A COHOMOLOGICAL INTERPRETATION OF THE REFLEXIVE BRAUER GROUP

by

Lieven LE BRUYN *

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(*) Research assistant of the NFWO

Department of Mathematics
University of Antwerp, U.I.A.
Universiteitsplein 1
2610 Wilrijk
Belgium

Throughout this note R will be a normal affine domain, i.e. R is the coordinate ring of a normal affine variety X, over an algebraically closed field of characteristic zero. Let K deote the field of fractions of R and let Σ be a central simple algebra over K. A subring Λ of Σ is said to be a reflexive Azumaya algebra over R iff :

(1) : the center of Λ is equal to R

(2) : Λ is f.g. and reflexive as R-module

(3) : Λ_p is an Azumaya algebra over the discrete valuation ring R_p for every height one prime p of R

Two reflexive Azumaya algebras (in possibly distinct central simple K-algebras) are said to be equivalent if there exist f.g. reflexive R-modules M and N such that

$$\Lambda \otimes_R' End_R(M) \simeq \Gamma \otimes_R' End_R(N)$$

as R-algebras, where the modified tensor-product $\otimes_R'$ is defined to be $(-\otimes_R -)^{**}$.

The set of equivalence classes of reflexive Azumaya algebras equipped with the modified tensor product as a multiplication rule is a group, the so called reflexive Brauer group $\beta(R)$ of R or of the corresponding normal affine variety X.Whereas the natural morphism

$$Br(R) \to Br(K)$$

between the (usual) Brauer groups is not always a monomorphism see e.g. [1] we obtain [4]

$$\beta(R) = \cap Br(R_p) \to Br(K)$$

where the intersection is taken over the set of all height one prime ideals of R. That is, one can view the reflexive Brauer group as being the subgroup of $Br(K)$ consisting of those central simple algebras which contain a maximal order Λ which is not ramified in any height one prime. Note that this is independent of the particular choice of the maximal order Λ.

There is a rather vaque connection between $Br(R)$ and $\beta(R)$, see [4]

$$0 \to Pic(R) \to Cl(R) \to BCl(R) \to Br(R) \to \beta(R)$$

where Pic(R) is the Picard group, Cl(R) the divisor classgroup and $BCl(R)$ stands for the so called Brauer-classgroup. It is defined to be the set of equivalence classes of reflexive R-modules M such that $End_R(M)$ is f.g. projective with respect to the relation

$$M\}N \text{ iff } M \otimes_R' P \simeq N \otimes_R' Q$$

for some f.g. projective R-modules P and Q. Note that $BCl(R)$ is a group under the modified tensor product and

$$[M]^{-1} = [M^*]$$

Another noteworthy property of reflexive Azumaya algebras is that projectivity as R-modules implies Azumaya.

In order to get a feeling for what a possible description the reflexive Brauer group might be let us investigate the low-dimensional cases first

$dim(R) = 1$, i.e. X is a curve. Then, of course, any height one prime is maximal so reflexive Azumaya implies Azumaya and hence $Br(R) \simeq \beta(R)$.

$dim(R) = 2$, i.e. X is a surface. Then it is no longer true that every reflexive Azumaya is Azumaya. For example, let $R = \mathbb{C}[x,y,z]/(x^2 - yz)$ be the affine cone, then $Cl(R) \simeq \mathbb{Z}/2\mathbb{Z}$ and is generated by a ruling P. THen

$$\Lambda = End_R(R \oplus P)$$

is a reflexive Azumaya algebra, but computing its Formanek center gives m^2 where $m = (x,y,z)$ so Λ is not Azumaya in the top.

Nevertheless, any reflexive Azumaya algebra is Azumaya on the open set of regular points since any reflexive module over a regular local ring of dimension ≤ 2 is free and free reflexive Azumaya algebras are Azumaya. So, $\beta(R) \simeq Br(X_{reg})$

$dim(R) \geq 3$. THen it is no longer true that a reflexive Azumaya algebra is Azumaya on the whole open set X_{reg} of regular points, even when R is regular. For example take $R = \mathbb{C}[x,y,z]$ and let M be the kernel of the natural R-module morphism

$$R.a \oplus R.b \oplus R.c \to R$$

sending $a \to x$, $b \to y$, $c \to z$. Then M is a reflexive R-module which is not projective in the origin. Therefore, its endomorphism ring is a reflexive Azumaya algebra which is not Azumaya in the origin.

Nevertheless, one has the following important result due to R. Hoobler

Theorem 0 (Hoobler) If R is a regular affine domain, then $\beta(R) \simeq Br(R)$.

This result may be regarded as a noncommutative analoque of the purity of the branch locus [3,I.3.7]:: if an Azumaya algebra ramifies then it does so on a set of pure codimension one.

Combining all this evidence, it is clear that the reflexive Brauer group is related to the Brauer group of the open set of regular points X_{reg}.

The main result of this note shows that this is , at least cohomologically, correct

Theorem : $\beta(R) \simeq H^2_{et}(X_{reg}, \mathbb{G}_m)$

Proof :

irst we will show that the reflexive Brauer group is the Brauer group of an open set $U \subset X_{reg}$.

