety $V^{irr}_{m,n}$ of dimension $(m-1)n^2+1$.

If one is not only interested in the semi-simple representations but also in their irreducible components one has to study a certain 'noncommutative algebraic variety' $U_{m,n}$. The trace ring of m generic n by n matrices is the subalgebra of $M_n(\mathcal{P}_{m,n})$ generated by $\mathcal{R}_{m,n}$ and $\mathbb{G}_{m,n}$ and will be denoted by $\mathbb{T}_{m,n}$.

The points of $U_{m,n}$ are the maximal twosided ideals of the noncommutative but affine p.i.-algebra $\mathbb{T}_{m,n}$. We can equip $U_{m,n}$ with the usual Zariski topology [Pr3], that is a typical open set consists of those maximal ideals not containing a given twosided ideal of $\mathbb{T}_{m,n}$. Since $\mathcal{R}_{m,n} \subset \mathbb{T}_{m,n}$ is a central extension, there is a canonical continuous map

$$i : U_{m,n} \to V_{m,n}$$

In [AS] the fibers of i were described in the following way. Let $\xi \in V_{m,n}$, then ξ corresponds to the equivalence class of a semi-simple representation

$$\phi = \phi_1 \oplus ... \oplus \phi_t$$

Let $\chi_1, ..., \chi_r$ be the distinct irreducible components, where χ_i is a k_i-dimensional representation occuring with multiplicity e_i in ϕ. That is, $\sum_{i=1}^{r} e_i = t$ and $\sum_{i=1}^{r} e_i.k_i = n$. We can always assume that $k_1 \geq k_2 \geq ... \geq k_r$ and then we say that ξ or ϕ is of representation-type $(e_1, k_1; e_2, k_2; ...; e_r, k_r)$.

The fiber $i^{-1}(\xi)$ consists of r points (ϕ, ξ_i) each corresponding to one of the distinct irreducible components. The morphism i is then given by sending a point $\xi = (\phi, \xi_i)$ to ξ.

II : THE VARIETIES $V_{m,n}$

In this chapter we aim to initiate the geometrical study of the varieties $V_{m,n}$. In the first section we will show that the different representation types give a finite stratification of $V_{m,n}$ into locally closed smooth subvarieties. In the second section we describe the étale local structure of $V_{m,n}$ in points corresponding to semi-simple representations with distinct irreducible components. In the last section we show that, except when $(m,n) = (2,2)$, $V_{m,n}$ is always singular and the singular locus is precisely the difference $V_{m,n} - V_{m,n}^{irr}$.

II.1 : STRATIFICATION OF $V_{m,n}$

Recall that a point $\xi \in V_{m,n}$ is said to be of representation-type $\tau = (e_1, k_1; ...; e_r, k_r)$ if the corresponding semi-simple n-dimensional representation has r distict irreducible components χ_i of dimension k_i and multiplicity e_i.

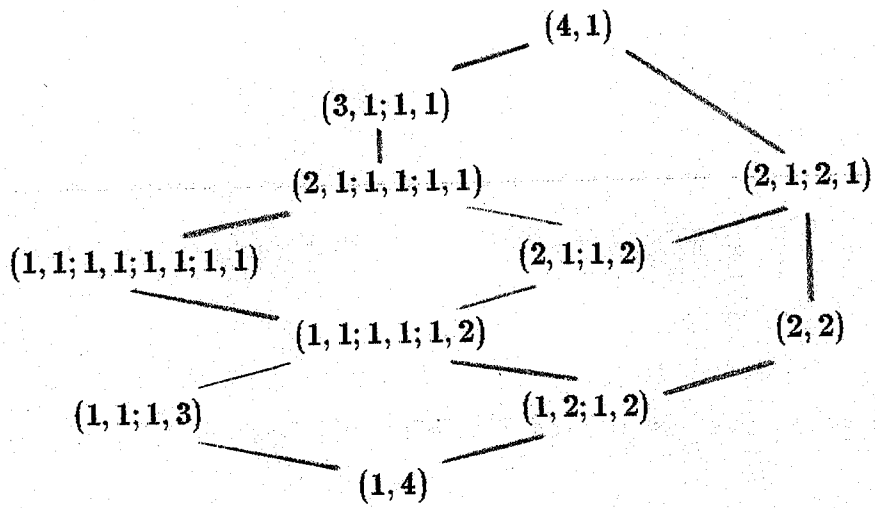
Another representation-type $\tau' = (e'_1, k'_1; ...; e'_{r'}, k'_{r'})$ is said to be a refinement of τ if there is a permutation σ on $\{1, ..., r'\}$ such that there exist natural numbers

$$j_0 = 1 \leq j_1 < j_2 < ... < j_r = r'$$

such that for every $1 \leq i \leq r$ we have

$$\begin{cases} e_i k_i = \displaystyle\sum_{j=j_{i-1}+1}^{j_i} e'_{\sigma(j)} k'_{\sigma(j)} \\ e_i \mid e'_{\sigma(j)} \text{ for all } j_{i-1} < j \leq j_i \end{cases}$$

This defines a partial ordering on the set of all representation- types for n-dimensional representations : RT_n. For example, RT_4 has the following Hasse-diagram :



For a representation-type $\tau \in T_n$, we denote by $V_{m,n}(\tau)$ the set of all points $\xi \in V_{m,n}$ of type τ. The main result of this section can now be stated as :

Theorem II.1.1 : With notations as above we have :
(1) : $\{V_{m,n}(\tau) : \tau \in RT_n\}$ is a finite stratification of $V_{m,n}$ into locally closed irreducible smooth algebraic subvarieties.
(2) : $V_{m,n}(\tau')$ lies in the closure of $V_{m,n}(\tau)$ if and only if τ' is a refinement of τ.

Proof :

Let $\tau = (e_1, k_1; ...; e_r, k_r)$ and $\xi \in V_{m,n}(\tau)$, then the fiber of ξ under the morphism $\pi_{m,n} : X_{m,n} \to V_{m,n}$ contains one closed orbit $T(\xi)$. In this orbit, one can find a point $x = (x_1, ..., x_m)$ where each n by n matrix x_i is of the form

$$\begin{pmatrix} m_1 \otimes 1_{e_1} & 0 & & \\ 0 & m_2 \otimes 1_{e_2} & 0 & \\ & & & \\ 0 & & m_r \otimes 1_{e_r} & \end{pmatrix} = x_i$$

where each $m_i \in M_{k_i}(\mathbb{C})$.

