

**Trace Rings of Generic 2 by 2 Matrices I :  
Invariant Theory**

by

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## I.1 : Introduction.

All results on the trace ring of  $m$  generic 2 by 2 matrices  $\mathbb{T}_{m,2}$ , which we will prove in the next four parts of these notes are based on the fact that  $\mathbb{T}_{m,2}$  is the ring of matrix concomitants. In this chapter we will prove this result and obtain a fairly precise description of  $\mathbb{T}_{m,2}$  from it.

In section two we recall some basic concepts from invariant theory. For more details, the reader is referred to the monograph of J. Dieudonné and J. Carrell [D-C]. Also, we define the ring of matrix invariants  $R_m$  and the ring of matrix-concomitants  $T_m$ . Both rings appear naturally in the study of finite dimensional representations of the free algebra  $\mathbb{F}_m = F \langle x_1, \dots, x_m \rangle$ , due to M. Artin [A] and C. Procesi [P1]. Both  $R_m$  and  $T_m$  can be viewed as fixed rings.

In section three we recall some results from the invariant theory of orthogonal and special orthogonal groups. These theorems were known from classical invariant theory, cfr. the book of H. Weyl [W]. Here, we follow the elegant, modern treatment due to C. De Conici and C. Procesi [D-P].

In section four, we prove that the ring of matrix invariants  $R_m$  is a polynomial ring over the ring  $R_m^0$  of invariants of  $SO_3(F)$ . Since we will frequently use this result in the sequel we gave here an explicit proof of this observation made by C. Procesi [P3]. Using this fact it is easy to prove that the ring of matrix invariants is equal to the center  $R_{m,2}$  of the trace ring of  $m$  generic 2 by 2 matrices. This fact was in another terminology of course, already known to J. Grace and A. Young [G-Y] as far back as 1903. For  $n$  by  $n$  matrices a similar result was proved by C. Procesi [P2].

In the last section we show that the trace ring  $\mathbb{T}_{m,2}$  is the ring of matrix concomitants  $T_m$ . Further, we give an explicit proof of the description of this ring due to C. Procesi [P2].

## I.2. : Invariant Theory.

Let  $E$  be a finite dimensional vector space over  $F$ . Let  $\Gamma$  be a group acting on  $E$ , that is, suppose there exists a map

$$\phi : \Gamma \times E \rightarrow E; (\sigma, x) \mapsto \sigma.x$$

with the properties :

- (1) :  $(\sigma.\tau).x = \sigma.(\tau.x)$  for all  $x \in E$ , all  $\sigma, \tau \in \Gamma$
- (2) :  $\epsilon.x = x$  for all  $x \in E$  where  $\epsilon$  is the identity of  $\Gamma$

An element  $x \in E$  is  $\Gamma$ -invariant if  $\sigma.x = x$  for all  $\sigma \in \Gamma$ . Suppose, that  $\Gamma$  acts on two finite dimensional vector spaces  $E$  and  $E'$ , then  $\Gamma$  acts on the set  $\mathcal{F}(E, E')$  of all mappings from  $E$  to  $E'$  in a natural way, i.e. if  $u \in \mathcal{F}(E, E')$  and  $\sigma \in \Gamma$ , then  $\sigma.u \in \mathcal{F}(E, E')$  is defined by  $(\sigma.u)(x) = \sigma(u(\sigma^{-1}, x))$ . Usually, one assumes that  $\Gamma$  acts linearly on the vector spaces, that is, for each  $\sigma \in \Gamma$  the map  $\mu_\sigma : E \rightarrow E$  sending  $x$  to  $\sigma.x$  is linear. Thus,  $\mu_\sigma$  is an element of  $GL(E)$  and the mapping  $\sigma \mapsto \mu_\sigma : \Gamma \rightarrow GL(E)$  is a linear representation of  $\Gamma$ .

Let  $\{e_1, \dots, e_m\}$  be a basis for  $E$  and  $\{f_1, \dots, f_n\}$  a basis for  $E'$ . A mapping  $u : E \rightarrow E'$  is said to be a polynomial concomitant if

$$u\left(\sum_j \alpha_j e_j\right) = \sum_k u_k(\alpha_1, \dots, \alpha_m) \cdot f_k$$

where each  $u_k$  is a polynomial function of  $\alpha_1, \dots, \alpha_m$ .

An invariant polynomial map  $u : E \rightarrow F$  is a polynomial concomitant for trivial  $\Gamma$  action on  $F$ .

If  $\Gamma$  acts linearly on an  $m$ -dimensional vectorspace  $E$  with basis  $\{e_1, \dots, e_m\}$  then this action extends to an action on the polynomial ring  $F[x_1, \dots, x_m]$  in the following way. Let  $\sigma \in \Gamma$  and  $\mu_\sigma(e_i) = \sum \alpha_{ij} e_j$ , then we associate to  $\sigma$  an automorphism  $\theta_\sigma$  of  $F[x_1, \dots, x_m]$  defined by

$$\theta_\sigma(x_i) = \sum_j \alpha_{ij} x_j$$

The determination of all invariant polynomial maps from  $E$  to  $F$  then clearly amounts to the study of the fixed ring

$$F[x_1, \dots, x_m]^\Gamma$$

In the next section we will review some results on the invariant theory of the (special) orthogonal groups, but in the rest of these notes we are primarily interested in the following situation :

$\Gamma$  will be the full linear group  $GL_2(F)$  of invertible 2 by 2 matrices over  $F$ . We make  $M_2(F)$  into a  $GL_2(F)$ -module, or equivalently, we define a linear action of  $GL_2(F)$  on  $M_2(F)$  in the following way :

$$\begin{aligned} \phi : GL_2(F) \times M_2(F) &\rightarrow M_2(F) \\ (\alpha, m) &\mapsto \alpha.m\alpha^{-1} \end{aligned}$$

The action of  $GL_2(F)$  on  $m$  copies of  $M_2(F)$  is, of course, given by defining it on every component.

The ring of matrix invariants  $R_m$  is the ring of polynomial maps

$$f : M_2(F) \oplus \dots \oplus M_2(F) \rightarrow F$$

from  $m$  copies of  $M_2(F)$  to  $F$  which are invariant under the action of  $GL_2(F)$ . In terms of fixed rings one can define  $R_m$  as follows : consider the polynomial ring

$$P_{m,2} = F[x_{11}(i), x_{12}(i), x_{21}(i), x_{22}(i) : 1 \leq i \leq m]$$

and an action  $\Phi$  of  $GL_2(F)$  on  $P_{m,2}$  which associates to any  $\alpha \in GL_2(F)$  the automorphism  $\Phi_\alpha$  sending the indeterminate  $x_{ij}(k)$  to the entry  $(i, j)$  of the matrix

$$\alpha.x_k.\alpha^{-1}$$

where  $x_k$  is the so-called generic matrix  $\begin{pmatrix} x_{11}(k) & x_{12}(k) \\ x_{21}(k) & x_{22}(k) \end{pmatrix}$ . Clearly,  $R_m$  is the fixed ring of  $P_{m,2}$  under this action.

We will briefly sketch the importance of the ring of matrix invariants to the study of 2-dimensional representations of the free algebra

$$\mathbb{F}_m = F \langle x_1, \dots, x_m \rangle$$



Note that a 2-dimensional representation of  $\mathbb{F}_m$ , i.e. an algebra morphism  $\phi : \mathbb{F}_m \rightarrow M_2(F)$ , is equivalent to giving  $m$  2 by 2 matrices  $\phi(x_1), \dots, \phi(x_m)$ . Therefore  $\text{REP}_2(\mathbb{F}_m)$ , the set of all two dimensional representations can be identified to the affine space  $A^{4m}$ , i.e.  $\text{Spec}(P_{m,2})$ . Two representations  $\phi_1$  and  $\phi_2$  are said to be equivalent if they differ up to an  $F$ -automorphism of  $M_2(F)$ ,  $\alpha$

$$\begin{array}{ccc} \mathbb{F}_m & \xrightarrow{\phi_1} & M_2(F) \\ & & \downarrow \alpha \\ \mathbb{F}_m & \xrightarrow{\phi_2} & M_2(F) \end{array}$$

So,  $\text{Aut}_F(M_2(F)) = PGL_2(F) = GL_2(F)/F^*$  acts on  $\text{REP}_2(\mathbb{F}_m) = A^{4m}$  and the orbits under this action are the equivalence classes of representations. To classify representations up to equivalence is therefore essentially the study of the orbit space of  $A^{4m}$  under the action of  $PGL_2(F)$ .

If  $\phi : \mathbb{F}_m \rightarrow M_2(F)$  is a representation,  $V = F \oplus F$  becomes an  $\mathbb{F}_m$ -module via  $\phi$ . If  $V$  is an irreducible  $\mathbb{F}_m$ -module,  $\phi$  is also called irreducible. If  $\phi$  is reducible, one can alter the basis in  $V$  in such a way that  $\phi$  has a matrix representation

$$\phi = \begin{pmatrix} \phi_1 & N \\ 0 & \phi_2 \end{pmatrix}$$

where the  $\phi_i : \mathbb{F}_m \rightarrow F$  are one-dimensional representations and  $N$  is the nilradical of  $\phi$ . When  $V$  is completely reducible, we call  $\phi$  semi-simple. To any reducible  $\phi$  one can associate a semi-simple representation  $\phi^{ss}$ .

$$\phi^{ss} = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}$$

Using the relation

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 & N \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \phi_1 & tN \\ 0 & \phi_2 \end{pmatrix}$$

proves that  $\phi^{ss}$  lies in the closure of the orbit of  $\phi$  under  $PGL_2(F)$ , i.e. we will not be able, in any type of quotient variety of  $A^{4m}$  by  $PGL_2(F)$ , to distinguish between  $\phi$  and  $\phi^{ss}$ . This motivates us to construct an affine variety whose  $F$ -points correspond to the equivalence classes of semi-simple representations of  $\mathbb{F}_m$

in  $M_2(F)$ . It is clear from the definition that the affine variety associated to the ring of matrix invariants is the required variety.

With  $T_m$  we will denote the set of polynomial concomitants

$$f : M_2(F) \oplus \dots \oplus M_2(F) \rightarrow M_2(F)$$

with  $GL_2(F)$ -module structure on each of the  $M_2(F)$  as defined above. Because  $M_2(F)$  is a ring, so is  $T_m$ . We can also view the ring of matrix concomitants,  $T_m$ , as a fixed ring. For, let  $\Phi'$  be the extension of the  $GL_2(F)$ -action  $\Phi$  on  $P_{m,2}$  to  $N_2(P_{m,2})$  by entrywise action. Now, define an action

$$\theta : GL_2(F) \times M_2(P_{m,2}) \rightarrow M_2(P_{m,2})$$

$$(\alpha, Y) \quad \mapsto \alpha^{-1} \cdot \Phi'_\alpha(Y) \cdot \alpha$$

Then, the ring of matrix concomitants is the fixed ring of  $M_2(P_{m,2})$  under this action.

### I.3. : The Orthogonal Groups.

In this section we aim to recall some classical results on the invariant theory of the (special) orthogonal groups. For proofs and further reference the reader is referred to [Weyl] and [De Conici, Procesi] for a modern treatment.

An  $n$  by  $n$  matrix  $A$  with entries in a field  $F$  is called an orthogonal matrix whenever

$$A.A^T = I_n$$

where  $I_n$  is the unit matrix and  $(-)^T$  denotes the transposed. The orthogonal group  $O_n(F)$  is the subgroup of  $GL_n(F)$  consisting of all  $n$  by  $n$  orthogonal matrices. A special orthogonal matrix  $A$  is an element of  $O_n(F)$  with  $\det(A) = 1$ . The group of all special orthogonal matrices is denoted by  $SO_n(F)$ , i.e. we have the exact sequence

$$1 \rightarrow SO_n(F) \rightarrow O_n(F) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

There is a canonical action of  $O_n(F)$  on the  $n$ -dimensional vectorspace  $V$  by left multiplication. The main target in the invariant theory of  $O_n(F)$  is to describe the ring of polynomial mappings

$$f : V \times \dots \times V \rightarrow F$$

from  $m$  copies from  $V$  to  $F$  which are invariant under the componentwise action on  $O_n(F)$ . The ring of polynomial maps is

$$S = F[u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$$

and the action of  $A \in O_n(F)$  on  $S$  is given by sending the variable  $u_{ij}$  to the  $j$ -th entry of the column vector

$$A.(u_{i1}, \dots, u_{in})^T$$

and one wants to describe the fixed rings  $S^{O_n(F)}$  and  $S^{SO_n(F)}$ . Let us denote with  $u_i$  the row vector  $(u_{i1}, \dots, u_{in})$  and define

$$(u_i, u_j) = \sum_{k=1}^n u_{ik} u_{jk}$$

Then the first fundamental theorem in the invariant theory of  $O_n(F)$  and  $SO_n(F)$  is, cfr. [Weyl] or [De Conici, Procesi, Th.5.6.]:

**THEOREM 3.1.** : With notations as above :

- (1) : The ring of invariants  $S^{O_n(F)}$  is the sub  $F$ -algebra of  $S$  generated by the elements  $(u_i, u_j) : 1 \leq i \leq j \leq m$ .
- (2) : The ring of invariants  $S^{SO_n(F)}$  is the sub  $F$ -algebra of  $S$  generated by the elements  $(u_i, u_j) : 1 \leq i \leq j \leq m$  and the elements  $[u_{i_1}, u_{i_2}, \dots, u_{i_n}]$  for  $1 \leq i_1 < i_2 < \dots < i_n \leq m$ .

Here,  $[u_{i_1}, \dots, u_{i_n}]$  stands for the determinant of the  $n$  by  $n$  matrix :

$$\begin{pmatrix} u_{i_1,1} & \dots & u_{i_n,1} \\ \vdots & & \vdots \\ u_{i_1,n} & \dots & u_{i_n,n} \end{pmatrix}$$

Further, one can verify that the product of such two elements

$$[u_{i_1}, \dots, u_{i_n}] \cdot [u_{j_1}, \dots, u_{j_n}] = \det(((u_{i_k}, u_{j_l}))_{k,l})$$

Next, one would like to describe explicitly an  $F$ -vector space basis for these rings of invariants. Let us first recall some definitions.

A double Young tableau is an array

$$T = \begin{pmatrix} a_{11} & \dots & a_{1m_1} & b_{11} & \dots & b_{1m_1} \\ a_{21} & \dots & a_{2m_2} & b_{21} & \dots & b_{2m_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{s1} & \dots & a_{sm_s} & b_{s1} & \dots & b_{sm_s} \end{pmatrix} = (A | B)$$

where the  $a_{ij}$ 's and  $b_{ij}$ 's are indices out of  $\{1, 2, \dots, m\}$  and one assumes that :

$$m_1 \geq m_2 \geq \dots \geq m_s$$

To any double Young tableau  $T$  as above one can associate the element in  $S^{O_n(F)}$  which is the product of the  $s$  determinants :

$$(a_{i1}, \dots, a_{im_i} | b_{i1}, \dots, b_{im_i}) = \det((u_{a_{ik}}, u_{b_{il}})_{k,l}) \quad (*)$$

Let us form from  $T$  the single Young tableau :

$$T' = \begin{array}{|c|} \hline a_{11} \quad \dots \quad a_{1m_1} \\ \hline b_{11} \quad \dots \quad b_{1m_1} \\ \hline a_{21} \quad \dots \quad a_{2m_2} \\ \hline b_{21} \quad \dots \quad b_{2m_2} \\ \hline \vdots \quad \quad \quad \vdots \\ \hline a_{s1} \quad \dots \quad a_{sm_s} \\ \hline b_{s1} \quad \dots \quad b_{sm_s} \\ \hline \end{array}$$

We will say that  $T$  is a double standard Young tableau, if the single Young tableau  $T'$  is standard.

Recall that a Young tableau

$$\begin{array}{|c|} \hline \alpha_{11} \quad \dots \quad \alpha_{1m_1} \\ \hline \alpha_{21} \quad \dots \quad \alpha_{2m_2} \\ \hline \vdots \quad \quad \quad \vdots \\ \hline \alpha_{k1} \quad \dots \quad \alpha_{km_k} \\ \hline \end{array}$$

is said to be standard whenever :

$$\begin{aligned} \alpha_{ij} &< \alpha_{ik} && \text{for all } k > j \\ \alpha_{ij} &\leq \alpha_{kj} && \text{for all } k \geq i \end{aligned}$$

**THEOREM 3.2.** (De Conici - Procesi. Th. 5.1.)

Using the interpretation rule (\*), there is a one-to-one correspondence between an  $F$ -vectorspace basis of  $S^{O_n(F)}$  and double standard Young tableau of length  $\leq n$  (i.e.  $m_1 \leq n$ ). Similarly, there is a one-to-one correspondence between an  $F$ -vectorspace basis of  $S^{SO_n(F)}$  and

- (1) : double standard Young tableau of length  $\leq n$
- (2) : products  $[u_{i_1}, \dots, u_{i_n}] \cdot T$  where  $T$  is a double tableau of length  $\leq n$  such that the single Young tableau

$$\begin{pmatrix} u_{i_1} & \dots & u_{i_n} \\ & T' & \end{pmatrix}$$

is standard.

If we give every  $u_{ij}$  degree one, then both  $S, S^{O_n(F)}$  and  $S^{SO_n(F)}$  are positively graded  $F$ -algebras. Remark that the degree of an element corresponding to a Young tableau as above is equal to the number of indices in this Young tableau.

Finally, let us recall the second fundamental theorem in the invariant theory of the orthogonal groups  $O_n(F)$ , cfr. [Weyl] or [De Conici, Procesi, Th. 5.7.].

**THEOREM 3.3.**

The ideal of relations among the generators  $(u_i, u_j)$  of  $S^{O_n(F)}$  is generated by the  $n + 1$  by  $n + 1$  minors of the symmetric  $m$  by  $m$  matrix  $((u_i, u_j))_{ij}$ .

#### I.4. : Description of the center.

In section two we defined the ring of matrix invariants  $R_m$  and indicated its importance in the study of equivalence classes of semi-simple 2-dimensional representations of the free  $F$ -algebra  $F \langle x_1, \dots, x_m \rangle$ . In this section we aim to show that  $R_m$  is actually the center  $R_{m,2}$  of the trace ring of  $m$  generic 2 by 2 matrices. A similar result holds for arbitrary  $n$  by  $n$  matrices [Procesi], but we give here a proof using only the material expounded above. Moreover we will frequently need the intermediate result, which provides a link between  $R_m$  (or  $R_{m,2}$ ) and the invariant theory of  $SO_3(F)$ , in the sequel.

As in the foregoing section,  $SO_3(F)$  acts on the polynomial ring  $S = F[u_{i1}, u_{i2}, u_{i3} : 1 \leq i \leq m]$  and we will denote the fixed ring by  $R_m^0$ .

#### THEOREM 4.1.:

The ring of matrix invariants  $R_m$  is isomorphic to the polynomial ring  $R_m^0[Tr(X_1), \dots, Tr(X_m)]$ .

#### PROOF.

$R_m$  is by definition the ring of polynomial maps

$$f : M_2(F) \oplus \dots \oplus M_2(F) \rightarrow F$$

from  $m$  copies of  $M_2(F)$  to  $F$  which are invariant under componentwise action by conjugation of  $GL_2(F)$ . As a  $GL_2(F)$ -module, each of the components  $M_2(F)$  decomposes into :

$$M_2(F) = F \oplus M^0$$

where  $M^0$  is the three dimensional vector space of trace zero matrices. Thus we can think that every matrix variables decomposes

$$X_i = \frac{1}{2}Tr(X_i) + X_i^0$$

where  $X_i^0$  is the 2 by 2 matrix

$$X_i^0 = \begin{pmatrix} \frac{1}{2}X_{11} - \frac{1}{2}X_{22} & X_{12} \\ X_{21} & \frac{1}{2}X_{22} - \frac{1}{2}X_{11} \end{pmatrix}$$

This observation entails that  $R_m$  is the free commutative polynomial ring in the variables  $Tr(X_1), \dots, Tr(X_m)$  over the ring of polynomial maps

$$f^0 : M^0 \oplus \dots \oplus M^0 \rightarrow F$$

from  $m$  copies of  $M^0$  to  $F$  which are invariant under componentwise conjugation by  $GL_2(F)$ . In order to prove that this ring is  $R_m^0$ , we will investigate the action of  $GL_2(F)$  on  $M^0$  in some detail. Provided that  $i = \sqrt{-1} \in F$ , a general element of  $M^0$  can be represented as a spinor matrix

$$m = \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix}$$

Now, let  $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a general element of  $GL_2(F)$ , then the image of  $m$  under the action of  $\alpha$  is :

$$\alpha.m.\alpha^{-1} = \frac{1}{\det(\alpha)} \begin{pmatrix} (ad + bc)x + (bd - ac)y + i(ac + bd)z & -2abx + (a^2 - b^2)y - i(a^2 + b^2)z \\ 2cdx + (d^2 - c^2)y + i(c^2 + d^2)z & -(ad + bc)x - (bd - ac)y - i(ac + bd)z \end{pmatrix}$$

This matrix can be brought into the spinor matrix form

$$\begin{pmatrix} x' & y' - iz' \\ y' + iz' & -x' \end{pmatrix}$$

where

$$\begin{aligned} x' &= (ad + bc)x + (bd - ac)y + i(bd + ac)z \\ y' &= (cd - ab)x + \frac{1}{2}(a^2 + d^2 - b^2 - c^2)y + \frac{1}{2}(c^2 + d^2 - a^2 - b^2)z \\ z' &= -i(ab + cd)x - \frac{i}{2}(b^2 + d^2 - a^2 - c^2)y + \frac{1}{2}(a^2 + b^2 + c^2 + d^2)z \end{aligned}$$

Therefore, if we represent the trace zero matrix  $m$  by the 3-dimensional column vector  $(x, y, z)^T$  then the action of  $\alpha$  on this vector can be represented by the 3 by 3 matrix :

$$S_\alpha = \frac{1}{\det(\alpha)} \begin{pmatrix} ad + bc & bd - ac & i(ac + bd) \\ cd - ab & \frac{1}{2}(a^2 - b^2 - c^2 + d^2) & -\frac{i}{2}(a^2 + b^2 - c^2 - d^2) \\ -i(ab + cd) & \frac{i}{2}(a^2 - b^2 + c^2 - d^2) & \frac{1}{2}(a^2 + b^2 + c^2 + d^2) \end{pmatrix}$$



An easy computation shows that  $S_\alpha$  is an element of  $SO_3(F)$ . This shows that the ring of invariant polynomial maps  $f^0$  is the ring of invariant polynomial maps

$$\tilde{f}^0 : F^{(3)} \oplus \dots \oplus F^{(3)} \rightarrow F$$

from  $m$  copies of standard 3-dimensional vectorspace  $F^{(3)}$  to  $F$ , which are invariant under componentwise action of  $SO_3(F)$ , i.e.  $\mathcal{R}_m^0$ .

