A NOTE ON CLIFFORD REPRESENTATIONS

L. Le Bruyn * and F. Van Oystaeyen
University of Antwerp, U.I.A. (Belgium)
* : research assistent of the NFWO.

0. Introduction

That twisted group rings provide a link between the theory of Clifford algebras and the theory of projective representations of finite groups is well-known , even if perhaps it was more of a gadget to physicists , cfr. [Ra] , than a tool to mathematicians. In the first section we establish that a projective representation of B_n , where B_n is the n-fold product of $\mathbb{Z}/2\mathbb{Z}$, is determined by an n-dimensional quadratic form together with some well described subgroup of B_n. Any twisted groupring of B_n is then a Clifford algebra over its center but we pay particular attention to the so called "Clifford representations" which yield Clifford algebras over the ground ring. In a short second section we introduce the notion of a Clifford group and provide some examples of these. We do not obtain a complete description of Clifford groups but we hope to create some interest in such a complete determination which may be the topic of some further investigations.

1. Clifford representations

In this section R is a commutative ring such that 2 is a unit in R and B_n , where $n \in \mathbb{N}$ stands for the abelian group $\mathbb{Z}/2\mathbb{Z} \times ... \times \mathbb{Z}/2\mathbb{Z}$, where n copies of $\mathbb{Z}/2\mathbb{Z}$ appear. A projective representation of a group G over k is a group morphism

$$\phi : G \to PGL_n(k)$$

where k is a field. Such a projective representation over a field k may also be determined by a ring homomorphism

$$\phi^c : kG_c \to M_n(k)$$

where $c \in H^2(G, k^*)$ and kG_c is the twisted group ring with respect to the factor system $c(\sigma, \tau)$ where $\sigma, \tau \in G$. First we generalize these concepts to the case where k is merely a ring. If P is a finitely generated projective R-module then $End_R(P)$ is an Azumaya algebra over R and its group of units $Aut_R(P)$ contains the trivial units $U(R)$ in its center because such a unit defines an automorphism of P. A **projective representation of G over R defined by P** is a group morphism $\pi : G \to Aut_R(P)/U(R)$. This may be interpreted as a map, again written π, $\pi : G \to Aut_R(P)$, satisfying the relations : $\pi(\sigma).\pi(\tau) = c(\sigma,\tau).\pi(\sigma.\tau)$ for all $\sigma, \tau \in G$, where $c(\sigma,\tau)$ is a factor set representing $c \in H^2(G, U(R))$ obtained by taking a transversal for $\pi(G)$ in $Aut_R(P)$. The twisted groupring RG_c is the free R-module $\bigoplus_{\sigma \in G} Ru_\sigma$ with multiplication induced by the rules

$$u_\sigma.u_\tau = c(\sigma,\tau).u_{\sigma.\tau}$$

for all $\sigma, \tau \in G$. The R-bilinear extension of the map $G \to Aut_R(P)$ determines a ringhomomorphism $\pi^e : RG_c \to End_R(P)$. For a given factor system $\{c(\sigma,\tau) : \sigma, \tau \in G\}$ we define $G^c = \{\sigma \in G : \forall \tau \in C_G(\sigma) : c(\sigma,\tau) = c(\tau,\sigma)\}$ where $C_G(x)$ is the centralizer of x in G.

Now , consider $G = B_n$. Since B_n is Abelian it is clear that the subring $(RB_n^c)_c$ coincides with the center of $(RB_n)_c$ (this follows for example immediatly from the fact that the center of $(RB_n)_c$ is B_n-graded.

Definition 1.1 : A projective representation of B_n determined by a factor system $\{c(\sigma,\tau) : \sigma, \tau \in G\}$ is a Clifford representation if B_n^c is minimal, i.e. $B_n^c = 0$ if n is even and $B_n^c \simeq \mathbb{Z}/2\mathbb{Z}$ if n is odd.

In order to see that this definition makes sense we should justify the discrepancy of the definition in the "n is odd" case. Assume that $R = k$ is a field, then $(kB_n)_c$ is a semi-simple k-algebra, i.e. $(kB_n)_c = A_1 \bigoplus \cdots \bigoplus A_t$ for simple k-algebras A_i. If $t \neq 1$ then the center of $(kB_n)_c$ is strictly larger than k. If $k = 1$, then the dimension of the center is also strictly greater than one since A_1 is of square dimension over its center.So, in both cases we arrive at $B_n^c \neq 0$, what explains that for odd n the minimal possibility for B_n^c is to be a copy of $\mathbb{Z}/2\mathbb{Z}$.