We will now compute the isotropy group in this point : $GL_n(\mathbb{C})_x$. An element $\alpha \in GL_n(\mathbb{C})$ leaves x fixed if and only if it commutes with each of the x_i. Therefore, $GL_n(\mathbb{C})_x$ is the multiplicative group of units of the centralizer of

$$\begin{pmatrix} M_{k_1}(\mathbb{C}) \otimes 1_{e_1} & & 0 \\ & & \\ 0 & & M_{k_r}(\mathbb{C}) \otimes 1_{e_r} \end{pmatrix}$$

which is the algebra generated by the x_i's by assumption. It is easy to verify that this group is equal to

$$GL_n(\mathbb{C})_x \simeq GL_{e_1}(\mathbb{C}) \times ... \times GL_{e_r}(\mathbb{C})$$

where the embedding in $GL_n(\mathbb{C})$ is given by

$$GL_{e_1}(\mathbb{C}).1_{k_1} \times ... \times GL_{e_r}(\mathbb{C}).1_{k_r}$$

Of course, a different choice of the element x in $T(\xi)$ gives a group conjugated to $GL_n(\mathbb{C})_x$ in $GL_n(\mathbb{C})$. Further, if $x' \in T(\xi')$ is chosen such that

$$GL_n(\mathbb{C})_{x'} = GL_{e'_1}(\mathbb{C}).1_{k'_1} \times ... \times GL_{e'_r}(\mathbb{C}).1_{k'_r}$$

and if $GL_n(\mathbb{C})_{x'}$ is conjugated to $GL_n(\mathbb{C})_x$ in $GL_n(\mathbb{C})$, then it is clear that $r = r'$ and there exists a permutation σ on $\{1, ..., r\}$ such that $(e_i, k_i) = (e'_{\sigma(i)}, k'_{\sigma(i)})$, i.e. ξ and ξ' belong to the same set $V_{m,n}(\tau)$.

The statement now follows immediatly from Theorem I.1.2.(1).

(2) : In the first part we have shown that the isotropy corresponding to $V_{m,n}(\tau)$, where $\tau = (e_1, k_1; ...; e_r, k_r)$, is the conjugacy class of

$$GL_{e_1}(\mathbb{C}).1_{k_1} x ... x GL_{e_r}.1_{k_r} \stackrel{\text{def}}{=} GL_n(\mathbb{C})_\tau$$

in $GL_n(\mathbb{C})$. From theorem I.1.2.(2) we know that $V_{m,n}(\tau')$ lies in the closure of $V_{m,n}(\tau)$ if and only if the group $GL_n(\mathbb{C})_\tau$ is conjugated to a subgroup of $GL_n(\mathbb{C})_{\tau'}$.It is easy to verify that this happens precisely when τ' is a refinement of τ.

For example, the closed subvariety of $V_{m,n}$ determined by the Formanek center of the trace ring of m generic n by n matrices (in out terminology $V_{m,n} - V_{m,n}(1, n)$) is in general reducible.

Each of its $\lceil \frac{n}{2} \rceil$ irreducible components contain an open set induced by $V_{m,n}(1, i; 1, n-i)$.

Finally, we note that the dimension of the subvariety $V_{m,n}(\tau)$ where $\tau = (e_1, k_1; ...; e_r, k_r)$ is equal to $(m-1)(k_1^2 + ... + k_r^2) + r$.

II.2 : LOCAL STRUCTURE OF $V_{m,n}$

According to the étale slice theorem,the local structure of the variety $V_{m,n} = X_{m,n}/GL_n(\mathbb{C})$ near a point ξ is isomorphic to that of the quotient of the slice representation near the origin , i.e. with $N_x/GL_n(\mathbb{C})_x$ where $x \in T(\xi)$ and N_x is the normal space in $X_{m,n}$ to the orbit $GL_n(\mathbb{C})(x)$.

Suppose ξ is a point of type $(e_1, k_1; ...; e_r, k_r)$, then we can take for $x = (x_1, ..., x_m) \in X_{m,n}$ such that each of the x_i has the form

$$x_i = \begin{pmatrix} m_1 \otimes 1_{e_1} & & \underline{0} \\ & \ddots & \\ \underline{0} & & m_r \otimes 1_{e_r} \end{pmatrix}$$

where $m_i \in M_{k_i}(\mathbb{C})$.

In the foregoing section we have calculated the isotropy group in such a point

$$GL_n(\mathbb{C})_x = GL_{e_1}(\mathbb{C}).1_{k_1} \times ... \times GL_{e_r}(\mathbb{C}).1_{k_r}$$

The tangent space $T_x(GL_n(\mathbb{C})(x))$ in $X_{m,n}$ to the orbit $GL_n(\mathbb{C})(x)$ is equal to the image of the linear map

$$M_n(\mathbb{C}) \to M_n(\mathbb{C}) \bigoplus ... \bigoplus M_n(\mathbb{C})$$

$$y \to [y, x_1] \bigoplus ... \bigoplus [y, x_m]$$

see for example [Mo].

The kernel of this map is clearly the centralizer of the subalgebra of $M_n(\mathbb{C})$ generated by $\{x_1, ..., x_m\}$.

So, we obtain an exact sequence of $GL_n(\mathbb{C})_x$-modules

$$0 \to C_x \to M_n(\mathbb{C}) \to T_x(GL_n(\mathbb{C})(x)) \to 0$$

where

$$C_x = \begin{pmatrix} M_{e_1}(\mathbb{C}) \otimes 1_{k_1} & & \underline{0} \\ & \ddots & \\ \underline{0} & & M_{e_r}(\mathbb{C}) \otimes 1_{k_r} \end{pmatrix}$$

But then, since $GL_n(\mathbb{C})_x$ is a reductive group (so every $GL_n(\mathbb{C})_x$- module is determined upto isomorphism by its irreducible components), the normal space N_x to $T_x(GL_n(\mathbb{C})(x))$ is isomorphic to the $GL_n(\mathbb{C})_x$-module

$$N_x = M_n(\mathbb{C}) \bigoplus ... \bigoplus M_n(\mathbb{C}) \bigoplus C_x$$

where we have $m-1$ copies of $M_n(\mathbb{C})$ and the action of the isotropy group $GL_n(\mathbb{C})_x$ is,of course, given by componentswise conjugation.

