Cohomology of $\mathbb{P}^m_{(2)}$.

by

Lieven Le Bruyn (*)

March 1985 85-07

Department of Mathematics
University of Antwerp, U.I.A.
Universiteitsplein 1
2610 Wilrijk
Belgium

0. Some motivation.

A basic theme in noncommutative ring theory is the attempt to generalize results of commutative ring theory to noncommutative rings. A good example of this theme is algebraic geometry for p.i. rings, which currently is quite popular.
This note can be viewed as an attempt to generalize Serre's result on the cohomology of projective space :

Theorem 0. (Serre, Cohomology of $\mathbb{P}^m$)

(1): $F[X_0, \ldots, X_m] \simeq \bigoplus H^0(\mathbb{P}^m, \underline{\vartheta}_{\mathbb{P}^m}(n))$

(2): $H^i(\mathbb{P}^m, \underline{\vartheta}_{\mathbb{P}^m}(n)) = 0$ for all n and for $0 < i < m$.

(3) : The dimension of the F-vectorspace

$$H^m(\mathbb{P}^m, \underline{\vartheta}_{\mathbb{P}^m}(-m-1-n))$$

is equal to $\binom{m+n}{n}$

to quaternionic projective space, or rather spaces. Of course, we have to define what we mean by this.

Let us start by considering the identities of 2 by 2 matrices, i.e.

$$\Lambda \simeq \mathbb{G}_{m,2}/I$$

for some twosided ideal I of $\mathbb{G}_{m,2}$. Here, $\mathbb{G}_{m,2}$ is the ring of m generic 2 by 2 matrices. It is the F-sub-algebra of

$$M_2(\mathcal{P}_m) = M_2(F[X_{11}(l), X_{12}(l), X_{21}(l), X_{22}(l) : 1 \leq l \leq m])$$

generated by the so called generic matrices

$$X_l = \begin{pmatrix} X_{11}(l) & X_{12}(l) \\ X_{21}(l) & X_{22}(l) \end{pmatrix}$$

We are interested in 2-dimensional representations of Λ, i.e. F-algebra morphisms

$$\varphi : \Lambda \to M_2(F)$$

This study can be seen as the description of the set of solutions in the matrix-variables $X_1, \ldots, X_m$ to the ideal of relations I, i.e. the topic of interest of what might be called (some day, hopefully) noncommutative algebraic geometry.
Clearly, we are not interested in all representations but more in a description for the equivalence classes of the relation

$$\varphi \sim \varphi' \quad \text{iff} \quad \exists \alpha \in \mathrm{Aut}_F(M_2(F)) : \varphi = \alpha' 0 \varphi$$

The main difficulty is that there is no scheme parametrizing these equivalence classes, cfr. [Ar]. This obstruction motivated Artin and Procesi to reformulate the object of study : characterize all equivalence classes of 2-dimensional semi-simple representations (and their irreducible components) of the affine Λ.
This can be done in the following way, see the work of Procesi [P1] and Artin-Schelter [A3].
Consider our algebra $\Lambda = \mathbb{G}_{m,2}/I$ then it is easily seen that

$$M_2(P_m)/M_2(P_m).I.M_2(P_m) \simeq M_2(S)$$

where $S = P_m/J$ where J is the ideal generated by the entries of the matrices in I. We have the situation :

$$\begin{array}{ccc} \mathbb{G}_{m,2} & \rightarrow & M_2(P_m) \\ \downarrow & & \downarrow \\ \Lambda & \xrightarrow{\pi} & M_2(S) \end{array}$$

With $T(\Lambda)$ we will denote the sub F-algebra of $M_2(S)$ generated by the image of Λ and $c(\Lambda)$ where $c(\Lambda)$ id the sub F-algebra of S generated by all coefficiënts of characteristic polynomials of elements of $\pi(\Lambda)$.
It is easy to see that there are F-algebras epimorphisms

$$\begin{array}{ccc} \mathbb{T}_{m,2} & \rightarrow & T(\Lambda) \\ \downarrow & & \Downarrow \\ R_{m,2} & \rightarrow & c(\Lambda) \end{array}$$

where $\mathbb{T}_{m,2}$ is $T(\mathbb{G}_{m,2})$ the so called trace ring of m generic 2 by 2 matrices.
For any affine F-algebra Γ we will denote by $\mathrm{Max}(\Gamma)$ (resp. $\mathrm{Spec}(\Gamma)$) the set of all twosided maximal (resp. prime) ideals of Γ. The main result can now be stated as follows

Theorem. (Artin - Schelter, Th. 3.20.)

(1) There is a one to one correspondence between $\mathrm{Max}(c(\Lambda))$ and equivalence classes of 2-dimensional semi-simple representations of Λ.

(2) $\mathrm{Max}(T(\Lambda))$ consists of couples (φ, φ_i) when $\varphi : \Lambda \rightarrow M_2(F)$ is a representant of a 2-dimensional semi-simple representation of Λ and φ_i is an irreducible factor of φ.

Therefore, the study of solutions in m 2 by 2 matrix variables to an ideal of relations I of $\mathbb{G}_{m,2}$ amants to the study of the maximal (or prime) ideal spectrum of $T(\mathbb{G}_{m,2}/I)$. The epimorphism

$$\mathbb{T}_{m,2} \to T(\mathbb{G}_{M,2}/I)$$

induces continuous maps :

$$\text{Spec}T(\Lambda) \to \text{Spec}\mathbb{T}_{m,2}$$

$$\text{Max}T(\Lambda) \to \text{Max}\mathbb{T}_{m,2}$$

So one can view $\text{Spec}\mathbb{T}_{m,2}$ as a quaternionic generalization of affine m-space. But, from the work of Procesi [P2] it follows that

$$\mathbb{T}_{m,2} = \mathbb{T}^0_m[Tr(X_1), \ldots, Tr(X_m)]$$

where $\mathbb{T}^0_m$ is the subalgebra generated by the generic trace zero matrices

$$X^0_i = X_i - \frac{1}{2}Tr(X_i)$$

So,

$$\text{Spec}\mathbb{T}_{m,2} = \text{Spec}\mathbb{T}^0_m \times A^m$$

and so, the real noncommutative (and hard) part of the problem is the description of Spec or Max of $\mathbb{T}^0_m$, which we will call quaternionic m-space and denote by $A^m_{(2)}$.

Clearly, one can also study the projective version of these questions. In that case one starts off with a positively graded F-algebra Λ satisfying the identities of 2 by 2 matrices. which is generated by a finite number of homogeneous elements of degree one, i.e. we have a gradation preserving epimorphism

$$\varphi : \mathbb{G}_{m+1,2} \to \Lambda$$

if we give every generic matrix X_i, $0 \leq i \leq m$, degree one. Then we want to study 2-dimensional projective representations, i.e. gradation preserving F-algebra morphisms

$$\varphi : \Lambda \to \Delta$$

where Δ is a graded central simple algebra of dimension 4 over its center. Since F is algebraically closed, it follows from [NV] that

$$\Delta \simeq M_2(F[y, y^{-1}])(\sigma_1, \sigma_2)$$

where $\deg(y) = 1$ or 2; $\sigma_1, \sigma_2 \in \mathbb{N}$ and $\mathbb{Z}$-gradation on Δ is defined by

$$\Delta_i = \begin{bmatrix} F[y, y^{-1}]_i & F[y, y^{-1}]_{i+\sigma_1-\sigma_2} \\ F[y, y^{-1}]_{i+\sigma_2-\sigma_1} & F[y, y^{-1}]_i \end{bmatrix}$$

We want to study equivalence classes, for the relation that $\varphi \sim \varphi$' is a gradation preserving automorphism between the target rings α s.t. $\varphi' = \alpha 0 \varphi$, of semisimple representations, i.e. such that $\varphi(\Lambda)$ is a graded semi-simple ring [NV].
Combining the argument of the affine case given above with ideas of [LVV], it is possible to show that this study amounts to a description of

$$\mathrm{Proj}(T(\Lambda))$$

where $T(\Lambda)$ is of course positively graded and Proj consists of all graded prime ideals of $T(\Lambda)$ not containing $T(\Lambda)_+ = \bigoplus_{i\geq 1} T(\Lambda)_i$.

Again, $\mathrm{Proj}(\mathbb{T}_{m+1,2})$ can be considered as a quaternionic generalization of projective space but since $\mathbb{T}_{m+1,2} = \mathbb{T}^0_{m+1}[Tr(X_0), \ldots, Tr(X_m)]$ we christen the hard, noncommutative part of this problem, i.e. $\mathrm{Proj}(\mathbb{T}^0_{m+1})$, quaternionic m- space and we denote it by $\mathbb{P}^m_{(2)}$.