**THEOREM 4.2.** : The ring of matrixinvariants is equal to the center of the trace ring of  $m$  generic 2 by 2 matrices,  $\mathcal{R}_{m,2}$ .

**PROOF :**

It is trivial to check that  $\mathcal{R}_{m,2} \subset \mathcal{R}_m$ . Using the results we have to check that every generator of  $\mathcal{R}_m^0$  can be written as a trace of a product of  $X_i$  (or  $X_i^0$ ). From section three we recall that  $\mathcal{R}_m^0$  is generated by the elements

$$(u_i, u_j) = u_{i1}u_{j1} + u_{i2}u_{j2} + u_{i3}u_{j3}$$

$$[u_i, u_j, u_k] = \det \begin{pmatrix} u_{i1} & u_{j1} & u_{k1} \\ u_{i2} & u_{j2} & u_{k2} \\ u_{i3} & u_{j3} & u_{k3} \end{pmatrix}$$

We will give an interpretation of these symbols in the  $X_i^0$ . Note that

$$X_i^0 = \begin{pmatrix} u_{i1} & u_{i2} - iu_{i3} \\ u_{i2} + iu_{i3} & -u_{i1} \end{pmatrix}$$

whence

$$u_{i1} = \frac{1}{2}x_{11}(i) - \frac{1}{2}x_{22}(i)$$

$$u_{i2} = \frac{1}{2}x_{12}(i) + \frac{1}{2}x_{21}(i)$$

$$u_{i3} = \frac{1}{2}x_{12}(i) - \frac{1}{2}x_{21}(i)$$

Using this, it is now fairly easy to compute that

$$(u_i, u_j) = \frac{1}{2}Tr(X_i^0 X_j^0)$$

$$[u_i, u_j, u_k] = Tr(X_i^0 X_j^0 X_k^0)$$

Finishing the proof.

We say that a Young tableau is of shape  $\sigma = 3^a 2^b 1^c$  if the array consists of  $a$  rows of length 3,  $b$  rows of length 2 and  $c$  rows of length one. An interpretation of the results of the foregoing section yields.

**THEOREM 4.3.** : There is a one-to-one correspondence between an  $F$ -vectorspace basis of  $R_m^0$  and standard Young tableaux of shape  $\sigma = 3^a 2^b 1^c$ ;  $a, b, c \in \mathbb{N}$ .

This correspondence is given using the dictionary :

$$\begin{array}{|c|c|c|} \hline i & j & k \\ \hline \end{array}$$

$$\text{Tr}(X_i^0 X_j^0 X_k^0)$$

$$\begin{array}{|c|c|} \hline i & j \\ \hline k & l \\ \hline \end{array}$$

$$\det \begin{pmatrix} \text{Tr}(X_i^0 X_k^0) & \text{Tr}(X_i^0 X_l^0) \\ \text{Tr}(X_j^0 X_k^0) & \text{Tr}(X_j^0 X_l^0) \end{pmatrix}$$

$$\begin{array}{|c|} \hline i \\ \hline j \\ \hline \end{array}$$

$$\text{Tr}(X_i^0 X_j^0)$$

### I.5. : Description of the Trace Ring.

In this section we will show that the trace ring of  $m$  generic 2 by 2 matrices,  $\mathbb{T}_{m,2}$ , is the ring of matrix concomitants. As always, let  $\Delta_{m,2}$  be the classical ring of quotients of  $\mathbb{K}_{m,2}$  (or  $\mathbb{T}_{m,2}$ ) i.e. the generic division algebra for  $m$  generic 2 by 2 matrices and  $P_{m,2}$  is the polynomial ring

$$P_{m,2} = F[x_{11}(i), x_{12}(i), x_{21}(i), x_{22}(i) : 1 \leq i \leq m]$$

One can embed  $P_{m,2}$  and  $\Delta_{m,2}$  naturally in  $P_{m+i,2}$  and  $\Delta_{m+i,2}$  respectively for any  $i$ .

**LEMMA 5.1.:**  $\mathbb{T}_{m,2} = \Delta_{m,2} \cap M_2(P_{m+2,2})$

**PROOF.**

The inclusion  $\mathbb{T}_{m,2} \subset \Delta_{m,2} \cap M_2(P_{m+2,2})$  is obvious. Conversely, let

$$Y \in \Delta_{m,2} \cap M_2(P_{m+2,2})$$

Consider the trace  $Tr(Y X_{m+i}) \in K_{m+i} \in K_{m+i,2} \cap P_{m+i,2}$  which is equal to  $R_{m+i,2}$ . By theorem 4.2.,  $R_{m+i,2}$  is generated by the traces of elements of  $\mathbb{K}_{m+i,2}$ . Since  $Tr(Y X_{m+i})$  is homogeneous of degree one in the variable  $X_{m+i}$ , we can express it in an  $F$ -linear combination :

$$Tr(Y X_{m+i}) = \sum_j a_j \cdot Tr(Y_1) \dots Tr(Y_l) \cdot Tr(Y_{l+1} X_{m+i})$$

for some  $l \in \mathbb{N}$ , with  $a_j \in F$  and  $\{Y_1, \dots, Y_l\}$  is a sequence of monomials in the generic matrices  $X_1, \dots, X_m$ . Therefore,  $Tr(Y X_{m+i}) = Tr(Y' X_{m+i})$  where

$$Y' = \sum_j a_j \cdot Tr(Y_1) \dots Tr(Y_l) \cdot Y_{l+1}$$

which lies in  $R_{m+i,2}$ .  $\mathbb{K}_{m+i,2} = \mathbb{T}_{m+i,2}$ . The nondegeneracy of the trace together with the fact that  $X_{m+1}$  and  $X_{m+2}$  span  $M_2(P_{m+2,2})$  over its center imply then that  $Y = Y'$ , done.

In section two we defined an action  $\theta$  of  $GL_2(F)$  on  $M_2(P_{m,2})$  which is compatible with the action  $\Phi$  of  $GL_2(F)$  on  $P_{m,2}$ . We defined the ring of matrix

concomitants to be the fixed ring  $M_2(P_{m,2})^{GL_2(F)}$ . Clearly,  $\theta$  extends to an action on  $M_2(F(X_{ij}(k)))$ .

**THEOREM 5.2. :**

- (1) : The fixed ring of  $M_2(F)(X_{ij} | l)$  under  $GL_2(F)$ - action is equal to  $\Delta_{m,2}$ .
- (2) : The ring of matrix concomitants is equal to  $\mathbb{T}_{m,2}$ .

**PROOF.**

Because  $\theta$  fixes every generic matrix,  $\Delta_{m,2} \subset M_2(F(X_{ij}(l)))^{GL_2(F)} = S$ . Therefore,  $S$  contains an  $F(X_{ij}(l))$ -basis for  $M_2(F(X_{ij}(l)))$  so its center consists of scalar matrices. But the actions of  $\theta$  and  $\Phi$  coincide on scalar matrices, i.e. :

$$Z(S) = K_{m,2} = Z(\Delta_{m,2})$$

Because  $S$  contains a basis for  $M_2(F(X_{ij}(l)))$  it is a prime p.i. ring, whence c.s.a. of dimension 4 over  $K_{m,2}$  since its center is a field. Now, both  $\Delta_{m,2}$  and  $S$  are of dimension 4 over  $K_{m,2}$ , finishing thr proof of (1). As for the second part of the theorem, note that :

$$\begin{aligned} M_2(P_{m,2})^{GL_2(F)} &= M_2(F(X_{ij}(l)))^{GL_2(F)} \cap M_2(P_{m,2}) \\ &= \Delta_{m,2} \cap M_2(P_{m,2}) \\ &= \mathbb{T}_{m,2} \end{aligned}$$

the last equality by lemma 5.1.

**THEOREM 5.3.**

- (1) : The trace ring of  $n$  generic 2 by 2 matrices is isomorphic to the polynomial ring  $\mathbb{T}_m^0[Tr(X_1), \dots, Tr(X_m)]$  where  $\mathbb{T}_m^0$  is the sub  $F$ -algebra of  $\mathbb{T}_{m,2}$  generated by the generic trace zero matrices  $X_1^0, \dots, X_m^0$ .
- (2) : There is a one-to-one correspondence between an  $F$ -vector space basis of  $\mathbb{T}_m^0$  and standard Young tableaux of shape  $\sigma = 3^a 2^b 1^c$  for all  $a, b, c \in \mathbb{N}$ .

**PROOF :**

Let  $\mathbb{T}_m^0$  be the ring of polynomial concomitants

$$f : M^0 \oplus \dots \oplus M^0 \rightarrow M_2(F)$$

from  $m$  copies of the trace zero matrices  $M^0$  to  $M_2(F)$ . Because the trace ring is generated by polynomials in the generic matrices  $X_i$  and by elements  $Tr(X_{i_1} \dots X_{i_k})$ , we can still separate traces and obtain that

$$\mathbb{T}_{m,2} \simeq \mathbb{T}_m^0 [Tr(X_1), \dots, Tr(X_m)]$$

Each polynomial concomitant  $f$  as above can be separated as  $f = f^0 + \frac{1}{2} Tr(f)$ , which corresponds to considering the trace in the ring  $\mathbb{T}_m^0$  and to decompose  $\mathbb{T}_m^0$  in a direct sum  $\mathbb{T}^0 \oplus R_m^0$  where  $\mathbb{T}^0$  is the space of all polynomial concomitants

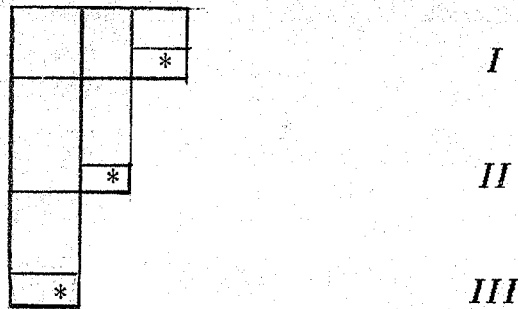
$$f^0 : M^0 \oplus \dots \oplus M^0 \rightarrow M^0$$

from  $m$  copies of  $M_0$  to  $M^0$ . The space  $\mathbb{T}^0$  can be described in the following way :

As in the proof of Theorem 5.1. one can prove that the map

$$Tr(-X_{m+1}) : \mathbb{T}_{m,2} \rightarrow R_{m+1,2}$$

is a linear injection. Moreover, it maps  $\mathbb{T}_{m,2}$  onto the subspace of elements which are linear in the generic matrix  $X_{m,1}$ . With all identifications as above, the image  $Tr(f^0 X_{m+1}^0)$  for any  $f^0 \in \mathbb{T}^0$  lies in the subspace of  $R_{m+1}^0$  consisting of invariants which are linear in the matrixvariable  $X_{m+1}^0$ . This means that  $Tr(f^0 X_{m+1}^0)$  can be written as a linear combination of standard tableaux of shape  $\sigma = 3^a 2^{2b} 1^{2c}$  filled with indices from 1 upto  $m+1$  with  $m+1$  appearing precisely one time in each Young tableau. But in a standard Young tableau,  $m+1$  can only be in one of the three starred cells :



In case I we have :

$$\boxed{i \mid j \mid m+1} = \text{Tr}(X_i^0 X_j^0 X_{m+1}^0) = \text{Tr}((X_i^0 X_j^0) \cdot X_{m+1}^0)$$

In case II we have :

$$\begin{array}{|c|c|} \hline i & j \\ \hline h & m+1 \\ \hline \end{array} = \text{Tr}(X_i^0 X_h^0) \text{Tr}(X_{m+1}^0) - \text{Tr}(X_i^0 X_{m+1}^0) \cdot \text{Tr}(X_j^0 X_h^0)$$

$$= \text{Tr}((\text{Tr}(X_i^0 X_h^0) \cdot X_j^0 - \text{Tr}(X_j^0 X_h^0) \cdot X_i^0) \cdot X_{m+1}^0)$$

Finally, in case III we get :

$$\boxed{\begin{array}{c} i \\ m+1 \end{array}} = \text{Tr}(X_i^0 X_{m+1}^0) = \text{Tr}((X_i^0) \cdot X_{m+1}^0)$$

These rules provide us with a method to reconstruct  $f^0$  out of a linear combination of standard Young tableaux describing  $\text{Tr}(f^0 \cdot X_{m+1}^0)$ . Moreover, by erasing the cell in which  $m+1$  occurs, we get a one-to-one correspondence between an  $F$ -vector space basis for  $\mathbb{T}^0$  and standard Young tableaux of shape

$$I : \sigma = 3^{a-1} \cdot 2^{2b+1} \cdot 1^{2c}$$

$$II : \sigma = 3^a \cdot 2^{2b-1} \cdot 1^{2c+1}$$

$$III : \sigma = 3^a \cdot 2^{2b} \cdot 1^{2c-1}$$

This, combined with  $\mathbb{T}_m^0 = \mathbb{T}^0 \oplus R_{m+1}^0$  and Theorem 4.3. proves part (2). In order to prove the remaining part of (1) we note that  $\mathbb{T}_m^0$  is generated by the  $X_i^0$  and the elements  $\text{Tr}(X_i^0 X_j^0)$  and  $\text{Tr}(X_i^0 X_j^0 X_k^0)$ . An easy computation shows :

$$\text{Tr}(X_i^0 X_j^0) = X_j^0 X_i^0 + X_i^0 X_j^0$$

$$\text{Tr}(X_i^0 X_j^0 X_k^0) = \frac{1}{3} \cdot S_3(X_i^0, X_j^0, X_k^0)$$

finishing the proof.

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**Trace Rings of Generic 2 by 2 Matrices II :  
Quadratic Forms.**

by

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## II.1. Introduction.

The theory of quadratic forms is, perhaps, as old as linear algebra. Historically, one was primarily interested in forms over  $\mathbb{Q}$  but like linear algebra, it can be developed fully and fruitfully over an arbitrary field, cfr. e.g. [Lam]. Since the work of Brandt it is known that there is a tight relation between the study of quadratic forms and quaternion algebras. So, it is only natural to seek for a connection between the trace ring of  $m$  generic 2 by 2 matrices,  $\mathbb{T}_{m,2}$ , and the algebraic theory of quadratic forms. In this preliminary section we aim to clarify this connection.

In the first part of these notes we have seen that

$$\mathbb{T}_{m,2} = \mathbb{T}_m^0[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$$

where  $\mathbb{T}_m^0$  is the  $F$ -subalgebra of  $\mathbb{T}_{m,2}$  generated by the generic trace-zero matrices  $X_i^0 = X_i - \frac{1}{2}\text{Tr}(X_i)$ . For any 2 by 2 matrices  $A$  and  $B$  of trace zero we know that

$$A.B + B.A = \text{Tr}(AB)$$

Therefore, if we define an iterated Öre-extension of  $F$

$$\Lambda_m = F[a_{ij} : 1 \leq i < j \leq m][a_1][a_2, \sigma_2, \delta_2] \dots [a_m, \sigma_m, \delta_m]$$

where  $\sigma_j(a_i) = -a_i$  and  $\delta_j(a_i) = 2a_{ij}$  and trivial actions on the other indeterminates, we may recover  $\mathbb{T}_m^0$  as an epimorphic image of this algebra of finite global dimension  $\Lambda_m$  by sending  $a_i$  to  $X_i^0$  and  $a_{ij}$  to  $\frac{1}{2}\text{Tr}(X_i^0 X_j^0)$ .

So, the study of homological properties is essentially the study of the homological study of the ring  $\mathbb{T}_m^0$ . This, in turn, is the description of a finite free resolution of  $\mathbb{T}_m^0$  as a left  $\Lambda_m$ -module. These questions will be tackled in the third part of these notes.

This chapter is concerned with a closer study of the ring  $\Lambda_m$ . It turns out that  $\Lambda_m$  is the generic Clifford algebra for  $m$ -ary quadratic forms over  $F$ . By this we mean that any Clifford algebra  $Cl(F, q)$  of an  $m$ -ary quadratic form  $q = \sum \alpha_{ij} X_i X_j$ ,  $\alpha_{ij} \in F$ , can be obtained as a specialization of  $\Lambda_m$ .

This important observation makes it possible to describe the prime ideal structure explicitly.  $\Lambda_m$  being a finite free extension of a polynomial subring of its center

$$S_m = F[a_{ij} : 1 \leq i < j \leq m][a_{11}, \dots, a_{mm}]$$

where  $a_{ii} = a_i^2$ , i.e. the homogeneous coordinate ring of the symmetric  $m$  by  $m$  matrices over  $F$ , this study is the description of the fibers

$$\phi : \text{Spec}(\Lambda_m) \rightarrow \text{Spec}(S_m)$$

We will show that the fibers  $\phi^{-1}(p)$  consist of at most two elements and the actual number depends on the rank of the symmetric  $m$  by  $m$  matrix :

$$\pi(A) = (\pi(a_{ij}))_{i,j}$$

where  $\pi : S_m \rightarrow S_m/p$  is the natural morphism.

In the last section we study the arithmetical theory of the maximal order  $\Lambda_m$ . It turns out that the normalizing class group is always trivial whereas the central class group is  $\mathbb{Z}/2\mathbb{Z}$  for even values of  $m$  and trivial otherwise. This entails, in particular, that  $\Lambda_m$  is a reflexive Azumaya algebra for  $m$  odd, and a  $\mathbb{Z}/2\mathbb{Z}$ -graded reflexive Azumaya algebra for  $m$  even.

In the next part, we will study the homological properties of certain quotients of  $\Lambda_m$ , including the ring  $\mathbb{T}_m^0$ .

## II.2. : Quadratic Forms and Clifford Algebras.

Throughout this chapter,  $F$  will be a field of characteristic different from two.

Let  $R$  be any commutative  $F$ -algebra. An  $m$ -ary quadratic form over the ring  $R$  is a polynomial  $f$  in  $m$  variables over  $R$  which is homogeneous of degree two. It has the general form

$$f(X_1, \dots, X_m) = \sum_{i,j=1}^m \alpha_{ij} X_i X_j \in R[X_1, \dots, X_m]$$

Replacing  $\alpha_{ij}$  by  $\frac{1}{2}(\alpha_{ij} + \alpha_{ji})$  if necessary, we may suppose that  $\alpha_{ij} = \alpha_{ji}$ . In this way,  $f$  determines uniquely a symmetric  $m$  by  $m$  matrix with coefficients in  $R$

$$M_f = (\alpha_{ij})_{i,j} \in M_m(R)$$

In matrix - notation we have

$$f(X_1, \dots, X_m) = X^T \cdot M_f X$$

where  $X = (X_1, \dots, X_m)^T$  is the  $m$  by 1 column vector of indeterminates.

Let  $f$  and  $g$  be  $m$ -ary quadratic forms over  $R$ . We say that  $f$  is equivalent to  $g$  and we denote  $f \simeq g$  if there exists an invertible matrix  $C \in GL_m(R)$  such that  $f(X) = g(CX)$ . This means that there exists a nonsingular homogeneous linear substitution of the variables  $X_1, \dots, X_m$ , which takes the form  $g$  to the form  $f$ . Since

$$g(CX) = (CX)^T \cdot M_g(CX) = X^T C^T M_g C X$$

the equivalence relation  $f(X) = g(CX)$  amounts to a matrix equation

$$M_f = C^T \cdot M_g \cdot C$$

Therefore, equivalence of forms amounts to congruence of the associated symmetric  $m$  by  $m$ -matrices.

Let  $R^m$  denote the standard free  $R$ -module of rank  $m$  with basis  $e_1, \dots, e_m$  the unit vectors. Any quadratic form,  $f$ , gives rise to a quadratic map

$$A_f : R^m \rightarrow R$$

defined by sending a column  $m$ -tuple  $x = (x_1, \dots, x_m)^T$  to  $Q_f(x) = x^T M_f x \in R$ . In terms of quadratic maps, equivalence of forms,  $f \simeq g$ , amounts to the existence of a linear  $R$ -automorphism  $C$  of  $R^m$  such that  $Q_f(x) = Q_g(Cx)$  for every  $x \in R^m$ . Note that the quadratic map  $Q_f$  determines uniquely the quadratic form  $f$ . If we set  $B_f(x, y) = \frac{1}{2}[Q_f(x, y) - Q_f(x, x) - Q_f(y, y)]$  then the map

$$B_f : R^m \times R^m \rightarrow R$$

is a symmetric bilinear pairing.

The radical of the quadratic form  $f$  is defined to be the set

$$\{x \in R^m : \forall y \in R^m; B_f(x, y) = 0\}$$

The quadratic form  $f$  is called regular if its radical contains only the zero-vector  $(0, \dots, 0)^T$ .

$f$  is said to be non-singular if  $M_f \in GL_m(R)$ , i.e. if  $\det(M_f) \in U(R)$ .