The étale slice is then the variety corresponding to the ring of invariant polynomial mappings from N_x to $\mathbb{C}$ under this action of $GL_n(\mathbb{C})_x$.

We will now describe this ring in the special case that all the irreducible components χ_i of the to ξ associated semi-simple representation are distinct, that is, ξ is of type $(1, k_1; ...; 1, k_r)$ where $\sum k_i = n$.

In this case, the isotropy group of x is the r-dimensional torus T_r which is embedded in $GL_n(\mathbb{C})$ as

$$T_r = \mathbb{C}^*.1_{k_1} \times ... \times \mathbb{C}^*.1_{k_r}$$

Clearly, T_r acts trivially on the following subspace N_1 of N_x

$$N_1 = \bigoplus_{i=1}^{m-1} \begin{pmatrix} M_{k_1}(\mathbb{C}) & & \\ \underline{0} & & \underline{0} \\ & & M_{k_r}(\mathbb{C}) \end{pmatrix} \bigoplus \begin{pmatrix} \mathbb{C}.1_{k_1} & & \\ \underline{0} & & \underline{0} \\ & & \mathbb{C}.1_{k_r} \end{pmatrix}$$

so, the étale slice is

$$N_x/T_r = \mathring{A}^d \times N_2/T_r$$

where $d = (m-1)(k_1^2 + ... + k_r^2) + r$ and

$$N_2 = \bigoplus_{i \neq j}^{r} V_{ij}$$

where V_{ij} is an $(m-1)k_ik_j$-dimensional vectorspace on which an element $t = (\alpha_1, ..., \alpha_r) \in T_r$ acts by sending an element $v \in V_{ij}$ to $\alpha_i \alpha_j^{-1} v$. Let

$$S = \mathbb{C}[N_2] = \mathbb{C}[v_{ij}(\alpha) : 1 \leq i \neq j \leq r; 1 \leq \alpha \leq (m-1)k_ik_j]$$

then the action of T_r on S is diagonal and is therefore determined by an r by s matrix with integer coefficients where $s = Kdim(S) = 2(m-1)\sum_{i \neq j}^{r} k_ik_j$.

The column corresponding to the variable $v_{ij}(\alpha)$ consists of zeroes except at the i-th row $+1$ and at the j-th row -1.

One easily verifies that the last row is a linear combination of the others so we can restrict attention to the $r-1$ by s matrix Φ which is obtained by erasing this last row. One verifies that $rk(\Phi) = r-1$.

The ring of invariants, $\mathbb{C}[N_2/T_r]$, is obtained from the set of integer solutions $\beta \in \mathbb{N}^s$ to $\Phi.\beta = 0$. As we have seen before, it suffices to consider the fundamental solutions. In this case, they are also completely fundamental. The corresponding monomials in $S = \mathbb{C}[N_2]$ are obtained by the following procedure :

Let $2 \leq k \leq r$ and let $(i_1, ..., i_k)$ be a cycle of k distinct elements from $\{1, ..., r\}$ s.t. its minimal element is i_1. Then,we get the invariants

$$V_{i_1i_2} \times V_{i_2i_3} \times ... \times V_{i_{k-1}i_k} \times V_{i_ki_1}$$

which are generated by the elements

$$v_{i_1i_2}(\alpha_1)...v_{i_{k-1}i_k}(\alpha_{k-1})v_{i_ki_1}(\alpha_k)$$

where the α_j run over all admissible values.

Finally, we note that

$$dim(N_2/T_r) = s - (r-1) = 2(m-1)\sum k_i k_j - r + 1$$

which is compatible with the fact that

$$(m-1)n^2 + 1 = dim(V_{m,n} = d + dim(N_2/T_r)$$

In the next section we will give a more precise description in the special case that $r = 2$. Let us summarize things in

Theorem II.2.1 :

If ξ is a point in $V_{m,n}$ of type $(1, k_1; ...; 1, k_r)$, then a neighbourhood of ξ is isomorphic to a neighborhood of the origin in $\mathring{A}^d \times N_2/T_r$ where $d = (m-1)(k_1^2 + ... + k_r^2) + r$, $N_2 = \bigoplus_{i \neq j}^r V_{ij}$ where V_{ij} is $(m-1)k_i k_j$-dimensional and T_r acts on it by $(\alpha_1, ..., \alpha_r).v = \alpha_i \alpha_j^{-1} v$.

Further, the coordinate ring $\mathbb{C}[N_2/T_r]$ is the subring of $\mathbb{C}[N_2] = \mathbb{C}[v_{ij}(\alpha) : 1 \leq i \neq j \leq r, 1 \leq \alpha \leq (m-1)k_i k_j]$ generated by all monomials of the form $v_{i_1 i_2}(\alpha_1)...v_{i_{k-1} i_k}(\alpha_{k-1}) v_{i_k i_1}(\alpha_k)$ where $(i_1, i_2, ..., i_k)$ is a cycle of length $2 \leq k \leq r$ of distinct elements from $\{1, ..., r\}$. Its Krull dimension is $2(m-1)\sum k_i k_j - r + 1$.

II.3 : SINGULAR LOCUS OF $V_{m,n}$

The main result of this section states that the closed subvariety of $V_{m,n}$ determined by the Formanek center of the trace ring of m generic n by n matrices (or,equivalently, the set of reducible semi-simple representations) is precisely the singular locus of $V_{m,n}$.

If m or n is equal to 1, $V_{m,n}$ is clearly nonsingular, so we may assume that m and $n \geq 2$.

Proposition II.3.1 : The variety $V_{m,n}$ is singular except when $(m,n) = (2,2)$.

Proof :

Assume that $V_{m,n}$ is nonsingular. Since $\mathcal{R}_{m,n}$ is a positively graded affine algebra, it has to be a polynomial ring in $(m-1)n^2+1$ variables over $\mathbb{C}$. So, the Brauer group $Br(V_{m,n})$ is just $Br(\mathbb{C}) = 1$.

R. Hoobler proved in [Hb] the Auslander-Goldman conjecture stating that the Brauer group of a smooth affine variety is determined by the codimension one irreducible subvarieties. Therefore, we have

$$Br(V_{m,n}) = \bigcap_{p \in X^{(1)}} Br((\mathcal{R}_{m,n})_p)$$

where $X^{(1)}$ is the set of all height one prime ideals of $\mathcal{R}_{m,n}$ and the intersection is taken in the Brauer group of the field of fractions $\mathcal{K}_{m,n}$.

