THE POINCARE SERIES OF $\mathbb{T}_{m,2}$

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0. Introduction.

Recently, E. Formanek extended a result of the author [13] to arbitrary n by n matrices.

Theorem 0 (Formanek, [5, Th. 21])
The Poincaré series of the trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$, satisfies the functional equation:

$$\mathcal{P}(\mathbb{T}_{m,n};\frac{1}{t_1},\ldots,\frac{1}{t_m}) = (-1)^d e_m^{n^2}.\mathcal{P}(\mathbb{T}_{m,n};t_1,\ldots,t_m)$$

where $m \geq n^2$, $d = Kdim(\mathbb{T}_{m,n}) = (m-1).n^2+1$ and e_m is the m-th elementary symmetric function $t_1 \ldots t_m$.

Despite its intrincsic beauty, this theorem should only be an intermediate result. For, ultimately one is interested in:

(1): finding a closed expression, i.e. rational form, of the Poincaré series either in multi- or single gradation.

(2): giving some homological explication for the existence of a functional equation, similar to Stanley's result in the commutative case.

Both these problems seem to be rather difficult for arbitrary n by n matrices. However, they have a satisfactory answer for 2 by 2 matrices, [14]. In this paper i would like to scetch three independent proofs for the functional equation of the Poincaré series of the trace rings of generic 2 by 2 matrices, each of them will provide us with an extra bit of information.

The first one, based on some ancient results of H. Weyl and I. Schur on rings of invariants under special orthogonal groups, will give us a rational expression in a multigradation.

The second proof, which was outlined to me by C. Procesi, provides us with a combinatorial method to calculate these rational forms in a single gradation.

The final proof gives us the explication for the existence of the functional equation, namely that the trace ring is a maximal order having trivial normalizing classgroup and that it is a Cohen-Macauley module over its center.

For details the interested reader is referred to [14].

1. Notation.

Throughout, F will be a field of characteristic zero, algebraically closed if necessary. Let $\mathcal{P}_{m,n}$ be the polynomial ring

$$\mathcal{P}_{m,n} = F[x_{ij}(\ell) : 1 \leq i,j \leq n; 1 \leq \ell \leq m]$$

and consider the so called generic matrices

$$X_\ell = (x_{ij}(\ell))_{i,j} \in M_n(\mathcal{P}_{m,n})$$

then $\mathbb{G}_{m,n}$, the ring of m generic n by n matrices, is the sub F-algebra of $M_n(\mathcal{P}_{m,n})$ generated by the elements $\{X_1,\ldots,X_m\}$.
H is well known that $\mathbb{G}_{m,n}$ is a domain, and if we localize it at the multiplicative set of all nonzero central elements we obtain a division ring $\Delta_{m,n}$ of dimension n^2 over its center $\mathcal{K}_{m,n}$. Let

$$Tr : \Delta_{m,n} \to \mathcal{K}_{m,n}$$

be the usual trace morphism, then the trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$, is the F-subalgebra of $\Delta_{m,n}$ generated by $\mathbb{G}_{m,n}$ and $Tr(\mathbb{G}_{m,n})$. We will denote the center of $\mathbb{T}_{m,n}$ by $\mathcal{R}_{m,n}$. Both $\mathbb{T}_{m,n}$ and $\mathcal{R}_{m,n}$ are $\mathbb{N}^{(m)}$-graded F-algebras by giving each generic matrix entry $x_{ij}(\ell)$

$$deg(x_{ij}(\ell)) = (0,\ldots,1,\ldots,0)$$

with 1 on spot ℓ. The corresponding multi-graded Poincaré series are then defined by

$$\mathcal{P}(\mathbb{T}_{m,n};t_1,\ldots,t_m) = \sum_{(i_1,\ldots,i_m)} .dim_F((\mathbb{T}_{m,n})_{(i_1,\ldots,i_m)}).t_1^{i_1}\ldots t_m^{i_m}$$

$$\mathcal{P}(\mathcal{R}_{m,n};t_1,\ldots,t_m) = \sum_{(i_1,\ldots,i_m)} .dim_F((\mathcal{R}_{m,n})_{(i_1,\ldots,i_m)}).t_1^{i_1}\ldots t_m^{i_m}$$

Clearly, they are also positively graded by giving each $x_{ij}(\ell)$ degree one. The corresponding Poincaré series are

$$\mathcal{P}(\mathbb{T}_{m,n};t) = \sum^{\infty} dim_F(\mathbb{T}_{m,n})_i.t^i$$

$$\mathcal{P}(\mathcal{R}_{m,n};t) = \sum_{i=0}^{\infty} dim_F(\mathcal{R}_{m,n})_i.t^i$$

Using the fact that both $\mathcal{R}_{m,n}$ and $\mathbb{T}_{m,n}$ are fixed rings under $GL_n(F)$, it is possible to compute these Poincaré series. This was done by Formanek [4], but his results are not relevant for the rest of this paper.
Finally, since $\mathcal{R}_{m,n}$ is an affine F-algebra and $\mathbb{T}_{m,n}$ is a finite $\mathcal{R}_{m,n}$- module, it is clear that all these Poincaré series are rational.
To my knowledge, the only rational form which is known for a $\mathbb{T}_{m,n}$, $n \geq 3$, is due to M. Van den Bergh [15]. He proves that

$$\mathcal{P}(\mathbb{T}_{2,3};t) = \frac{1}{(1-t)^4(1-t^2)^4(1-t^3)^2}$$

For $n = 2$, we have a complete information about this rational form.