An important difference with the commutative case is that the rings $\mathbb{T}^0_{m+1}$ are allmost never regular, [L1]. Moreover, M. Van den Bergh has shown that for $m > 3$, $\mathbb{T}^0_{m+2}$ cannot be obtained as an epimorphic image of a positively graded F-algebra of finite global dimension satisfying the identities of 2 by 2 matrices.
If we drop the assumption on having the same p.i.-degree, such an F- algebra does exist [L2].
For, take the iterated Öre extension

$$Cl_{m+1} = F[a_{ij} : 0 \leq i < j \leq m][a_0][a_1, \sigma_1, \delta_1]\ldots[a_m, \sigma_m, \delta_m]$$

where for each $i < j, \delta_j(a_i) = 2a_{ij}$ and $\sigma_j(a_i) = -a_1$ and trivial action on the other variables.
Then, sending a_i to X^0_i and a_{ij} to $\frac{1}{2}Tr(X^0_i X^0_j)$ we obtain an epimorphism

$$\varphi_{m+1} : Cl_{m+1} \rightarrow \mathbb{T}^0_{m+1}$$

Therefore, we can view $\mathrm{Proj}(Cl_{m+1})$ as a sort of regular quaternionic m-space, $\mathbb{P}^m_{reg}$.
A pleasant property is that, whereas prime ideals of Cl_{m+1} can split up wildly over primes of the central subring

$$S_{m+1} = F[a_{ij} : 0 \leq i \leq j \leq m]$$

where $a_{ii} = (a_i)^2$, graded prime ideals lie uniquely i.e. $\mathbb{P}^m_{reg}$ is homeomorphic, as topological spaces to projective $\binom{m+1}{2} - 1$-space.
In this note we aim to construct structure sheafs and to compute the cohomology of both spaces. The main result will be

Theorem 8. (cohomology of $\mathbb{P}^m_{(2)}$)

(1) : $\mathbb{T}^0_{m+1} \simeq \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n))$

(2) : $H^i(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n)) = 0$ for all n and all $0 < i < 3m-1$

(3) : The dimension of the F-vector space

$$H^{3m-1}\Big(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(-3m-3-n)\Big)$$

is equal to the number of standard Young tableaux of length ≤ 3 having n boxes. The proof of it relies on three facts :

(1) : We can cover $\mathbb{P}^m_{(2)}$ and $\mathbb{P}^m_{\text{reg}}$ with affine open sets such that the sections of the structure sheaves contain units of degree one.

(2) : the ring of generic trace zero 2 by 2 matrices, $\mathbb{T}^0_{m+1}$, is a Cohen-Macauley module, cfr. [L3].

(3) : The Poincaré series of $\mathbb{T}^0_{m+1}$ satisfies a functional equation, [L1].

We believe that from this point on, it should be possible to develop a "quaternionic geometry". Modesty forces us to leave this topic to people with geometrical expertise. Hopefully, these results will be of some value to them.

Acknowledgement.

This paper is a continuation of my talk forin the séminaire P. Dubreuil, M.P. Malliavin in november '84 on Cohen-Macauleyness of $\mathbb{T}_{m,2}$. I like to thank M.P. Malliavin for her kind invitation.

This paper was written while nursing our baby, which I hereby thank for her patience.

1. The scheme structure.

In this section we will calculate the local structure of the schemes $\mathbb{P}^m_{(2)}$ and $\mathbb{P}^m_{\text{reg}}$. We will start off with the last one. First, we recall some properties of the so called generic Clifford algebras. For proofs and more details, the reader is referred to [L3] or [L4].

(a) : Generic Clifford Algebras.

Let R be any commutative f-algebra. An m-ary quadratic form over the ring R is a polynomial f in m variables over R which is homogeneous of degree two, i.e.

$$f = \Sigma \alpha_{ij} X_i X_j \in R[X_1, \ldots, x_m]$$

with $\alpha_{ij} = \alpha_{ij} \in R$. So, f determines uniquely a symmetric m by m matrix with coefficients in R

$$M_f = (\alpha_{ij})_{ij} \in M_m(R)$$

Let R^m denote the standard free R-module of rank m with basis $e_1, \ldots, e_m$ the unit vectors. The quadratic form f givens rise to a quadratic map

$$Q_f : R^m \to R$$

defined by sending a column m-tuple $x = (x_1, \ldots, x_m)^\tau$ to $Q_f(x) = x^\tau . M_f . x$. To any m-ary quadratic form f over R one can associate its Clifford algebra $Cl(R, f)$ which is defined to be the quotient of the tensor algebra of the R-module R^m modulo the twosided ideal generated by all elements

$$x \otimes x - Q_f(x)$$

where $x \in R^m$. If we give the tensor algebra the usual $\mathbb{Z}$- gradation, then $x \otimes x$ is homogeneous of degree 2 whereas $_f(x)$ is of degree zero. This entails that the Clifford algebra $Cl(R, f)$ has an induced $\mathbb{Z}/2\mathbb{Z}$- gradation, i.e.

$$Cl(R, f) = C_0 \bigoplus C_1$$

with $C_i . C_j \subset C_k$ where $k \equiv i + j \bmod 2$.
We will now introduce a noncommutative F-algebra Cl_m which is generic in the sense that every Clifford algebra of an m-ary quadratic form over F can be obtained as a specialization of Cl_m.
Let S_m be the homogeneous coordinate ring of the variety of symmetric m by m matrices with entries in F, i.e. S_m is the commutative polynomial ring

$$F[a_{ij} : 1 \leq i \leq j \leq m]$$

in $\binom{m+1}{2}$ indeterminates. By f_m we will denote the following m-ary regular quadratic form over S_m

$$f_m(X_1,\ldots,X_m) = \sum_{i,j=1}^{m} a_{ij}X_iX_j$$

The mth generic Clifford algebra over F, Cl_m, is defined to be the Clifford algebra $Cl(S_m, f_m)$.
If $f = \Sigma\alpha_{ij}X_iX_j$ is any m-ary quadratic form over F, then specializinga_{ij} to α_{ij} gives an F- algebra epimorphism

$$\pi_f : Cl_m \to Cl(F, f)$$

It is possible to give a more concrete description of Cl_m. Consider the iterated Öre-extension

$$\Lambda_m = F[a_{ij} : 1 \leq i < j \leq m][a_1][a_2, \sigma_m, \delta_m]$$

where one defines for each $i < j$ that $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = 2a_{ij}$ and trivial actions of σ_j and δ_j on the other indeterminates.
Using the universal property of Clifford algebras it is easy to check that $\Lambda_m \simeq Cl_m$.
Moreover, giving each of the variables a_{ij} degree 2 and the a_i degree one, one checks that Cl_m is a positively graded F- algebra generated by the m elements of degree one : $a_1, \ldots, a_m$.
Further, Cl_m has finite global dimension equal to $\binom{m+1}{2}$ and the p.i.-degree of the generic Clifford algebra Cl_m is equal to 2^α is the largest natural number $\leq \frac{m}{2}$.
Also, Cl_m is a maximal order and its center Z_m is equal to

$$\begin{array}{cl} S_m & \text{if } m \text{ is even} \\ S_m \bigoplus S_m.d & \text{if } m \text{ is odd} \end{array}$$

where

$$d = S_m(a_1, \ldots, a_m) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) a_{\sigma(1)} \ldots a_{\sigma(m)}$$

where s_m is the permutation group on m elements. Of course, one has $a_i^2 = a_{ii}$.
We will need a concrete description of Spec(Cl_m), i.e. the set of all twosided prime ideals of Λ equipped with the usual Zariski topology. Clearly, intersecting with $\bar{S}_m$ yields a continuous map

$$\varphi : \operatorname{Spec}(Cl_m) \to \operatorname{Spec}(S_m)$$

and since Cl_m is a finite module over S_m (even free of rank 2^m). This map is surjective.

The prime ideal spectrum of a commutative polynomial ring (such as S_m) may be assumed to be relatively well known. Therefore, describing Spec(Cl_m) essentially amounts to describing the fibers of φ.

Proposition 1. [13] If p is any prime ideal of S_m, the fiber $\varphi^{-1}(p)$ contains at most two elements.