Now, we know that the localization $(\mathbb{T}_{m,n})_p$ of $\mathbb{T}_{m,n}$ at any height one prime ideal p of $\mathcal{R}_{m,n}$ is Azumaya except for $(m,n) = (2,2)$ and $p = (X_1X_2 - X_2X_1)^2$.

For, the dimension of the closed subvariety determined by p is equal to $(m-1)n^2$. Suppose that the corresponding localization of $\mathbb{T}_{m,n}$ is not Azumaya, then the points lying on this closed subvariety correspond to reducible semi-simple representations.

It follows from our stratification result (Th.II.1.1) that such a variety of maximal dimension has an open subset consisting of semi-simple representations with two irreducible components of dimensions r and $n-r$. Therefore, the dimension of such a variety is at most $(m-1)[r^2+(n-r)^2]+2$. Clearly, the equation

$$(m-1)[r^2+(n-r)^2]+2 = (m-1)n^2$$

has only an integer solution if $(m,n) = (2,2)$ and $r = 1$. It is well known from 19-th century algebra that $\mathcal{R}_{2,2} = \mathbb{C}[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1X_2)]$. Therefore, the class of the generic division algebra $\Delta_{m,n}$ in $Br(\mathcal{K}_{m,n})$ belongs to $\bigcap_{p \in X^{(1)}} Br((\mathcal{R})_p)$ provided $(m,n) \neq (2,2)$, and so we obtain a contradiction.

We will now investigate when $V_{m,n}$ is smooth in a point ξ corresponding to an equivalence class of a semi-simple representation having two distinct irreducible components.

Proposition II.3.2 : The étale slice of $V_{m,n}$ in a point ξ of type $(1,r;1,n-r)$ is $\mathring{A}^d \times W$ where $d = (m-1)[r^2 + (n-r)^2] + 2$ and

$$\mathbb{C}[W] = \mathbb{C}[t_{ij} : 1 \leq i,j \leq (m-1)r(n-r)]/I_2$$

where I_2 is the ideal generated by all 2 by 2 minors of the generic matrix $(t_{ij})_{i,j}$.

Proof :

By the calculations of the foregoing section we know that the slice in ξ is equal to

$$\mathring{A}^d \times N_2/T_2$$

where $N_2 = V_{12} \bigoplus V_{21}$ and both components are $(m-1)r(n-r)$-dimensional and an element $(\alpha,\beta) \in T_2$ acts on a generator $x_i \in S(V_{12})$ (resp. $y_j \in S(V_{21})$) by sending it to $\alpha\beta^{-1}x_i$ (resp. $\alpha^{-1}\beta y_j$). Therefore, the invariant ring $\mathbb{C}[N_2/T_2]$ is generated by the monomials $x_i y_j$ for all admissible values for i and j. The relations v monomials $x_i y_j$ for all admissible values for i and j. The relations among these invariants are easily seen to be generated by the 2 by 2 minors of the matrix $(x_i y_j)_{i,j}$. Sending the indeterminate t_{ij} to $x_i y_j$ we get the required statement.

Proposition II.3.3 : If ξ is a point of $V_{m,n}$ of type $(1,r;1,n-r)$, then $V_{m,n}$ is singular in ξ except when $(m,n) = (2,2)$.

Proof :

By étale descent, it suffices to show that the étale slice is singular in the origin. This follows from the fact that I_2 is a nontrivial ideal (if $(m,n) \neq (2,2)$) not generated by degree one elements (in the obvious gradation on $\mathbb{C}[t_{ij}]$.

Using this fact and the stratification result of II.1, we can now prove the main result of this section :

Theorem II.3.4 : The singular locus of the variety $V_{m,n}$ coincides with the complement $V_{m,n} - V_{m,n}^{irr}$ except when $(m,n) = (2,2)$.

Proof :

Let $F\mathbb{T}_{m,n}$ be the Formanek center of the trace ring of m generic n by n matrices, i.e. the ideal defining the open set $V_{m,n}^{irr}$. Let $\overline{V_{m,n}}$ be the variety defined by $\mathbb{C}[\overline{V_{m,n}}] = \mathcal{R}_{m,n}/F\mathbb{T}_{m,n}$, then by theorem II.1.1 we know that each of the irreducible components of $\overline{V_{m,n}}$ has an open set determined by semi-simple representations having two distinct irreducible components.

Suppose that $V_{m,n}^{reg}$, the open set of all regular points in $V_{m,n}$, is strictly larger than $V_{m,n}^{irr}$, then $V_{m,n}^{reg}$ induces a proper open subvariety in at least one of the irreducible components of $\overline{V_{m,n}}$. This entails that $V_{m,n}^{reg}$ contains points corresponding to semi-simple representations having two distinct irreducible components, but this is impossible by proposition II.3.3.

III : TRACE RINGS OF GENERIC MATRICES.

In this chapter we will investigate the étale local structure of the trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$. If $\xi \in V_{m,n}^{irr}$, it is well known that this étale local structure is just n by n matrices over a commutative (regular) domain. We will describe explicitly the structure when ξ is a point corresponding to a semi-simple representation with distinct irreducible components. It will turn out that the 'noncommutative slice' in such a point is Cohen-Macaulay and its Poincaré series satisfies a certain functional equation. In the final section, these results are applied to solve the regularity problem for trace rings, i.e. $gldim(\mathbb{T}_{m,n}) < \infty$ iff m or n is 1 or $(m,n) = (2,2), (2,3)$ or $(3,2)$.