To any  $m$ -ary quadratic form  $f$  over  $R$  one can associate its Clifford algebra  $Cl(R, f)$  which is defined to be the quotient of the tensor algebra of the  $R$ -module  $R^m$  modulo the twosided ideal generated by all elements

$$x \otimes x - Q_f(x)$$

where  $x \in R^m$ . If we give the tensor algebra the usual  $\mathbb{Z}$ -gradation, then  $x \otimes x$  is homogeneous of degree 2 whereas  $Q_f(x)$  is of degree zero. This entails that the Clifford algebra  $Cl(R, f)$  has an induced  $\mathbb{Z}/2\mathbb{Z}$ -gradation, i.e.

$$Cl(R, f) = C_0 \oplus C_1$$

with  $C_i C_j \subset C_k$  where  $k \equiv i + j \pmod{2}$ .

We will now recall some structure results on Clifford algebras. For proofs and more details the reader is referred to [Lam], [Bass] and [Cassels].

**THEOREM 2.1.** (Bass, [Bass]).

If  $f$  is a non-singular  $m$ -ary quadratic form over  $R$ , then  $Cl(R, f)$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebra over  $R$ . In particular, every twosided  $\mathbb{Z}/2\mathbb{Z}$ -graded ideal of  $Cl(R, f)$  is generated by its intersection with  $R$ .

Since we will only need the consequence in the sequel, we will not go into the theory of  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebras, here.

If  $R$  is a principal ideal domain, one can reduce the study of the Clifford algebra of an arbitrary quadratic form to that of regular ones.

**THEOREM 2.2.** : ([Cassels], § 7.4.).

If  $f$  is an  $m$ -ary quadratic form over the principal ideal domain  $R$  and if the radical of  $F$  is a free  $R$ -module of rank  $0 \leq n \leq m$ , then

$$Cl(R, f) \simeq Cl(R, g) \hat{\otimes} \Lambda(F)$$

where  $g$  is an  $(m - n)$ -ary regular quadratic form,  $\Lambda(F)$  is the exterior  $R$ -algebra of the free  $R$ -module  $F$  of rank  $n$ , with the usual  $\mathbb{Z}/2\mathbb{Z}$ -gradation and  $\hat{\otimes}$  denotes the graded tensor product.

If  $A$  and  $B$  are two  $\mathbb{Z}/2\mathbb{Z}$ -graded  $R$ -algebras, then  $A \hat{\otimes} B$  is the  $R$ -module  $A \otimes B$  equipped with the gradation

$$\begin{aligned} (A \hat{\otimes} B)_0 &= A_0 \otimes B_0 + A_1 \otimes B_1 \\ (A \hat{\otimes} B)_1 &= A_1 \otimes B_0 + A_0 \otimes B_1 \end{aligned}$$

and multiplication defined by the rule

$$(a \hat{\otimes} b) \cdot (a' \hat{\otimes} b') = (-1)^{\partial b \cdot \partial a'} \cdot a a' \hat{\otimes} b b'$$

for homogeneous elements  $b$  and  $a'$  with degrees  $\partial b$  and  $\partial a'$ .

In case  $R = K$  is a field, regular quadratic forms are clearly non-singular. In view of Theorem 2.2., the study of Clifford algebras over  $K$  reduces essentially to the study of Clifford algebras of non-singular quadratic forms.

**THEOREM 2.3.** ([Lam], Ch. 5)

Let  $f$  be a non-singular  $m$ -ary quadratic form over a field  $K$ .

Let  $\delta = (-1)^{\frac{m(m-1)}{2}} \cdot \det(M_f) \in K^*$ , then :

(1) : If  $m$  is even,  $Cl(K, f)$  is a central simple algebra of dimension  $2^m$  over  $K$ .

(2) : If  $m$  is odd and if  $\delta \notin (K^*)^2$ , then  $Cl(K, f)$  is a central simple algebra of dimension  $2^{m-1}$  over the field  $K(\sqrt{\delta})$ .

(3) : If  $m$  is odd and if  $\delta \in (K^*)^2$ , then  $Cl(K, f)$  is the direct sum of two isomorphic central simple algebras each of dimension  $2^{m-1}$  over  $K$

### II.3 : Generic Clifford Algebras.

In this section we aim to introduce and study a noncommutative  $F$ -algebra  $Cl_m$  which is generic in the sense that every Clifford algebra of an  $m$ -ary quadratic form over  $F$  can be obtained as a specialization of  $Cl_m$ .

Let  $S_m$  be the homogeneous coordinate ring of the symmetric  $m$  by  $m$  matrices with entries in  $F$ , i.e.  $S_m$  is the commutative polynomial ring

$$F[a_{ij} : 1 \leq i \leq j \leq m]$$

in  $\frac{m(m+1)}{2}$  indeterminates. With  $f_m$  we will denote the  $m$ -ary regular quadratic form over  $S_m$

$$f_m(X_1, \dots, X_m) = \sum_{i,j=1}^m a_{ij} X_i X_j$$

The  $m$ -th generic Clifford algebra over  $F$ ,  $Cl_m$ , is then defined to be the Clifford algebra  $Cl(S_m, f_m)$ .

If  $f = \sum \alpha_{ij} X_i X_j$  is any  $m$ -ary quadratic form over  $F$ , then specializing  $a_{ij}$  to  $\alpha_{ij}$  gives us an  $F$ -algebra epimorphism

$$\pi_f : Cl_m \rightarrow Cl(F, f)$$

We will now study some ringtheoretical properties of these generic Clifford algebras.

Let  $\Lambda$  be an  $F$ -algebra and  $\sigma$  an  $F$ -automorphism of  $\Lambda$ . A map  $\delta : \Lambda \rightarrow \Lambda$  is called a  $\sigma$ -derivation of  $\Lambda$  if

$$\delta(\lambda.\lambda') = \delta(\lambda).\lambda' + \sigma(\lambda).\delta(\lambda')$$

for all  $\lambda$  and  $\lambda'$  in  $\Lambda$ . For such a couple  $(\sigma, \delta)$  one can define the Öre-extended  $\Lambda[x, \sigma, \delta]$  of  $\Lambda$  which is the set of all polynomials with coefficients in  $\Lambda$  equipped with the multiplication rule

$$x.\lambda = \sigma(\lambda).x + \delta(\lambda)$$

for all  $\lambda \in \Lambda$ . Allowing ourselves one recursive definition, we say that iterated Öre-extensions are Öre-extensions or Öre-extensions of iterated Öre-extensions.

**THEOREM 3.1.**

The generic Clifford algebra  $Cl_m$  is an iterated Öre-extension.

**PROOF.**

Consider the  $F$ -algebra

$$\Lambda_m = F[a_{ij} : 1 \leq i \leq j \leq m][a_1][a_2, \sigma_2, \delta_2] \dots [a_m, \sigma_m, \delta_m]$$

where one defines for each  $i < j$  that  $\sigma_j(a_i) = -a_i$  and  $\delta_j(a_i) = 2a_{ij}$  and trivial actions of  $\sigma_j$  and  $\delta_j$  on the other indeterminates. In order to verify that  $\Lambda_m$  is an iterated Öre-extension one has to verify that every  $\sigma_k$  is an automorphism (which is trivial) and that  $\delta_k$  is a  $\sigma_k$ -derivation of the subalgebra :

$$\Lambda_{m,k} = F[a_{ij} : 1 \leq i \leq j \leq m][a_1][a_2, \sigma_2, \delta_2] \dots [a_{k-1}, \sigma_{k-1}, \delta_{k-1}]$$

Since we defined  $\delta_k$  on a generating set, it is defined on  $\Lambda_{m,k}$ . So we only need to verify that it preserves the commutation rules, i.e. we have to check all  $i < j < k$  that

$$\delta_k(a_i a_j + a_j a_i) = \delta_k(2a_{ij}) = 0$$

Now,  $\delta_k(a_i a_j) = a_{ik} a_j - a_i a_{jk}$  and  $\delta_k(a_j a_i) = a_{jk} a_i - a_j a_{ik}$  and since each  $a_{ij}$  is central the relation follows.

Defining  $a_{ii} = a_i^2$ ,  $\Lambda_m$  becomes an  $S_m$ -algebra. Further, sending the  $i$ -th unit vector  $e_i$  of  $(S_m)^m$  to  $a_i$  we get an  $S_m$ -module morphism

$$\phi : (S_m)^m \rightarrow \Lambda_m$$

compatible with the quadratic form, in the sense that

$$\phi(x)^2 = Q_{f_m}(x)$$

for every  $x \in (S_m)^m$ . Further, for any  $S_m$ -algebra  $A$  which is compatible with  $f_m$  with defining morphism

$$\chi : (S_m)^m \rightarrow A$$

one can define an  $S_m$ -algebra morphism from  $\Lambda_m$  to  $A$  by sending  $a_i$  to  $\chi(e_i)$ . Since the Clifford algebra  $Cl(S_m, f_m)$  is defined by this universal property we get



that

$$Cl_m = Cl(S_m, f_m) \simeq \Lambda_m$$

finishing the proof.

An  $F$ -algebra  $\Lambda$  has finite global dimension  $n$ , denoted  $\text{gldim}(\Lambda) = n$ , if every finitely generated left  $\Lambda$ -module  $M$  has a projective resolution of length  $\leq n$ .

**PROPOSITION 3.2.** : The generic Clifford algebra  $Cl_m$  has finite global dimension  $\frac{m(m+1)}{2}$ .

**PROOF.**

Using an iterated version of a result due to Fields [Fields] and the description of  $Cl_m$  as an iterated Öre-extension  $\Lambda_m$ , we get that  $\text{gldim}(Cl_m) \leq \frac{m(m+1)}{2}$ .

On the other hand, it is clear from the definition of  $\Lambda_m$  that  $Cl_m$  is a free module of rank  $2^m$  over the polynomial ring  $S_m$ . Then, using a result of McConnell [McConnell] we get that  $\text{gldim}(Cl_m) \geq \text{gldim}(S_m) = \frac{m(m+1)}{2}$ , yielding the result.

It follows from the description of  $Cl_m$  as an iterated Öre-extension, that  $Cl_m$  is a domain. We will now calculate its p.i.-degree, i.e. the smallest natural number  $n$  such that there is an  $F$ -algebra embedding of  $Cl_m$  into  $M_n(A)$  for some commutative  $F$ -algebra  $A$ . Or, alternatively, such that

$$s_{2n}(x_1, \dots, x_{2n}) = \sum_{\sigma \in s_{2n}} \text{sgn}(\sigma) x_{\sigma(1)}, \dots, x_{\sigma(2n)}$$

vanishes for all substitutions with elements from  $Cl_m$ .

**PROPOSITION 3.3.** : The p.i. degree of the generic Clifford algebra  $Cl_m$  is  $2^\alpha$  where  $\alpha$  is the largest natural number  $\leq \frac{m}{2}$ .

**PROOF.**

With  $K_m$  we will denote the field of fractions of the domain  $S_m$ . Then,  $Cl_m \otimes K_m$  is the Clifford algebra of the  $m$ -ary nonsingular quadratic form  $f_m$  over  $K_m$ .

If  $m$  is even, it follows from Theorem 2.3. that  $Cl_m \otimes K_m$  is central simple over  $K_m$  of dimension  $2^m$ , i.e.

$$\text{p.i. deg}(Cl_m) = \text{p.i. deg}(Cl_m \otimes K_m) = \sqrt{2^m} = 2^{\frac{m}{2}}$$

If  $m$  is odd, then since  $\det(M) \notin (K_m^*)^2$  it follows that  $Cl_m \otimes K_m$  is central simple over  $K_m(\sqrt{\delta})$  of dimension  $2^{m-1}$ , i.e.

$$\text{p.i. deg}(Cl_m) = \text{p.i. deg}(Cl_m \otimes K_m) = \sqrt{2^{m-1}} = 2^{\frac{m-1}{2}}$$

#### II.4. The Prime Ideal Structure.

For any  $F$ -algebra  $\Lambda$  we will denote with  $\text{Spec}(\Lambda)$  the set of all two-sided prime ideals of  $\Lambda$  equipped with the Zariski topology, i.e. a typical closed set is of the form  $V(I) = \{P \in \text{Spec}(\Lambda) : I \subset P\}$  for some two-sided ideal  $I$  of  $\Lambda$ . In this section we will describe the prime ideal spectrum of the generic Clifford algebra  $Cl_m$ . Clearly, intersecting with  $S_m$ , which is a subring of the center, gives a continuous map :

$$\phi : \text{Spec}(Cl_m) \rightarrow \text{Spec}(S_m)$$

and since  $Cl_m$  is a finite module over  $S_m$ , this map is surjective.

The prime ideal spectrum of a commutative polynomial ring (such as  $S_m$ ) may be assumed to be relatively well known. Therefore, describing  $\text{Spec}(Cl_m)$  essentially amounts to describing the fibers of  $\phi$ .

**THEOREM 4.1.** : If  $p$  is any prime ideal of  $S_m$ , the fiber  $\phi^{-1}(p)$  contains at most two elements.

#### PROOF.

Let us denote  $S = S_m/p$  and  $\pi : S_m \rightarrow S$  the natural morphism.  $S$  being a domain, it has a field of fractions  $K$ . Now, studying the fiber  $\phi^{-1}(p)$  is the same as describing the fiber of the zero prime ideal of  $S$  under the map

$$\phi_\pi : \text{Spec}(Cl_m \otimes_\pi S) \rightarrow \text{Spec}(S)$$

The fiber  $\phi_\pi^{-1}(0)$  is in natural one-to-one correspondence with :

$$\text{Spec}(Cl_m \otimes_\pi S \otimes_\pi K) = \text{Spec}(Cl_m \otimes_\pi K)$$

Now,  $Cl_m \otimes_\pi K$  is the Clifford algebra over the field  $K$  associated to the  $m$ -ary quadratic form :

$$\pi(f_m) = \sum_{i,j} \pi(a_{ij})X_iX_j$$

In view of Theorem 2.2. we know that

$$Cl_m \otimes_\pi K \simeq Cl(K, g) \hat{\otimes} \Lambda(F)$$

where  $F$  is an  $n$ -dimensional  $K$ -vectorspace,  $n$  being the dimension of the radical of  $\pi(f_m)$ , and  $g$  is a non-singular  $(m - n)$ -ary quadratic form over  $K$ . The exterior algebra  $\Lambda(F)$  has an argumentation morphism

$$\epsilon : \Lambda(F) \rightarrow F$$

and it is easy to verify that  $1 \hat{\otimes} \text{Ker} \epsilon$  is contained in the prime radical of  $Cl_m \otimes_{\pi} K$ . This entails that there is a one-to-one correspondence between  $\text{Spec}(Cl_m \otimes_{\pi} K)$  and  $\text{Spec}(Cl(K, g))$ .

Finally,  $g$  being a non-singular quadratic form over the field  $K$ , it follows from Theorem 2.3. that  $Cl(K, g)$  has at most two prime ideals.

Moreover, it is easy to see whether  $\phi^{-1}(p)$  has one or two elements. For, let :

$$\pi(\mathcal{A}) = (\pi(a_{ij}))_{i,j}$$

be the symmetric  $m$  by  $m$ -matrix over  $S = S_m/p$ . If the rank of  $\pi(\mathcal{A})$  is even then  $Cl(K, g)$  is central simple, whence  $\phi^{-1}(p)$  has just one element.

If the rank of  $\pi(\mathcal{A})$  is odd,  $\pi(\mathcal{A})$  is congruent over the field of fractions of  $S, K$  to a matrix of the form :

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

where  $A$  is a symmetric invertible  $n$  by  $n$ -matrix over  $K$ ,  $n = \text{rank}(\pi(\mathcal{A}))$ . Let  $\delta = (-1)^{\frac{n(n-1)}{2}} \cdot \det(A)$ , then  $\phi^{-1}(p)$  has one element if  $\delta \notin (K^*)^2$  and two elements if  $\delta \in (K^*)^2$ .

For instance, let  $F$  be an algebraically closed field, then the set of maximal ideals of  $S_m$  corresponds bijectively to the set of symmetric  $m$  by  $m$  matrices over  $F$ . In this case, the number of maximal ideals of  $Cl_m$  lying over a maximal ideal corresponding to a matrix  $(\alpha_{ij})_{i,j}$  is equal to  $1 + (\text{rank}(\alpha_{ij}) \bmod 2)$ .

**PROPOSITION 4.2.** : If  $p$  is a prime ideal of  $S_m$  and if  $n = \text{rank}(\pi(\mathcal{A}))$ , then the p.i.-degree of the quotient  $Cl_m/P$ , where  $P$  is a prime ideal lying over  $p$ , is equal to  $2^\alpha$ , where  $\alpha$  is the largest natural number  $\leq \frac{n}{2}$ .

**PROOF :**

With notations as in the proof of Theorem 4.1. we have that the p.i.-degree of the quotient  $Cl_m/P$  is equal to the p.i.-degree of  $Cl(K, g)$  (or its components). Using Theorem 2.3., the propositions follows immediately.

For any  $F$ -algebra  $\Lambda$  one defines  $\text{Spec}_n(\Lambda)$  to be the subset of  $\text{Spec}(\Lambda)$  consisting of those prime ideals  $P$  such that the p.i.-degree of the quotient  $\Lambda/P$  is smaller or equal to  $n$ .  $\text{Spec}_n(\Lambda)$  is always a closed subset of  $\text{Spec}(\Lambda)$ , i.e. of the form  $V(I_n)$  for some ideal  $I_n$  of  $\Lambda$  but it is usually hard to compute these ideals.

**COROLLARY 4.3.** : For any natural number  $2^{a-1} < n \leq 2^a$ , the ideal  $I_n$  is generated by the  $2a$  by  $2a$  minors of the matrix  $A = (a_{ij})_{i,j}$ .

**PROOF :**

Follows immediately from the foregoing proposition.

## II.5. Classgroups.

In [A-G] Auslander and Goldman introduced the notion of maximal orders. By an order over an integrally closed Noetherian domain  $R$  we mean a subring  $\Lambda$  of a central simple algebra  $\Sigma$  over the field of fractions  $K$  of  $R$  such that  $\Lambda$  is a finitely generated  $R$ -module which spans  $\Sigma$  over  $K$ . An order  $\Lambda$  in the central simple algebra  $\Sigma$  is said to be maximal if  $\Lambda$  is not properly contained in any order of  $\Sigma$ . Later, Fossum [F] generalized these notions over a Krull domain  $R$ , replacing the finitely generated condition by the fact that orders should be reflexive  $R$ -lattices. In [Ch] M. Chamarie proved that a prime p.i. ring  $\Lambda$  with classical ring of quotients  $\Sigma$  is a maximal order over its center  $R$ , which is then a Krull domain, if and only if for every twosided ideal  $I$  of  $\Lambda$  :

$$(I :_l I) = (I :_r I) = (\Lambda) \quad (*)$$

where for any subsets  $A$  and  $B$  of  $\Sigma$  one denotes :

$$(A :_l B) = \{x \in \Sigma : xA \subset B\}$$

$$(A :_r B) = \{x \in \Sigma : Ax \subset B\}$$

**THEOREM 5.1.** : The generic Clifford algebra  $Cl_m$  is a positively graded maximal order.

### PROOF.

M. Chamarie [Ch] proved that an Öre-extension  $\Lambda[x, \sigma, \delta]$  of a maximal order satisfies the condition (\*) for all its two-sided ideals  $I$ . So, from the fact that  $Cl_m$  is an iterated Öre extension and a domain of finite p.i.- degree, it follows that  $Cl_m$  is a maximal order.

Giving each of the indeterminates  $a_{ij}$  degree two and the indeterminates  $a_i$  degree one, one verifies easily that  $Cl_m$  is positively graded since the defining relations :

$$a_i a_j + a_j a_i = 2a_{ij}$$

are all homogeneous. This gradation is compatible with the  $\mathbb{Z}/2\mathbb{Z}$ -gradation on  $Cl_m$ .

This result shows that the center of  $Cl_m$ , which we will denote from now on with  $Z_m$ , is completely integrally closed. In order to describe it completely, let us denote

$$d = s_m(a_1, \dots, a_m) = \sum_{\sigma \in s_m} \text{sgn}(\sigma) a_{\sigma(1)} \dots a_{\sigma(m)}$$

where  $s_m$  is the group of permutations on  $m$  elements.

**PROPOSITION 5.2. :**

- (1) : If  $m$  is even ,  $Z_m = S_m$
- (2) : If  $m$  is odd,  $Z_m = S_m \oplus S_m.d$

**PROOF.**

(1) Look at  $Cl_m \otimes K_m$ . This is a central simple algebra with center  $K_m$ . Therefore,  $Z_m = K_m \cap Cl_m = S_m$ .

(2) In this case,  $Cl_m \otimes K_m$  is a central simple algebra with center  $K_m(\sqrt{\delta})$ . Now it is fairly easy to verify that  $d^2$  is equal to  $\delta$  upto multiplication with some power of 2. This, combined with the fact that  $S_m$  is a factorial domain in which 2 is invertible, entails that the integral closure of  $S_m$  in  $K_m(\sqrt{\delta})$  is equal to  $S_m \oplus S_m.d$ . Because  $d \in Cl_m$ , this finishes the proof.

Studying the arithmetical theory of a maximal order  $\Lambda$  over a normal domain  $R$  amounts to describing the normalizing classgroup,  $NCl(\Lambda)$ , and the normalizing classgroup,  $CCL(\Lambda)$ . Let us recall their definition.

A twosided  $\Lambda$ -submodule  $A$  of the central simple  $K$ - algebra  $\Sigma$  is said to be a fractional ideal of  $\Lambda$  if  $r.A \subset \Lambda$  for some non-zero element  $r \in R$ . A fractional ideal  $A$  is said to be divisorial if

$$(A :_l \Lambda) :_r A = (A :_r \Lambda) :_l A = A$$

or, equivalently, that  $\cap A_p = A$ , where the intersection is taken over all height one prime ideals  $p$  of  $R$ .