Moreover, it is easy to see whether $\varphi^{-1}(p)$ has one or two elements. For let

$$\Pi(A) = (\Pi(a_{ij}))_{i,j}$$

be the symmetric m by m matrix over the domain $S = S_m/p$. If the rank of $\Pi(A)$ is even, then $\varphi^{-1}(p)$ has just one element. If the rank of $\pi(A)$ is odd, $\pi(A)$ is congruent over the field of fractions of S, K to a matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is a symmetric invertible k by k matrix over K, $k = \text{rank}(\Pi(A))$. Let

$$\delta = (-1)^{\binom{k}{2}}.\det(A)$$

then $\varphi^{-1}(n)$ has one element if $\delta \notin (K^*)^2$ and has two elements if $\delta \in (K^*)$? In our situation where F is an algebraically closed field, the set of maximal ideals of S_m corresponds bijectively to the set of symmetric m by m matrices over F. In this case, the number of maximal ideals of Cl_m lying over a maximal ideal corresponding to a matrix $(\alpha_{ij})_{i,j}$ is equal to $1 + (\text{rank}(\alpha_{ij}) \text{mod} 2)$.

There is a main automorphism on Cl_m defined by sending a_i to $-a_i$ or in terms of $\mathbb{Z}/2\mathbb{Z}$-graded algebras by sending an element

$$x = x_0 \bigoplus x_1$$

$x \in Cl_m, x_i \in C_i$, to

$$x_0 \bigoplus (-x_1)$$

It is easy to verify, using the classical structure results of Clifford algebras over a field, see e.g. [Lam], that this automorphism fixes every prime ideal of Cl_m lying uniquely over S_m and permutes the two primes in the other case.

(b) : Local structure of $\mathbb{P}^m_{\text{reg}}$.

The generic Clifford algebras being positively graded F- algebras one can define

$$\mathbb{P}^m_{\text{reg}} = \text{Proj}(Cl_{m+1})$$

where Cl_{m+1} is the generic Clifford algebra generated by $a_0, \ldots, a_m$ and Proj (Cl_{m+1}) is the set of all twosided graded prime ideals of Cl_{m+1} not containing the positive part $(Cl_{m+1})_+ = \bigoplus_{i \geq 1}(Cl_{m+1})_i$ equipped with the induced Zariski topology.
Our first aim will be to prove that $\mathbb{P}^m_{\text{reg}}$ has the structure of a scheme, i.e. it can be covered by open sets which are homeomorphic to the prime ideal spectrum of certain F-algebras. Later we will define structure sheaves on $\mathbb{P}^m_{\text{reg}}$. For any central homogeneous element f of Cl_{m+1}, we denote

$$X_+(f) = \{P \in \mathbb{P}^m_{\text{reg}} : f \notin P\}$$

The open sets we choose are

$$\begin{array}{lcl} I: & X_+(a_{ii}) & 0 \leq i \leq m \\ II: & X_+(2a_{ij} + a_{ii} + a_{jj}) & 0 \leq i < j \leq m \end{array}$$

We will treat every case separately.

Case I : By C_i we denote the graded localization of Cl_{m+1} at the central multiplicative set of homogeneous elements

$$\{a_{ii}^n : n \in \mathbb{N}\}$$

Then it is clear that C_i contains an homogeneous unit of degree one : a_i. Therefore :

$$C_i \simeq (C_i)_0[a_i, a_i^{-1}, \sigma]$$

where σ is the F-automorphism of $(C_i)_0$ determined by conjugation with a_i in C_i, i.e. for every $x \in (C_i)_0$:

$$\sigma(x) = a_i.x.a_i^{-1}$$

We will now calculate the part of degree zero.

Lemma 2: $(C_i)_0 \simeq Cl_m[x_1, \ldots, x_m]$

Proof.
Clearly, $(C_i)_0$ is generated by the elements

$$a_{kl}.a_{ii}^{-1}; a_k.a_i^{-1}$$

for all $0 \leq k <\leq m$ different from i. Let us compute the relations between them.

$$\begin{cases} a_k.a_i^{-1}.a_l a_i^{-1} = -a_k a_l a_{ii}^{-1} + a_{li} a_k^{-1}.a_i^{-1} \\ a_l.a_i^{-1}.a_k.a_i^{-1} = -a_l a_k a_{ii}^{-1} + a_{ki}.a_{ii}^{-1}.a_l.a_i^{-1} \end{cases}$$

Therefore, if we denote

$$\begin{cases} A_k = a_k.a_i^{-1} & \text{for all } 0 \leq h \leq m \text{ different from } i \\ A_{kl} = \frac{1}{2} a_{kl}.a_{ii}^{-1} & \text{for all } 0 \leq h < l \leq m \text{ different from } i \\ A_{ki} = a_{ki}.a_{ii}^{-1} & \text{for all } 0 \leq h \leq m \text{ different from } i \end{cases}$$

we get the following relation for all $k < l$

$$(*) : A_k.A_l + A_l.A_k = -A_{kl} + A_{ki}A_l + A_{li}.A_k$$

Now, change the variables in the following way :

$$\begin{cases} B_k = \frac{1}{2}A_{ki} - A_k & \text{for all } 0 \leq h \leq m \text{ different from } i \\ B_{kl} = \frac{1}{2}A_{ki}A_{li} - A_{kl} & \text{for all } 0 \leq h <\leq m \text{ different from } i \end{cases}$$

then we obtain that

$$\begin{cases} B_k B_l = \frac{1}{4}A_{ki}A_{li} - \frac{1}{2}A_{ki}A_l - \frac{1}{2}A_{li}A_k + A_k A_l \\ B_l B_k = \frac{1}{4}A_{ki}A_{li} - \frac{1}{2}A_{ki}A_l - \frac{1}{2}A_{li}A_k + A_l A_k \end{cases}$$

whence

$$B_k B_l + B_l B_k = \frac{1}{2}A_{ki}A_{li} - A_{ki}A_l - A_{li}A_k + (A_k A_l + A_l A_k)$$

and using the relation $(*)$ as above, this gives us

$$B_k B_l + B_l B_k = \frac{1}{2}A_{ki}A_{li} - A_{kl} = B_{ij}$$

Therefore,

$$(C_i)_0 \simeq F[B_{kl} : 0 \leq k <\leq m, \neq i][B_0][B_1, \sigma_1, \delta_1]\ldots[B_i]\ldots[B_m, \sigma_m, \delta_m]$$

where for all $k < l$ we have

$$\sigma_l(B_k) = B_k \text{ and } \delta_l(B_k) = B_{kl}$$

and trivial actions on the other variables, i.e.

$$(C_i)_0 \simeq Cl_m[A_{1i}, \ldots, A_{ii}, \ldots, A_{mi}]$$

finishing the proof.

Therefore, $C_i \simeq Cl_{\bar{m}}[x_1, \ldots, x_{\bar{m}}][a_i, a_i^{-1}, \sigma]$. Whereas σ acts trivially on each x_j and B_{kl}, it has a nontrivial action on the B_k, for,

$$\begin{aligned} \sigma(B_k) &= a_i.B_k.a_i^{-1} \\ &= \tfrac{1}{2}A_{ki} - a_i A_k a_i^{-1} \\ &= \tfrac{1}{2}A_{ki} + a_k a_i^{-1} - a_{ki}a_{ii}^{-1} = A_k - \tfrac{1}{2}A_{ki} = -B_k \end{aligned}$$

That is, σ is the main automorphism on Cl_m. So, from the discussion above it follows that

$$X_+(a_{ii}) \simeq \mathrm{Spec}_\sigma(Cl_m[X_1,\ldots,X_m])$$

where Spec_σ denotes the set of all σ-prime ideals. In (a) we have seen that the main automorphism σ on Cl_m permits the elements in a fiber consisting of two elements, i.e.

$$\begin{aligned}\mathrm{Spec}_\sigma(Cl_m[x_1,\ldots,x_m]) &\simeq \mathrm{Spec}(S_m[x_1,\ldots,x_m]) \\ &= A_F^{\frac{m+1}{2}+m}\end{aligned}$$

So, the open set $X_+(a_{ii})$ is homeomorphic to affine space.

Case II : We will write $X_{ij} = za_{ij} + a_{ii} + a_{jj}$ and C_{ij} will be the graded localization of Cl_{m+1} at the central homogeneous multiplicative system

$$\{X_{ij}^n : n \in \mathbb{N}\}$$

Again, C_{ij} contains a homogeneous unit of degree one : $a_i + a_j$. Therefore,

$$C_{ij} = (C_{ij})_0[a_i + a_j, (a_i + a_j)^{-1}, \sigma]$$

where σ is the F-automorphism on $\Gamma_{ij})_0$ given by conjugation in C_{ij} with $a_i + a_j$.