2. Rational Expression.

Since we will be working only with the case that $n = 2$, we denote $\mathbb{T}_m = \mathbb{T}_{m,2}$ and $\mathcal{R}_m = \mathcal{R}_{m,2}$. By a result of Procesi's [17], the center $\mathcal{R}_m$ is the ring of the polynomial maps from m copies of $M_2(F)$ to F

$$M_2(F) \oplus \ldots \oplus M_2(F) \to F$$

which are invariant under componentswise action by conjugation of $GL_2(F)$. Of course, each of the factors decomposes as a $GL_2(F)$-module

$$M_2(F) = F \oplus M^\circ$$

where N° is the 3-dimensional vector space of trace zero matrices over F. The funny thing about working with 2 by 2 matrices is that the action of $GL_2(F)$ on M° is equivalent to the action of the special orthogonal 3-dimensional group, $SO_3(F)$, on standard three dimensional vector space $F^{(3)}$ (at least if $\sqrt{-1} \in F$).
Combining these two remarks we get that

$$\mathcal{R}_m = \mathcal{R}^\circ_m[Tr(x_1), \ldots, Tr(x_m)]$$

where

$$\mathcal{R}^\circ_m = F[u_{1i}, u_{2i}, u_{3i} : 1 \leq i \leq m]$$

The second remark we make is that there is a linear injection

$$Tr(-x_{m+1}) : \mathbb{T}_m \to \mathcal{R}_{m+1}$$

whose image consists precisely of those elements which are homogeneous of degree one in t_{m+1}, i.e. the variable corresponding to the generic matrix x_{m+1}.

If we translate this fact to the level of Poincaré series in a multi-gradation, we get:

$$\mathcal{P}(\mathbb{T}_m;t_1,\ldots,t_m) = \frac{\partial}{\partial t_{m+1}} \mathcal{P}(\mathcal{R}_{m+1};t_1,\ldots,t_{m+1}) \mid_{t_{m+1}=0}$$

and since $\mathcal{R}_{m+1}$ is a polynomial ring over $\mathcal{R}^\circ_{m+1}$ we have

$$\mathcal{P}(\mathcal{R}_{m+1};t_1,\ldots,t_{m+1}) = \mathcal{P}(\mathcal{R}^\circ_{m+1};t_1,\ldots,t_{m+1}).\frac{1}{\Pi_{i=1}^{m+1}(1-t_i)}$$

H. Weyl and I. Schur ($\pm$ 1925) have calculated the rational expression of the Poincaré series of rings of invariants under special orhtogonal groups, so in particular of R°_{m+1}. They got that $\mathcal{P}(\mathcal{R}^o_{m+1};t_1,...,t_{m+1})$ is equal to

$$(\Pi_{i<k}^{m+1}(t_k - t_i).\Pi_{i\leq k}^{m+1}(1 - t_i t_k))^{-1} \quad \begin{bmatrix} \cdots & 1+t_i^{2m-1} & \cdots \\ \cdots & t_i + t_i^{2m-2} & \cdots \\ \cdots & t_i^{m-i} + t_i^{m+1} & \cdots \\ \cdots & t_i^{m-i} & \cdots \\ \cdots & t_i^m & \cdots \end{bmatrix}$$

Where the numerator is the determinant of an $m+1$ by $m+1$ matrix, $1 \leq i \leq m+1$. So, we have reduced the problem of finding a rational expression to easy but boring calculations. One gets

Theorem 1. [13]
The Poincaré series of the trace ring of in generic z by z matrices has the following rational expression:

$$\mathcal{P}(\mathbb{T}_m;t_1,\ldots,t_m) = \frac{e_m\Delta_1 - (e_m + e_1 e_m + e_{m-1})\Delta_2}{e_m^2.\pi_{j=1}^m(1-tj)\pi_{i\leq k}^m(t_k - t_i).\pi_{i\leq k}^m(1-t_i t_k)}$$

where e_i is the i-th elementary symmetric function and the Δ_j are determinants of m by m matrices, having as its i-th column:

$$\Delta_1 = \begin{bmatrix} & 1+t_i^{2m-1} & \\ & t_i^2 + t_i^{2m-3} & \\ & t_i^3 + t_i^{2m-4} & \\ \cdots & t_i^{m-1} + t_i^{m+1} & \cdots \\ & t_i^{m-1} & \\ & t_i^m & \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} & t_i + t_i^{2m-2} & \\ & t_i^2 + t_i^{2m-3} & \\ \cdots & t_i^{m-1} + t_i^m & \cdots \\ & t_i^{m-1} & \end{bmatrix}$$

Since clearly

$$e_m^{2m-1}.\Delta_1(\frac{1}{t_1},\ldots,\frac{1}{t_m}) = -\Delta_1(t_1,\ldots,t_m)$$

$$e_m^{2m-1}.\Delta_2(\frac{1}{t_1},\ldots,\frac{1}{t_m}) = -\Delta_2(t_1,\ldots,t_m)$$

$$e_m^2.[(e_m + e_1e_m + e_{m-1})(\frac{1}{t_1},\ldots,\frac{1}{t_m})] = e_m + e_me_1 + e_{m-1}$$

we get as an immediate corollary of Theorem 1 our first proof of the functional equation.

Corollary 2.
The Poincaré series of the trace ring of m generic 2 by 2 matrices satisfies the functional equations:

$$\mathcal{P}(\mathbb{T}_m;\frac{1}{t_1},\ldots,\frac{1}{t_m}) = -e_m^4.\mathcal{P}(\mathbb{T}_m;t_1,\ldots,t_m)$$

So, the first proof has the advantage of giving a closed formula for the Poincaré series but also the disadvantedge that it is allmost impossible to calculate this rational expression explicitly for a given m.