The set of all divisorial  $\Lambda$ -ideals, equipped with the product :

$$A * B = (AB :_l \Lambda) :_r \Lambda$$

is the free Abelian group generated by the height one prime ideals of  $\Lambda$ . We will denote this group by  $\mathbb{D}(\Lambda)$  and call it the divisor group of  $\Lambda$ .

With  $\mathbb{N}(\Lambda)$  (resp.  $\mathbb{C}(\Lambda)$ ) we denote the subgroup of  $\mathbb{D}(\Lambda)$  consisting of those divisorial  $\Lambda$ -ideals which are generated by a normalizing (resp. central) element. The normalizing (resp. central) classgroup of  $\Lambda$  are then defined to be the quotient groups :

$$\begin{aligned} NCl(\Lambda) &= \mathbb{D}(\Lambda)/\mathbb{N}(\Lambda) \\ CCl(\Lambda) &= \mathbb{D}(\Lambda)/\mathbb{C}(\Lambda) \end{aligned}$$

If the maximal order  $\Lambda$  is a  $\mathbb{Z}$ -graded algebra, one can similarly define the graded counterparts of these notions.

$\mathbb{D}_g$  will be the group generated by homogeneous divisorial height one prime ideals of  $\Lambda$ .  $\mathbb{N}_g(\Lambda)$  (resp.  $\mathbb{C}_g(\Lambda)$ ) will be the subgroup of  $\mathbb{D}_g(\Lambda)$  consisting of all homogeneous divisorial  $\Lambda$ -ideals generated by an homogeneous normalizing (resp. central) element. The corresponding (graded) class group will be denoted by  $NCl^g(\Lambda)$  and  $CCl^g(\Lambda)$ .

**THEOREM 5.3. :**

- (1) : If  $m$  is odd,  $CCl(Cl_m) \simeq 0$
- (2) : If  $m$  is even,  $CCl(Cl_m) \simeq \mathbb{Z}/2\mathbb{Z}$

**PROOF.**

With  $Q^g(Cl_m)$  we denote the localization of  $Cl_m$  at the multiplicative system of all homogeneous non-zero elements. Clearly,  $Q^g(Cl_m)$  is an Azumaya algebra over a factorial center. Therefore,  $CCl(Q^g(Cl_m)) \simeq 0$ . Hence, from the localization sequence :

$$0 \rightarrow CCl^g(Cl_m) \rightarrow CCl(Cl_m) \rightarrow CCl(Q^g(Cl_m)) \rightarrow 0$$

it follows that  $CCl^g(Cl_m) \simeq CCl(Cl_m)$ . So, we may restrict attention to  $\mathbb{Z}$ -homogeneous divisorial ideals.

First consider the case that  $m$  is odd. Let  $D_1$  (resp.  $D_2$ ) be the determinant of the  $m-1$  by  $m-1$  top left (resp. bottom right) corner minor of  $M = (a_{ij})_{i,j}$ .



These two elements are algebraically independent over  $F$ , so every  $\mathbb{Z}$ -graded height one prime ideal of  $Cl_m, P$ , must exclude one of these elements. But then, by the results of the previous section,  $\text{p.i.-deg}(Cl_m/P) = \text{p.i.-deg}(Cl_m)$ . So, the localization of  $Cl_m$  at  $P \cap Z_m$  is an Azumaya algebra. This entails that  $CCL(Cl_m) = CCl^g(Cl_m) = Cl^g(Z_m) = 0$ .

Now, let  $m$  be even. If  $P$  is a height one prime ( $\mathbb{Z}$ -graded) of  $Cl_m$  not containing  $D = \det(a_{ij})_{i,j}$ , then by an argument as in the preceding paragraph,  $P$  is generated by a central element since  $Z_m = S_m$  is factorial. So  $CCL^g(Cl_m)$  is the group generated by the one prime ideal  $P$  containing  $D$ . We claim that  $P^2$  is generated by  $D$ , i.e.  $CCL^g(CCL_m) \simeq \mathbb{Z}/2\mathbb{Z}$  since  $\sqrt{D} \notin Z_m$ . Let  $R$  be the quotient  $S_m/(D)$ , which is a domain with field of fractions, say  $K$ . Then

$$Cl_m \otimes K \simeq Cl(K, g) \hat{\otimes} K[x]/(x^2)$$

by the results of the foregoing section. Let  $\epsilon$  be the argumentation of  $K[x]/(x^2)$ , i.e. sending  $x \mapsto 0$ , then  $P$  is the kernel of

$$Cl_m \rightarrow Cl_m \otimes K \xrightarrow{1 \otimes \epsilon} Cl(K, g)$$

and the claim follows.

A maximal order  $\Lambda$  over a normal domain  $R$  is said to be a reflexive Azumaya algebra whenever the natural extension of the map

$$\Lambda \otimes_R \Lambda^{opp} \rightarrow \text{End}_R(\Lambda)$$

to

$$(\Lambda \otimes_R \Lambda^{opp})^{**} \rightarrow \text{End}_R(P)$$

is an isomorphism,  $( )^{**}$  denoting the bidual  $R$ -module. Note that whenever  $F$  is a field of characteristic zero, then  $\Lambda$  is a reflexive Azumaya algebra if and only if  $CCL(\Lambda) \simeq Cl(R)$ .

**COROLLARY 5.4.** : If  $F$  is a field of characteristic zero, then  $Cl_m$  is a reflexive Azumaya algebra whenever  $m$  is odd.

Further, note that in this case  $Cl_m$  cannot be a free module over its center  $Z_m$  since this would entail that  $Cl_m$  is an Azumaya algebra and  $Cl_m$  has clearly commutative epimorphic images.

Let us now study the normalizing class group :

**THEOREM 5.5.** : For all  $m \in \mathbb{N}$ ,  $NCl(Cl_m) = 0$ .

**PROOF** :

Because the normalizing classgroup is an epimorphic image of the central classgroup, it suffices to show that the ramified prime ideal  $P$ , for  $m$  even, is generated by a normalizing element. Since  $CCL(Cl_m) \simeq \mathbb{Z}/2\mathbb{Z}$  in this case, it is sufficient to prove that there exists a non-central normalizing element of  $Cl_m$ . The generic Clifford algebra,  $Cl_m$ , is  $\mathbb{Z}/2\mathbb{Z}$ -graded

$$Cl_m = C_0 \oplus C_1$$

and therefore it admits an automorphism  $\alpha$  sending  $c_0 \oplus c_1$  to  $c_0 \oplus (-c_1)$ . Since  $Z_m$  reduces in this case to  $S_m \subset C_0$ ,  $\alpha$  leaves the center elementwise fixed, so  $\alpha$  is given by conjugation with a non-central normalizing element by the Skolem-Noether theorem.

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**Trace Rings of Generic 2 by 2 Matrices III :  
Homological Algebra**

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### III.1 : Introduction

In this chapter we will prove some homological properties of quotients of the generic Clifford algebra  $Cl_m$  and of the trace ring of  $m$  generic 2 by 2 matrices,  $\mathbb{T}_{m,2}$ .

In the second section we recall the relevant commutative definitions and results on graded Cohen-Macaulay and Gorenstein rings. Moreover, we define the noncommutative counterparts of these notions in the following way. A positively graded quotient  $\Lambda$  of  $Cl_m$  is said to be Cohen-Macaulay if the projective dimension of  $\Lambda$ , i.e. the least  $t$  such that there is a resolution

$$0 \rightarrow F_t \rightarrow F_{t-1} \rightarrow \cdots \rightarrow F_0 = Cl_m \rightarrow \Lambda \rightarrow 0$$

with every  $F_i$  finitely generated graded free  $Cl_m$ -module and all morphisms gradation preserving, is precisely the difference between the Krull dimension of  $Cl_m$  and that of  $\Lambda$ . It turns out that Cohen-Macaulay quotients are finitely generated graded free modules over a polynomial subring  $F[\theta_1, \dots, \theta_\alpha]$  of its center. A Cohen-Macaulay quotient is said to be Gorenstein if its canonical module  $\Omega^{nc}(\Lambda)$  is isomorphic as a  $\Lambda$ -module to  $\Lambda$ .

In the third section we show that there is only one prime ideal  $J_{H,n}$  of  $Cl_m$  lying over certain prime ideals  $I_{H,n}$  of  $S_m$  determined by minors of the generic symmetric  $m$  by  $m$  matrix  $A = (a_{ij})_{i,j}$ . The most interesting of these ideals are those determined by all  $k$  by  $k$  minors of  $A$ . So, there is a generic  $F$ -algebra for Clifford algebras of  $m$ -ary quadratic forms of rank smaller than  $k$ .

In section four we prove the main result which states that for many of these ideals, the quotient  $Cl_{H,n} = Cl_m/J_{H,n}$  are, in fact, Cohen-Macaulay. Further, for the ideal corresponding to the prime ideal of  $S_m$  generated by all  $k$  by  $k$  minors of  $A$ , we show that the quotient is a maximal order and even a reflexive Azumaya algebra.

In the final section we will apply all these results to the trace ring of  $m$  generic 2

by 2 matrices,  $\mathbb{T}_{m,2}$ . In the first chapter we have seen that

$$\mathbb{T}_{m,2} \simeq \mathbb{T}_m^o[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$$

where  $\mathbb{T}_m^o$  is the  $F$ -subalgebra of  $\mathbb{T}_{m,2}$  generated by the generic trace zero matrices  $X_i^o = X_i - \frac{1}{2}\text{Tr}(X_i)$ . It turns out that  $\mathbb{T}_m^o$  is the generic  $F$ -algebra for Clifford algebras of  $m$ -ary quadratic forms of rank smaller or equal to 3. Clearly, homological properties of  $Cl_m$  with respect to a resolution of it as a  $Cl_m$ -module are identical to the homological properties of  $\mathbb{T}_{m,2}$  as a module over the positively graded  $F$ -algebra with finite global dimension

$$\Gamma_m = Cl_m[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$$

So, in particular we get that  $\mathbb{T}_{m,2}$  is a free module of finite rank over a polynomial subring of its center. Further, we show that the center of  $\mathbb{T}_{m,2}$ ,  $\mathcal{R}_{m,2}$ , is a unique factorization Gorenstein domain entailing that the normalizing classgroup of  $\mathbb{T}_{m,2}$  (and  $\mathbb{T}_m^o$ ) is trivial. This entails that its canonical module, which is for every maximal order Cohen-Macaulay quotient a divisorial twosided ideal, is isomorphic to  $\mathbb{T}_m^o$ . Therefore,  $\mathbb{T}_m^o$  and  $\mathbb{T}_{m,2}$  are Gorenstein.

In the next chapter we will show that this fact is the explication for the functional equation of the Poincaré series of  $\mathbb{T}_{m,2}$ .

### III.2 : Some homological algebra

In this section, we will recall some basic facts from commutative (graded) homological algebra. For more details we refer the reader to [Rot] or [St]. Further, we propose a noncommutative generalization of Cohen-Macaulay and Gorenstein rings.

Throughout, let  $R$  be a positively graded  $F$ -algebra. As usual, the Krull dimension of  $R$ ,  $Kdim(R)$ , is the length of the longest chain of prime ideals of  $R$ . If  $M$  is a finitely generated graded  $R$ -module, we define the Krull dimension of  $M$ ,  $Kdim(M)$ , to be  $Kdim(R/Ann(M))$ .

A partial homogeneous system of parameters for  $M$  is a sequence  $\theta_1, \theta_2, \dots, \theta_r$  of homogeneous elements of  $R_+ = \bigoplus_{i \geq 1} R_i$  such that

$$Kdim(M/(\theta_1.M + \dots + \theta_r.M)) = Kdim(M) - r$$

If  $r = Kdim(M)$ , we call  $\{\theta_1, \theta_2, \dots, \theta_r\}$  an homogeneous system of parameters. Equivalently,  $\{\theta_1, \dots, \theta_r\}$  is an homogeneous system of parameters if and only if  $r = Kdim(M)$  and  $M$  is a finitely generated  $F[\theta_1, \dots, \theta_r]$ -module.

A set  $\{\theta_1, \dots, \theta_r\}$  of homogeneous elements from  $R_+$  is a homogeneous  $M$ -sequence if  $\theta_{i+1}$  is a non-zero divisor on  $M/(\theta_1.M + \dots + \theta_i.M)$  for all  $0 \leq i < r$ . Equivalently,  $\theta_1, \dots, \theta_r$  are algebraically independent over  $F$  and  $M$  is a free (but not necessarily finitely generated)  $F[\theta_1, \dots, \theta_r]$ -module.

We define the depth of the  $R$ -module  $M$ ,  $depth(M)$ , to be the length of the longest homogeneous  $M$ -sequence. It is clear that  $depth(M) \leq Kdim(M)$ . The case of equality, i.e. when some homogeneous system of parameters is an  $M$ -sequence, is of particular importance. We call  $M$  a Cohen-Macaulay module if  $depth(M) = Kdim(M)$ , or equivalently,  $M$  is a finitely generated and free  $F[\theta_1, \dots, \theta_r]$ -module for some (equivalently, every) homogeneous system of parameters  $\{\theta_1, \dots, \theta_r\}$ . Note that the property of being a Cohen-Macaulay module is independent of the ring



$R$  over which  $M$  is finitely generated.  $R$  is called a Cohen-Macaulay ring if it is a Cohen-Macaulay module over itself.

In invariant theory, there are many Cohen-Macaulay rings. For, let  $F$  be algebraically closed and let  $G$  be a linear algebraic group, i.e. a subgroup of some  $GL_n(F)$  which is defined by the condition that the entries of the matrices  $\sigma = (a_{ij})$  in  $G$  satisfy certain polynomial equations over  $F$ . We call a group  $G$  linearly reductive if every finite dimensional representation is completely reducible as a direct sum of irreducible representations. We say that  $G$  acts  $F$ -rationally on a finite dimensional  $F$ -vector space  $V$  if  $V$  is a  $G$ -module and if the map  $G \rightarrow GL_n(F)$  is both a groupmorphism and an  $F$ -morphism of varieties. Clearly, this action extends to the symmetric algebra  $S(V)$  of  $V$ , i.e. a polynomial ring in  $\dim_F(V)$  variables.

**Theorem 2.1** : (Hochster-Roberts [H-R])

Let  $F$  be a field and let  $G$  be a linearly reductive, linear algebraic group over  $F$  acting  $F$ -rationally on a finite dimensional vector space  $V$ , then the fixed ring of the extended action on the symmetric algebra,  $S(V)$ , is a Cohen-Macaulay ring.

Let us return to our exposition of the homological properties of a finitely generated graded module  $M$  over a positively graded affine  $F$ -algebra  $R$ . If  $x_1, \dots, x_s$  are homogeneous elements of  $R_+$  which generated  $R$  as an  $F$ -algebra, then there is an epimorphism

$$A = F[y_1, \dots, y_s] \rightarrow R$$

by sending  $y_i$  to  $x_i$ . This is a gradation preserving map if we define  $\deg(y_i) = \deg(x_i)$ .  $M$  is a finitely generated graded  $A$ -module, whence there exists a finite free resolution of  $M$  as a graded  $A$ -module

$$0 \rightarrow \Lambda_t \rightarrow \Lambda_{t-1} \rightarrow \dots \rightarrow \Lambda_1 \rightarrow \Lambda_0 \rightarrow M \rightarrow 0 \quad (*)$$

where the  $\Lambda_i$  are finitely generated graded free  $A$ -modules and all morphisms  $\phi_i : \Lambda_i \rightarrow \Lambda_{i-1}$  are gradation preserving. The projective (or homological) dimension of

$M$ ,  $pd_A(M)$ , is the minimal number  $t$  possible in such a resolution. By a result of Auslander and Buchsbaum we know that

$$pd_A(M) = s - depth(M)$$

The finite free resolution (\*) can be dualized by applying the functor  $Hom_A(-, A) = (-)^*$  to it. One obtains a sequence

$$0 \rightarrow \Lambda_0^* \rightarrow \Lambda_1^* \rightarrow \dots \rightarrow \Lambda_{t-1}^* \rightarrow \Lambda_t^*$$

which is a complex because  $\phi_{i+1}^* \circ \phi_i^* = 0$ , but in general it is not exact. The homology of the dualized free resolution of an  $R$ -module  $M$ , considered as an  $A$ -module, is one of the fundamental functors of homological algebra and it is independent of the chosen free resolution

$$Ext_A^i(M, A) = Ker(\phi_{i+1}^* / Im(\phi_i^*))$$

The functors  $Ext^i$  can be calculated using local cohomology with respect to the irrelevant ideal  $R_+$ . For any  $R$ -module  $M$  define

$$L(M) = \{m \in M \mid (R_+)^n \cdot m = 0 \text{ for some } n > 0\}$$

It is easy to check that  $L$  is a left exact additive functor, so we can define the right derived functors  $R^i L$ , see for example [Rot], to define

$$H^i(M) = R^i L(M)$$

Local cohomology is depth sensitive in the following sense :  $H^i(M) = 0$  unless  $e = depth(M) \leq i \leq Kdim(M) = d$  and  $H^e(M) \neq 0, H^d(M) \neq 0$ . The local cohomology can be calculated explicitly using a version of Cech-cohomology, see for example [St] or [Ha].

The injective hull of  $F$  as an  $A = F[y_1, \dots, y_s]$ -module is  $E_A(F) = F[y_1^{-1}, \dots, y_s^{-1}]$ . This given, we define the Matlis dual module of any  $\mathbb{Z}$ -graded  $A$ -module  $M$  by  $M^{(\cdot)} = Hom_A(M, E_A(F))$ . The functors  $Ext^i$  and  $(-)^{(\cdot)}$  are related to local

cohomology by the local duality theorem, cfr. [Ha,ch.6]

$$\text{Ext}_A^i(M, A)^{(\cdot)} = H^{s-i}(M)$$

Now, let  $M$  be a Cohen-Macaulay module of dimension  $d$  with a minimal free resolution (\*) and define

$$\Omega(M) = \text{Coker}(\phi_t^*) = \Lambda_t^*/\text{Im}(\phi_t^*) = \text{Ext}_A^{s-d}(M, A)$$

Because  $M$  is Cohen-Macaulay,  $H^i(M) \neq 0$  only for  $i = d$  and hence by the local duality theorem  $\text{Ext}_A^i(M, A) \neq 0$  only for  $i = s - d = \text{pd}_A(M) = t$ . So, the dual complex of (\*)

$$0 \rightarrow \Lambda_0^* \rightarrow \dots \rightarrow \Lambda_t^* \rightarrow \Omega(M) \rightarrow 0$$

is an exact sequence and is, in fact, a minimal resolution of  $\Omega(M)$ .  $\Omega(M)$  is called the canonical module of  $M$  and it can be shown that it is, as an  $R$ -module, independent of the choice of  $A$ .

A Cohen-Macaulay ring  $R$  is said to be Gorenstein if and only if  $\Omega(R)$  is a graded free  $R$ -module of rank one. It can be shown that Gorenstein rings are precisely the rings with finite self-injective dimension. By a result of Murthy, see for example [Fo], we know that a unique factorization domain which is Cohen-Macaulay is, in fact, Gorenstein.

We will now say what we mean by a Cohen-Macaulay and Gorenstein quotient of the generic Clifford algebra  $Cl_m$ . Let  $I$  be a two-sided  $\mathbb{Z}$ -graded ideal of  $Cl_m$  and let  $\Lambda = Cl_m/I$ , then  $\Lambda$  has a finite free resolution as a graded (left)  $Cl_m$ -module

$$0 \rightarrow F_t \rightarrow F_{t-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 = Cl_m \rightarrow \Lambda \rightarrow 0 \quad (1)$$

with every  $F_i$  a finitely generated graded free (left)  $Cl_m$ -module and every morphism  $\phi_i : F_i \rightarrow F_{i-1}$  gradation preserving. Again, the projective dimension of  $\Lambda$ ,  $\text{pd}(\Lambda)$ , is defined to be the minimal  $t$  possible in such a resolution. In accordance with the commutative theory we define

**Definition 2.2** :  $\Lambda$  is said to be a Cohen-Macaulay quotient of the generic Clifford algebra  $Cl_m$  if and only if  $pd(\Lambda) = Kdim(Cl_m) - Kdim(\Lambda)$ .

**Proposition 2.3** : If  $\Lambda$  is a Cohen-Macaulay quotient of  $Cl_m$  then  $\Lambda_m$  is a Cohen-Macaulay module over  $S_m$ , i.e.  $\Lambda$  is a finitely generated graded free module over a polynomial subring of its center, and  $Ext_{Cl_m}^i(\Lambda, Cl_m) = 0$  unless  $i = Kdim(Cl_m) - Kdim(\Lambda)$ .

**Proof** :

Because  $Cl_m$  is a graded free module of finite rank over  $S_m$  it is clear that the Krull dimension of the  $S_m$ -module  $\Lambda$  is equal to  $Kdim(\Lambda)$  expressed in terms of chains of twosided prime ideals of  $\Lambda$ , and of course  $Kdim(Cl_m) = Kdim(S_m)$ .  $Cl_m$  being free over  $S_m$  we have

$$Kdim(S_m) - depth(\Lambda) = pd_{S_m}(\Lambda) \leq pd(\Lambda) = Kdim(Cl_m) - Kdim(\Lambda)$$

and since  $depth(\Lambda) \leq Kdim(\Lambda)$  we get that  $depth(\Lambda) = Kdim(\Lambda)$ , so  $\Lambda$  is a Cohen-Macaulay  $S_m$ -module. Therefore,

$$Ext_{S_m}^i(\Lambda, S_m) \neq 0 \text{ iff } i = Kdim(S_m) - Kdim(\Lambda)$$

Because  $Cl_m$  is finitely generated and free over  $S_m$  we have

$$Ext_{S_m}^i(\Lambda, S_m) \simeq Ext_{Cl_m}^i(\Lambda, Hom_{S_m}(Cl_m, S_m))$$

by [Rot,Th.11.66]. Since we are in characteristic zero we can identify  $Hom_{S_m}(Cl_m, S_m)$  with

$$\{x \in Cl_m \otimes K_m \mid T(x.Cl_m) \subset S_m\}$$

where  $T$  is the composition of the reduced trace map of the central simple algebra  $Cl_m \otimes K_m$  to its center  $Z_m \otimes K_m$  and the trace map from  $Z \otimes K_m$  to  $K_m$ .