Let us recall the definition of Clifford algebras. Let S be any commutative ring, V a free S-module of rank n equipped with a nonsingular quadratic form $Q = \alpha_1 X_1^2 + \cdots + \alpha_n X_n^2$, where $\alpha_i \in U(S)$. The Clifford algebra $C(V,Q)$ is defined to be the quotient of the tensor algebra $T(V)$ with respect to the twosided ideal generated by the elements $v \otimes v - Q(v)$ where $v \in V$. More specifically,

$$C(V,Q) = \bigoplus_{\epsilon_i \in \{0,1\}} R.e_1^{\epsilon_1} \cdots e_n^{\epsilon_n}$$

with multiplication defined by the rules $e_i^2 = \alpha_i \in U(S)$ and $e_i.e_j = -e_j.e_i$ for $i \neq j$. The following theorem explains why a representation with minimal B_n^c is called a Clifford representation.

Theorem 1.2 : If the factor system $\{\bar{c}(\bar{\sigma},\bar{\tau}) : \bar{\sigma}, \bar{\tau} \in G\}$ where $G \equiv B_n$, determines a Clifford representation , then RG_c is isomorphic to a Clifford algebra of a nonsingular quadratic form over R.

Proof :

Since $RG_c \simeq RG_{c'}$ if the factor systems c and c' are equivalent, i.e. when $c = c' \in H^2(G, U(R))$ we may assume that $c(\sigma, \tau)$ is normalized so that $c(\sigma, 0) = c(0, \sigma) = 1$ for all $\sigma \in B_n$. The first part of the proof consists in describing the cohomology group $H^2(B_n, U(R))$. We claim that

$$(*) : H^2(B_n, U(R)) \simeq \prod_{i=1}^{n} H^2(\mathbb{Z}/2\mathbb{Z}, U(R)) \times \prod_{i<j} M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))$$

where each $i - j$-mixterm $M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))$ equals $H^2_\mu(\mathbb{Z}/2\mathbb{Z}, U(R))$ which is the group of equivalence classes of 2-cocycles satisfying the relations :

$$f(\sigma_1\sigma_2, \tau) = f(\sigma_1, \tau) f(\sigma_2, \tau); f(\sigma, \tau_1\tau_2) = f(\sigma, \tau_1) f(\sigma, \tau_2)$$

The isomorphism (*) may be easily verified by (a) : restricting the 2-cocycles to the cyclic components $\mathbb{Z}/2\mathbb{Z}$ of B_n and (b) : associating to basis elements e_i and e_j an element f of $M_{ij}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, U(R))$

$$f : \mathbb{Z}/2\mathbb{Z}.e_i \times \mathbb{Z}/2\mathbb{Z}.e_j \to U(R); (\sigma, \tau) \to c(\sigma, \tau).c(\tau, \sigma)^{-1}$$

Furthermore, we have that $H^2(\mathbb{Z}/2\mathbb{Z}, U(R)) \simeq U(R)/U(R)^2$, the isomorphism being given by the correspondence $f \to f(1,1)$, and also $H^2_\mu(\mathbb{Z}/2\mathbb{Z}, U(R)) = \mu_2 = \{+1, -1\}$. So, we finally obtain :

$$H^2(B_n, U(R)) = \prod_{i=1}^{n} U(R)/U(R)^2 \times \prod_{i<j} \mu_2$$

The foregoing states that each factor system corresponds (up to equivalence) in an unambiguous way to a quadratic form , viewed as an element of $\prod_{i=1}^n U(R)/U(R)^2$, together with some element ς in $\prod_{i<j} \mu_2$. There are Clifford representations such that ς is not equal to $\prod_{i<j}(-1)$. For example, consider B_3 as a $\mathbb{Z}/2\mathbb{Z}$-vectorspace with basis-elements e_1, e_2, e_3. We may determine a 2-cocycle $c(\sigma, \tau)$ by the following table

$c(e_i, e_j)$	e_1	e_2	e_3
e_1	1	−1	1
e_2	−1	1	1
e_3	1	1	1

It is clear that $B_3^c = \mathbb{Z}/2\mathbb{Z}.e_3$ and hence the given cocycle describes a Clifford representation. On the other hand, it is possible to change the $\mathbb{Z}/2\mathbb{Z}$-basis for B_n as follows $(e_1, e_2, e_3) \to (e_1, e_2, e_1e_2e_3)$ then the new table for c is the following

c	e_1	e_2	$e_1e_2e_3$
e_1	1	-1	-1
e_2	-1	1	-1
$e_1e_2e_3$	-1	-1	1

The corresponding element in $\prod_{i<j} \mu_2$ is $(-1,-1,-1)$. In the next lemma we will prove that this is possible in general.