3. A Combinatorial Computation.

As stated in the introduction, the results in this section were outlined to me by C. Procesi.
We have seen in 2. that the center $\mathcal{R}_m$ is a polynomial extension in the traces of the generic matrices over a subring $\mathcal{R}_m^\circ$. A simular result is also valid for the trace ring itself. One can show that [17]

$$\mathbb{T}_m = \mathbb{T}_m^\circ[Tr(x_1),\ldots,Tr(x_m)]$$

where $\mathbb{T}_m^\circ$ is the subalgebra of $\mathbb{T}_m$ generated by the so called generic trace zero matrices

$$x_i^\circ = x_i - \frac{1}{2}Tr(x_i)$$

For this reason, we call $\mathbb{T}_m^\circ$ the ring of generic trace zero matrices. Procesi has given in [17] an F-vectorspace basis for this ring:

Theorem 3. ([17]):
There is a natural one-to-one correspondence between:
(a): an F-vectorspace basis of $\mathbb{T}_m^\circ$

standard Young tableaux of shape $\sigma = 3^a2^b1^c$, where $(a,b,c) \in \mathbb{N}^3$, and filled with numbers from 1 upto m.

Moreover, the degree of an element corresponding to a certain Young diagram is equal to the number of cells in that diagram. Therefore, the Poincaré series is (in single gradation)

$$\mathcal{P}(\mathbb{T}_m^\circ;t) = \sum_{(a,b,c)} \mathcal{L}_{3^a2^b1^c}.t^{3a+2b+c}$$

where $\mathcal{L}_{3^a2^b1^c}$ is the number of standard Young tableaux of shape $\sigma = 3^a2^b1^c$.
The crucial observation to make at this point is that

$$\sum_{(a,b,c)} \mathcal{L}_{3^a2^b1^c}.t^{3a+2b+c} = \frac{1}{(1-t)^m}\sum_{i=0}^{\infty} \mathcal{L}_{2^i}.t^{2i}$$

This is a sort of Pieri-formula, see e.g. [0, Cor. IV.2.6.]. Now, luckily we know what the power series $\sum \mathcal{L}_{2^i}.t^{2i}$ look like. It is the Poincaré series of the homogeneous coördinate ring of the Grassmann variety $G(2,m)$ of 2-dimensional subspaces of m-dimensional space. We will denote this ring by G_m^2 in the sequel. It can be defined in the following way.
Consider the polynomial ring

$$T = F[Z_{1i}, Z_{2i} : 1 \leq i \leq m]$$

i.e. with as many indeterminates as there are entries in a generic 2 by m matrix which we call Z.
The ring G_m^2 is then the subring generated by the 2 by 2 minors of Z, i.e. by the Plücker coördinates

$$\lambda_{ij} = \det\begin{pmatrix} Z_{1i} Z_{2i} \\ Z_{1j} Z_{2j}\end{pmatrix}$$

Another way of introducing G_m^2 is to note that it is the fixed ring of T under the action of $GL_2(F)$ by left multiplication on the generic 2 by m matrix Z.
Therefore, by the Hochster-Roberts theorem we know that G_m^2 is a Cohen-Macauley domain, i.e. it is a free module of finite rank over a polynomial subring. Further, one can explicitly determine the generators Θ_i of this polynomial subring. To do so, let $\mathcal{H}_m$ be the set of the $\frac{m(m-1)}{2}$ Plücker coordinates λ_{ij} for $1 \leq i < j \leq m$. One can make $\mathcal{H}_m$ into a partially ordered set by defining

$$\lambda_{ij} \leq \lambda_{kl} \text{ iff } i \leq k \text{ and } j \leq l$$

$rk(7)$	:	λ_{45}			
$rk(6)$	:		λ_{35}		
$rk(5)$	:	λ_{34}		λ_{25}	
$rk(4)$	:		λ_{24}		λ_{15}
$rk(3)$	:	λ_{23}		λ_{14}	
$rk(2)$	:		λ_{13}		
$rk(1)$	:	λ_{12}			

For general m, $\mathcal{H}_m$ is a ranked partially ordered set of rank $2m-3$. Combining Th.8.1 and Th.11.1 of [3] one can prove that G^2_m is a free module of finite rank over the polynomial ring

$$\mathcal{F}_m = F[\theta_1, \cdots, \theta_{2m-3}]$$

where

$$\theta_k = \sum_{rk(\lambda_{ij})=k} \lambda_{ij}$$

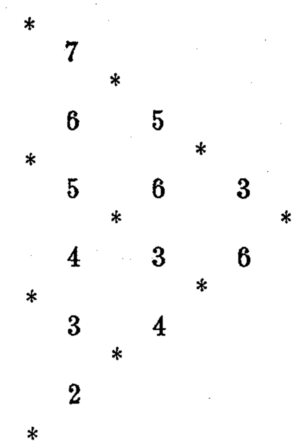
Moreover, one can give an homogeneous basis for G^2_m over $\mathcal{F}_m$. To begin, note that $\mathcal{H}_m$ is graded, i.e. it is finite, bounded (it has a minimal and maximal element) and it is pure (all maximal chains have the same lenght). Further, $\mathcal{H}_m$ is lexicographic shellable, i.e. one can assign to every edge in the Hasse diagram of $\mathcal{H}_m$, $\lambda_{ij} \prec \lambda_{kl}$, a natural number $\mu(\lambda_{ij}, \lambda_{kl})$ such that in every interval $[\lambda_{ij}, \lambda_{kl}]$ of $\mathcal{H}_m$ there is a unique unrefinable chain

$$\lambda_{ij} = \lambda_{i_0j_0} \prec \lambda_{i_1j_1} \prec \cdots \prec \lambda_{i_nj_n} = \lambda_{kl}$$

which is rising, i.e. such that

$$\mu(\lambda_{i_0j_0}, \lambda_{i_1j_1}) \leq \mu(\lambda_{i_1j_1}, \lambda_{i_2j_2}) \leq \cdots \leq \mu(\lambda_{i_{n-1}j_{n-1}}, \lambda_{i_nj_n})$$

One assigns to an edge in the n-th main diagonal corridor the number $2n$ and to an edge in the other n-th diagonal corridor the number $2n+1$. For example, for $m = 5$ we obtain the following picture



There is a one-to-one correspondence between a basis of G^2_m over $\mathcal{F}_m$ and the set of maximal chains in $\mathcal{H}_m$, cfr. e.g. [2] or [6] for more details.