This means, that as a  $Cl_m$ -module,  $Hom_{S_m}(Cl_m, S_m)$  can be identified with a twosided fractional  $Cl_m$ -ideal which is even divisorial since it is clearly reflexive as an  $S_m$ -module. We have seen in II.5 that the normalizing classgroup of  $Cl_m$  is trivial, whence  $Hom_{S_m}(Cl_m, S_m) \simeq Cl_m$  as  $Cl_m$ -modules and therefore

$$Ext_{Cl_m}^i(\Lambda, Cl_m) \simeq Ext_{S_m}^i(\Lambda, S_m)$$

finishing the proof.

A Cohen-Macaulay quotient  $\Lambda$  of  $Cl_m$  with minimal free resolution (1) has a dualizing complex

$$0 \rightarrow F_0^* \rightarrow F_1^* \rightarrow \dots \rightarrow F_t^* \rightarrow \Omega^{nc}(\Lambda) \rightarrow 0$$

with  $F_i^* = Hom_{Cl_m}(F_i, Cl_m)$  and  $\Omega^{nc}(\Lambda) = F_t^*/Im(\phi_t^*)$  which is exact by the foregoing proposition.

**Definition 2.4** : A Cohen-Macaulay quotient  $\Lambda$  of  $Cl_m$  is said to be Gorenstein if the canonical module

$$\Omega^{nc}(\Lambda) = F_t^*/Im(\phi_t^*) = Ext_{Cl_m}^t(\Lambda, Cl_m)$$

is isomorphic to  $\Lambda$  as a  $\Lambda$ -module.

We have seen above that

$$\Omega^{nc}(\Lambda) = Ext_{Cl_m}^t(\Lambda, Cl_m) \simeq Ext_{S_m}^t(\Lambda, S_m) = \Omega(\Lambda)$$

and this last term does not depend on the commutative basering. So, we may take this ring to be the commutative polynomial subring  $S = F[\theta_1, \dots, \theta_d]$  of the center of  $\Lambda$  over which it is a graded free module of finite rank. Over this ring

$$\Omega(\Lambda) = Hom_S(\Lambda, S)$$

Moreover, if the Cohen-Macaulay quotient  $\Lambda$  is a maximal order, one may identify  $\Omega^{nc}(\Lambda)$  with a two-sided divisorial  $\Lambda$ -ideal, by an argument as in the proof of Proposition 2.3.

Of course, it is possible to define these notions for quotients of any positively graded  $F$ -algebra  $\Gamma$  which has finite global dimension,  $\Gamma_0 = F$  and  $\Gamma$  is a graded free module of finite rank over a polynomial subring of its center.

### III.3 : Describing sets for ideals

In the first part of this section we will recall some results of Kutz [Ku] on certain prime ideals of  $S_m$ . Further, we will show that there is only one prime ideal of  $Cl_m$  lying over these prime ideals. In the next section we will show that the corresponding quotients are Cohen-Macaulay.

For any couple of natural numbers  $h = (t, i) \in \mathbb{N} \times \mathbb{N}$  we will denote by  $I_h$  the ideal of  $S_m$  generated by the  $t + 1$  by  $t + 1$  minors of the rightmost  $i$  columns of the symmetric  $m$  by  $m$  matrix  $A = (a_{ij})_{i,j}$ . Further, if  $H$  is a subset of  $\mathbb{N} \times \mathbb{N}$  we will denote

$$I_H = \sum_{h \in H} I_h$$

The set  $H$  is then said to be a describing set for  $I_H$ . Two sets  $H$  and  $H'$  are called equivalent if and only if  $I_H = I_{H'}$ . Hochster and Eagon [H-E] proved that every describing set is equivalent to one of the form

$$H = \{(0, s_0); (1, s_1); \dots; (k, s_k)\}$$

where  $s_0 < s_1 < \dots < s_k = m$ . Such a describing set is said to be a standard description and will be abbreviated by

$$H = \{s_0, \dots, s_k\}$$

Remark that for such a standard description, the corresponding ideal  $I_H$  is generated by

- (0) : the entries of the last  $s_0$  columns of  $A$
- (1) : the 2 by 2 minors of the last  $s_1$  columns of  $A$
- ...
- (k-1) : the  $k$  by  $k$  minors of the last  $s_{k-1}$  columns of  $A$
- (k) : the  $k + 1$  by  $k + 1$  minors of  $A$

Further, for any  $H$  in standard description and  $n \in \mathbb{N}$  we define

$$I_{H,n} = I_H + (a_{1,m-n+1}, \dots, a_{1,m})$$

i.e.  $I_{H,n}$  is the ideal of  $S_m$  generated by  $I_H$  as described above and the entries of the last  $n$  columns of the first row of  $A$ .

Troughout this chapter, we will assume that  $s_0 = 0$  and  $n < m$ . Further, note that in the special case that  $H = \{0, 1, \dots, k-2, m\}$  and  $n = 0$ , then  $I_{H,n}$  is the ideal of  $S_m$  generated by all  $k$  by  $k$  minors of the symmetric  $m$  by  $m$  matrix  $A$ .

We will give another description of the ideals  $I_{H,n}$ , where  $H = \{s_0, \dots, s_k\}$  is a standard description such that  $s_0 = 0$ ,  $s_k = m$ ,  $n < m$  and  $n = s_l < m$  for some  $l$ , which will be more convenient in the sequel.

To simplify notation, let  $w_i = m - i + 1$  for all  $1 \leq i \leq m$ . We will define another symmetric  $m$  by  $m$  matrix  $U$  in the following way. The last column of  $U$  will be filled with the  $m - 1$  indeterminates

$$u_{2,w_1}; \dots; u_{m,w_1}$$

The entry  $(1, m)$  will be zero if  $n > 0$  and an indeterminate  $u_{1,w_1}$  otherwise. Because  $U$  is supposed to be symmetric, this also determines the last row of  $U$ . Now, we can fill the remaining rightmost  $s_1$  columns with entries in

$$F(u_{i,w_1} \mid 1 \leq i \leq m)$$

such that all 2 by 2 minors vanish. For, take an entry  $(i, j)$  in this region, then

$$\det \begin{pmatrix} * & u_{i,m} \\ u_{m,j} & u_{m,m} \end{pmatrix} = 0$$

entails that this entry should be  $u_{i,m} \cdot u_{j,m} \cdot u_{m,m}^{-1}$ . Clearly, the bottom  $s_1$ -rows are then determined by symmetry.

Let us assume by induction that the rightmost  $s_i$  columns and bottom  $s_i$  rows of  $U$  are determined. First of all, we fill the  $w_{s_i+1}$ -th column with indeterminates

$$u_{2,w_{s_i+1}}; \dots; u_{m-s_i,w_{s_i+1}}$$



and the entry  $(1, w_{s_i+1})$  will be zero if  $i \leq h$  or an indeterminate  $u_{1, w_{s_i+1}}$  if  $i > h$ . So, we have filled  $U$  up to but not including the crosshatched area

$$U = \begin{pmatrix} & s_{i+1} & s_i + 1 & s_j + 1 & s_1 + 1 & s_0 + 1 \\ \begin{array}{c} \text{crosshatched} \\ \text{area} \end{array} & & & & & \\ & * & \cdots & * & \cdots & * & * \\ & \vdots & & \vdots & & \vdots & \vdots \\ & * & \cdots & * & \cdots & * & * \\ & \vdots & & \vdots & & \vdots & \vdots \\ & * & \cdots & * & \cdots & * & * \\ & & & & & & \\ & * & \cdots & * & \cdots & * & * \end{pmatrix}$$

Let  $\Delta_i$  be the (symmetric)  $i + 1$  by  $i + 1$  submatrix of  $U$  consisting of the starred entries (which are all indeterminates). So,  $\Delta_i$  is an invertible matrix over the field  $F(u_{i, w_{s_j+1}} \mid i, j)$ . But then one can determine every entry  $(k, l)$  in the crosshatched area by demanding that the  $i + 2$  by  $i + 2$  minor

$$\begin{pmatrix} (k, l) & u_{k, w_{s_i+1}} & \cdots & u_{k, m} \\ u_{w_{s_i+1}, l} & & & \\ \vdots & & \Delta_i & \\ u_{m, l} & & & \end{pmatrix}$$

must vanish. By symmetry, the bottom  $s_{i+1}$  rows are determined. Continuing this process, we finally get a symmetric  $m$  by  $m$  matrix  $U$  over  $F(u_{i, j})$ . Now, define the  $F$ -algebra morphism

$$\phi_{H, n} : S_m \rightarrow F[U] \rightarrow F(u_{i, j})$$

where  $F[U]$  is the sub  $F$ -algebra of  $F(u_{i, j})$  generated by the entries of  $U$  and  $\phi_{H, n}$  is defined by sending an indeterminate  $a_{ij}$  to the entry  $(i, j)$  of  $U$ .

Kutz [Ku] has proved that the kernel of  $\phi_{H, n}$  is equal to the ideal  $I_{H, n}$ , which is by this a prime ideal of height

$$ht(I_{H, n}) = Kdim(S_m) - Kdim F[U]$$

$$\begin{aligned}
&= \binom{m+1}{2} - \text{trdeg}_F(F(u_{i,j})) \\
&= \binom{m+1}{2} - k \cdot m + l + s_1 + s_2 + \cdots + s_{k-1}
\end{aligned}$$

Further, Kutz [Ku,Th.1] proves that the quotients  $S_{H,n} = S_m/I_{H,n}$  are Cohen-Macaulay domains. Using this result one can mimic the proof of Hochster and Eagon [H-E,12] and arrive at the normality of  $S_{H,n}$ .

We will now show that the couple  $(H, n)$  also describes a unique prime ideal of the generic Clifford algebra  $Cl_m$ .

**Theorem 3.1** : If  $(H, n)$  is a couple such that

- (1) :  $H = \{s_0, \dots, s_k\}$  is a standard description.
- (2) :  $s_0 = 0$  and  $n = s_l < m = s_k$  for some  $l$

Then there is only one prime ideal of  $Cl_m$  lying over  $I_{H,n}$ .

**Proof** :

Consider the canonical morphism

$$\phi_{H,n} : S_m \rightarrow F[\mathcal{U}] = S_m/I_{H,n}$$

with notations as above and let  $F(\mathcal{U})$  be the field of fractions of  $F[\mathcal{U}]$ . Then

$$Cl_m \otimes F(\mathcal{U})$$

is the Clifford algebra of the  $m$ -dimensional quadratic vector space over  $F(\mathcal{U})$  with associated symmetric matrix  $\mathcal{U}$ . By a rearrangement of rows and columns of  $\mathcal{U}$ ,  $\mathcal{U}$  is congruent to an  $m$  by  $m$  symmetric matrix over  $F(\mathcal{U})$  of the form

$$u' = \begin{pmatrix} \Delta_{k-1} & A \\ A^T & B \end{pmatrix}$$

where  $\Delta_{k-1}$  is the symmetric  $k$  by  $k$  matrix of indeterminates described above. Since  $\mathcal{U}$  (and therefore  $u'$ ) is of rank  $k$ , we can find an invertible  $m$  by  $m$  matrix

$V$  over  $F(U)$  such that

$$u' = V^r \cdot \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \cdot V$$

where  $Z \in GL_k(F(U))$ . If we write  $V$  in the same block decomposition as  $u'$ , i.e.

$$V = \begin{pmatrix} V_0 & V_1 \\ V_2 & V_3 \end{pmatrix}$$

then we obtain the following equation

$$\Delta_{k-1} = V_0^r \cdot Z \cdot V_0$$

Since both  $\Delta_{k-1}$  and  $Z$  belong to  $GL_k(F(U))$ , so does  $V_0$  and therefore  $\Delta_{k-1}$  and  $Z$  are congruent over  $F(U)$ . Further, since clearly

$$\det(\Delta_{k-1}) \notin (F(U))^*{}^2$$

the same is true for  $\det(Z)$ . Using the results of II.4, this finishes the proof.

Let us denote this prime ideal of the generic Clifford algebra  $Cl_m$  by  $J_{H,n}$ . Note that, since  $I_{H,n}$  is a  $\mathbb{Z}$ -graded prime ideal,  $J_{H,n}$  is also homogeneous. Now, let us consider the other case, i.e.  $H = \{s_0, \dots, s_k\}$  is a standard description such that  $s_0 = 0$ ,  $s_k = m$ ,  $k < m$  and  $n \in \mathbb{N}$  such that  $s_l < n < s_{l+1}$  for some  $0 \leq l < k$ . We define another standard description

$$H' = \{s_0, \dots, s_{l-1}, n, s_{l+1}, \dots, s_k\}$$

and let  $n' = s_{l+1}$ . Kutz [Ku] has proved that in this case

$$I_{H,n} = I_{H',n} \cap I_{H,n'}$$

By theorem 3.1 there is a uniquely determined prime ideal  $J_{H',n}$  (resp.  $J_{H,n'}$ ) of  $Cl_m$  lying over  $I_{H',n}$  (resp.  $I_{H,n'}$ ). We will define

$$J_{H,n} = J_{H',n} \cap J_{H,n'}$$

In the next section we will show that  $Cl_m/J_{H,n}$  is a Cohen-Macaulay quotient whenever  $n = s_l$  or  $n = s_l + 1$  for some  $l$ .

### III.4 : Cohen-Macaulay quotients.

Troughout, we will use the same notation as in the previous section. We will first prove two technical lemmas :

**Lemma 4.1** : If  $H = \{s_0, \dots, s_k\}$  is a standard description such that  $s_0 = 0, s_k = m, k < m$  and  $n \in \mathbb{N}$  such that  $n = s_l + 1$  for some  $0 \leq l < k$ , then

$$J_{H',n'} = J_{H',n} + J_{H,n'}$$

**Proof** :

By definition of  $H', n'$  and the ideals  $I_{H,n}$ , it is clear that

$$I_{H',n'} = I_{H,n'} + I_{H',n}$$

in  $S_m$ . Moreover, from the foregoing section we recall that each term is a prime ideal of  $S_m$ . Further, from the calculation of the height of these ideals it follows that

$$ht(I_{H,n'}) + 1 = ht(I_{H',n}) + 1 = ht(I_{H',n'})$$

Therefore, under the canonical epimorphism

$$\phi_{H',n} : S_m \rightarrow S_{H',n} = S_m/I_{H',n}$$

the images of  $I_{H',n'}$  and  $I_{H,n'}$  coincide and give an height one prime ideal of  $S_{H',n}$  which we will denote by  $p$ . Let us consider, as in the previous section, the description  $F[U]$  of  $S_{H',n}$  (replacing, of course,  $H$  by  $H'$ ). Then, over  $S_{H',n}$  the symmetric  $m$  by  $m$  matrix  $U$  is congruent to one of the form

$$U' = \begin{pmatrix} \Delta_{k-1} & A \\ A^T & B \end{pmatrix}$$

Consider the localization of  $S_{H',n}$  at  $p$ ,  $D$ , which is a discrete valuation ring because  $S_{H',n}$  is normal. Further, the determinant of  $\Delta_{k-1}$  does clearly not

belong to  $p$ , i.e.  $\Delta_{k-1} \in GL_k(D)$ .  $D$  being a principal ideal domain, there is an invertible matrix  $V \in GL_m(D)$  such that

$$U' = V^T \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} V$$

Decomposing  $V$  in the same block form as  $U'$  we get the equation

$$\Delta_{k-1} = V_0^T \cdot Z \cdot V_0$$

and since  $\Delta_{k-1} \in GL_k(D)$ , so are  $V_0$  and  $Z$ . This entails that the localization of  $Cl_{H',n} = Cl_m/J_{H',n}$  at the prime ideal  $p$  is isomorphic to the Clifford algebra over  $D$  of the quadratic free module of rank  $k$  associated to the invertible symmetric  $k$  by  $k$  matrix  $Z$ . Thus, this localization is a  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebra.

Because  $J_{H',n'}$  and  $J_{H,n'}$  are  $\mathbb{Z}$ -homogeneous prime ideals of  $Cl_m$ , their images under the canonical morphism

$$\phi_{H',n} : Cl_m \rightarrow Cl_{H',n}$$

are  $\mathbb{Z}/2\mathbb{Z}$ -graded, so are their localizations at  $p$ . Since all  $\mathbb{Z}/2\mathbb{Z}$ -graded twosided ideals in  $Cl_{H',n} \otimes D$  are extended from  $D$ , it follows that

$$\phi_{H',n}(J_{H,n'})_p = \phi_{H',n}(J_{H',n'})_p$$

giving

$$J_{H',n'} = J_{H,n'} + J_{H',n}$$

finishing the proof.

**Lemma 4.2** : If  $H = \{s_0, \dots, s_k\}$  is a standard description such that  $s_0 = 0$ ,  $s_k = m$ ,  $k < m$  and  $n \in \mathbb{N}$  such that  $n = s_l$  for some  $l < k$ , then

$$J_{H,n+1} = J_{H,n} + Cl_m \cdot a_{1,m-n}$$

**Proof** :

It follows from the description of the ideals  $I_{H,n}$  that

$$I_{H,n+1} = I_{H,n} + S_m \cdot a_{1,m-n}$$

Let us first consider the case that  $n+1 = s_{l+1}$ , then the image of  $I_{H,n+1}$  under the canonical morphism

$$\phi_{H,n} : S_m \rightarrow S_{H,n}$$

is an height one prime ideal  $p$  of  $S_{H,n}$ . Localizing  $Cl_{H,n}$  at this prime ideal  $p$  we obtain, as in the proof of the foregoing lemma, a  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebra over a discrete valuation ring  $D = (S_{H,n})_p$ . Again, using the one-to-one correspondence between twosided  $\mathbb{Z}/2\mathbb{Z}$ -graded ideals of  $(Cl_{H,n})_p$  and ideals of  $D$  we get

$$\phi_{H,n}(J_{H,n+1})_p = (Cl_{H,n})_p \cdot \phi_{H,n}(a_{1,m-n})$$

yielding the desired result.

Now, suppose that  $n+1 < s_{l+1}$ , then using the result of Kutz, see section 3, we have

$$I_{H,n+1} = I_{H',n+1} \cap I_{H,n'}$$

where  $n' = s_{l+1}$  and  $H' = \{s_0, \dots, s_{l-1}, n+1, s_{l+1}, \dots, s_k\}$ . Consider the domain  $D$  which is the intersection of the localization of  $S_{H,n}$  at the height one prime ideals  $\phi_{H,n}(I_{H',n+1})$  and  $\phi_{H,n}(I_{H,n'})$ . Since  $D$  is a normal domain having only two height one prime ideals,  $D$  is a principal ideal domain. We can mimic the appropriate part of the proof of lemma 4.1 to obtain that the corresponding localization of  $Cl_{H,n}$  is a  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebra over  $D$ . Finally, using the one-to-one correspondence of graded ideals with ideals of  $D$ , we get the desired result.

We are now in a position to state and prove the main result of this section :

**Theorem 4.3** : If  $H = \{s_0, \dots, s_k\}$  is a standard description such that  $s_0 = 0, s_k = m, k < m$  and  $n \in \mathbb{N}$  such that either  $n = s_l$  or  $n = s_l + 1$  for

some  $0 \leq l < k$ , then

$$Cl_{H,n} = Cl_m / J_{H,n}$$

is a Cohen-Macaulay quotient of the generic Clifford algebra  $Cl_m$ .

**Proof** :

Let  $\mathcal{L}$  be the set of all ideals  $J_{H,n}$  of  $Cl_m$  such that the couple  $(H, n)$  satisfies the conditions of the theorem.

Suppose that the theorem is false, then we can find, by Noetherian induction, an ideal  $J_{H,n}$  in  $\mathcal{L}$  maximal with respect to the property that  $Cl_{H,n}$  is not a Cohen-Macaulay quotient.

First, we claim that we may reduce to the case that  $n < m$ . For, otherwise

$$Cl_{H,n} \simeq (Cl_{m-1}/J_{H'}) \oplus (Cl_{m-1}/J_{H'}) \cdot x$$

where  $x^2 = 0$  and  $\bar{a}_i \cdot x = -x \cdot \bar{a}_i$  for all  $1 \leq i \leq m-1$ . By induction on  $m$  we may assume that  $Cl_m/J_{H'}$  is Cohen-Macaulay, hence so is  $Cl_{H,n}$ . The only case which might cause some problems is when  $k = m-1$ . But, in this case

$$Cl_{m-1}/J_{H'} \simeq (Cl_{m-1}/J_{H^*}) / (Cl_{m-1}/J_{H^*}) \cdot s_{m-1}(\bar{a}_1, \dots, \bar{a}_{m-1})$$

where  $H^* = \{0, 1, \dots, m-3, m-1\}$ . Using the fact that  $s_{m-1}(\bar{a}_1, \dots, \bar{a}_{m-1})$  is normalizing in  $Cl_{m-1}$  and in  $Cl_{m-1}/J_{H^*}$  we obtain that

$$pd(Cl_{m-1}/J_{H^*}) = pd(Cl_{m-1}/J_{H'}) - 1$$

and induction finishes the proof of our claim.

Now, suppose that  $n = s_l + 1 < m$ . By definition we have

$$J_{H,n} = J_{H',n} \cap J_{H,n'}$$

Maximality of  $J_{H,n}$  entails that both  $Cl_{H',n}$  and  $Cl_{H,n'}$  are Cohen-Macaulay quotients of  $Cl_m$ . Lemma 4.1 learns us that  $J_{H',n'} = J_{H',n} + J_{H,n'}$  and again  $Cl_{H',n'}$  is Cohen-Macaulay by maximality of  $J_{H,n}$ .