III.1 : LOCAL STRUCTURE OF $\mathbb{T}_{m,n}$

Recall from [Pr2] that the trace ring of m generic n by n matrices is the ring of equivariant maps :

$$\phi : X_{m,n} = M_n(\mathbb{C}) \bigoplus \cdots \bigoplus M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

i.e. polynomial maps such that for every $\alpha \in GL_n(\mathbb{C})$ the following diagram is commutative :

$$\begin{array}{ccc} X_{m,n} & \xrightarrow{\phi} & M_n(\mathbb{C}) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X_{m,n} & \xrightarrow{\phi} & M_n(\mathbb{C}) \end{array}$$

where the action of $GL_n(\mathbb{C})$ on $M_n(\mathbb{C})$ is given by conjugation. Now, let ξ be any point in $V_{m,n}$ and $x \in T(\xi)$, then we know that the diagram below is defined and commutative in a neighborhood of x

$$\begin{array}{ccc} GL_n(\mathbb{C}) \times^{GL_n(\mathbb{C})_x} N_x & \longrightarrow & N_x/GL_n(\mathbb{C})_x \cong (GL_n(\mathbb{C}) \times^{GL_n(\mathbb{C})_x} N_x)/GL_n(\mathbb{C}) \\ \downarrow & & \downarrow \\ X_{m,n} & \longrightarrow & V_{m,n} \end{array}$$

where the morphism $N_x \to X_{m,n}$ is defined by sending a point n to $x + n$. There exists an open affine $GL_n(\mathbb{C})_x$ stable neighborhood N_x^o of the origin of N_x and an open affine $GL_n(\mathbb{C})$ stable neighborhood $X_{m,n}^o$ of x in $X_{m,n}$ so that

$$GL_n(\mathbb{C}) \times^{GL_n(\mathbb{C})_x} N_x^o \cong X_{m,n}^o \times_{U_{m,n}} (N_x^o/GL_n(\mathbb{C})_x)$$

From this the following can be easily proved : let B be the coordinate ring of $N_x^o/GL_n(\mathbb{C})_x$ which is an algebra over $\mathbb{C}[V_{m,n}]$ then $\mathbb{T}_{m,n} \otimes_{\mathbb{C}[V_{m,n}]} B$ (the non-commutative ring $\mathbb{T}_{m,n}$ localized in the given étale neighborhood of ξ) is isomorphic

to the ring of equivariant maps from $GL_n(\mathbb{C}) \times^{GL_n(\mathbb{C})_x} N_x^o$ to $M_n(\mathbb{C})$. Furthermore, we can assume that N_x^o is the set of elements of N_x where an invariant polynomial f (under $GL_n(\mathbb{C})_x$) on N_x is not zero.

Then, if R is the ring of equivariant maps from

$$GL_n(\mathbb{C}) \times^{GL_n(\mathbb{C})_x} N_x$$

to $M_n(\mathbb{C})$ we have that

$$\mathbb{T}_{m,n} \bigotimes_{\mathbb{C}[V_{m,n}]} B \cong R[\frac{1}{f}]$$

The ring R can be called the noncommutative slice (in the point ξ).

We will now restrict attention to the case that ξ is of representation type $(1, k_1; ...; 1, k_r)$, that is when $GL_n(\mathbb{C})_x = T_r$. Then we have to describe the ring of equivariant maps

$$f : M_n(\mathbb{C}) \bigoplus ... \bigoplus M_n(\mathbb{C}) \bigoplus C_x \to M_n(\mathbb{C})$$

where T_r acts on every component by conjugation. This study is essentially the study of all polynomial maps

$$g : M_n(\mathbb{C}) \bigoplus ... \bigoplus M_n(\mathbb{C}) \bigoplus C_x \bigoplus M_n(\mathbb{C})^* \to \mathbb{C}$$

which are invariant under T_r and homogeneous of degree one in the indeterminates corresponding to the component $M_n(\mathbb{C})^*$. As a T_r-module, $M_n(\mathbb{C})^*$ decomposes into a direct sum of one-dimensional vectorspaces $\bigoplus \mathbb{C}e_{ij}$. If

$$k_1 + ... + k_s < i \leq k_1 + ... + k_{s+1}$$

$$k_1 + ... + k_t < j \leq k_1 + ... + k_{t+1}$$

then $\tau = (u_1, ..., u_r)$ acts on $\mathbb{C}e_{ij}$ by sending e_{ij} to $u_s^{-1} u_t e_{ij}$.

This allows us to determine the part of the invariant ring which is homogeneous of degree one in the variable corresponding to $e_{ij} : z_{ij}$. A typical element is of the form $h.z_{ij}$ where h is a semiinvariant on N_x with respect to the character

$$\chi_{st} : T_r \to \mathbb{C}^*$$

determined by sending $\tau = (u_1, ..., u_r)$ to $u_s.u_t^{-1}$. With notations as in II.2 this module of semi-invariants is

$$\mathbb{C}[E_{\Phi,\alpha}][y_1, ..., y_d]$$

where $d = (m-1)(k_1^2 + ... + k_r^2) + r$ and α is the $(r-1)$-dimensional column vector with a -1 on the i-th row, $+1$ at the j-th row and zeroes elsewhere. Therefore, $\mathbb{C}[E_{\Phi,\alpha}]$ is the subvectorspace of $\mathbb{C}[N_2]$ consisting of all polynomials h such that $h.v_{ts}(a) \in \mathbb{C}[N_2/T_r]$ for any $1 \leq a \leq k_s k_t (m-1)$. These observations prove

Theorem III.1.1 : The noncommutative slice of the trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$, in a point $\xi \in V_{m,n}$ corresponding to a semi-simple representation of type $(1, k_1; ...; 1, k_r)$ is isomorphic to a polynomial ring in $(m-1)(k_1^2 + ... + k_r^2) + r$ indeterminates over the $\mathbb{C}$- algebra

$$\Gamma_{m,n}(\xi) = \begin{pmatrix} M_{k_1}(\mathbb{C}[N_2/T_r]) & W_{12} & ... & W_{1r} \\ W_{21} & M_{k_2}(\mathbb{C}[N_2/T_r]) & ... & W_{2r} \\ & & & \\ W_{r1} & W_{r2} & ... & M_{k_r}(\mathbb{C}[N_2/T_r]) \end{pmatrix}$$

where W_{ij} is a k_i by k_j block of $\mathbb{C}[N_2/T_r; \chi_{ij}]$ which is the module of semi-invariants on N_2 with respect to the character $\chi_{ij} : T_r \to \mathbb{C}^*$ or, equivalently, the subvectorspace of $\mathbb{C}[N_2]$ consisting of polynomials h such that $h.v_{ji}(a) \in \mathbb{C}[N_2/T_r]$ for all $1 \leq a \leq (m-1)k_ik_j$.

III.2 : THE FUNCTIONAL EQUATION

In [Le] it was shown that the trace ring of m generic 2 by 2 matrices is always a Cohen-Macaulay module over its center , i.e. a (graded) free module of finite rank over a polynomial subring of the center.

Unfortunately, the étale-slice machinary cannot be used to prove Cohen-Macaulayness for arbitrary p.i.-degree. The reason is that the study of the noncommutative slice in a point of type $(n,1)$ is as hard as the study of the trace ring in the origin. Nevertheless, we can give some weight to the conjecture that trace rings are Cohen-Macaulay by proving that the Cohen-Macaulay locus is large.