Take a maximal chain

$$\lambda_{12} = \lambda_{i_0j_0} \prec \lambda_{i_1j_1} \prec \cdots \lambda_{i_{2m-4}j_{2m-4}} = \lambda_{m-1\,m}$$

then the element of G^2_m corresponding to it is

$$\prod_{k\in S} \lambda_{i_kj_k}$$

where S is the subset of $\{0, ..., 2m-4\}$ consisting of the indices k such that

$$\mu(\lambda_{i_{k-1}j_{k-1}}, \lambda_{i_kj_k}) > \mu(\lambda_{i_kj_k}, \lambda_{i_{k+1}j_{k+1}})$$

In the case that $m = 5$ we get

chains	*elements*
234567	1
234657	λ_{25}
243567	λ_{14}
243657	$\lambda_{14}.\lambda_{25}$
246357	λ_{15}

Combining all this information we get

Theorem 4. :
The Poincaré series of the homogeneous coordinate ring of the Grassmann variety of 2-planes in m-space is

$$\mathcal{P}(G_m^2;t) = \sum_{i=0}^{\infty} L_{2^i}.t^{2i} = \frac{g_m(t^2)}{(1-t^2)^{2m-3}}$$

where $g_m(t^2)$ is a polynomial in $\mathbb{N}[t]$. The coefficient of t^{2j} in $g_m(t^2)$ is the number of maximal chains in $\mathcal{H}_m$ having precisely j descents.

In our example, we have

$$\mathcal{P}(G_m^2;t) = \frac{1+3t^2+t^4}{(1-t^2)^7}$$

Another immediate consequence of the theory expounded above is the following elegant proof of the functional equation due to C.Procesi.

Since G_m^2 is a Gorenstein,Cohen-Macauley domain , cfr. [16], its Poincaré series satisfies the functional equation

$$\mathcal{P}(G_m^2;\frac{1}{t}) = -t^{\alpha}.\mathcal{P}(G_m^2;t)$$

for some $\alpha \in \mathbb{Z}$ since $Kdim(G_m^2) = 2m-3$ is odd, cfr. [16]. Finally, since

$$\mathcal{P}(\mathbb{T}_{m,2};t) = \frac{1}{(1-t)^{2m}}.\mathcal{P}(G_m^2;t)$$

the functional equation follows.

Further, it is easy to verify that

$$\alpha = 4m - 6 - deg(g_m) = 2m$$

since $deg(g_m)$ is equal to twice the maximal number of descents possible for a chain in $\mathcal{H}_m$. The unique chain having a maximal number of descents can be visualized as

```
0
   0
*     0
   0     *
*     0     *
:  :  :  :  :
*     0     *
   0     *
*     0
   0
0
```


so it has precisely $m-3$ descents. This finishes the proof of the fact that $\mathcal{P}(\mathbb{T}_{m,2};t)$ satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,2};\frac{1}{t}) = -t^{4m}.\mathcal{P}(\mathbb{T}_{m,2};t)$$

The two preceeding proofs of the functional equation, as well as Formanek's proof, are obtained from rather formal operations on the Poincaré series and they do not give us any ringtheoretical reason (or insight) for the existence of the functional equation.

4. Homological Explication.

Let us first recall where functional equations come from in commutative algebra. There, we have the result of Stanley [16]

Theorem 5.:
[16] Let R be a positively graded Cohen-Macauley domain, then the two statements are equivalent:
(a): R is Gorenstein, i.e. $\Omega(R) \simeq R$ or equivalently, R has finite self-injective dimension.
(b): The Poincaré series of R satisfies the functional equation $\mathcal{P}(R;\frac{1}{t}) = (-1)^d.t^{\alpha}.\mathcal{P}(R;t)$ where $d = Kdim(R)$ and $\alpha \in \mathbb{Z}$.

So, let us first consider the question whether $\mathbb{T}_m$ is Cohen-Macauley in some suitable definition.
Even in the commutative case it is fairly exceptional that one proves Cohen-Macauleyness of a ring by describing explicitly a polynomial subring and a free set of generators. Usually, one proves Cohen-Macaulayness of a ring R by choosing a set of generators, i.e. an epimorphism

$$S = F[x_1,\ldots,x_n] \to R$$

and then showing that $pd_s(R) = Kdim(S) - Kdim(R)$. We will follow the same approach for the trace rings $\mathbb{T}_m$, or rather for the ring of generic trace zero matrices $\mathbb{T}_m^{\circ}$. So, we first have to find a positively graded F-algebra having finite global dimension s.t. $\mathbb{T}_m^{\circ}$ can be obtained from it as an epimorphic image.
Consider the polynomial ring

$$S_m = F[a_{ij} : 1 \leq i \leq j \leq m]$$

i.e. the homogeneous coordinate ring of the variety of all symmetric m by m matrices. Over this ring we have a sort of "generic quadratic form":