The calculation of the Krull dimensions of the quotients  $S_{H',n}, S_{H,n'}$  and  $S_{H',n'}$  shows that

$$pd(Cl_{H',n}) = pd(Cl_{H,n'}) = pd(Cl_{H',n'}) - 1$$

Call this common value  $\alpha$ . In order to obtain a contradiction, we have to verify that  $pd(Cl_{H,n}) \leq \alpha$ . Let  $A$  be the  $Cl_m$ -module

$$A = J_{H',n'}/J_{H',n} \simeq (J_{H',n} + J_{H,n'})/J_{H',n} \simeq J_{H,n'}/(J_{H',n} \cap J_{H,n'}) \simeq J_{H,n'}/J_{H,n}$$

We have the exact sequence of (left)  $Cl_m$ -modules

$$0 \rightarrow A \rightarrow Cl_{H',n} \rightarrow Cl_{H',n'} \rightarrow 0$$

so we have for every left  $Cl_m$ -module  $B$  the long exact sequence

$$\dots \rightarrow Ext^{\alpha+1}(Cl_{H',n}; B) \rightarrow Ext^{\alpha+1}(A; B) \rightarrow Ext^{\alpha+2}(Cl_{H',n'}; B) \rightarrow \dots$$

Because  $\alpha = pd(Cl_{H',n}) = pd(Cl_{H',n'}) - 1$  we get that  $Ext^{\alpha+1}(A; B) = 0$  for every left  $Cl_m$ -module  $B$ , i.e.  $pd(A) \leq \alpha$ .

Now, consider the exact sequence of left  $Cl_m$ -modules

$$0 \rightarrow A \rightarrow Cl_{H,n} \rightarrow Cl_{H,n'} \rightarrow 0$$

then  $pd(Cl_{H,n}) \leq \max\{pd(A), pd(Cl_{H,n'})\} = \alpha$ . This shows that

$$pd(Cl_{H,n}) = Kdim(Cl_m) - Kdim(Cl_{H,n}) = \alpha$$

a contradiction. Finally, consider the case that  $n = s_l < m$ . Then, we know from lemma 4.2 that

$$J_{H,n} + Cl_m \cdot a_{1,m-n} = J_{H,n+1}$$

The element  $a_{1,m-n}$  is homogeneous of degree two in  $Cl_m$  and a non-zero divisor on  $Cl_{H,n}$ . Therefore

$$0 \rightarrow Cl_{H,n} \rightarrow Cl_{H,n} \rightarrow Cl_{H,n+1} \rightarrow 0$$



is exact where  $\mu : Cl_{H,n} \rightarrow Cl_{H,n}$  is multiplication with  $a_{1,m-n}$ . From the maximality of  $J_{H,n}$  we may assume that

$$\alpha = pd(Cl_{H,n+1}) = Kdim(Cl_m) - Kdim(Cl_{H,n+1})$$

Therefore, we obtain the long exact sequence

$$\begin{aligned} Tor_{\alpha+1}(Cl_{H,n}; F) &\rightarrow Tor_{\alpha+1}(Cl_{H,n}; F) \rightarrow Tor_{\alpha+1}(Cl_{H,n+1}; F) \\ &\rightarrow Tor_{\alpha}(Cl_{H,n}; F) \rightarrow Tor_{\alpha}(Cl_{H,n}; F) \end{aligned}$$

Because  $a_{1,m-n}$  is a central (normalizing would be sufficient) element of  $(Cl_m)_+$  and  $F = Cl_m / (Cl_m)_+$ , the first and last morphism in this sequence is the zero map. So we have

$$Tor_{\alpha+1}(Cl_{H,n+1}; F) \simeq Tor_{\alpha}(Cl_{H,n}; F)$$

and therefore  $Tor_{\alpha}(Cl_{H,n}; F) = 0$  because  $pd(Cl_{H,n+1}) = \alpha$ . Consider an exact sequence of  $\mathbb{Z}$ -graded left  $Cl_m$ -modules with gradation preserving morphisms

$$0 \rightarrow Q \rightarrow P_{\alpha-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 = Cl_m \rightarrow Cl_{H,n} \rightarrow 0$$

where every  $P_i$  is graded free of finite rank. Because  $Tor_1(Q; F) = Tor_{\alpha}(Cl_{H,n}; F) = 0$  and  $Q$  is a finitely generated graded left  $Cl_m$ -module, we obtain that  $Q$  is graded free, yielding that

$$pd(Cl_{H,n}) \leq \alpha - 1 = Kdim(Cl_m) - Kdim(Cl_{H,n})$$

a contradiction, finishing the proof of the theorem.

Of prime interest to us is the special case when  $n = 0$  and

$$H = \{0, 1, \dots, k-2, m\}$$

then  $I_{H,n}$  is the ideal of  $S_m$  generated by all  $k$  by  $k$  minors of  $\mathcal{A} = (a_{ij})_{i,j}$  and the quotient  $S_m^k = S_{H,n}$  is the homogeneous coordinate ring of the variety of all symmetric  $m$  by  $m$  matrices with entries in  $F$  and of rank smaller than  $k$ . The rings  $S_m^k$  may be viewed as fixed rings in the following way (see also I.3). Let  $Z = (z_{ij})_{i,j}$  be a  $(k-1)$  by  $m$  matrix of indeterminates. The orthogonal group  $O_{k-1}(F)$  acts on the polynomial ring

$$S = F[z_{ij} : 1 \leq i < k; 1 \leq j \leq m]$$

by sending, for an orthogonal matrix  $A \in O_{k-1}(F)$ , the variable  $z_{ij}$  to the entry  $(i, j)$  of  $A \cdot Z$ . The ring of invariants under this action is equal to  $F[(Z^r \cdot Z)_{i,j}] \subset S$ . Now, consider a map

$$S_m \rightarrow S^{O_{k-1}(F)}$$

by sending the variable  $a_{ij}$  to the entry  $(i, j)$  of  $Z^r \cdot Z$ . The second fundamental theorem of the invariant theory of the orthogonal group tells us that the kernel of this morphism is equal to the ideal generated by all  $k$  by  $k$  minors of  $\mathcal{A} = (a_{ij})_{i,j}$ , i.e.  $S^{O_{k-1}(F)} \simeq S_m^k$ .

From the discussion in II.4 we recall that there is only one prime ideal of the generic Clifford algebra  $Cl_m$  lying over the ideal of  $k$  by  $k$  minors of  $\mathcal{A}$ . Call the corresponding quotient  $Cl_m^k$ , then this  $\mathbb{Z}$ -graded  $F$ -algebra is generic with respect to the Clifford algebras of  $m$ -ary quadratic forms over  $F$  of rank smaller than  $k$ .

From theorem 4.3 we retain that  $Cl_m^k$  is Cohen-Macaulay. The next result is a noncommutative counterpart for the normality of  $S_m^k$ .

**Theorem 4.2** :  $Cl_m^k$  is a maximal order for all  $k \leq m$ .

**Proof** :

Since  $Cl_m^k$  is a Cohen-Macaulay quotient of  $Cl_m$ , it is a free module of finite rank over a polynomial subring of  $S_m^k$ . This entails that  $Cl_m^k$  is a reflexive module over the normal domain  $S_m^k$ . Therefore, it is sufficient to show that the localization of  $Cl_m^k$  at every height one prime ideal of  $S_m^k$  is a maximal order over its center (these may be different from the corresponding localization of  $S_m^k$ ). Further, since  $Cl_m^k$  is  $\mathbb{Z}$ -graded one may restrict attention to  $\mathbb{Z}$ -graded height one prime ideals.

Let  $\Delta_1$  be the  $(k-1)$  by  $(k-1)$  minor of the top left corner of  $\mathcal{A}$  and  $\Delta_2$  the  $(k-1)$  by  $(k-1)$  minor of the bottom right corner of  $\mathcal{A}$ , then  $\Delta_1$  and  $\Delta_2$  are algebraically independent over  $F$ . Therefore, an height one prime ideal  $p$  of  $S_m^k$  cannot contain both  $\Delta_1$  and  $\Delta_2$ .

But this means that the localization of  $Cl_m^k$  at  $p$  is isomorphic to a  $\mathbb{Z}/2\mathbb{Z}$ -graded Azumaya algebra over the discrete valuation ring  $D = (S_m^k)_p$ .

Because of the one-to-one correspondence between ideals of  $D$  and twosided  $\mathbb{Z}/2\mathbb{Z}$ -graded ideals of  $(Cl_m^k)_p$ , we have for every  $\mathbb{Z}$ -graded twosided ideal  $I$  of  $(Cl_m^k)_p$  that

$$(I :_l I) = (I :_r I) = (Cl_m^k)_p$$

finishing the proof.

Further, by a similar argument, no height one prime ideal of  $Cl_m^k$  can contain all  $(k-1)$  by  $(k-1)$  minors of the image of  $\mathcal{A}$  in  $M_m(S_m^k)$ , i.e. the p.i.-degree of the quotient  $Cl_m^k/P$  is equal to that of  $Cl_m^k$  for every height one prime ideal of  $Cl_m^k$ . Therefore,

**Corollary 4.5** :  $Cl_m^k$  is a reflexive Azumaya algebra over its center.

### III.5 : Trace rings revisited.

In this section, we will update our knowledge on the trace ring of  $m$  generic 2 by 2 matrices,  $\mathbb{T}_{m,2}$ , taking into account the foregoing results. In the first chapter we have seen that  $\mathbb{T}_{m,2}$  is the free polynomial ring in the commuting variables  $Tr(X_1), \dots, Tr(X_m)$  over the ring  $\mathbb{T}_m^o$ , which is the  $F$ -subalgebra of  $\mathbb{T}_{m,2}$  generated by the generic trace zero matrices  $X_i^0 = X_i - \frac{1}{2}Tr(X_i)$ .

Now, for any 2 by 2 matrices  $A$  and  $B$  of trace zero we have

$$A.B + B.A = Tr(A.B)$$

This entails that there is an  $F$ -algebra morphism

$$\phi_m : Cl_m \rightarrow \mathbb{T}_m^o$$

by sending  $a_i$  to  $X_i^0$  and  $a_{ij}$  to  $\frac{1}{2}Tr(X_i^0.X_j^0)$ . Further, because  $\mathbb{T}_m^o$  is generated by the  $X_i^0$ ,  $\phi_m$  is an epimorphism.

The center of  $\mathbb{T}_m^o$ ,  $\mathcal{R}_m^0$ , turned out to be the fixed ring of  $F[u_{i1}, u_{i2}, u_{i3} : 1 \leq i \leq m]$  under the canonical action of  $SO_3(F)$ , see I.3. It follows from the exact sequence

$$1 \rightarrow SO_3(F) \rightarrow O_3(F) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

that there is an induced  $\mathbb{Z}/2\mathbb{Z}$ -action on  $\mathcal{R}_m^0$  whose fixed ring is the fixed ring of  $F[u_{i1}, u_{i2}, u_{i3} : 1 \leq i \leq m]$  under action of the full orthogonal group  $O_3(F)$ . This fixed ring is, by the results of I.2, equal to  $S_m^4$ , the coordinate ring of the variety of all symmetric  $m$  by  $m$  matrices with entries in  $F$  of rank smaller or equal to 3. This  $\mathbb{Z}/2\mathbb{Z}$ -action of  $\mathcal{R}_m^0$  can be made explicit in the following way it sends an element  $x$  corresponding to a standard Young tableau of shape  $\sigma = 3^a 2^{2b} 1^{2c}$  to  $(-1)^a.x$ . Therefore, we obtain the following situation

$$\begin{array}{ccc} Cl_m & \rightarrow & \mathbb{T}_m^o \\ & & \uparrow \\ & & \mathcal{R}_m^0 \\ & & \uparrow \\ S_m & \rightarrow & S_m^4 \end{array}$$

i.e. the kernel of  $\phi_m$  is the ideal  $I_{H,n}$  with  $n = 0$  and  $H = \{0, 1, 2, m\}$ . Since  $\mathbb{T}_m^o$  is a prime ring, the kernel of  $\pi_m$  is a prime ideal of  $Cl_m$  lying over this  $I_{H,n}$ , so  $Ker(\pi_m) = J_{H,n}$ .

We can therefore summarize our present knowledge about  $\mathbb{T}_{m,2}$  in the following result.

**Theorem 5.1** :

- (a) :  $\mathbb{T}_{m,2} \simeq Cl_m^A[Tr(X_1), \dots, Tr(X_m)]$
- (b) :  $\mathbb{T}_{m,2}$  is a free module of finite rank over a commutative polynomial subring of its center.
- (c) :  $\mathbb{T}_{m,2}$  is a maximal order and it is a reflexive Azumaya algebra over its center if  $m > 2$ .

**Proof** :

- (a) : Follows from the discussion above and theorem 3.1.
- (b) : Follows from theorem 4.3 and the results of section two.
- (c) : If  $m \geq 4$ , the result follows from theorem 4.4 and corollary 4.5. If  $m = 2$  or 3 it is easy to check that  $Kdim(Cl_m) = Kdim(\mathbb{T}_m^o)$ , for,  $Kdim(Cl_m) = m + \binom{m}{2}$  and  $Kdim(\mathbb{T}_m^o)$  can be computed using the formula of  $Kdim(F[U])$  given before theorem 3.1, i.e.  $Kdim(\mathbb{T}_m^o) = 3m - 3$ . Equality of Krull dimensions entails that  $\pi_2$  and  $\pi_3$  are isomorphisms. The result then follows from theorem II.5.1 and corollary II.5.4.

We will now study the arithmetical theory of  $\mathbb{T}_{m,2}$ , i.e. we will calculate its classgroups. Let us start by proving that the center of  $\mathbb{T}_{m,2}$  is a unique factorization domain. The next result is well known for finite groups  $G$  but since we did not find a reference for the general case, we include the proof.

**Lemma 5.2** : Let  $B$  be a unique factorization domain and  $G$  a group

of automorphisms of  $B$  such that  $H^1(G, U(B)) = 1$ , where  $U(B)$  is the group of units of  $B$ . Let  $A$  be the fixed ring of  $B$  under this action of  $G$ , then

(a) :  $A \subset B$  satisfies no blowing up.

(b) :  $A$  is a unique factorization domain.

**Proof** :

(a) : Let  $P = B.p$  be an height one prime ideal of  $B$  such that  $P \cap A \neq 0$ . Then  $P$  has a finite orbit under  $G$ , say  $\{Bp, Bp_1, \dots, Bp_k\}$ . For, take an element  $0 \neq a \in P \cap A$  and write it as a product of irreducible elements in  $B$ , say

$$a = p^{l_0} \cdot p_1^{l_1} \cdot p_k^{l_k}$$

then for every  $\sigma \in G$  we have that  $\sigma(B.p)$  belongs to the finite set  $\{Bp, Bp_1, \dots, Bp_k\}$ . This shows that there exists a unit  $f_\sigma \in U(B)$  for every  $\sigma \in G$  such that

$$\sigma(p \cdot p_1 \dots p_k) = f_\sigma p \cdot p_1 \dots p_k$$

Now,  $\{f_\sigma : \sigma \in G\}$  is clearly a 1-cocycle so, by assumption, there exists a unit  $\alpha \in U(B)$  such that  $f_\sigma = \sigma(\alpha) \cdot \alpha^{-1}$  for every  $\sigma \in G$ . Replace  $p$  by  $p' = \alpha^{-1} \cdot p$  then  $p' \cdot p_1 \dots p_k \in A$ . Therefore, any nonzero element  $a \in P \cap A$  can be written as

$$a = (p' \cdot p_1 \dots p_k)^{l_0} \cdot q_1^{m_1} \dots q_l^{m_l}$$

i.e.  $a \in (p' \cdot p_1 \dots p_k) \cdot A$  because  $\sigma(q_1^{m_1} \dots q_l^{m_l}) = q_1^{m_1} \dots q_l^{m_l}$  for all  $\sigma \in G$ . Therefore,  $P \cap A = (p' \cdot p_1 \dots p_k) \cdot A$  whence  $A \subset B$  satisfies no blowing up.

(b) : By part (a) we know that there is a natural groupmorphism between the classgroup  $Cl(A) \rightarrow Cl(B) = 0$ . Suppose that  $Q$  is an height one prime ideal of  $A$ , then

$$(BQ)^{**} = B \cdot p_1^{l_1} \dots p_z^{l_z}$$

for irreducible elements  $p_i \in B$ . Clearly,  $Q = Bp_i \cap A$  and as in the proof of (a) one shows that this intersection is principal.

**Theorem 5.3** : The center of the trace ring of  $m$  generic 2 by 2 matrices,  $\mathcal{R}_{m,2}$ , is a Gorenstein unique factorization domain.

**Proof** :

In chapter I we have seen that  $\mathcal{R}_{m,2}$  is the fixed ring of the unique factorization domain

$$\mathcal{P}_{m,2} = F[x_{11}(l), x_{12}(l), x_{21}(l), x_{22}(l) : 1 \leq l \leq m]$$

under an action of  $GL_2(F)$ , or better, of  $PGL_2(F)$ . Further, this group acts trivially on  $F$ , so

$$H^1(PGL_2(F).U(\mathcal{P}_{m,2})) = H^1(PGL_2(F), F^*) = Hom(PGL_2(F), F^*)$$

which is trivial, because  $PGL_2(F)$  is a simple algebraic group. By the foregoing result,  $\mathcal{R}_{m,2}$  is a unique factorization domain. Because  $GL_2(F)$  is a linearly reductive group, it follows from the Hochster-Roberts theorem [H-R] that  $\mathcal{R}_{m,2}$  is a Cohen-Macaulay domain. Finally, since  $\mathcal{R}_{m,2}$  is factorial and affine, it follows from a result of Murthy, see for example Fossums monograph [Fo], that  $\mathcal{R}_{m,2}$  is Gorenstein.

In contrast,  $S_m^4$ , of which ring  $\mathcal{R}_m^0$  is a quadratic extension, is Cohen-Macaulay but it is Gorenstein only for even values of  $m$ , see the work of [J-P-W].

**Theorem 5.4** :

- (1) : For all  $m \geq 3$ ,  $NCl(\mathbb{T}_{m,2}) \simeq CCl(\mathbb{T}_{m,2}) \simeq 0$ .
- (2) : For  $m = 2$ ,  $CCl(\mathbb{T}_{2,2}) \simeq \mathbb{Z}/2\mathbb{Z}$ ,  $NCl(\mathbb{T}_{2,2}) \simeq 0$ .

**Proof** :

(1) : Because  $\mathbb{T}_{m,2}$  is a reflexive Azumaya algebra over its center  $\mathcal{R}_{m,2}$  we know that

$$CCl(\mathbb{T}_{m,2}) \simeq Cl(\mathcal{R}_{m,2}) \simeq 0$$

by the foregoing theorem. The normalizing classgroup, being a factor of the central classgroup, is also trivial.

(2) : In this case we have that  $\mathbb{T}_{2,2} = Cl_2[Tr(X_1), Tr(X_2)]$  and both the normalizing and central classgroup remain unchanged for a base ring which is a domain, the result follows from theorem II.5.3 and theorem II.5.5.

As an immediate consequence of this result, we obtain a new proof of a result due to S. Montgomery [M].

**Theorem 5.5** : Every  $F$ -automorphism of  $\mathbb{K}_{m,2}$ , the ring of  $m$  generic 2 by 2 matrices, which leaves the center elementwise fixed, is the identity.

**Proof** :

By the Skolem-Noether theorem, such an automorphism is given by conjugation with a normalizing element of  $\mathbb{K}_{m,2}$ , say  $h$ .  $\mathbb{K}_{m,2} \subset \mathbb{T}_{m,2}$  being a central extension,  $h$  is also a normalizing element of  $\mathbb{T}_{m,2}$ , i.e.  $\mathbb{T}_{m,2}.h$  is a divisorial ideal.

If  $m \geq 3$ , this ideal must be centrally generated, i.e.  $h = \gamma.c$  for some  $\gamma \in U(\mathbb{T}_{m,2}) = F^*$  and  $c \in \mathbb{K}_{m,2}$ , done.

If  $m = 2$ , the only noncentral normalizing element of  $\mathbb{T}_{2,2}$  is  $X_1X_2 - X_2X_1$ . It is easy to verify that this element does not normalize  $\mathbb{K}_{2,2}$ , done.

Another immediate consequence of theorem 5.3 is

**Theorem 5.6** : The trace ring of  $m$  generic 2 by 2 matrices is Gorenstein.



**Proof** :

Of course, this time the regular  $\mathbb{Z}$ -graded  $F$ -algebra  $\Gamma$  of which ring  $\mathbb{T}_{m,2}$  is an epimorphic image is

$$\Gamma = Cl_m[Tr(X_1), \dots, Tr(X_m)]$$

We have seen in section two that  $\Omega^{nc}(\mathbb{T}_{m,2})$  is a twosided divisorial ideal of  $\mathbb{T}_{m,2}$ . So, the result follows from theorem 5.4.

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**Trace Rings of Generic 2 by 2 Matrices IV :  
Poincaré Series**

by

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## IV.1 : Introduction

In this chapter we aim to study the formal power series

$$\mathcal{P}(\mathbb{T}_{m,2}; t) = \sum_{i=0}^{\infty} \dim_F((\mathbb{T}_{m,2})_i) t^i$$

the so called Poincaré series of the positively graded  $F$ -algebra  $\mathbb{T}_{m,2}$ .

In the second section we recall some standard results on Poincaré series of finite graded modules over positively graded commutative affine  $\bar{F}$ -algebras. As a consequence of them, we get that  $\mathcal{P}(\mathbb{T}_{m,2}; t)$  is a rational function, the pole of which is in  $t = 1$  equal to the Krull dimension of  $\mathbb{T}_{m,2}$ , i.e.  $4m - 3$ .