Theorem III.2.1 : The trace ring of m generic n by n matrices is a Cohen-Macaulay module in a point ξ of $V_{m,n}$ of type $(1,k_1;...;1,k_r)$.

Proof :

By étale descent , it is sufficient to prove Cohen-Macaulayness of the noncommutative slice in ξ. By the Hochster-Roberts theorem and theorem III.1.1 this amounts to showing that $\mathbb{C}[N_2/T_r;\chi_{ij}]$ is a Cohen-Macaulay module. In the foregoing section we have seen that $\mathbb{C}[N_2/T_r;\chi_{ij}] = \mathbb{C}[E_{\Phi,\alpha}]$ where $\alpha(i) = -1$,$\alpha(j) = 1$ and $\alpha(k) = 0$ elsewhere.By proposition I.2.1 we know that $\mathbb{C}[E_{\Phi,\alpha}]$ is Cohen- Macaulay if there exists a rational solution $\beta = (\beta_1,...,\beta_s)^T$ to $\Phi.\beta = \alpha$ satisfying $-1 < \beta_i \leq 0$. It is easy to see that such a solution always exists (with entries $\frac{1}{2}$ or 0) if $n > 2$ or $m > 2$.

The remaining case, $\mathbb{T}_{2,2}$ is easily seen to be a free module of rank four generated by $\{1, x_1, x_2, x_1x_2\}$ over its center which is a polynomial ring.

We will now study the Poincaré series of the noncommutative slice, or equivalently that of $\Gamma_{mn,n}(\xi)$.Usually, it is rather hard to determine the power series $P(\Gamma_{m,n}(\xi);t)$. There is, however, one important exception :

Proposition III.2,2 : Let $\xi \in V_{m,n}$ be a point of type $(1,k;1,l)$, then the noncommutative slice in ξ has the following Poincaré series

$$\frac{1}{(1-d)^d}\cdot\{2kl.\sum_{s>0}^{\infty}\binom{(m-1)kl+s-1}{s}\binom{(m-1)kl+s-1}{s-1}t^{2s-1}$$
$$+(k^2+l^2)\sum_{s=0}^{\infty}\binom{(m-1)kl+s-1}{s}^2 t^{2s}\}$$

Proof :

We will first determine the Poincaré series of the ring of all polynomial maps

$$M_n(\mathbb{C}) \bigoplus \ldots \bigoplus M_n(\mathbb{C}) \bigoplus \begin{pmatrix} \mathbb{C} \otimes 1_k & 0 \\ 0 & \mathbb{C} \otimes 1_l \end{pmatrix} \bigoplus M_n(\mathbb{C})^* \to \mathbb{C}$$

which are invariant under T_2. Here we give the indeterminates corresponding to the first m factors degree t and those to the last factor degree x. This invariant ring is a polynomial ring in $(m-1)(k^2+l^2)+2$ indeterminates of degree t and k^2+l^2 indeterminates of degree x over the ring of invariants of

$$(V_{12} \bigoplus W_{21}) \bigoplus (V_{21} \bigoplus W_{12})$$

where $dim(V_{12}) = dim(V_{21}) = (m-1)kl$ and $dim(W_{12}) = dim(W_{21}) = kl$ and $(\alpha, \beta) \in T_2$ acts on the first component by multiplication with $\alpha\beta^{-1}$ and with $\alpha^{-1}\beta$ on the second. The ring of invariants is then the subring of

$$\mathbb{C}[x_i, y_i : 1 \leq i \leq kl(m-1); u_j, v_j : 1 \leq j \leq kl]$$

generated by all products $x_i y_i, x_i v_j, y_i u_j, y_i v_j$. The ideal of relations between them is generated by the 2 by 2 minors of the matrix

$$\begin{pmatrix} x_1y_1 & \ldots & x_1y_{kl(m-1)} & x_1v_1 & \ldots & x_1v_{kl} \\ \\ x_{kl(m-1)}y_1 & \ldots & x_{kl(m-1)}y_{kl(m-1)} & x_{kl(m-1)}v_1 & \ldots & x_{kl(m-1)}v_{kl} \\ y_1u_1 & \ldots & y_{kl(m-1)}u_1 & u_1v_1 & \ldots & u_1v_{kl} \\ \\ y_1u_{kl} & \ldots & y_{kl(m-1)}u_{kl} & u_{kl}v_1 & \ldots & u_{kl}v_{kl} \end{pmatrix}$$

So, all relations are homogeneous in the (t,x)-gradation. It is then fairly easy to see that the Poincaré series of this invariant ring is (use Plethysm formula)

$$\sum_{s=0}^{\infty} S^s(\mathbb{C}^{klm}) \bigotimes S^s(\mathbb{C}^{klm})$$

where $\mathbb{C}^{klm} = U \bigoplus V$ with $dim(U) = (m-1)kl$, $dim(V) = kl$ and the coordinates of U (resp. V) have degree t (resp. x). So,

$$S^s(U \bigoplus V) = \sum_{i=0}^{s} S^i(U) \bigotimes S^{s-i}(V)$$

$$= \sum_{i=0}^{s} \binom{(m-1)kl+i-1}{i} \binom{kl+s-i-1}{s-i} t^i x^{s-i}$$

Therefore, the Poincaré series in the (t,x)-gradation of the invariant ring is equal to

$$\sum_{s=0}^{\infty} \left(\sum_{i=0}^{s} \binom{(m-1)kl+i-1}{i} \binom{kl+s-i-1}{s-i} t^i x^{s-i} \right)^2$$

and therefore, this expression multiplied with

$$(1-t)^{-((m-1)(k^2+l^2)+2)}(1-x)^{-(k^2+l^2)}$$

is the Poincaré series of the total ring of invariants in the (t,x)-gradation. The Poincaré series of the noncommutative slice is then the partial derivative of this expression with respect to x and evaluated at $x=0$, which gives us the claimed expression.

We have seen in the foregoing section that $\mathbb{C}[N_2/T_r;\chi_{ij}] = \mathbb{C}E_{\Phi,\alpha}$, for certain α, is a Cohen-Macaulay module which is clearly of dimension $2(m-1)\sum k_ik_j - r + 1 = h$.