Lemma 3 : $(C_{ij})_0 \simeq Cl_m[X_1,\ldots,x_m]$

Proof.

If we denote $Z_{ij} = a_i + a_j$, then $(C_{ij})_0$ is generated by the elements :

$$\begin{cases} a_{kl}.x_{ij}^{-1} & 0 \leq h < \leq m \\ a_k.Z_{ij}^{-1} & k \neq j \end{cases}$$

Let us calculate the commutation rules provided neither k nor $l \neq i$

$$\begin{cases} a_k Z_{ij}^{-1}.a_l Z_{ij}^{-1} = -a_k a_l X_{ij}^{-1} + a_{il} a_k X_{ij}^{-1} Z_{ij}^{-1} + a_{jl} a_k X_{ij}^{-1} Z_{ij}^{-1} \\ a_l Z_{ij}^{-1}.a_k Z_{ij}^{-1} = -a_l a_k X_{ij}^{-1} + a_{ik} a_l X_{ij}^{-1} Z_{ij}^{-1} + a_{jk} a_l X_{ij}^{-1} Z_{ij}^{-1} \end{cases}$$

Therefore, if we denote

$$\begin{aligned} A_k &= a_k Z_{ij}^{-1} \quad \text{for all} \quad 0 \leq k \neq i,j \leq m \\ A_{kl} &= a_{kl}.X_{ij}^{-1} \quad \text{for} \quad (k,l) \neq (i,j) \end{aligned}$$

we obtain the commutation relation

$$A_k A_l + A_l A_k = -A_{kl} + (A_{il} + A_{jl})A_k + (A_{ik} + A_{jk})A_l \qquad (*)$$

Change the variables in the following way

$$\begin{cases} B_k = \frac{1}{2}(A_{ik} + A_{jk} - A_k & 0 \leq h \neq i, j \leq m \\ B_{kl} = \frac{1}{2}(A_{ik} + A_{jk})A_{il} + A_{jl} - A_k & k \neq i \end{cases}$$

Then we obtain using $(*)$ above

$$B_k B_l + B_l B_k = B_{kl}$$

Now, consider the special case that $k = i$, then

$$\begin{cases} a_i Z_{ij}^{-1} a_l Z_{ij}^{-1} = -a_i a_l x_{ij}^{-1} + a_{il} a_i X_{ij}^{-1} Z_{ij}^{-1} + a_{jl} a_i X_{ij}^{-1} Z_{ij}^{-1} \\ a_l Z_{ij}^{-1} a_i Z_{ij}^{-1} = -a_l a_i X_{ij}^{-1} + a_{ii} a_l X_{ij}^{-1} Z_{ij}^{-1} + a_{ij} a_l X_{ij}^{-1} Z_{ij}^{-1} \end{cases}$$

So, if $A_i = a_i Z_{ij}^{-1}$ and $A_{ij} = a_{ij}^{-1}$, the

$$A_1 A_l + A_l A_i = -A_{il} + (A_{il} + A_{jl})A_i + (A_{ii} + A_{ij})A_l$$

And define

$$\begin{cases} B_i = \frac{1}{2}(A_{ii} + A_{ij}) - A_i \\ B_{il} = \frac{1}{2}(A_{ii} + A_{ij})(A_{il} + A_{jl}) - A_{il} \end{cases}$$

then

$$B_1 B_l + B_l B_i = B_{il}$$

finishing the proof, since

$$(C_{ij})_0 = F[B_{kl} : 0 \leq k < l \leq m][B_0][B_1, \sigma_1, \delta] \ldots [\check{B}_j] \ldots [B_m . \sigma_m . \delta_m]$$

where for all $k < l$

$$\sigma_l(B_k) = -B_k \text{ and } \delta_l(B_k) = B_{kl}$$

and trivial actiuons on the other indeterminates.

Now, let us compute the action of σ. If $k \neq i$, then

$$\begin{aligned} \sigma(B_k) &= (a_i + a_j) B_k (a_i + a_j)^{-1} \\ &= \frac{1}{2}(A_{ik} + A_{jk}) + a_k Z_{ij} X_{ij}^{-1} - a_{jk} X_{ij}^{-1} \\ &= A_k - \frac{1}{2}(A_{ik} + A_{jk}) = -B_k \end{aligned}$$

$$\begin{aligned} \sigma(B_i) &= Z_{ij} B_i Z_{ij}^{-1} \\ &= \frac{1}{2}(A_{ii} + A_{ij}) - Z_{ij} A_i Z_{ij}^{-1} \\ &= \frac{1}{2}(A_{ii} + A_{ij}) + A - i Z_{ij} Z_{ij}^{-2} - a_{ii} X_{ij}^{-1} - a_{ij} X_{ij}^{-1} \\ &= -B_i \end{aligned}$$

And as in case I we obtain that

$$X_T(2a_{ij}+a_{ii}+a_{jj}) \simeq \mathrm{Spec}_\sigma(Cl_m[X_1,\ldots,X_m])$$
$$\simeq A_F^{\binom{m+1}{2}+m}$$

Clearly, the open sets $X_+(a_{ii})$ and $X_+(2a_{ij}+a_{ii}+a_{jj})$ cover the whole of $\mathbb{P}^m_{\mathrm{reg}}$. This completes the proof of.

Theorem 4 : As a topological space, $\mathbb{P}^m_{\mathrm{reg}}$ is homeomorphic to the projective space

$$\mathbb{P}_F^{\binom{m+2}{2}-1}$$

We needed the rather lengthly calculations given above in order to define the structure sheaves later. There is another, easier way of deriving Theorem 4 : intersecting with S_m gives a morphism

$$\begin{array}{ccc} \mathbb{P}^m_{\mathrm{reg}} = \mathrm{Proj}\, Cl_{m+1} & \overset{\varphi'}{\rightarrow} & \mathrm{Proj}\, S_{m+1} = \mathbb{P}_F^{\binom{m+2}{2}-1} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, Cl_{m+1} & \overset{\varphi}{\rightarrow} & \mathrm{Spec} S_{m+1} \end{array}$$

and we need to prove that the fibers $\varphi^{-1}(p)$ of any $p \in \mathrm{Proj}(S_{m+1})$ consist of one element.
Now, $\varphi^{-1}(p)$ is homeomorphic to

$$\mathrm{Spec}(Cl_{m+1} \bigotimes_{S_{m+1}} K)$$

where K is the field of fractions of S_{m+1}/p. This algebra is the Clifford algebra associated to the m-ary quadratic form over K

$$\sum \pi(a_{ij})X_iX_j$$

and is therefore isomorphic (as $\mathbb{Z}/2\mathbb{Z}$-graded algebras) to

$$Cl(K,g)\hat{\bigotimes}\Lambda(W)$$

where g is a regular k-ary quadratic form over K and w is $m-k+1$ dimensional, $k = \mathrm{rank}\,\pi(a_{ij})$. Dividing out the kernel of the augmentation map on $\Lambda(W)$ we have a one-to-one correspondence between $\varphi^{-1}(p)$ and $\mathrm{Spec}Cl(K,q)$. Moreover, the map

$$Cl_{m+1} \rightarrow Cl(K,g)$$

is $\mathbb{Z}/z\mathbb{Z}$-graded. If φ^{-1} consists of two elements, then

$$Cl(K,g) \simeq b \oplus b'$$

where b and b' are isomorphic central simple algebras but they are not $\mathbb{Z}/2\mathbb{Z}$-graded. So $\varphi^{-1}(p)$ does not consist out of $\mathbb{Z}/2\mathbb{Z}$-graded prime ideals, so certainly they are not $\mathbb{Z}$-graded, a contradiction. Therefore, φ' is one-to-one. This argument gives also a simplified proof for Theorem III.3.1. of [23].