Lemma 1.3 : Consider $f \in \prod_{i<j} \mu_2$ such that the twisted groupring $(RB_n)_f = \bigoplus R.e_1^{\epsilon_1}\cdots e_n^{\epsilon_n}$ with $e_i^2 = 1$, $e_ie_j = f_{ij}e_je_i$ (i.e. the cocycle corresponding to the quadratic form $(1,\cdots,1)$ and the element f given above), is a Clifford representation. Then one can change the basis of B_n such that in this new basis the element in $\prod_{i<j} \mu_2$ corresponding to it is $(-1,\cdots,-1)$.

Proof :

The proof is by induction on n, since there is nothing to prove for $n = 1, 2$. First consider the case where n is odd and $n \geq 3$. By our assumption, $B_n^f \simeq \mathbb{Z}/2\mathbb{Z}$. Let σ be a generator of this subgroup. We have that $B_n/B_n^f \simeq B_{n-1}$ and let $H \simeq B_{n-1}$ be the complement for $< \sigma >$ in B_n and let g be the factor system induced by f on H. It is clear that $H^g = 0$ and therefore g determines a Clifford representation of $H \simeq B_{n-1}$. By induction we may assume that a basis for H, say $d_1, \cdots, d_{n-1}$, has been chosen such that the corresponding element in $\prod_{i<j}^{n-1} \mu_2$ equals $(-1,\cdots,-1)$. Now, we may construct a basis for B_n by taking $\{d_1,\cdots,d_{n-1},d\}$ where $d = \sigma d_1\cdots d_{n-1}$. It is easily verified that the element of $\prod_{i<j} \mu_2$ corresponding to the selected basis is exactly equal to $(-1,\cdots,-1)$ as claimed. Next, we consider the case where n is even. Let $H \simeq B_{n-1}$ be the complement of $< e_n >$ in B_n and let g be the induced factor system of f on H. Let us check that g determines a Clifford representation of B_{n-1}. First, note that $H^g \neq 0$ because $n-1$ is odd. On the other hand, H^g cannot contain some $\mathbb{Z}/2\mathbb{Z}.\sigma \bigoplus \mathbb{Z}/2\mathbb{Z}.\tau$ with $\sigma, \tau \in H$ because, in case $f(\sigma, e_n) = 1$ or $f(\tau, e_n) = 1$ then $B_n^f \neq 0$, a contradiction, while in the other case $f(\sigma, e_n) = f(\tau, e_n) = -1$ and we find that $\sigma\tau \in B_n^f$, again a contradiction. Applying the induction hypothesis we may choose a basis $c_1, \cdots, c_{n-1}$ of H such that the corresponding element in $\prod_{i<j}^{n-1} \mu_2$ is $(-1,\cdots,-1)$. Also by induction and the first part of the proof, we may assume that the central element in $(RH)_g$ equals $c_1\cdots c_{n-1} = \varsigma$. Since $H^f = 0$ we have $\varsigma e_n = -e_n\varsigma$ and so e_n anti-commutes with an odd number of the c_i say $c_{i_1}, \cdots, c_{i_k}$. Take as the new n-th basis vector for B_n the

element $\omega = e_n c_{i_1} \cdots c_{i_k}$. It is easily checked that $\omega c_j = -c_j \omega$ for all $j \leq n-1$ and this finishes the proof of the lemma.

Now, this also finishes the proof of the theorem because, in the new basis we clearly see that the relations $e_i^2 = \alpha_i \in U(R)$ and $e_i e_j = -e_j e_i$ for $i \neq j$ do hold indeed.

Remark 1.4 : For an arbitrary projective representation of B_n determined by a factor system $\{c(\sigma, \tau) : \sigma, \tau \in B_n\}$ it is always true that $(RB_n)_c$ is a Clifford algebra over the ring $(RB_n^c)_c$ (i.e. over its center which is determined by the ray classes with respect to c).