Assume that $\theta_1,...,\theta_h$ is a homogeneous system of parameters for $\mathbb{C}[N_2/T_r;\chi_{ij}]$ and let $S = \mathbb{C}[\theta_1,...,\theta_h]$, then the canonical module of $\mathbb{C}[N_2/T_r;\chi_{ij}]$ is defined to be

$$\Omega(\mathbb{C}[N_2/T_r;\chi_{ij}] = HOM_S(\mathbb{C}[N_2/T_r;\chi_{ij}],S)$$

which is $\mathbb{Z}$-graded in the obvious way.

Form [St,th 4.4] we retain that $\Omega(\mathbb{C}[N_2/T_r;\chi_{ij}])$ is the subvectorspace of $\mathbb{C}[N_2]$ spanned by all monomials $\prod v_{ij}(k)^{b_{ij}(k)}$ s.t. $(\beta_{ij}(k)) \in \overline{E_{\Phi,\alpha}}$ (for notation see I.2) and $\Gamma_{-\beta_{ij}(k)} = \emptyset$.

But, if $(\beta_{ij}(k)) \in \overline{E_{\Phi,\alpha}}$ then $(-\beta_{ij}(k)) \in \overline{E_{\Phi,-\alpha}}$ and since $\mathbb{C}[E_{\Phi,-\alpha}] = \mathbb{C}[N_2/T_r;\chi_{ji}]$ we have again a solution to $\Phi.\beta = -\alpha$ satisfying $-1 < \beta_i \leq 0$ for all i, whence we can apply the argument at the end of I.2 to ensure that $\Gamma_{(-\beta_{ij}(k))} = \emptyset$ iff $-\beta_{ij}(k) < 0$ for all i,j,k.

So, we have shown that

$$\Omega(\mathbb{C}[N_2/T_r;\chi_{ij}]) = \bigoplus\{\mathbb{C}\prod v_{ij}(k)^{\beta_{ij}(k)} : \beta_{ij}(k) > 0\}$$

Luckily, there is a unique strictly positive solution to $\Phi.\beta = 0$ namely $\beta_i = 1$ for all $1 \leq i \leq 2(m-1)\sum k_ik_j$. Therefore,

$$\Omega(\mathbb{C}[N_2/T_r;\chi_{ij}]) = \prod v_{ij}(k).\mathbb{C}[N_2/T_r;\chi_{ij}]$$

Translating this equality to Poincaré series gives us

$$P(\Omega(\mathbb{C}[N_2/T_r;\chi_{ij}]);t) = t^s.P(\mathbb{C}[N_2/T_r;\chi_{ij}];t)$$

where $s = 2(m-1)\sum k_ik_j$.

Applying [St2,p.58] we get the functional equation

$$P(\mathbb{C}[N_2/T_r;\chi_{ij}];\frac{1}{t}) = (-1)^h.t^s.P(\mathbb{C}[N_2/T_r;\chi_{ij}];t)$$

where $h = 2(m-1)\sum k_ik_j - r + 1$.

This concludes the proof of the following

Theorem III.2.3 : Let $\xi \in V_{m,n}$ be of representation type $(1, k_1; ...; 1, k_r)$, then the Poincaré series of the noncommutative slice R at ξ satisfies the following functional equation

$$P(R; \frac{1}{t}) = (-1)^d . t^{(m-1)n^2+r} . P(R; t)$$

where $d = Kdim(\mathbb{T}_{m,n}) = (m-1)n^2 + 1$.

III.3 : REGULARITY OF $\mathbb{T}_{m,n}$

When m or n is equal to one, $\mathbb{T}_{m,n}$ is a commutative polynomial ring and hence has finite global dimension. In [SS] L.Small and J.T. Stafford proved that $gldim(\mathbb{T}_{2,2}) = 5$. In [LV] it was shown that for $n \leq 4$, $gldim(\mathbb{T}_{m,n}) < \infty$ if and only if $(m,n) = (2,2); (2,3)$ or $(3,2)$. In this section we aim to show that these are the only noncommutative trace rings of generic matrices having finite global dimension.

The strategy of the proof is the following : choose a suitable point ξ in $V_{m,n}$. If $gldim(\mathbb{T}_{m,n}) < \infty$, then the global dimension of the noncommutative slice in ξ has to be finite,too. Suppose that ξ is chosen in such a way that all indecomposable graded projective modules of the noncommutative slice have the same Poincaré series, then finite global dimension can be tested by the fact that the Poincaré series of such a projective module needs to have the rational form $\frac{k}{f(t)}$ for some $f(t) \in \mathbb{Z}[t]$ and $k \in \mathbb{N}$, by the standard argument.

Theorem III.3.1 : The trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$, has finite global dimension if and only if

(1) : m or n is equal to one

(2) : $(m,n) = (2,2); (2,3)$ or $(3,2)$

Proof :

Consider a point $\xi \in V_{m,n}$ corresponding to a sum of n distinct one-dimensional irreducible representations. By theorem II.2.1 , the central slice is a polynomial ring in $m.n$ variables over the invariant ring $\mathbb{C}[N_2/T_n]$ described in II.2.1.

By theorem III.1.1 , the noncommutative slice is also a polynomial ring in $m.n$ variables over the ring $\Gamma_{m,n}(\xi)$ described in III.1.1 . From this description it is clear that every indecomposable graded projective left module over the noncommutative slice is of the form

$$P_i = \bigoplus_{j=1}^{n} \mathbb{C}[N_2/T_r; \chi_{ji}] \underset{\mathbb{C}[N_2/T_r]}{\otimes} \mathbb{C}[N_x/T_r]$$

Further, by symmetry it is clear that the Poincaré series of all $\mathbb{C}[N_2/T_r; \chi_{ij}]$, $i \neq j$, are equal. So, all indecomposable graded projectives have the same Poincaré series and we have to verify when $\wp(P_i; t)$

$$= \frac{1}{(1-t)^{mn}} \cdot \wp(\mathbb{C}[N_2/T_n]; t) + (n-1)\wp(\mathbb{C}[N_2/T_n; \chi_{12}]; t)]$$

has a rational expression of the form $(f(t))^{-1}$ where $f(t) \in \mathbb{Z}[t]$.