(c) : Sheaves over $\mathbb{P}^m_{\mathrm{reg}}$

We will noew introduce the structure sheaf of a graded module over Cl_{m+1} on $\mathbb{P}^m_{\mathrm{reg}}$. We remind tha reader that F. Van Oystaeyen and A. Verschoren introduced such structure sheaves in [VV]. However, we prefer here to follow a different approach, mainly for two reasons. First, we believe that for p.i.-algebras there is no real need to introduce the machinery of abstract symmetric or bimodule localization theory, but that central localization usually suffices. For one thing, these artificial localizations are allmost never computable except when they agree with central localization, e.g. if the ring is Zariski central or Azumaya. The second, and more fundamental reason, is that their projectivity schemes are allmost never schemes (in the definition of [VV]) except in the trivial cases such as Zariski central rings. A typical example of what might go wrong is $\mathbb{P}^m_{\mathrm{reg}}$. It is possible to find an open cover for it, all opens being homeomorphic to the affine spectrum of an F-algebra but this algebra is allmost never the part of degree zero of the corresponding graded localization since only σ-prime ideals extend.
To remedy this, we associate to an affine p.i. algebra Λ several structure schemes, one for each subring R of the center over which Λ is a finite module, namely

$$(\mathrm{Spec}(R), \underline{\vartheta}_\Lambda) = \underline{\mathrm{Spec}}_R(\Lambda)$$

where $\underline{\vartheta}_\Lambda$ is the usual structure sheaf of the R-module Λ. Similarly, the R-structure sheaf on a left Λ-module M will be

$$(\mathrm{Spec}(R), \underline{\vartheta}_M$$

It is clear that these concepts work only well for affine F-algebras which are finite modules over their center, but since we aim to study only quotients of trace rings of generic matrices this condition is always satisfied. Working with a subring of the center rather that with the center itself provides this theory with some extra (and necessary) flexibility. This approach was motivated by ideas of M. Van den Bergh.
If we stick to this framework, one can define a structure sheaf on $\mathbb{P}^m_{\mathrm{reg}}$ such that this ringed space is a scheme. For consider an open set $X_+(a_{ii})$, then this

is homeomorphic to the affine spectrum of

$$R_i - \text{Spec}\, F[B_{kl} : 0 \leq k \leq \underset{\neq i}{l} \leq m][A_{1i}, \ldots, \check{A}_{ii}, \ldots, A_{mi}]$$

and $(C_i)_0$ is a finite module over this subring of its center. The structure sheaf $\underline{\vartheta}_{\mathbb{P}^m_{reg}}$ will be defined by

$$\underline{\vartheta}_{\mathbb{P}^m_{reg}} \mid X_+(a_{ii}) = \underline{\text{Spec}}_{R_i}((C_i)_0)$$

and similarly for the open sets $X_+(2a_{ij}+a_{ii}+a_{jj})$. Then this open is homeomorphic to the affine spectrum of

$$R_{ij} = F[B_{kl} : 0 \leq k \leq \underset{\neq j}{\leq} m][A_{1i}, \ldots, \check{A}_{jj}, \ldots \check{A}_{mi}]$$

and we define

$$\underline{\vartheta}_{\mathbb{P}^m_{reg}} \mid X_+(2a_{ij} + a_{ii} + a_{pj}) = \text{Spec}\, _{R_{ij}}((C_{ij})_0)$$

It is trivial to verify that $\underline{\vartheta}_{\mathbb{P}^m_{reg}}$ is a sheaf of F-algebras on $\mathbb{P}^m_{reg}$.
Similarly, one can define a structure sheaf $\underline{\theta}_M$ over $\mathbb{P}^m_{reg}$ for any graded left Cl_{m+1}-module M. For take the graded localization M_i of M at the central homogeneous multiplicative set

$$\{T_i^n : n \in \mathbb{N}\}$$

for $T_i \in \{a_{ii}; 2a_{ij} + a_{ii} + a_{jj}\}$. Then M_i is generated by its part of degree zero which is a module over $\Gamma(X_+(Z_i), \underline{\vartheta}_{\mathbb{P}^m_{reg}})$. Then, we define

$$\underline{\vartheta}_M \mid X_+(Z_i) = (\text{Spec}(R_i), \underline{\vartheta}_{(M_i)_0})$$

In particular we denote for any $n \in \mathbb{Z}$ that the structure sheaf of $Cl_{m+1}(n)$, i.e. the graded module whose part of degree i is equal to $(Cl_{m+1})_{n+i}$, is $\underline{\vartheta}_{\mathbb{P}^m_{reg}}(n)$. One verifies easily that

$$\bigoplus_{n \in \mathbb{Z}} \Gamma(X_+(Z_i); \underline{\vartheta}_{\mathbb{P}^m_{reg}}(n)) \simeq (Cl_{m+1})_i$$

as $\mathbb{Z}$-graded modules. Now, it is about time to turn attention to $\mathbb{P}^m_{(2)}$.

(d) : Rings of generic trace zero matrices

We have seem above that the trace ring of m generic 2 by 2 matrices is the free polynomial ring

$$\mathbb{T}_{m,2} = \mathbb{T}^0_m[Tr(X_1), \ldots, Tr(X_m)]$$

where $\mathbb{T}^0_m$ is the F-subalgebra generated by the generic trace zero-matrices

$$X_i^0 = X_i - \frac{1}{2} Tr(X_i)$$

For any 2 by 2 matrices A and B having trace zero, one knows that

$$A.B + B.A = Tr(AB)$$

Therefore, sending a_i to X_i^0 and a_{ij} to $\frac{1}{2}Tr(X_i^0 X_j^0)$ we obtain an epimorphism

$$\vartheta_m : Cl_m \to \mathbb{T}^0_m$$

We will now indicate what Ker(φ_m) look like. The center of $\mathbb{T}^0_m, R^0_m$, turns out to be the fixed ring of $F[u_{i1}, u_{i2}, u_{i3} : 1 \leq i \leq m]$ under the canonical action of $SO_3(F)$. It follows from the exact sequence

$$1 \to SO_3(F) \to O_3(f) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

that there is an induced $\mathbb{Z}/2\mathbb{Z}$ action on R^0_m whose fixed ring is the fixed ring of $F[u_{i1}, u_{i2}, u_{i3}]$ under action of the full orthogonal group $O_3(F)$.
This ring is by classical invariant theory equal to S^4_m, the homogeneous coordinate ring of the variety of all symmetric m by m matrices with entries in F of rank smaller than or equal to 3. Therefore, we obtain the situation

$$\begin{array}{ccccc} Cl_m & \to & \mathbb{T}^0_m & & \\ & & & \nwarrow & \\ \uparrow & & & & R^0_m \\ & & & \nearrow & \\ S_m & \to & S^4_m & & \end{array}$$

i.e. kernel of φ_m is a (graded) prime ideal of Cl_m lying over the kernel of $\pi_m : S_m \to S^4_m$, i.e. the ideal generated by all 4 by 4 minors of the generic symmetric m by m matrix

$$\mathcal{A} = (a_{ij})_{ij} \in M_m(S_m)$$

Since Ker π_m is a graded prime of S_m, there is only one prime of Cl_m lying over it which is Kerφ_m.

Example :
1. If $m = 2$ or $m = 3$ then φ_M is an isomorphism
2. If $m = 4$, then Kerφ_m is generated by the normalizing element $S_4(a_1, a_2, a_3, a_4)$, i.e. we have an exact sequence

$$0 \to Cl_4.s_4(a_1, a_2, a_3, a_4) \to Cl_4 \to \mathbb{T}^0_4 \to 0$$

(e) Sheaves over $\mathbb{P}^m_{(2)}$

By $\mathbb{P}^m_{(2)}$ we will denote the projective spectrum of the positively graded F-algebra $\mathbb{T}^0_{m+1}$. The epimorphism φ_{m+1} gives a closed immersion

$$\tilde{\varphi}.\mathbb{P}^m_{(2)} \to \mathbb{P}^m_{reg}$$

To economize all our definitions a little we observe that a graded left $\mathbb{T}^0_{m+1}$-module M, is a graded Cl_{m+1}-module, so one has a structure sheaf

$$\underline{\vartheta}_M \text{over } \mathbb{P}^m_{reg}$$

The structure sheaf of M over $\mathbb{P}^m_{(2)}$ is then, of course, defined to be

$$\underline{\vartheta}^{(2)}_M = \tilde{\varphi}^* \underline{\vartheta}_M$$

If it is clear that we are working over $\mathbb{P}^m_{(2)}$ we forget the superscript (2). In particular we denote

$$\underline{\vartheta}_{\mathbb{P}^m_{(2)}} = \underline{\vartheta}^{(2)}_{Cl_{m+1}}$$
$$\underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n) = \underline{\vartheta}^{(2)}_{Cl_{m+2}(n)} \qquad \text{for all } n \in \mathbb{Z}$$

Again, one can cover $\mathbb{P}^m_{(2)}$ by affine open sets namely $X_+(Z_i)$ where

$$Z_i \in \{\varphi_{m+1}(a_{ii}); \varphi_{m+1}(2a_{ij} + a_{ii} + a_{jj})\}$$

and on each such set, one can verify directly

$$\bigoplus_{n \in \mathbb{Z}} \Gamma(X_+(Z_i); \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n)) \simeq (Cl_{m+1})_{Z_i}$$

as graded modules. where the right hand side denotes the graded localization of Cl_{m+1} at $\{Z_i^n : n \in ?\}$.