In the proof of theorem 1.2 we have established that a Clifford representation of B_n , say $(RB_n)_c$ is isomorphic to the Clifford algebra associated to the quadratic form $< c(e_1, e_1), \cdots, c(e_n, e_n) >$ (in diagonal form) where $e_1, \cdots, e_n$ is a suitable $\mathbb{Z}/2\mathbb{Z}$-basis for B_n. It is now easy to study the splitting problem for a Clifford representation. Let S be the free extension of R obtained by adjoining $\sqrt{c(e_i, e_i)}$ for $1 \leq i \leq n$, then we obtain $(SB_n)_c \simeq C(S^n, < 1, \cdots, 1 >) \simeq M_{2^m}(S)$ when n is even and $m = \frac{n}{2}$ or $(SB_n)_c \simeq M_{2^t}(S) \bigoplus M_{2^t}(S)$ when n is odd and $t = \frac{n-1}{2}$.

It is easy to verify that $(SB_n)_c$ is an epimorphic image of the groupring SG_n where G_n is a finite central extension of B_n . In the case considered here we can give a complete description of such a group. Put $G_n = < a_1, \cdots, a_n; b >$ with $a_i = 1$ and $[a_i, a_j] = b$ for all $i \neq j$. An immediate consequence is that $b^2 = 1$ and that b is central in G_n. We may now define a ringepimorphism $\phi : SG_n \rightarrow (SB_n)_c$ by sending b to -1 and a_i to $\frac{1}{\sqrt{c(e_i, e_i)}} u_{e_i}$. So, we may view the (projective) Clifford representations of B_n in a generic way as common representations of G_n up to splitting the representation by passing to an extension S of R in the well-described way above.

2. Clifford groups, an introduction

In classical representation theory, projective representations appear in Clifford's theorem describing a representation of a group in terms of a projective representation of some subgroup(s). After the foregoing section, the natural question is to investigate then has to be : which groups G have the property that the groupring RG splits into Clifford representations over the respective centra, i.e.

$$RG = \bigoplus_{i=1}^{r} (R_i B_{n_i})_{c_i}$$

in particular, when RG split in Clifford representations over R, i.e.

$$RG = \oplus_{i=1}^{r}(RB_{n_i})_{c_i}$$

Groups with the first property are called **general Clifford groups**, groups with the second property are called **Clifford groups**. Since every abelian group is a general Clifford group the second concept is much more restrictive; for general Clifford groups G one should be more interested in $G/Z(G)$.

In the sequel we will always assume that $\mid G \mid^{-1} \in R$.

Proposition 2.1 : If G is a (general) Clifford group and H is a normal subgroup of G, then G/H is a (general) Clifford group.

Proof : Let ω_H be the augmentation ideal of H in RG. Since $\mid G \mid$ is a unit in R, RG is an Azumaya algebra and hence the canonical epimorphism

$$\pi_H : RG \to R(G/H)$$

maps the center onto the center. If RG decomposes as $A_1 \oplus \cdots A_n$, where each A_i is a Clifford algebra (over its center) , then a similar result holds for $R(G/H) = \overline{A_1} \oplus \cdots \oplus \overline{A_n}$ (note that each A_i is Azumaya, even if it is possible that 2 does not divide $\mid G \mid$.

Proposition 2.2 : Any group G such that the commutator subgroup G' is central and such that $G/G' \simeq B_n$ for some n is a general Clifford group.

PROOF : Since RG' is central in RG we may put $RG' = R'$ and view RG as $R'(G/G')_c$ where c is some 2-cocycle obtained by a selection of a transversal of G' in G. By remark 1.4 , $R'(G/G')_c \simeq (R'B_n)_c$ is a Clifford algebra over its center, hence the result follows.

Example 2.3 : The groups G_n constructed at the end of the foregoing section are Clifford groups.

Clearly, the more difficult and as yet unsolved problem is to describe the Clifford groups more completely. This problem may not be very easy, for example the fact that the product of Clifford groups need not be a Clifford group (verify for $D_4 \times D_4$ where D_4 is the dihaeder group of order 8) is already an obstruction.

[Ra] : Ramakrishnan ; Clifford algebra,its generalizations and applications,Matscience,Madras India