It follows from the description of $\mathbb{C}[N_2/T_n]$ and from the fact that $\mathbb{C}[N_2/T_n; \chi_{12}]$ is a finitely generated module over $\mathbb{C}[N_2/T_n]$ that their Poincaré series have the

rational form

$$\wp(\mathbb{C}[N_2/T_n];t) = \frac{p(t)}{(1-t)^{\alpha_1}...(1-t^n)^{\alpha_n}}$$

$$\wp(\mathbb{C}[N_2/T_n;\chi_{ij}];t) = \frac{q(t)}{(1-t)^{\alpha_1}...(1-t^n)^{\alpha_n}}$$

for some $p(t), q(t) \in \mathbb{Z}[t]$ and $\alpha_i \in \mathbb{N}$. Therefore, $f(t)$ has to be a product of irreducible factors of $1-t^i$ where $1 \leq i \leq n$.

Further , the residue of the pole in $t = 1$ must be equal to $Kdim(P_i) = Kdim(\mathbb{C}[N_x/T_n]) = (m-1)n^2 + 1 \doteq d$. Therefore,

$$f(t) = (1-t)^d \prod (\text{ irreducible factors } \neq 1-t)$$

In any case, $f(t)^{-1}$ satisfies the functional equation

$$(1) : f(\frac{1}{t}) = (-1)^d t^{def(f)} f(t)$$

On the other hand, we have seen in the proof of theorem III.2.3 that the Poincaré series of $\mathbb{C}[N_x/T_n]$ and $\mathbb{C}[N_2/T_n;\chi_{12}] \bigotimes \mathbb{C}[N_x/T_n]$ satisfy the functional equation

$$(2) : \wp(-;\frac{1}{t}) = (-1)^d t^{(m-1)n^2+n} \wp(-;t)$$

Therefore, if the noncommutative slice has finite global dimension, or if $\mathbb{C}$ has finite projective dimension, we get from a combination of (1) and (2) that

$$deg(f) = (m-1)n^2 + n$$

whence $f(t) = (1-t)^d(1 + a_1t + a_2t^2 + ... + a_{n-1}t^{n-1})$. This entails that the first terms in the power series expansion of $f(t)^{-1}$ are

$$(3) : 1 + (d - a_1)t + (\frac{d(d+1)}{2} - da_1 + a_1^2 - a_2)t^2 + ...$$

On the other hand, it follows from the description of $\mathbb{C}[N_2/T_n]$ and $\mathbb{C}[N_2/T_n;\chi_{12}]$ that

$$\wp(\mathbb{C}[N_2/T_n];t) = 1 + (m-1)\frac{n(n-1)}{2}t^2 + ...$$

$$\wp(\mathbb{C}[N_2/T_n;\chi_{12}];t) = (m-1)t + (n-2)(m-1)^2t^2 + ...$$

Therefore, $\wp(P_i;t)$ is equal to

$$(4) : \frac{1}{(1-t)^n}1 + (n-1)(m-1)t + (n-1)(m-1)(\frac{n}{2} + (m-1)(n-2))t^2 + ...$$

Comparing the coefficient of t in (3) and (4) gives the equation

$$a_1 = (n-1)[(n-1)(m-1)-1]$$

But we know that a_1 has to be smaller or equal to the number of irreducible factors in the remaining part of $f(t)$.Therefore, $a_1 \leq n-1$ giving the inequality

$$(n-1)(m-1)-1 \leq 1$$

It is clear that the only integer solutions with m or $n > 1$ are

$$(m,n) = (2,2), (2,3) \text{ or } (3,2)$$

So, in all other cases $\mathbb{C}$ is of infinite projective dimension over the noncommutative slice R, and so also R/R_+. In view of the slice theorem, we have to verify that

$$\mathbb{T}_{m,n} \underset{\mathbb{C}[V_{m,n}]}{\otimes} B \simeq R[\frac{1}{f}]$$

has infinite global dimension. Now, assume $R[\frac{1}{f}]$ has global dimension d . If $k > d$ and M is an R-module we have

$$0 = Ext^k_{R[\frac{1}{f}]}((R/R_+)[\frac{1}{f}], M[\frac{1}{f}])$$

$$= Ext^k_R(R/R_+, M)[\frac{1}{f}] = Ext^k_R(R/R_+, M)$$

since $f(0) \neq o$ and R/R_+ is annihilated by R_+.So, we obtain a contradiction.

In view of [LV,Prop 1,Prop 2,Prop 7] this finishes the proof.

We will conclude this paper with an example. Consider the trace ring of 3 generic 2 by 2 matrices and let $\xi \in V_{3,2}$ of representation type $(1,1;1,1)$. Then, $\mathbb{C}[N_2] = \mathbb{C}[v_1, v_2, v_3, v_4]$ and $\mathbb{C}[N_2/T_2]$ is the subring generated by the monomials $v_i v_j$, $i \neq j$. Hence, $\mathbb{C}[N_2/T_2] = \mathbb{C}[x,y,z,t]/(xy - zt)$ and therefore

$$\wp(\mathbb{C}[N_2/T_2]; t) = \frac{1+t^2}{(1-t^2)^3}$$

Let P be the height one prime ideal of $\mathbb{C}[N_2/T_2]$ generated by (x,z), then

$$\Gamma_{3,2}(\xi) \cong \begin{pmatrix} \mathbb{C}[N_2/T_2] & P \\ P^{-1} & \mathbb{C}[N_2/T_2] \end{pmatrix}$$

Since $v_1.\mathbb{C}[N_2/T_2;\chi_{12}] = P$ and the quotient $\mathbb{C}[N_2/T_2]/P$ is the polynomial ring in z and t we obtain

$$t\wp(\mathbb{C}[N_2/T_2;\chi_{12}];t) = \frac{1+t^2}{(1-t^2)^3} - \frac{1}{(1-t^2)^2} = \frac{2t^2}{(1-t^2)^3}$$

and

$$\wp(P_i;t) = \frac{1}{(1-t)^6}\wp(\mathbb{C}[N_2/T_2];t) + \wp(\mathbb{C}[N_2/T_2;\chi_{12}];t)$$

$$= \frac{1+2t+t^2}{(1-t)^6(1-t^2)^3} = \frac{1}{(1-t)^8(1-t^2)}$$

Since $\mathbb{T}_{3,2}$ has finite global dimension, $\Gamma_{3,2}(\xi)$ is regular too, giving an example of a reflexive Azumaya algebra of global dimension three having an height one prime ideal $(\Gamma_{3,2}(\xi)(x,z))^{**}$ which is not generated by a normalizing element since (x,z) is the generator of $Cl(\mathbb{C}[N_2/T_2]) \cong \mathbb{Z}$.

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