We refer the reader to $[L - V - V]$ for a description of the scheme- structure of $\mathbb{P}^m_{(2)}$ on the Azumaya open sets.

$$X_+((X_i^0 X_j^0 - X_j^0 X_i^0)^2)$$

Now, it is about time to calculate some cohomology groups.

2. The cohomology.

In this section we will compute the cohomology of $\mathbb{P}^m_{(2)}$ and $\mathbb{P}^m_{reg}$. Both results can be viewed as a quaternionic generalization of Serre's classical result.

Theorem 5 : (Cohomology of $\mathbb{P}^m_{reg}$)

(1) : $Cl_{m+1} \simeq \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^m_{reg}, \underline{\vartheta}_{\mathbb{P}^m_{reg}}(n))$

(2) : $H^i(\mathbb{P}^m_{reg}; \underline{\vartheta}_{\mathbb{P}^m_{reg}}(n)) = 0$ for all n and all $0 < i < \binom{m+2}{2} - 1$

(3) : For all $n \in \mathbb{N}$, the dimension of the F-vector space

$$H^{\binom{m+2}{2}-1}(\mathbb{P}^m_{reg} l \underline{\vartheta}_{\mathbb{P}^m_{reg}}(-\binom{m+2}{2} - n))$$

is equal to

$$\sum_{i+2j=n} \binom{m+i}{i} . \binom{\binom{m+1}{2}+j}{j}$$

Proof.

With $\mathcal{F}$ I will denote the quasi-coherent sheaf

$$\bigoplus_{n\in\mathbb{Z}} \underline{\vartheta}_{\mathbb{P}^m_{reg}}(n)$$

Since cohomology commutes with arbitrary direct sums on a Noetherian topological space, the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology of the shaves $\underline{\vartheta}_{\mathbb{P}^m_{reg}}(n)$. Therefore, we aim to compute the cohomology of $\mathcal{F}$ and keep track of the grading by n, so that we can sort out the gradings at the end.
We cover $\mathbb{P}^m_{reg}$ with

$$\mathcal{U} = \{X_+(y_i) : 0 \leq i \leq \binom{m+2}{2} - 1\}$$

where the y_i are the elements from the set

$$\{a_{ii} 1 \leq i \leq m; 2a_{ij} + a_{ii} + a_{jj} : 0 \leq i \leq j \leq m\}$$

For any set of indices $i_i, \ldots, i_p$ between 0 and $\binom{m+2}{2} - 1$ the open set

$$U_{i_0 \ldots i_p} = X_+(y_{i_o}) \cap \ldots \cap X_+(y_{i_p}) = X_+(y_{i_0} \ldots y_{i_p})$$

and by the results of the foregoing section we know that the sections of $\mathcal{F}$ on $U_{i_o,\dots,i_p}$ is equal to the graded localization of Cl_{m+1} at the central homogeneous multiplicative set $\{(y_{i_0}\cdots y_{i_p})^n : n \in \mathbb{N}\}$, and the grading by n on $\mathcal{F}$ is the same as the grading of this localization.
Therefore the Čech complex is

$$G^o(\mathcal{U}, \mathcal{F}) : 0 \to Cl_{m+1} \to^{\delta_1} \prod_{i_o}(Cl_{m+1})_{y_{i_0}} \to^{\delta_2} \prod_{i_0,\dots i_1}(cl_{m+1})_{y_{i_0}y_{i_1}} \to$$

$$\dots \to (Cl_{m+1})_{y_0,\dots y_\alpha} \to 0$$

where $\alpha = \binom{m+2}{2} - 1$ and all the modules have a natural gradation compatible with that on $\mathcal{F}$, so

$$\bigoplus_{n\in\mathbb{Z}} H^2(\mathbb{P}^m_{reg}, \vartheta_{\mathbb{P}^m_{reg}}(n)) \simeq \mathrm{Ker}(\delta_{i+2}/Im(\delta_{i+1})$$

The right hand side of this expression can be computed using local cohomology with respect to the irrelevant ideal $(S_{m+1})_+ = \bigoplus_{i\geq 1}(S_{m+1})_i$. Let us define for every graded left S_{m+1}-module M

$$L(M) = \{m \in M \mid \exists n > 0 : (S_{m+1})^n_+ .m = 0\}$$

It is well known that L is a left-exact additive functor so we can take the right derived functors R^iL and one has, cfr. e.g. [St]

$$R^{i+1}L(M_{m+1}) = H^{i+1}(Cl_{m+1}) = \mathrm{Ker}\delta_{i+2}/Im\delta_{i+1}$$

and so it will suffice these local cohomology modules. From [St, p. 43-44], we recall that $H^i(M) = 0$ unless $e = \mathrm{depth}(M) \leq i \leq \dim(M) = d$, for $i > 0$, and that $H^e(M) \neq 0, H^d(M) \neq 0$.
Now, Cl_{m+1}, is a graded free module of rank 2^{m+1} over the polynomial subring of its center S_{m+1}, i.e. Cl_{m+1} is a Cohen-Macauley, i.e.

$$\mathrm{depth}(Cl_{m+1} = \dim(Cl_{m+1}) = \binom{m+2}{2}$$

Therefore, if $0 < i < \binom{m+2}{2} - 1$, then

$$\bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^m_{reg}, \vartheta_{\mathbb{P}^m_{reg}}(n)) \simeq H^{i+1}(Cl_{m+1}) = 0$$

and clearly,

$$\bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^m_{reg}, \vartheta_{\mathbb{P}^m_{reg}}(n)) \simeq H^1(Cl_{m+1}) = 0$$

So, we are left to prove $H^{\binom{m+2}{2}}(Cl_{m+1})$. From [St, Th. 6.4] and Cohen-Macauleyness of Cl_{m+1} we retain that

$$(-1)^{\binom{m+2}{2}}.\mathcal{P}(H^{\binom{m+2}{2}}(Cl_{m+2});t) = \mathcal{P}(Cl_{m+2};t)_{\infty}$$

where we denote for every graded left Cl_{m+1}-module M its Poincaré series

$$\mathcal{P}(M;t)\sum_{n=0}^{\infty}\dim_F(M_n).t^n$$

and where $\mathcal{P}(Cl_{m+1};t)_{\infty}$ signifies that the Poincaré series of Cl_{m+1} is to be expanded as a Laurent series around ∞.
The Poincaré series of Cl_{m+1} is easy to calculate, since as a graded F-vector space, Cl_{m+1} is isomorphic to the commutative polynomial ring

$$F[a_{ij} : 0 \leq i < j \leq m][a_0, \ldots, a_m]$$

with deg $a(_{ij}) = 2$ and deg$(a_i) = 1$. Therefore,

$$\mathcal{P}(Cl_{m+1};t) = \frac{1}{(1-t)^{m+1}(1-t^2)^{\binom{m+1}{2}}}$$

and its Laurent series expansion ∞ is equal to

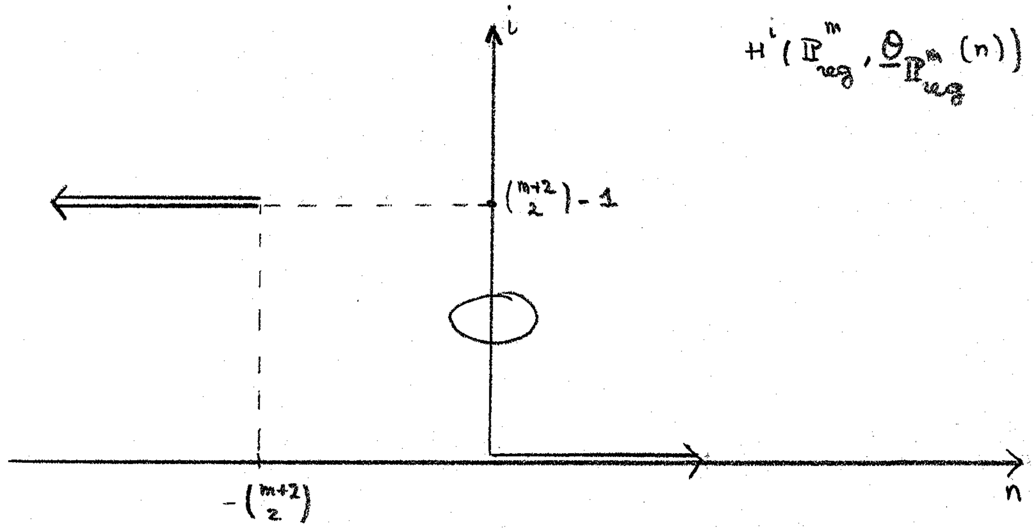
$$(-1)^{\binom{m+2}{2}}.\frac{t^{-\binom{m+2}{2}}}{(1-t^{-1})^{m+1}.(1-t^{-2})^{\binom{m+1}{2}}}$$
$$= (-1)^{\binom{m+2}{2}}.\sum_{n\in\mathbb{N}} t^{-\binom{m+2}{2}-n}.\dim_F(Cl_{m+1})_n$$