$$q_m = \sum^{m} a_{ij}\ X_iX_j$$

and we define the generic Clifford algebra Cl_m to be the Clifford algebra associated to this form, which is the quotient of the tensor-algebra of a free S_m-module of rank m by the ideal generated by elements of the form $x \otimes x - q_m(x)$ as x runs through the elements of the free module.
For ringtheorists, the next description of the generic Clifford algebra is probably more convenient: Consider the F-algebra

$$F[a_{ij} : 1 \leq i < j \leq m][a_1][a_2, \sigma_2, \delta_2] \ldots [a_m, \sigma_m, \delta_m]$$

where one definis for each $i < j$ that $\sigma_j(a_i) = -a_i$, $\delta_j(a_i) = 2a_{ij}$ and trivial actions of σ_j and δ_j on the other indeterminates. Defining $a_i^2 = a_{ii}$, it is clear that this algebra is an S_m-algebra and it can be shown that it is precisely Cl_m.
Immediate consequences of this description are e.g. Cl_m is a maximal order, it has finite global dimension equal to $\frac{m(m+1)}{2}$, it is positively graded if we define $deg(a_{ij}) = 2$ and $deg(a_i) = 1$. Moreover, it is easy to describe its Poincaré series

$$P(Cl_m; t) = \frac{1}{(1-t)^m (1-t^2)^{\frac{m(m-1)}{2}}}$$

because, as a graded F-vectorspace, Cl_m is isomorphic to the polynomial ring $F[a_{ij}; a_i]$.
Further, localizing at the multiplicative set of non-zero elements of S_m one gets the Clifford algebra of a regular quadratic form over the field of fractions K_m. Using the classical structure results of Clifford algebras over fields one gets that the p.i.-degree of Cl_m is the 2^α where α is the largest natural number $\leq \frac{m}{2}$. It is then also easy to prove that the center of Cl_m is equal to S_m whenever m is even and equal to the quadratic extension $S_m[\sqrt{\det(a_{ij})}]$ if m is odd. Finally, we mention that it is possible to describe the prime ideal stucture of Cl_m. In particular we find that there is precisely one (graded) prime ideal of Cl_m lying over a graded prime of S_m.

In the generic Clifford algebra we have the commutation relations

$$a_i a_j + a_j a_i = 2a_{ij}$$

Therefore, sending a_i to X_i° and a_{ij} to $\frac{1}{2}Tr(X_i^\circ X_j^\circ)$ we define an epimorphism:

$$\phi_m : Cl_m \to \mathbb{T}_m^\circ$$

Of course; we are ultimately interested in describing a resolution of $\mathbb{T}_m^\circ$ considered as a (say left) module oer the regular algebra Cl_m. To begin, let us see whether one can describe the kernel of ϕ_m. Aboe we have seen that the center of $\mathbb{T}_m^\circ$, $\mathcal{R}_m^\circ$, is the ring of invariants under $SO_3(F)$. Therefore, there is an induced $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{R}_m^\circ$ whose fixed ring is the ring of invariants under the full orthogonal group

coordinate ring of the variety of symmetric m by m matrices of rank ≤ 3, i.e. S_m^4 the quotient of S_m by the ideal generated by the 4 by 4 minors of the generic symmetric matrix $\mathcal{A} = (a_{ij}), I_4$. So, we have the following situation

$$\begin{array}{ccccc} & & Cl_m & \xrightarrow{\phi_m} & \mathbb{T}_m^\circ \\ & & & & \\ & & & & \quad\quad \mathcal{R}_m^\circ \\ & & & & \\ I_4 & \to & S_m & \to & S_m^4 \end{array}$$

The kernel of ϕ_m is clearly a prime ideal which is graded and lies over the graded prime ideal I_4. We know that there is only one such prime, i.e.

$$Ker\phi_m = radCl_M.I_4$$

Let us look at the simplest non-trivial example, i.e. when $m = 4$. Then

$$Ker\phi_4 = Cl_4.S_4(a_1, a_2, a_3, a_4)$$

and one verifies that $S_4(a_1, a_2, a_3, a_4)$ is a normalizing element of Cl_4.
As mentioned at the beginning of this section, we would like to prove that

$$pd_{Cl_m}(\mathbb{T}_m^0) = KdimCl_m - Kdim\mathbb{T}_m^0 = \frac{m(m+1)}{2} - (3m-3) = \frac{(m-2)(m-3)}{2}$$

To this end, it is quite instructive to see how Kutz [11] proves Cohen-Macauleyness of the rings

$$S_m^k = S_m / I_k$$

where I_k is the ideal generated by all k by k minors of $\mathcal{A}$. To any sequence of natural numbers

$$M : 0 = s_0 < s_1 < \ldots < s_k = m$$

he associates the ideal I_M of S_m which is generated by

(1): the 2 by 2 minors of the last s_1 columns of $\mathcal{A}$.
(2): the 3 by 3 minors of the last s_2 columns of $\mathcal{A}$.
⋮
(k-1): the k by k minors of the last s_{k-1} columns of $\mathcal{A}$.
(k): the $k+1$ by $k+1$ minors of the last $s_k = m$ columns of $\mathcal{A}$.

Furthermore, for any $n \in \mathbb{N}$ he defines $I_{M,n}$ to be the ideal of S_m generated by I_M and the last n entries of the first row of $\mathcal{A}$, i.e.