To this end, note that X_{reg} is determined by an ideal I of R which is clearly of $ht(I) \geq 2$ since R is normal. Therefore, we can find elements $u_1, u_2 \in I$ such that

$ht(Ru_1+Ru_2) = 2$. Let U be the open set determined by the ideal Ru_1+Ru_2, then U can be covered with the two affine open sets U_1 and U_2 where U_i is determined by Ru_i.

Since the reflexive Brauer group is determined by the Brauer groups in the height one prime ideals we get that $\beta(R)$ is the pullback

$$\begin{array}{ccc} \beta(R) & \rightarrow & \beta(R_{u_1}) \\ \downarrow & & \downarrow \\ \beta(R_{u_2}) & \rightarrow & \beta(R_{u_1u_2}) \end{array}$$

Now from the proof of Hoobler's result [2] we see that we may replace the condition 'affine' by 'localization of an affine' in the statement of Theorem 0. Therefore the reflexive Brauer group is the pullback

$$\begin{array}{ccc} \beta(R) & \rightarrow & Br(U_1) \\ \downarrow & & \downarrow \\ Br(U_2) & \rightarrow & Br(U_1 \cap U_2) \end{array}$$

By Gabber's result, cfr. e.g. [2], the Brauer group of any affine scheme is equal to $H^2_{ét}(-,\mathbb{G}_m)_{tors}$. Since all schemes occuring are regular it follows that $H^2_{ét}(-,\mathbb{G}_m)$ is a torsion group because of the natural injection

$$H^2_{ét}(-,\mathbb{G}_m) \rightarrow H^2_{ét}(K,\mathbb{G}_m) = Br(K)$$

Therefore we obtain the pullback-diagram :

$$\begin{array}{ccc} \beta(R) & \rightarrow & H^2_{ét}(U_1,\mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^2_{ét}(U_2,\mathbb{G}_m) & \rightarrow & H^2_{ét}(U_1 \cap U_2,\mathbb{G}_m) \end{array}$$

Or, in other words :

$$\beta(R) = H^2_{ét}(U,\mathbb{G}_m) \simeq Br(U)$$

The last isomorphism is by the general Gabber result and torsionness of the cohomology-group (regular !).

Now we can invoke Grothendieck's result on the cohomological purity of the Brauer group [1,III.Th.6.1] to the situation $U \subset X_{reg}$ in order to get that

$$H^2_{ét}(U,\mathbb{G}_m) \simeq H^2_{ét}(X_{reg},\mathbb{G}_m)$$

finishing the proof of the theorem.

Remark :

(1) : The intermediate result that $\beta(R) \simeq Br(U)$ is quite surprising. It is not difficult to extend a given reflexive Azumaya algebra to an Azumaya algebra on

an open set determined by an ideal of $ht \geq 2$, e.g. because projectivity is an open condition, but to find a common open set for all reflexive Azumaya algebras (up to equivalence) is not so clear at first sight.

(2) : A natural question to ask now is whether $\beta(R) = Br(X_{reg}$, i.e. can every reflexive Azumaya algebra be extended (up to equivalence) to an Azumaya algebra on the open set of regular points. The author would be surprised if this result holds. Possible candidates may be quotients of generic Clifford algebras determined by minors. They turn out to be reflexive Azumaya algebras and one can easily compute the codimension of the non-Azumaya stalks.

(3) : At this point I would like to make the following conjecture : let R be a Gorenstein domain of dimension n and let Λ be a reflexive Azumaya algebra such that the open set of Azumaya stalks is determined by an ideal of $ht \geq [\frac{n}{2}]$, then Λ should be a Cohen-Macauley module over R (?!). Of course such a result would be interesting for trace rings of m generic n by n matrices.

(4) : Let us denote by Z the closed subvariety $X - X_{reg}$, then we have a long exact sequence [3,III.1.25] if we denote with $H^i_Z(X,-)$ the cohomology groups with support on Z, see [3,p.91] :

$$\begin{array}{ccccccc}
H^1_Z(X,\mathbb{G}_m) & \to & H^1_{\acute{e}t}(X,\mathbb{G}_m) & \to & H^1_{\acute{e}t}(X_{reg},\mathbb{G}_m) & \to & H^2_Z(X,\mathbb{G}_m) \\
 & & \simeq & & \simeq & & \uparrow \\
 & & Pic(R) & \to & Cl(R) & \to & BCl(R)
\end{array}$$

$$\begin{array}{ccccccc}
\to & H^2_{\acute{e}t}(X,\mathbb{G}_m) & \to & H^2_{\acute{e}t}(X_{reg},\mathbb{G}_m) & \to & H^3_Z(X,\mathbb{G}_m) \\
 & \uparrow & & \simeq & & \\
\to & Br(R) & \to & \beta(R) & &
\end{array}$$

Here, $H^1_{\acute{e}t}(X_{reg},\mathbb{G}_m) \simeq Cl(R)$ because every regular local ring is factorial and $H^1_{\acute{e}t}(-,\mathbb{G}_m)$ classifies locally invertible modules upto isomorphism. Here, $Br(R) \simeq H^2_{\acute{e}t}(X,\mathbb{G}_m)_{tors}$ by Gabber's result.

By diagram-chasing one can therefore obtain also a cohomological interpretation of the rather obscure Brauer-classgroup.

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