Combining all this information, we get that the dimension of

$$H^{\binom{m+2}{2}-1}(\mathbb{P}^m_{reg}, \underline{\vartheta}_{\mathbb{P}^m_{reg}}(-\binom{m+2}{2}-n))$$

is equal to $\dim_F(Cl_m)_n = \sum_{i+2j=n}\binom{m+i}{i}\binom{\binom{m+2}{2}+j}{j}$ finishing the proof.

Pictorially, we get



Because we have computed the local structure of $\mathbb{P}^m_{reg}$ explicitly, it might be possible to mimic Serre's proof as in [Ha] to obtain Theorem 5. We have chosen this approach because it clarifies the two main ingredients necessary for a computation of the cohomology of $\mathbb{P}^m_{(2)}$ for which the local structure is not so easy to give.
These two ingredients are :

(A) : a proof that $\mathbb{T}^0_{m+1}$ is a Cohen-Macauley module.

(B) : a functional equation for the Poincaré series of $\mathbb{T}^0_{m+1}$ in order to calculate $\mathcal{P}(\mathbb{T}^0_{m+1};t)_\infty$.

In my talk before the séminaire in Paris, november '84, I outlined the proof of (A). Since details of this proof will appeare elswhere, e.g. [L3] or [L3], I will sketch here only the main ideas.
First, note that for $m = 2$ of 3, $\mathbb{T}^0_m$ is Cohen-Macauley in view of the isomorphism with Cl_m, mentioned above.
To prove Cohen-Macauleyness for $m \geq 4$, we mimic the argument of Kutz [Ku] to prove Cohen-MacAuleyness of the rings S^h_m, i.e. the homogeneous coordinate ring of symmetric m by m matrices of rank $< k$. He constructs ideals $I_{H,n}$ where

$$H = \{s_0 < s_1 < \ldots < s_l : s_i \in [0,m]\}$$

and $n \in \mathbb{N}$ to be generated by

(0) : the entries of the last s_0 columns of $\mathcal{A} = (a_{ij})$

(1) : The 2 by 2 minors of the last s_1 columns of $\mathcal{A}$

$\vdots$

(l) : The $l+1$ by $l+1$ minors matrices of the last s_l columns of $\mathcal{A}$

(*) : The entries of the last n columns of the first row of $\mathcal{A}$

Remark that the ideal generated by all k by k minors (i.e. $\text{Ker}(S_m \to S_m^k)$) is of the form with

$$H = \{0 < 1 < 2 \ldots < k-2 < m\}; n = 0$$

Kutz shows that in case $n = s_p$ for some p, then $I_{H,n}$ is a prime ideal and if $s_p < n < s_{p+1}$, then

$$I_{H,n} = I_{H',n} \cap I_{H,n'}$$

where $n' = s_{p+1}$ and $H' = \{s_o < \ldots < s_{p-1} < n < s_{p+1} \ldots\}$ where both $I_{H',n}$ and $I_{H,n'}$ are (graded) prime ideals of S_m.
There is a unique prime ideal $J_{H,n}$ of Cl_m lying over such a prime and we define for $s_p < n < s_{p+1}$

$$J_{H,n} = J_{H',n} \cap J_{H,n'}$$

Using some structure theory of Clifford algebras one can show that if $n = s_p+1$, then

$$J_{H',n'} = J_{H',n} + J_{H,n'}$$

and if $n = s_p$, then

$$J_{H,n+1} = J_{H,n} + Cl_m.a_{1,m-n}$$

Now, using these two relations one can show by an induction argument that

$$pd_{Cl_m}(Cl_m/J_{H,n}) = \text{Kdim}(Cl_m) - \text{Kdim}(Cl_m/J_{H,n})$$

Whenever $n = s_p$ or $n = s_p + 1$ for some p.
In particular

$$\begin{aligned} pd_{Cl_m}(\mathbb{T}^0_m) &= \text{Kdim}(Cl_m) - \text{Kdim}(\mathbb{T}^0_m) \\ &= \binom{m+1}{2} - 3m + 3 \\ &= \frac{(m-2)(m-3)}{2} \end{aligned}$$

Finally, using that Cl_m is a free module of finite rank over S_m and that Cl_m is a maximal order having trivial normalizing class group one can show that

$$\begin{aligned} \text{Ext}^i_{S_m}(\mathbb{T}^0_m, S_m) &\simeq \text{Ext}^i_{Cl_m}(\mathbb{T}^0_m, \text{Hom}_{S_m}(Cl_m, S_m)) \\ &\simeq \text{Ext}^i_{Clm}(\mathbb{T}^0_m, Cl_m) \end{aligned}$$

and so $pd_{S_m}(\mathbb{T}^0_m) = \dim S_m - \dim \mathbb{T}^0_m$ whence $\mathbb{T}^0_m$ is a Cohen-Macauley module over S_m. For more details, see [L3].

Now, let us consider objective (B). The map

$$Tr(-X_{m+1})\mathbb{T}_{m,2} \rightarrow \mathcal{R}_{m+1,2}$$

is a linear injection onto the subspace of elements which are homogeneous of degree one in X_{m+1}. In terms of multilinear Poincaré series this means that

$$\mathcal{P}(\mathbb{T}_{m,2}; t_1, \ldots, t_m) = \frac{\partial}{\partial t_{m+1}} \mathcal{P}(\mathcal{R}_{m+1,2}; t_1, \ldots, t_{m+1}) \Big|_{t_{m+1}=0}$$

Now, $\mathcal{R}_{m+1,2} = \mathcal{R}^0_{m+1}[Tr(X_1), \ldots, Tr(X_{m+1})]$ where $\mathcal{R}^0_{m+1}$ is the ring of invariants under action of $SO_3(F)$ whose multilinear Poincaré series were calculated by e.g. H. Weyl.
Combining all this, we gave a rational expression for the Poincaré series of $\mathbb{T}_{m,2}$ in [1].

Proposition 6 [L1]

The Poincaré series of the trace ring of m generic 2 by 2 matrices has the following rational expression

$$\mathcal{P}(\mathbb{T}_{m,2}; t_1, \ldots, t_m) = \frac{e_m \Delta_1 - (e_m + e_1 e_m + e_{m-1})\Delta_2}{e_m^2 \prod_{j=1}^m (1 - tj) \prod_{i<k}^m (T_k - t_i) \prod_{i \leq k}^m (1 - t_i t_k)}$$

Here, the e_i are the i-th elementary symmetric function in m variables and the Δ_i are defined by

$$\Delta_1 = \det \begin{bmatrix} 1 + t_1^{2m-1} & \ldots & 1 + t_m^{2m-1} \\ t_1^2 + t_1^{2m-3} & \ldots & t_m^2 + t_m^{2m-3} \\ t_1^3 + t_1^{2m-4} & \ldots & t_m^3 + t_m^{2m-n} \\ \vdots & & \vdots \\ t_1^{m-2} + t_1^{M+1} & \ldots & t_m^{m-2} + t_m^{m+1} \\ t^{m-1} & \ldots & t_m^{m-1} \\ t_1^m & \ldots & t_m^m \end{bmatrix}$$

$$\Delta_2 = \det \begin{bmatrix} t_1 + t_1^{2m-2} & \ldots & t_m + t_m^{2m-2} \\ t_1^2 + t_1^{2m-3} & \ldots & t_m^2 + t_m^{2m-3} \\ \vdots & & \vdots \\ t_1^{m-2} + t_1^{m+2} & \ldots & t_m^{m-2} + t_m^{m+2} \\ t_1^{m-1} & \ldots & t_m^{m-1} \\ t_1^m & \ldots & t_m^m \end{bmatrix}$$

and as an immediate consequence of this, one obtains :

Corollary 7 [L1] (Functional equation)

The Poincaré series of the trace ring of m-generic 2 by 2 matrices satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,2}; \frac{1}{t} = -t^{4m}.\mathcal{P}(\mathbb{T}_{m,2}; t)$$

Another, more ringtheoretocal proof of this fact using Cohen-Macauleybess of $\mathbb{T}_{m,2}$ and that $\mathbb{T}_{m,2}$ is a maximal order having trivial normalizing classgroups can be found in [L3].
We have now all material at our disposal to calculate the cohomology of $\mathbb{P}^m_{(2)}$.