Note that, in particular, we recover the ideals I_k by setting

$$M = 0 < 1 < 2 < \ldots < k-1 < m$$

and $n = 0$. If $n = s_e$ for some ℓ, Kutz proves that $I_{M,n}$ is a prime ideal of S_m and, in the other case, i.e. when $s_\ell < n < s_{\ell+1}$ he shows that

$$I_{M,n} = I_{M',n} \cap I_{M,n'}$$

where $n' = s_{\ell+1}$ and

$$M' = 0 = s_0 < \ldots < s_{\ell-1} < n < s_{\ell+1} < \ldots < s_k = m$$

so both $I_{M',n}$ and $I_{M,n'}$ one prime ideals of S_m whence $I_{M,n}$ is radical. His main result is then

<u>Theorem 5.</u> (Kutz [11])
If $n = s_\ell$ or $n = s_\ell + 1$, then the quotient

$$S_{M,n} = S_m / I_{M,n}$$

is a Cohen-Macauley ring, i.e. $pd_{S_m}(S_{M,n}) = KdimS_m - KdimS_{M,n}$

Now, let us return to the homological study of certain quotients of the generic Clifford algebra Cl_m. Consider a couple (M, n) as before and suppose that $n = s_\ell$ for some ℓ, then $I_{M,n}$ is a graded prime ideal of S_m, so there exists precisely one prime ideal of Cl_m lying over it. We will denote this prime ideal by $J_{M,n}$. In the other case, i.e. when $s_\ell < n < s_{\ell+1}$, then $I_{M,n}$ was the intersection of the graded prime ideals $I_{M',n}$ and $I_{M,n'}$. By definition we set

$$J_{M,n} = J_{M',n} \cap J_{M,n'}$$

where the ideals on the right one determined as above. Using the ideals of the proof of Kutz, as well as the structure theory of H. Bass [1] for Clifford algebras of non-degenerate quadratic modules, more in particular that they are $\mathbb{Z}/2\mathbb{Z}$-graded Azumaya algebras whence there is a natural one-to-one corresponcence between $\mathbb{Z}/2\mathbb{Z}$-graded ideals of the Clifford algebra and ideals of the commutative base ring, one can prove:

<u>Theorem 7.</u> ([14, III.4.3.])
If $n = s_\ell$ or $n = s_\ell + 1$ for some ℓ, then the quotient

$$Cl_{M,n} = Cl_m / J_{M,n}$$

is a Cohen-Macauley quotient. By this we mean that

$$pd(Cl_{M,n}) = KdimCl - KdimCl$$

We note that this result immediately entails that for these couples (M, n), the quotient $Cl_{M,n}$ is a Cohen-Macauley module over S_m, i.e. $Cl_{M,n}$ is a free module of finite rank over a polynomial subring of the center.
To see this, first note since Cl_m is a free S_m-module of finite rank we have

$$pd_{S_m}(Cl_{M,n}) \leq pd_{Cl_m}(Cl_{M,n}) = KdimCl_m - KdimCl_{M,n}$$

Further, $KdimCl_m = KdoiimS_m$ and $KdimCl_{M,n}$ is equal to the dimension of $Cl_{M,n}$ as an S_m-module. By the Auslander-Buchsbaum result this implies that $dephts_{S_m}(Cl_{M,n}) = dim(Cl_{M,n})$ whence $Cl_{M,n}$ is an S_m Cohen-Macauley module. So, in particular we get

$$Ext^i_{S_m}(Cl_{M,n}; S_m) = 0 \quad \text{whenever} \quad i \neq dimS_m - dimCl_{M,n}$$

and because Cl_m is free over S_m this entails

$$Ext^i_{Cl_m}(Cl_{M,n}, Hom_{S_m}(Cl_m, S_m)) = 0$$

for these values of i. Finally, since $Hom_{S_m}(Cl_m, S_m)$ is a twosided divisorial ideal of Cl_m and the normalizing classgroup of Cl_m is trivial we get that $Hom_{S_m}(Cl_m, S_m) \simeq Cl_m$ and therefore

$$Ext^i_{Cl_m}(Cl_{M,n}; Cl_m) = 0 \quad \text{whenever} \quad i \neq KdimCl_m - KdimCl_{M,n}$$

Returning to the study of $\mathbb{T}^\circ_m$ which was the quotient $Cl_{M,n}$ for $M = 0 < 1 < 2 < 3 < m$ and $n = 0$ we get

$$Ext^i_{Cl^i_m}(\mathbb{T}^\circ_m, Cl_m) = 0 \quad \text{for} \quad i \neq \frac{(m-2)(m-3)}{2}$$

So, if we take a resolution of $\mathbb{T}^\circ_m$ as a left Cl_m-module

$$0 \to F_h \to \ldots \to F_1 \to F_0 = Cl_m \to \mathbb{T}^\circ_m \to 0$$

we know that $h = \frac{(m-2)(m-3)}{2}$. Further we may take every F_i to be a graded free Cl_m-module and all morphisms gradation preserving.
So, on the level of Poincaré series we have

$$\mathcal{P}(\mathbb{T}^\circ_m; t) = \sum_{i=0}^{h} (-1)^i \mathcal{P}(F_i; t) \tag{1}$$

and if F_i has a basis of homogeneous elements $f_{i1}, \ldots, F_{i\beta_i}$ with $deg(f_{ij}) = g_{ij}$ we have

$$\mathcal{P}(F_i; t) = (\sum^{\beta_i} t^{g_{ij}}).\mathcal{P}(Cl_m; t) \tag{2}$$

$$P(\Omega^{nc}(\mathbb{T}^\circ_m);t) = \sum_{i=0}^{h} (-1)^{h-i}.P(F^*_{h-i};t) \qquad (3)$$

and

$$P(F^*_i;t) = (\sum_{j=0}^{\beta_i} t^{-g_{ij}}).P(Cl_m;t)$$

which follows immediately from the rational expression of $P(Cl_m;t)$ given before, we find that