In section three we give a computer-algorithm to calculate the rational expression of  $\mathcal{P}(\mathbb{T}_{m,2}; t)$  modulo a given prime number  $p$ . This algorithm is based on the description of  $\mathbb{T}_m^o$  in terms of standard Young tableaux of length smaller or equal to 3, cfr. I.5, and on the fact that  $\mathbb{T}_m^o$  is an epimorphic image of the iterated Ore extension  $Cl_m$ .

The computations of these rational expressions gave us some numerical evidence for the conjecture that  $\mathcal{P}(\mathbb{T}_{m,2}; t)$  satisfies the functional equation

$$\mathcal{P}\left(\mathbb{T}_{m,2}; \frac{1}{t}\right) = -t^{4m} \cdot \mathcal{P}(\mathbb{T}_{m,2}; t)$$

The first proof of this equation is based on some 60 years old results of H. Weyl and I. Schur on the Poincaré series of the ring of invariants of  $SO_3(F)$ .

The second proof we give, uses only results proved in these notes. It turns out that the underlying motivation for the functional equation is that the trace ring of  $m$  generic 2 by 2 matrices is a Gorenstein ring.

The main result of section four states that  $\mathbb{T}_{m,2}$  has finite global dimension if and only if  $m \leq 3$ . That  $\mathbb{T}_{2,2}$  has finite global dimension was known by [S-S] or [L-V]. The fact that  $\mathbb{T}_{3,2}$  has finite global dimension has some nice  $K$ -theoretical consequences as S.C. Coutinho [Co] shows.

In the final section we scetch the contents of a letter from C. Procesi [P2]. In it he gives an elegant proof of the functional equation, using the Gorensteinness of the Grassmann variety of lines in  $\mathbb{P}^m$  and a sort of Pieri's formula. Moreover, his approach makes it possible to compute the rational expression of  $\mathcal{P}(\mathbb{T}_{m,2}; t)$  by combinatorial methods.

## IV.2 : Poincaré series

Throughout, let  $R$  be a positively graded  $F$ -algebra

$$R = \bigoplus_{i=0}^{\infty} R_i$$

which is commutative and affine, i.e. finitely generated as an  $F$ -algebra by homogeneous elements. Moreover, we will assume that  $R_0 = F$ . A graded module

$$M = \bigoplus_{i=0}^{\infty} M_i$$

over  $R$  will always be positively graded and finitely generated. Each homogeneous component  $M_i$  is then a finite dimensional vector space over  $F$ . In the special (geometrical) case where  $R$  is generated by homogeneous elements of degree one,  $M$  has a Hilbert polynomial

$$\mathcal{H}_M(x) = \frac{e(M)}{(n-1)!} \cdot x^{n-1} + \alpha_2 \cdot x^{n-2} + \dots + \alpha_n$$

such that for sufficiently large  $i$

$$\mathcal{H}_M(i) = \dim_F(M_i)$$

For a nonzero  $R$ -module  $M$ ,  $d(M) = n$  and  $e(M)$  are nonnegative integers, well known and important invariants of  $M$ . We call  $d(M)$  the dimension of  $M$  and  $e(M)$  the multiplicity of  $M$ .

Unfortunately, if  $R$  is not generated by elements of degree one, a graded  $R$ -module  $M$  does not necessarily have a Hilbert polynomial. In the general case, one would like to have a substitute for the Hilbert polynomial. It turns out that the so called Poincaré series

$$\mathcal{P}(M; t) = \sum_{i=0}^{\infty} \dim_F(M_i) \cdot t^i$$

of the module is a good substitute. In the classical situation, where the Hilbert polynomial exists, the relation between  $\mathcal{H}_M(x)$  and  $\mathcal{P}(M; t)$  is such that  $\mathcal{H}_M(x)$  is of

degree at most  $n - 1$  if and only if  $(1 - t)^n \cdot \mathcal{P}(M; t)$  is a polynomial in  $t$ . Moreover, if  $\mathcal{H}_M(x)$  is of degree precisely  $n - 1$ , then the multiplicity  $e(M)$  of  $M$  is the value of  $(1 - t)^n \cdot \mathcal{P}(M; t)$  for  $t = 1$ . Thus in the classical case, both dimension and multiplicity are found easily from the Poincaré series. In [Sm], W. Smoke showed how the Poincaré series gives rise to corresponding invariants in the general case.

Let  $\mathcal{G}(R)$  be the category of (finitely generated, positively) graded  $R$ -modules and homogeneous  $R$ -module homomorphisms of degree zero. To this category, one can associate its Grothendieck group  $K_0^g(R)$ . In particular,  $\mathcal{G}(F)$  is the category of finite dimensional grade vector spaces over  $F$ , whence  $K_0^g(F)$  is simply the polynomial ring  $\mathbb{Z}[t]$ .

If  $M$  belongs to  $\mathcal{G}(R)$ , then  $Tor_i^R(F, M)$  belongs to  $\mathcal{G}(F)$  and one can define the generalized Euler characteristic

$$\chi_R(M) = \sum_{i=0}^{\infty} [Tor_i^R(F, M)]$$

which turns out to be a well defined element of the formal power series ring  $\mathbb{Z}[[t]]$  since the  $j$ -th component of the graded vector space  $Tor_i^R(F, M)$  is zero for  $j < i$ , cfr. [Sm]. He calls  $\chi_R(M)$  the generalized multiplicity of  $M$ . It follows from the universal property of the Grothendieck group that this  $\chi_R$  defines a group homomorphism

$$\chi_R : K_0^g(R) \rightarrow \mathbb{Z}[[t]]$$

and the Euler-Poincaré principle gives the relation

$$\chi_R(M) = \chi_R(F) \cdot \mathcal{P}(M; t)$$

between the Poincaré series and the generalized multiplicity. Consider the special case of a polynomial ring  $S = F[x_1, \dots, x_n]$  with  $deg(x_i) = d_i$ , then for all  $M$  in  $\mathcal{G}(S)$ ,  $Tor^S(F, M)$  is a finite complex, i.e.  $\chi_S$  has values in  $\mathbb{Z}[t]$ . Assigning to every graded  $F$ -vector space its total dimension gives a ring morphism

$$dim : K_0^g(F) \simeq \mathbb{Z}[t] \rightarrow \mathbb{Z}$$

and composing  $\chi_S$  with  $\dim$  gives a homomorphism

$$e_S : K_0^g(S) \rightarrow \mathbb{Z}$$

where

$$e_S(M) = \sum_{i=0}^{\infty} \dim_F(\text{Tor}_i^S(F, M))$$

is the multiplicity of the  $S$ -module  $M$ . The existence of the Koszul complex [Sm, Prop.4.1] shows that

$$\chi_S(F) = \prod_{i=1}^n (1 - t^{d_i})$$

so the formula of the generalized multiplicity becomes

$$\chi_S(M) = \prod_{i=1}^n (1 - t^{d_i}) \cdot \mathcal{P}(M; t)$$

and both sides are polynomials in  $t$ . Every positively graded affine  $F$ -algebra  $R$  is the quotient of a graded polynomial algebra  $S$ , and we may regard an  $R$ -module  $M$  as an  $S$ -module. The Poincaré series of  $M$  is independent of the  $F$ -algebra, so

$$\prod_{i=1}^n (1 - t^{d_i}) \cdot \mathcal{P}(M; t) \quad (*)$$

is a polynomial in  $t$ , entailing that  $\mathcal{P}(M; t)$  and  $\chi_R(M)$  are rational functions.

Now, let us turn attention to the dimension of a nonzero module  $M$  in  $\mathcal{G}(R)$ . Let  $d(M)$  be the least  $n$  such that there exist positive integers  $d_1, \dots, d_n$  such that the expression (\*) is a polynomial in  $t$ . Let  $s(M)$  be the least  $n$  such that there are homogeneous elements  $y_1, \dots, y_n$  of  $R$  such that  $M$  is finitely generated over  $F[y_1, \dots, y_n]$ . Finally, let  $K \dim(M)$  be the supremum of the integers  $n$  such that there exists a strictly increasing chain

$$p_1 \subset p_2 \subset \dots \subset p_n$$

of graded prime ideals in  $R$  containing the annihilator of  $M$ . Then, [Sm, Th.5.5] entails that

$$d(M) = s(M) = K \dim(M)$$



which justifies naming the common value the dimension of  $M$ .

In the rest of this chapter we are interested in the following situation.  $R$  will be the positively graded  $F$ -algebra  $\mathcal{R}_{m,2}$ , the center of the trace ring of  $m$  generic 2 by 2 matrices. Note that  $\mathcal{R}_{m,2}$  is affine by I.4. The graded polynomial ring  $S$  will be in this situation

$$S = F[a_{ij} : 1 \leq i \leq j \leq m][b_1, \dots, b_m]$$

where  $\deg(a_{ij}) = 2$  and  $\deg(b_i) = 1$ . Because  $\mathcal{R}_{m,2}$  is a Cohen-Macaulay domain which is Gorenstein by the results of III.5, we know from [St] that the Poincaré series of  $\mathcal{R}_{m,2}$  satisfies the functional equation

$$\mathcal{P}(\mathcal{R}_{m,2}; \frac{1}{t}) = -t^\alpha \cdot \mathcal{P}(\mathcal{R}_{m,2}; t)$$

for some  $\alpha \in \mathbb{Z}$ , since  $K \dim(\mathcal{R}_{m,2}) = 4m - 3$  is odd. Our main aim will be to deduce and explain a similar functional equation for the Poincaré series of  $\mathbb{T}_{m,2}$ , the trace ring of  $m$  generic 2 by 2 matrices, which is a finitely generated and positively graded module over  $\mathcal{R}_{m,2}$  by the description given in I.5.

Further, we will give the rational expression of the Poincaré series of  $\mathbb{T}_{m,2}$ , both in a multigradation (section three) and as a function in  $t$  (section five).

### IV.3 : The functional equation

In this section we will study the Poincaré series of the trace ring of  $m$  generic  $2$  by  $2$  matrices,  $tt_{m,2}$ . Because

$$\mathbb{T}_{m,2} \simeq \mathbb{T}_m^o[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$$

and the degree of each of the  $\text{Tr}(X_j)$  is one, one has the equality

$$\mathcal{P}(\mathbb{T}_{m,2}; t) = \frac{1}{(1-t)^m} \cdot \mathcal{P}(\mathbb{T}_m^o; t)$$

Further, it is easy to see that every graded quotient of  $Cl_m$  has a rational Poincaré series. For, take a resolution

$$0 \rightarrow F_r \rightarrow \dots \rightarrow F_0 = Cl_m \rightarrow Cl_m/I \rightarrow 0$$

where every  $F_i$  is a graded free left  $Cl_m$ -module of finite rank and all morphisms are gradation preserving. Then,

$$\mathcal{P}(Cl_m/I; t) = \sum_{i=0}^r \mathcal{P}(F_i; t)$$

and if  $F_i$  is graded free with basis  $\{f_{i1}, \dots, f_{ir_i}\}$  such that  $\text{deg}(f_{ij}) = \alpha_{ij}$ , then

$$\mathcal{P}(F_i; t) = (t^{\alpha_{i1}} + \dots + t^{\alpha_{ir_i}}) \cdot \mathcal{P}(Cl_m; t)$$

So, we get that

$$\mathcal{P}(Cl_m/I; t) = f(t) \cdot \mathcal{P}(Cl_m; t)$$

for some polynomial  $f(t) \in \mathbb{Z}[t]$ . Finally, the Poincaré series of the generic Clifford algebra is

$$\mathcal{P}(Cl_m; t) = \frac{1}{(1-t)^m \cdot (1-t^2)^{\frac{m(m-1)}{2}}}$$

This follows from the description of  $Cl_m$  as an iterated Öre extension, cfr. theorem II.3.1, which entails that  $Cl_m$  is isomorphic to

$$F[a_{ij} : 1 \leq i < j \leq m; a_i : 1 \leq i \leq m]$$

as a graded  $F$ -vectorspace with  $\deg(a_{ij}) = 2$  and  $\deg(a_i) = 1$ . In particular,

$$\mathcal{P}(\Pi_m^o; t) = \frac{f_m(t)}{(1-t)^m \cdot (1-t^2)^{\frac{m(m-1)}{2}}}$$

We will briefly scetch an algorithm to compute the polynomial  $f_m(t) \in \mathbb{Z}[t]$  explicitly. A Pascal program for it is listed in the appendix.

In 1.5 we have seen that there is a one-to-one correspondence between an  $F$ -vectorspace basis for  $\Pi_m^o$  and standard Young tableaux of shape  $\sigma = 3^a 2^b 1^c$  for all  $a, b, c \in \mathbb{N}$ . Further, the degree of an element corresponding to a Young tableau  $\sigma = 3^a 2^b 1^c$  is equal to  $3a + 2b + c$ , i.e. the number of cells of  $\sigma$ . Therefore

$$\mathcal{P}(\Pi_m^o; t) = \sum_{a,b,c \in \mathbb{N}} \mathcal{L}_{3^a 2^b 1^c} \cdot t^{3a+2b+c}$$

where  $\mathcal{L}_{3^a 2^b 1^c}$  is the number of standard Young tableaux of shape  $\sigma = 3^a 2^b 1^c$  filled with indices from 1 to  $m$ . This number was computed by H. Weyl for  $m \geq 4$

$$\mathcal{L}_{3^a 2^b 1^c} = (1+b) \cdot (1+c) \cdot \left(1 + \frac{b+c}{2}\right) \cdot \prod_{j=3}^{m-1} \left(1 + \frac{a+b+c}{j}\right) \cdot \prod_{j=2}^{m-2} \left(1 + \frac{a+b}{j}\right) \cdot \prod_{j=1}^{m-3} \left(1 + \frac{a}{j}\right)$$

Using this formula, it is possible to calculate the power series expansion of  $\mathcal{P}(\Pi_m^o; t)$ . For computational reasons, the program listed in the appendix calculates the coefficients modulo a given prime number  $p$ .

Further, it is easy to calculate the power series expansion of  $\mathcal{P}(Cl_m; t)$  and from a comparison of these two power series one can compute the coefficients of the polynomial  $f_m(t)$ , modulo a given prime number  $p$ . Let us give a few examples

$$f_4(t) = 1 + t^4$$

$$f_5(t) = 1 - 5t^4 + 5t^6 - t^{10}$$

$$f_6(t) = 1 - 15t^4 + 35t^6 + 80t^8 + 80t^{10} + 35t^{12} - 15t^{14} + t^{18}$$

$$f_7(t) = 1 + 66t^4 + 39t^6 + 13t^8 - 11t^{10} + 28t^{12} + 31t^{14} + 28t^{16} - 11t^{18}$$

$$- 88t^{20} + 39t^{22} + 66t^{24} + t^{28}$$

$$f_8(t) = 1 + 31t^4 + 16t^6 + 65t^8 + 51t^{10} + 51t^{12} - 83t^{14} - 68t^{16} - 26t^{18}$$

$$+ 26t^{22} + 68t^{24} - 18t^{26} + 50t^{28} + 30t^{30} + 36t^{32} + 85t^{34} - 31t^{36} - t^{40}$$

$$f_9(t) = 1 - 25t^4 + 40t^6 - 31t^8 - 51t^{10} + 83t^{12} + 26t^{14} + 20t^{16} + 58t^{18}$$

$$+ 19t^{20} - 37t^{22} + 76t^{24} + 77t^{26} + 24t^{28} + 25t^{30} + 37t^{32} + 82t^{34} - 58t^{36}$$

$$+ 81t^{38} - 26t^{40} + 18t^{42} + 51t^{44} + 31t^{46} + 61t^{48} + 25t^{50} - t^{54}$$

$$f_{10}(t) = 1 - 8t^4 + 88t^6 + 8t^8 - 26t^{10} + 16t^{12} + 51t^{14} + 34t^{16} + 50t^{18}$$

$$- 2t^{20} + 11t^{22} - 3t^{24} + 25t^{26} + 86t^{28} + 64t^{30} + 41t^{32} + 69t^{34} + 69t^{36}$$

$$+ 41t^{38} + 64t^{40} + 86t^{42} + 25t^{44} + 98t^{46} + 11t^{48} + 99t^{50} + 50t^{52} + 34t^{54}$$

$$+ 51t^{56} + 16t^{58} + 75t^{60} + 8t^{62} - 13t^{64} - 8t^{66} + t^{70}$$

All coefficients are computed modulo 101. Combining these numerical calculations with the fact that  $\mathcal{P}(\mathbb{T}_{m,2}; t) = (1-t)^{-m} \cdot \mathcal{P}(\mathbb{T}_m^o; t)$  one can raise the question whether the Poincaré series of  $\mathbb{T}_{m,2}$  always satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,2}; \frac{1}{t}) = -t^{4m} \cdot \mathcal{P}(\mathbb{T}_{m,2}; t) \quad (FE)$$

In this chapter we will give three proofs of this fact. The first one is based on some 60 years old results of H. Weyl and I. Schur which enable us to compute the rational expression of  $\mathcal{P}(\mathbb{T}_{m,2}; t)$ . The second follows from the fact that  $\mathbb{T}_{m,2}$  is Cohen-Macaulay and has a trivial normalizing classgroup. The last proof, due to C. Procesi, is based on the Gorensteinness of some Grassmann varieties and a sort of Pieri's formula. Procesi's proof will be outlined in the last section.

First, we will give a rational expression of the Poincaré series of  $\mathbb{T}_{m,2}$  in a multi-gradation, i.e. by giving each generic matrix-entry  $x_{ij}(l)$  degree  $(0, \dots, 0, 1, 0, \dots, 0)$  with 1 on spot  $l$ . Then,  $\mathbb{T}_{m,2}$  is an  $\mathbb{N}^{(m)}$ -graded  $F$ -algebra and the Poincaré series

is defined to be

$$\mathcal{P}(\mathbb{T}_{m,2}; t_1, \dots, t_m) = \sum_{(i_1, \dots, i_m)} \dim_F((\mathbb{T}_{m,2})_{(i_1, \dots, i_m)}) \cdot t_1^{i_1} \dots t_m^{i_m}$$

In I.5 we have seen that the map sending an element  $f \in \mathbb{T}_{m,2}$  to  $\text{Tr}(f \cdot X_{m+1})$  defines a monomorphism from  $\mathbb{T}_{m,2}$  onto the subspace of  $\mathcal{R}_{m+1,2}$  consisting of all elements which are homogeneous of degree one in  $X_{m+1}$ , i.e.

$$\sum_{(i_1, \dots, i_m)} (\mathcal{R}_{m+1,2})_{(i_1, \dots, i_m, 1)}$$

Translating this fact to multivalued Poincaré series we obtain that  $\mathcal{P}(\mathbb{T}_{m,2}; t_1, \dots, t_m)$  is the coefficient of  $t_{m+1}$  in the power series expansion of  $\mathcal{P}(\mathcal{R}_{m+1,2}; t_1, \dots, t_m, t_{m+1})$ . Or,

$$\mathcal{P}(\mathbb{T}_{m,2}; t_1, \dots, t_m) = \frac{\partial}{\partial t_{m+1}} \mathcal{P}(\mathcal{R}_{m+1,2}; t_1, \dots, t_{m+1}) \Big|_{t_{m+1}=0}$$

Further, in I.4 we have seen that

$$\mathcal{R}_{m+1,2} \simeq \mathcal{R}_{m+1}^o [\text{Tr}(X_1), \dots, \text{Tr}(X_{m+1})]$$

where  $\mathcal{R}_{m+1}^o$  is the fixed ring of  $F[u_{i1}, u_{i2}, u_{i3} : 1 \leq i \leq m+1]$  under canonical action of  $SO_3(F)$ . The computation of the Poincaré series in the multigradation of the ring  $\mathcal{R}_{m+1}^o$  was carried out by H. Weyl [W, p.17] and I. Schur [S]. They obtained the rational expression

$$\mathcal{P}(\mathcal{R}_{m+1}^o; t_1, \dots, t_{m+1}) = \frac{[1 + t_1^{2m-1}, \dots, t_1^{m-2} + t_1^{m+1}, t_1^{m-1}, t_1^m]}{\prod_{i < k}^{m+1} (t_k - t_i) \cdot \prod_{i \leq k}^{m+1} (1 - t_i \cdot t_k)}$$

where the numerator in this expression denotes the determinant of the following  $m+1$  by  $m+1$  - matrix

$$\begin{pmatrix} 1 + t_1^{2m-1} & 1 + t_2^{2m-1} & \dots & 1 + t_{m+1}^{2m-1} \\ t_1 + t_1^{2m-2} & t_2 + t_2^{2m-2} & \dots & t_{m+1} + t_{m+1}^{2m-2} \\ \vdots & \vdots & & \vdots \\ t_1^{m-2} + t_1^{m+1} & t_2^{m-2} + t_2^{m+1} & \dots & t_{m+1}^{m-2} + t_{m+1}^{m+1} \\ t_1^{m-1} & t_2^{m-1} & \dots & t_{m+1}^{m-1} \\ t_1^m & t_2^m & \dots & t_{m+1}^m \end{pmatrix}$$

Combining these facts we get that  $\mathcal{P}(\mathbb{T}_{m,2}; t_1, \dots, t_m)$  is equal to

$$\frac{\partial}{\partial t_{m+1}} \frac{[1 + t^{2m-1}, \dots, t^{m-2} + t^{m+1}, t^{m-1}, t^m]}{\prod_{j=1}^{m+1} (1 - t_j) \cdot \prod_{i < k}^{m+1} (t_k - t_i) \cdot \prod_{i \leq k}^{m+1} (1 - t_i t_k)} \Big|_{t_{m+1}=0}$$