Theorem 8 : (Cohomology of $\mathbb{P}^m_{(2)}$)

(1) : $\mathbb{T}^0_{m+1} \simeq \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n))$

(2) : $H^i(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n)) = 0$ for all n and $0 < i < 3m - 1$

(3) : There is a one-to-one correspondence between a basis of the F-vectorspace

$$H^{3m-1}(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(-3m - 3 - n))$$

and standard young tableaux of length ≤ 3 filled with entries from 0 to m having n boxes.

Proof.

Of course we take as an affine cover of $\mathbb{P}^m_{(1)}$

$$Tl = \{y_i : 0 \leq i \leq \binom{m+2}{2} - 1\}$$

where the y_i are the elements from the set

$$\{\phi(a_{ii}) : 0 \leq i \leq m; \phi(2a_{ij} + a_{ii} + a_{jj}) : 0 \leq i < j \leq m\}$$

where $\phi : Cl_{m+2} \to \mathbb{T}^0_{m+1}$ is the natural epimorphism. As in the proof of Theorem 5 we consider the quasi-coherent sheaf

$$\bigoplus_{n\in\mathbb{Z}} \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n)$$

and similarly one deduces that

$$\bigoplus_{n\in\mathbb{Z}} H^i(\mathbb{P}^m_{(2)}, \underline{\vartheta}_{\mathbb{P}^m_{(2)}}(n)) \simeq H^{i+1}(\mathbb{T}^0_{m+1})$$

We know that $\mathbb{T}^0_{m+1}$ is a Cohen-Macauley module module over S_{m+1} of dimension $3(m+1)-3=3m$, so part (2) of the theorem is proved.
As for part (3), we know that

$$\mathcal{P}(H^{3m}(\mathbb{T}^0_{m+1});t) = \mathcal{P}(\mathbb{T}^0_{m+1};t)_\infty.(-1)^{3m}$$

From the functional equation for the Poincaré series of $\mathbb{T}_{m+1,2}$ and the fact that $\mathbb{T}_{m+1,2} = \mathbb{T}^0_{m+1}[Tr(X_0),\ldots,Tr(X_m)]$ i.e.

$$\mathcal{P}(\mathbb{T}_{m+1,2};t) = \frac{1}{(1-t)^{m+1}}.\mathcal{P}(\mathbb{T}^0_{m+1};t)$$

we find that

$$\mathcal{P}(\mathbb{T}^e_{m+1};\frac{1}{t}) = (-1)^{3m}t^{3m+3}\mathcal{P}(\mathbb{T}^0_{m+1};t)$$

i.e.

$$\mathcal{P}(\mathbb{T}^0_{m+1};t)_\infty = (-1)^{3m}t^{-3m-3}.\mathcal{P}(\mathbb{T}^0_{m+1};\frac{1}{t})$$

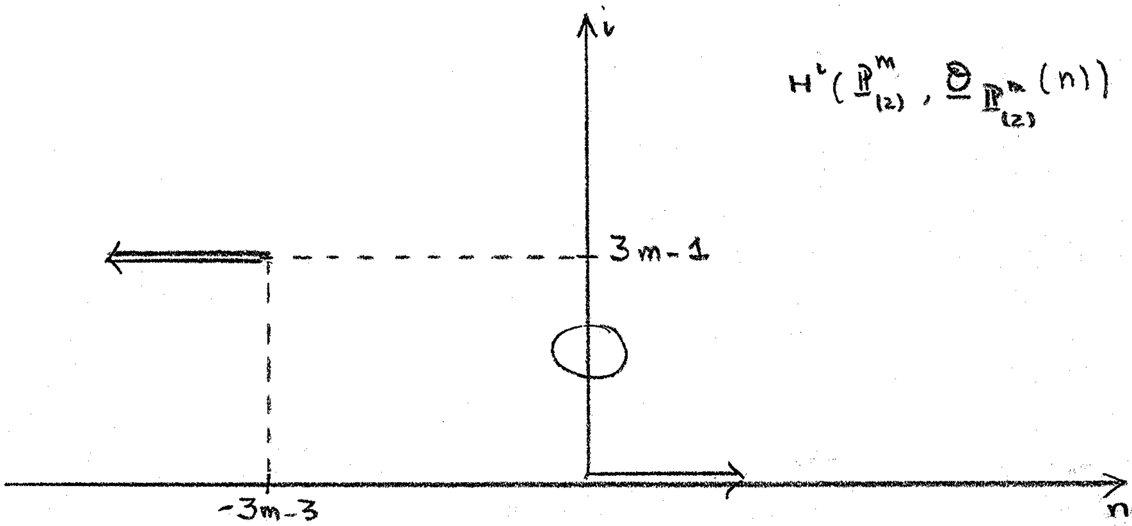
and therefore

$$\mathcal{P}(H^{3m}(\mathbb{T}^0_{m+1});t = \sum_{n\in\mathbb{N}} t^{-3m-3-n}.\dim_F(\mathbb{T}^0_{m+1})_n$$

and from the work of C. Procesi, [P2], we retain that there isa a one-to-one correspondence between standard Young tableaux of shape $\sigma = 3^a2^b1^c$ for $a,b,c,\in\mathbb{N}$. i.e. a diagram consisting of a rows of length 3, b rows of length 2 and c rows of length 1

filled with indices from 0 tot m such that the numbers in every row strictly increase and that the numbers in every column do not decrease; and an F-vector space basis of $\mathbb{T}^0_{m+1}$. Moreover, the degree of an element corresponding to a standard Young tableau of shape $\sigma = 3^a2^b1^c$ is equal to the number of boxes in the diagram, i.e. is $3a+2b+c$.
This finishes the proof of the theorem.

Pictorially, we have the situation :



From this point on, it is easy to derive as in the commutative case a Grothendieck-Serre duality result for coheremnt sheaves over $\mathbb{P}^m_{(2)}$ or $\mathbb{P}^m_{(reg)}$. We leave this as an (easy) exercise to the reader, cfr. [A-K] for the commutative proof.

References :

[A-K] : Altman, Kleiman : Introduction to Grothendieck duality, Springer Verlag LNM.

[Ar] : Artin : Azumaya algebras and finite dimensional representations of rings. J. Alg. II (1969), 532-563.

[A-S] : Artin, Schelter : Integral ring morphisms, Adv. in Math. 39, (1981), 289-329.

[L1] : Le Bruyn : Functional equation of the Poincaré series of $\mathbb{T}_{m,2}$; Israel J. Math. (1985).

[L2] : Le Bruyn : Homological properties of trace rings of generic matrices, Trans. Am. Math. Soc. (1985).

[L3] : Le Bruyn : Trace rings of generic 2 by 2 matrices, thesis UIA '85. Memoirs of the AMS (1986 ?)

[L4] : Le Bruyn : The Poincaré series of $\mathbb{T}_{m,2}$; Proceeding Antwerp Conference, April '85.

[LVV] : Le Bruyn, Van den Bergh, Van Oystaeyen : Proj. of generic matrices and trace rings, to appear.

[NV] : Năstăsescu, Van Oystaeyen : graded ring theory, North Holland.

[St] : Stanley : Combinatorial methods in commutative ring theory, Birkhäuser.

[P1] : Procesi : Finite dimensional representations of algebra, Israel J. Math. 19, (1974), 169-184.

[P2] : Procesi : Computing with 2 by 2 matrices, J of Algebra 87, (1984), 342-359.

[Lam] : Lam : Algebraic theory of quadratric forms, Benjamin, (1973).

[VV] : Van Oystaeyen, Verschoren : Noncommutative algebraic geometry. Springer LNM 887.

[Ha] : Hartshorne : Algebraic geometry, Springer Verlag

[Ku] : Kutz : Cohen-Macauley rings and ideal theory in rings of invariants of algebraic groups, Trans. Am. Math. Soc. 197, (1974), 115-129.