$$\begin{aligned} &P(\Omega^{nc}(\mathbb{T}^\circ_m);\frac{1}{t}) \\ &= P(Cl_m;\frac{1}{t}).(\sum_{i=0}^{h}(-1)^{h-i}.\sum_{j=1}^{\beta_i} t^{g_{ij}}) \\ &= (-1)^{\frac{m(m+1)}{2}}.t^{m^2}.P(Cl_m;t).(\sum_{i=0}^{h}(-1)^{h-i}.\sum_{j=1}^{\beta_i} t^{g_{ij}}) \\ &= (-1)^{\frac{m(m+1)}{2}-h}.t^{m^2}.P(\mathbb{T}^\circ_m;t) \end{aligned}$$

and since $h = \frac{(m-2)(m-3)}{2}$ this gives us

$$P(\Omega^{nc}(\mathbb{T}^\circ_m);\frac{1}{t}) = (-1)^{3m-3}.t^{m^2}.P(\mathbb{T}^\circ_m;t)$$

Now, look at $\Omega^{nc}(\mathbb{T}^\circ_m) = Ext^h_{Cl_m}(\mathbb{T}^\circ_m;Cl_m)$. If $\mathbb{T}^\circ_m$ is free of finite rank over the polynomial subring $\mathcal{R}$ of $\mathcal{R}^\circ_m$ we know that

$$\Omega^{nc}(\mathbb{T}^\circ_m) \simeq Hom_R(\mathbb{T}^\circ_m;R)$$

as $\mathbb{T}^\circ_m$-modules. Therefore, $\Omega^{nc}(\mathbb{T}^\circ_m)$ is a graded twosided divisorial ideal of the maximal order $\mathbb{T}^\circ_m$. Since $\mathbb{T}^\circ_m$ is a reflexive Azymaya algebra (i.e. all localizations at height one prime ideals of the center are Azumaya and the ring is a reflexive module) over its center $\mathcal{R}^\circ_m$ which is a unique factorization domain (follows from $\mathcal{R}_m = \mathcal{R}^\circ_m[Tr(X_1),\ldots,Tr(X_m)]$ being the fixed ring under a single reductive group) we find that

$$\Omega^{nc}(\mathbb{T}^\circ_m) \simeq \mathbb{T}^\circ_m$$

as graded $\mathbb{T}^\circ_m$-modules, so

$$P(\Omega^{nc}(\mathbb{T}^\circ_m);t^\epsilon) = t^\beta.P.(\mathbb{T}^\circ_m;t^\epsilon)$$

for some $\beta \in \mathbb{Z}$ and $\epsilon = \pm 1$. Finally, using that $\mathbb{T}_m = \mathbb{T}^\circ_m[Tr(X_1),\ldots,Tr(X_m)]$ we get

Theorem 8. (Functional equation)
The Poincaré series of $\mathbb{T}_m$ satisfies the functional equation

$$P(\mathbb{T}_m;\frac{1}{t}) = -t^\alpha.P(\mathbb{T}_m;t)$$

for some $\alpha \in \mathbb{Z}$.

5. Odds and Ends.

In this final section we would like to mention some of the remaining questions on trace rings of generic matrices. First, consider 2 by 2 matrices. In the third section we have scetched Procesi's proof of the functional equation, linking the study of $\mathbb{T}_{m,2}$ to the rather extensive theory of Grassmannians.

Question 1.
Is there a ringtheoretical connection between $\mathbb{T}_{m,2}$ and $G(2,m)$?

Looking at their Poincaré series it is possible that $\mathbb{T}_{m,2}$ is an iterated Öre-extension of $G(2,m)$. But the author is rather sceptical about this possibility. We have shown that $\mathbb{T}^\circ_m$ is a Cohen-Macauley quotient of Cl_m. Ultimately, one would like to solve:

Question 2.
Give an explicit resolution of $\mathbb{T}^\circ_m$ as a (left) module over Cl_m.

In commutative algebra it is usually very difficult to find resolutions for Cohen-Macauley rings. However, in the case of the rings S^k_m, i.e. symmetric m by m matrices of rank smaller than k, Jozeficek, Pragacz and Weymann [9] succeeded in giving such a resolution. In a subsequent paper we hope to extend their method in order to solve question 2.
Along the same lines it would be very intersting to know the solution to

Question 3.
Give an explicit description of $\mathbb{T}_{m,2}$, i.e. find the generators of the polynomial subring R of $\mathcal{R}_{m,2}$ and a free bais of $\mathbb{T}_{m,2}$ over R.

For $m = 4$, this problem can be solved quite easily. The general case seems to be quite hard, however.

Of course, a solution to question 1 would immediately yield solutions to the other questions.

Now, let us turn attention to the case of n by n matrices where $n > 2$. As mentioned in the introduction, E. Formanek proved the functional equation for $\mathbb{T}_{m,n}$ where $m \geq n^2$. This result may be viewed as an indication for a positive solution to

Question 4.
Is the trace ring of m generic n by n matrices always a Cohen-Macauley module over its center?

For, one can prove

Proposition 9.
If the trace ring $\mathbb{T}_{m,n}$ is a Cohen-Macauley module over its center, then the Poincaré series satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,n}; \frac{1}{t}) = (-1)^d.t^\alpha.\mathcal{P}(\mathbb{T}_{m,n}; t)$$

where $d = Kdim(\mathbb{T}_{m,n}) = (m-1)n^2 + 1$ and $\alpha \in \mathbb{Z}$.

Proof
If $\mathbb{T}_{m,n}$ is a Cohen-Macauley module, one can use a result of Stanley's [16] to get:

$$\mathcal{P}(\Omega(\mathbb{T}_{m,n}); \frac{1}{t}) = (-1)^d.\mathcal{P}(\mathbb{T}_{m,n}; t)$$

where $\Omega(\mathbb{T}_{m,n})$ is the canonical module of the $\mathcal{R}_{m,n}$-module $\mathbb{T}_{m,n}$. Considered as a $\mathbb{T}_{m,n}$-module, one can show that it is a twosided divisorial ideal of the maximal order $\mathbb{T}_{m,n}$. Since $\mathbb{T}_{m,n}$ is a reflexive Azumaya algebra over its center $\mathcal{R}_{m,n}$ which is a unique factorization domain, cfr. e.g. [12], we have

$$\Omega(\mathbb{T}_{m,n}) \simeq \mathbb{T}_{m,n}$$

one this isomorphism is clearly graded, therefore

$$\mathcal{P}(\Omega(\mathbb{T}_{m,n}); t^\epsilon) = t^\alpha \mathcal{P}(\mathbb{T}_{m,n}; t^\epsilon)$$

for $\epsilon = \pm 1$, finishing the proof.

Because $\mathbb{T}_{m,n}$ is the fixed F-algebra under action of the reductive linear algebraic group $GL_n(F)$ on the F-algebra of finite global dimension

$$M_n(F[x_{ij}(\ell) : 1 \leq i, j \leq n, 1 \leq \ell \leq m])$$

question 4 could be answered affirmatively, provided, one has a noncommutative version of the famous Hochster-Roberts theorem. At present, it is not clear how one might prove such a result since the main tool of the commutative proof (of Hochster-Roberts [7] or Kempf [10]), i.e. reduction to finite characteristic and investigation of the Frobenius morphism on local cohomology, clearly does not generalize (directly) to the noncommutative setting.
As was suggested to me by J.L. Colliot-Thélène, it might be possible to generalize Boutot's proof. Apparantly, this proof exists only in preprint and I was not yet able to obtain a copy. But, from Hochster's talk in [8], it seems to boil down to the following. Look at the resolution of singularities

$$g : X \rightarrow Y = Spec(\mathcal{R}_{m,n})$$

then a special case of Boutot's result implies that g is a rational resolution, i.e.

(1): $\Theta_y \rightarrow g_*\Theta_X$ is an isomorphism
(2): $R^i g_* \Theta_X$ is zero for all $i > 0$
(3): $R^i g_* \Omega_X^x$ is zero for all $i > 0$

where (3) is always satisfied by the Grauert-Riemenschneider generalization of the Kodeira vanishing-result. Probably, one can show Cohen-Macauleyness of $\mathbb{T}_{m,n}$ provided one has an answer to the rather vague.

Question 5.
What happens to $\mathbb{T}_{m,n}$ under resolution of the central singularities? More precisely, what is the structure of $g^*.\Theta_{\mathbb{T}_{m,n}}$ where $\Theta_{\mathbb{T}_{m,n}}$ is the structure sheaf of the $\mathcal{R}_{m,n}$-module $\mathbb{T}_{m,n}$ over $y = Spec(\mathcal{R}_{m,n})$.

If we denote $Z = g^*\Theta_{\mathbb{T}_{m,n}}$, one can prove Cohen-Macauleyness provided one can show

(1): $R^i g_* Z = 0$ for all $i > 0$
(2): $R^i g_*(Z^y \otimes \Omega_X^X) = 0$ for all $i > 0$

I hope to come back to these questions in a future publication.

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An Explicit Description of $\mathbb{T}_{3,2}$.

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Let F be a field of characteristic different from 2. Consider the polynomial ring

$$P_{m,n} = F[X_{ij}(l) : 1 \leq i,j \leq n, i \leq l \leq m]$$

and the so called generic n by n matrices :

$$X_l = (X_{ij}(l))_{ij} \in M_n(P_{m,n})$$

The ring of m generic n by n matrices, $G_{m,n}$, is the F-subalgebra of $M_n(P_{m,n})$ generated by $\{X_1, \ldots, X_m\}$. The trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$ is the F-subalgebra of $M_n(P_{m,n})$ generated by $G_{m,n}$ and the elements $Tr(Y)$ where $Y \in G_{m,n}$.

Herstein [1] and Formanek, Halpin and Lie [2] have given an explicit description of the trace ring of 2 generic 2 by 2-matrices. It turned out that $R_{2,2}$, the center of $\mathbb{T}_{2,2}$, is the polynomial ring

$$R_{2,2} = F[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1X_2)]$$

and $\mathbb{T}_{2,2}$ is the free $R_{2,2}$-module of rank four with generators $\{1, X_1, X_2, X_1X_2\}$. In [5] Small and Stafford proved that $\mathbb{T}_{2,2}$ has finite global dimension. We give here a shorter proof of this result :

PROPOSITION : $\text{gldim}(\mathbb{T}_{2,2}) = 5$

PROOF : It is sufficient to prove that for any maximal ideal m in $R_{2,2}$, $\text{gldim}((\mathbb{T}_{2,2})_m) \leq 5$. . Consider first the case that m contains $(X_1X_2 - X_2X_1)^2$. It is easy to verify that $X_1X_2 - X_2X_1$ is a normalizing element of $\mathbb{T}_{2,2}$ and the quotient

$$\mathbb{T}_{2,2}/\mathbb{T}_{2,2}(X_1X_2 - X_2X_1) = F[\overline{X}_1, \overline{X}_2, \overline{Tr(X_1)}, \overline{Tr(X_2)}]$$