Calculating the numerator of this expression gives

$$\begin{aligned} & \prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot (-1)^m \cdot \prod_{j=1}^m t_j \cdot \prod_{i \leq k}^m (1 - t_i t_k) \cdot M_1 - \\ & \left\{ - \prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot (-1)^m \cdot \prod_{j=1}^m t_j \cdot \prod_{i \leq k}^m (1 - t_i t_k) \right. \\ & + \prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot \left( \sum_{j=1}^m (-1)^{m-1} \cdot t_1 \dots \check{t}_j \dots t_m \right) \cdot \prod_{i \leq k}^m (1 - t_i t_k) \\ & \left. + \prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot (-1)^m \cdot \prod_{j=1}^m t_j \cdot \prod_{i \leq k}^m (1 - t_i t_k) \cdot \left( \sum_{j=1}^m -t_j \right) \right\} \cdot M_2 \end{aligned}$$

where  $M_1$  and  $M_2$  are defined to be

$$M_1 = \det \begin{pmatrix} 1 + t_1^{2m-1} & \dots & 1 + t_m^{2m-1} & 0 \\ \vdots & & \vdots & \vdots \\ t_1^{m-2} + t_1^{m+1} & \dots & t_m^{m-2} + t_m^{m+1} & 0 \\ t_1^{m-1} & \dots & t_m^{m-1} & 0 \\ t_1^m & \dots & t_m^m & 0 \end{pmatrix} = (-1)^{m+2} \cdot \Delta_1$$

$$M_2 = \det \begin{pmatrix} 1 + t_1^{2m-1} & \dots & 1 + t_m^{2m-1} & 1 \\ \vdots & & \vdots & \vdots \\ t_1^{m-2} + t_1^{m+1} & \dots & t_m^{m-2} + t_m^{m+1} & 0 \\ t_1^{m-1} & \dots & t_m^{m-1} & 0 \\ t_1^m & \dots & t_m^m & 0 \end{pmatrix} = (-1)^{m+1} \cdot \Delta_2$$

where  $\Delta_1$  and  $\Delta_2$  are the obvious  $m$  by  $m$  minors. Therefore, the numerator is equal to

$$\prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot \prod_{i \leq k}^m (1 - t_i t_k) \cdot [e_m \cdot \Delta_1 - (e_m + e_{m-1} + e_1 \cdot e_m) \cdot \Delta_2]$$

where  $e_i$  denotes the  $i$ -th elementary symmetric function in  $m$  variables. This concludes the proof of

**Theorem 3.1** : The Poincaré series of the trace ring of  $m$  generic 2 by 2 matrices has the following rational expression

$$\mathcal{P}(\mathbb{T}_{m,2}; t_1, \dots, t_m) = \frac{e_m \cdot \Delta_1 - (e_m + e_1 \cdot e_m + e_{m-1}) \cdot \Delta_2}{e_m^2 \cdot \prod_{j=1}^m (1 - t_j) \cdot \prod_{i < k}^m (t_k - t_i) \cdot \prod_{i \leq k}^m (1 - t_i t_k)}$$

This result extends the computation of E. Formanek [F] in the case that  $m = 4$ . For, then one gets

$$\mathcal{P}(\mathbb{T}_{4,2}; t_1, t_2, t_3, t_4) = \frac{1 - t_1 t_2 t_3 t_4}{\prod_{i=1}^4 (1 - t_i)^2 \cdot \prod_{i < j}^4 (1 - t_i t_j)}$$

Moreover, it is now quite easy to prove the functional equation

**Theorem 3.2** : The Poincaré series of the trace ring of  $m$  generic 2 by 2 matrices satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,2}; \frac{1}{t}) = -t^{4m} \cdot \mathcal{P}(\mathbb{T}_{m,2}; t)$$

**Proof** :

An easy verification shows that

$$e_m^{2m-1} \cdot \Delta_1(\frac{1}{t_1}, \dots, \frac{1}{t_m}) = -\Delta_1(t_1, \dots, t_m)$$

$$e_m^{2m-1} \cdot \Delta_2(\frac{1}{t_1}, \dots, \frac{1}{t_m}) = -\Delta_2(t_1, \dots, t_m)$$

$$e_m^2 \cdot ((e_m + e_1 \cdot e_m + e_{m-1}) (\frac{1}{t_1}, \dots, \frac{1}{t_m})) = e_m + e_1 \cdot e_m + e_{m-1}$$

Substitution of this information in the rational expression yields

$$\begin{aligned} \mathcal{P}(\mathbb{T}_{m,2}; \frac{1}{t_1}, \dots, \frac{1}{t_m}) = \\ \frac{-e_m^{-2m-1} \cdot (e_m \cdot \Delta_1 - (e_m + e_1 \cdot e_m + e_{m-1}) \cdot \Delta_2)}{e_m^{-2m-5} \cdot (\prod_{j=1}^m (t_j - 1) \cdot \prod_{i < k}^m (t_i - t_k) \cdot \prod_{i \leq k}^m (t_i t_k - 1))} = \end{aligned}$$

$$-e_m^4 \cdot (-1)^{m^2+m} \cdot \mathcal{P}(\mathbb{T}_m^o; t_1, \dots, t_m)$$

Finally, specializing  $t_1 = t_2 = \dots = t_m = t$  we get the desired result.

Of course, the hard part of this proof is contained in the computation of the rational expression of the Poincaré series of the ring  $\mathcal{R}_{m+1}^o$ . Similarly, Procesi's proof of the functional equation, given in section five, relies heavily on some fairly deep results on Grassmann varieties.

We will now present a proof of the functional equation which uses only results proved in these notes. Moreover, it clarifies the underlying reason for the Poincaré series to satisfy a functional equation of this form : the Gorensteinness of  $\mathbb{T}_{m,2}$ , cfr. III.5.

**Theorem 3.3** : The Poincaré series of the trace ring of  $m$  generic 2 by 2 matrices satisfies the functional equation

$$\mathcal{P}(\mathbb{T}_{m,2}; \frac{1}{t}) = -t^\alpha \cdot \mathcal{P}(\mathbb{T}_{m,2}; t)$$

for some  $\alpha \in \mathbb{Z}$ .

**Proof** :

Take a finite free resolution of  $\mathbb{T}_m^o$  as a (left)  $Cl_m$ -module

$$0 \rightarrow F_h \rightarrow \dots \rightarrow F_1 \rightarrow F_0 = Cl_m \rightarrow \mathbb{T}_m^o \rightarrow 0 \quad (1)$$

where each  $F_i$  is graded free with a basis of homogeneous elements  $x_{i1}, \dots, x_{i\beta_i}$  with  $deg(x_{ij}) = g_{ij}$ , and all morphisms are gradation preserving, i.e.

$$\mathcal{P}(F_i; t) = \left( \sum_{j=1}^{\beta_i} t^{g_{ij}} \right) \cdot \mathcal{P}(Cl_m; t) \quad (2)$$

and we have the equation

$$\mathcal{P}(\mathbb{T}_m^o; t) = \mathcal{P}(F_0; t) - \mathcal{P}(F_1; t) + \dots + (-1)^h \cdot \mathcal{P}(F_h; t) \quad (3)$$



To the sequence (1) we can apply the functor  $\text{Hom}_{Cl_m}(-, Cl_m) = (-)^*$ . Because  $\mathbb{T}_m^o$  is a Cohen-Macaulay quotient of  $Cl_m$  by the results of III.5. So, we have

$$\text{Ext}_{Cl_m}^i(\mathbb{T}_m^o, Cl_m) = 0 \text{ for } i \neq \frac{(m-2)(m-3)}{2}$$

Therefore, we obtain the exact sequence

$$0 \rightarrow F_0^* \rightarrow F_1^* \rightarrow \cdots \rightarrow F_h^* \rightarrow \Omega^{nc}(\mathbb{T}_m^o) \rightarrow 0 \quad (4)$$

where  $\Omega^{nc}(\mathbb{T}_m^o) \simeq \text{Ext}_{Cl_m}^{\frac{(m-2)(m-3)}{2}}(\mathbb{T}_m^o, Cl_m)$ . Let  $x_{i1}^*, \dots, x_{i\beta_i}^*$  be the basis of  $F_i^*$  dual to the basis  $x_{i1}, \dots, x_{i\beta_i}$  of  $F_i$ . If we define  $\text{deg}(x_{ij}^*) = -g_{ij}$ , then the homomorphisms of the sequence (4) are all degree preserving. As before, we have

$$\mathcal{P}(F_i^*; t) = \left( \sum_{j=1}^{\beta_i} t^{-g_{ij}} \right) \cdot \mathcal{P}(Cl_m; t) \quad (5)$$

and from (4) we obtain

$$\mathcal{P}(\Omega^{nc}(\mathbb{T}_m^o); t) = \mathcal{P}(F_h^*; t) - \mathcal{P}(F_{h-1}^*; t) + \cdots + (-1)^h \mathcal{P}(F_0^*; t) \quad (6)$$

Substituting (2) in (3) and (5) in (6) and using the equation

$$\mathcal{P}(Cl_m; \frac{1}{t}) = (-1)^{\frac{m(m+1)}{2}} \cdot t^{m^2} \cdot \mathcal{P}(Cl_m; t) \quad (7)$$

we get that

$$\begin{aligned} \mathcal{P}(\Omega^{nc}(\mathbb{T}_m^o); \frac{1}{t}) &= \mathcal{P}(Cl_m; \frac{1}{t}) \cdot \left( \sum_{i=0}^h (-1)^{h-i} \cdot \sum_{j=1}^{\beta_i} t^{g_{ij}} \right) \\ &= (-1)^{\frac{m(m+1)}{2}} \cdot t^{m^2} \cdot \mathcal{P}(Cl_m; t) \cdot \left( \sum_{i=0}^h (-1)^{h-i} \sum_{j=1}^{\beta_i} t^{g_{ij}} \right) \\ &= (-1)^{\frac{m(m+1)}{2} - h} \cdot t^{m^2} \cdot \mathcal{P}(\mathbb{T}_m^o; t) \end{aligned}$$

Because  $\mathbb{T}_m^o$  is a Cohen-Macaulay quotient of  $Cl_m$ ,  $h = K \dim(Cl_m) - K \dim(\mathbb{T}_m^o)$  and we get

$$\mathcal{P}(\Omega^{nc}(\mathbb{T}_m^o); \frac{1}{t}) = (-1)^{3m-3} \cdot t^{m^2} \cdot \mathcal{P}(\mathbb{T}_m^o; t)$$

Since  $\mathbb{T}_m^o$  is Gorenstein, we know from the results of III.2 and III.5 that

$$\Omega^{nc}(\mathbb{T}_m^o) \simeq \mathbb{T}_m^o$$

as left  $\mathbb{T}_m^o$ -modules. In particular,

$$\mathcal{P}(\Omega^{nc}(\mathbb{T}_m^o); t^\epsilon) = t^\beta \cdot \mathcal{P}(\mathbb{T}_m^o; t^\epsilon)$$

for  $\epsilon = 1$  or  $-1$  and some  $\beta \in \mathbb{Z}$ . Finally, using the fact that  $\mathbb{T}_{m,2}^o \simeq \mathbb{T}_m^o[\text{Tr}(X_1), \dots, \text{Tr}(X_m)]$  we obtained the required functional equation.

#### IV.4 : Regularity

Having proved that the trace rings  $\mathbb{T}_{m,2}$  are Cohen-Macaulay, one can ask for stronger homological properties. In particular we would like to determine all  $m$  for which the global dimension of  $\mathbb{T}_{m,2}$  is finite. The only result on regularity of trace rings existing in the literature is due to L.Small and T.Stafford [S-S], they proved that  $gldim(\mathbb{T}_{2,2}) = 5$ . We will give here a short proof, due to M. Van den Bergh and the author [L-V], of this result.

**Proposition 4.1** :  $gldim(\mathbb{T}_{2,2}) = 5$ .

#### Proof

It is sufficient to prove that  $gldim((\mathbb{T}_{2,2})_m) \leq 5$  for any maximal ideal  $m$  of  $\mathcal{R}_{2,2}$ . Consider first the case that  $m$  contains  $(X_1.X_2 - X_2.X_1)^2$ . It is easy to verify that  $X_1.X_2 - X_2.X_1$  is a normalizing element of  $\mathbb{T}_{2,2}$  and the quotient

$$\mathbb{T}_{2,2}/\mathbb{T}_{2,2}(X_1.X_2 - X_2.X_1) \simeq F[\overline{X_1}, \overline{X_2}, \overline{Tr(X_1)}, \overline{Tr(X_2)}]$$

because  $\overline{D(X_i)} = \overline{Tr(X_i).X_i} - \overline{X_i^2}$  and  $\overline{Tr(X_1.X_2)} = 2.\overline{X_1.X_2} + \overline{Tr(X_1).Tr(X_2)} - \overline{Tr(X_1).X_2} - \overline{Tr(X_2).X_1}$ . Therefore,

$$gldim(\mathbb{T}_{2,2}/\mathbb{T}_{2,2}(X_1.X_2 - X_2.X_1)) = 4$$

and by a standard argument one derives that  $gldim((\mathbb{T}_{2,2})_m) = 5$ . For a maximal ideal  $m$  not containing  $(X_1.X_2 - X_2.X_1)^2$ ,  $(\mathbb{T}_{2,2})_m$  is an Azumaya algebra since  $\mathcal{R}_{2,2}(X_1.X_2 - X_2.X_1)^2$  is the Formanek center of  $\mathbb{T}_{2,2}$ . Therefore,

$$gldim((\mathbb{T}_{2,2})_m) = gldim((\mathcal{R}_{2,2})_m) = 5$$

since  $\mathcal{R}_{2,2}$  is a regular domain.

A necessary condition for a positively graded, left Noetherian  $F$ -algebra  $\Lambda$  such that  $\Lambda_0 = F$  to have finite global dimension is that the Poincaré series should

have the form

$$\mathcal{P}(\Lambda; t) = \frac{1}{f(t)}$$

for some polynomial  $f(t) \in \mathbb{Z}[t]$ . For, dividing out the graded maximal ideal  $\Lambda_+ = \bigoplus_{i \geq 1} \Lambda_i$  we obtain  $F$ . So, there must be a graded free resolution with gradation preserving morphisms

$$0 \rightarrow F_r \cdots \rightarrow F_1 \rightarrow F_0 = \Lambda \rightarrow F \rightarrow 0$$

and by an argument as in the foregoing section we obtain

$$1 = \mathcal{P}(F; t) = f(t) \cdot \mathcal{P}(\Lambda; t)$$

finishing the proof of the claim.

For example, if  $m = 2$  we obtain from the description given in the introductory chapter that

$$\mathcal{P}(\mathbb{T}_{2,2}; t) = (1 + 2t + t^2) \cdot \mathcal{P}(\mathcal{R}_{2,2}; t) = \frac{1}{(1-t)^4 \cdot (1-t^2)}$$

And for  $m = 3$  we have

$$\begin{aligned} \mathcal{P}(\mathbb{T}_{3,2}; t) &= (1 + 3t + 3t^2 + t^3) \cdot \mathcal{P}(\mathcal{R}; t) \\ &= \frac{1}{(1-t)^6 \cdot (1-t^2)^3} \end{aligned}$$

On the other hand, we obtain from the rational expression for  $m = 4$ , i.e.

$$\mathcal{P}(\mathbb{T}_{4,2}; t) = \frac{1-t^4}{(1-t)^8 \cdot (1-t^2)^6} = \frac{1+t^2}{(1-t)^8 \cdot (1-t^2)^5}$$

that  $\mathbb{T}_{4,2}$  has infinite global dimension. In general we have

**Theorem 4.2** : The trace ring of  $m$  generic 2 by 2 matrices,  $\mathbb{T}_{m,2}$ , has finite global dimension if and only if  $m \leq 3$ .

**Proof :**

Because  $\mathbb{T}_{m,2}$  is a polynomial ring over  $\mathbb{T}_m^o$ , it is clearly sufficient to prove the result for  $\mathbb{T}_m^o$ . We know that

$$\mathcal{P}(\mathbb{T}_m^o; t) = \frac{f_m(t)}{(1-t)^m \cdot (1-t^2)^{\frac{m(m-1)}{2}}}$$

for some polynomial  $f_m(t) \in \mathbb{Z}[t]$ . Now, suppose that  $\mathbb{T}_m^o$  has finite global dimension, then its Poincaré series must be of the form

$$\mathcal{P}(\mathbb{T}_m^o; t) = \frac{1}{(1-t)^\alpha \cdot (1+t)^\beta}$$

for some  $\alpha, \beta \in \mathbb{N}$ . The first terms in the power series expansion of this expression are

$$1 + (\beta - \alpha) \cdot t + \left( \frac{\alpha(\alpha+1)}{2} - \alpha \cdot \beta + \frac{\beta(\beta+1)}{2} \right) \cdot t^2 + \dots$$

On the other hand we know that

$$\mathcal{P}(\mathbb{T}_m^o; t) = \sum_{(a,b,c)} \mathcal{L}_{3a+2b+c} \cdot t^{3a+2b+c}$$

and using Weyl's formula we can calculate the first terms

$$\mathcal{L}_1 = m$$

$$\mathcal{L}_2 = \frac{m(m-1)}{2}; \mathcal{L}_{1^2} = \frac{m(m+1)}{2}$$

$$\mathcal{L}_3 = \frac{m(m-1)(m-2)}{6}; \mathcal{L}_{21} = \frac{m(m+1)(m-1)}{3}; \mathcal{L}_{1^3} = \frac{m(m+1)(m+2)}{6}$$

so the first terms in  $\mathcal{P}(\mathbb{T}_m^o; t)$  become

$$1 + m \cdot t + m^2 \cdot t^2 + \frac{m(2m^2+1)}{3} \cdot t^3 + \frac{m(m+1)(3m^2-m+2)}{8} \cdot t^4 + \dots$$

Comparing both series,  $\alpha$  and  $\beta$  should be solutions of the set of equations

$$\beta - \alpha = m$$

$$\alpha(\alpha + 1) - 2\alpha\beta + \beta(\beta + 1) = 2m^2$$

Therefore,  $\alpha = \frac{m(m-1)}{2}$  and  $\beta = \frac{m(m+1)}{2}$ . So, if  $\mathbb{T}_m^o$  has finite global dimension, its Poincaré series should be equal to

$$\mathcal{P}(\mathbb{T}_m^o; t) = \frac{1}{(1-t)^m \cdot (1-t^2)^{\frac{m(m-1)}{2}}} = \mathcal{P}(Cl_m; t)$$

But this implies that the kernel of the canonical epimorphism

$$\pi_m : Cl_m \rightarrow \mathbb{T}_m^o$$

should be zero, i.e.  $Cl_m \simeq \mathbb{T}_m^o$ . Now, let us compute the Krull dimensions of both rings

$$Kdim(Cl_m) = \frac{m(m+1)}{2}$$

$$Kdim(\mathbb{T}_m^o) = 3m - 3$$

Therefore,  $m$  has to be a solution of the quadratic equation

$$m^2 - 5m + 6 = 0$$

leaving  $m = 2$  and  $m = 3$  as the only solutions. Conversely, if  $m = 2$  or  $m = 3$  then we have calculated the Poincaré series and they were equal to those of  $Cl_2$  and  $Cl_3$  respectively. Therefore, equality of the Krull dimensions entails that  $\pi_2$  and  $\pi_3$  are isomorphisms and the result follows from II.3.2.

In the introductory chapter we described the center of  $\mathbb{T}_{3,2}$  which is not regular as a computation of its Poincaré series shows. So,  $\mathbb{T}_{3,2}$  is a natural example of a maximal order having finite global dimension such that its center is not regular.

S.C. Coutinho [Co] has used our result that  $\mathbb{T}_{3,2}$  is regular and hence  $K_i(\mathbb{T}_{3,2}) \simeq K_i(F)$  to prove that  $K_0(\mathbb{G}_{3,2}) \not\simeq \mathbb{Z}$ .

We will make a short excursion to the ring of  $m$  generic 2 by 2 matrices,  $\mathbb{G}_{m,2}$ . Procesi [P] found a relation between the Poincaré series of  $\mathbb{G}_{m,2}$  and  $\mathbb{T}_{m,2}$  :

$$\mathcal{P}(\mathbb{G}_{m,2}; t) = \frac{1}{(1-t)^m} \cdot \left[ \mathcal{P}(\mathbb{T}_m^o; t) - \binom{m}{3} - 1 + \frac{1}{(1-t)^m} \right]$$

**Proposition 4.3** : The rational expression of the Poincaré series of the ring of  $m$  generic 2 by 2 matrices,  $\mathbb{G}_{m,2}$ , can never be a pure inverse.

**Proof** :

Above we have seen that

$$\mathcal{P}(\mathbb{G}_{m,2}; t) = \frac{f_m(t)}{(1-t)^m \cdot (1-t^2)^{\frac{m(m-1)}{2}}}$$

for some polynomial  $f_m(t) \in \mathbb{Z}[t]$ . If we substitute this information in (\*) above, we get that whenever  $\mathcal{P}(\mathbb{G}_{m,2}; t)$  is of the form  $\frac{1}{g(t)}$ , then

$$g(t) = (1+t)^\alpha \cdot (1-t)^\beta$$

for some natural numbers  $\alpha$  and  $\beta$ . Computing the first terms in the power series expansion of (\*) gives us

$$\mathcal{P}(\mathbb{G}_{m,2}; t) = 1 + mt + m^2 t^2 + m^3 t^3 + \dots$$

Another way of deriving this result is to note that the minimal degree of an identity of  $\mathbb{G}_{m,2}$  is 4. So, we have to solve the same set of equations as in the proof of Theorem 4.2, giving

$$\alpha = \frac{m(m-1)}{2} \text{ and } \beta = \frac{m(m+1)}{2}$$

as the only solution. Comparing the coefficient of  $t^3$  in the power series expansion of

$$\frac{1}{(1+t)^{\frac{m(m-1)}{2}} \cdot (1-t)^{\frac{m(m+1)}{2}}}$$

with that of  $\mathcal{P}(\mathbb{G}_{m,2}; t)$  gives us the equation

$$m^3 = \frac{1}{3} \cdot m \cdot (2m+1)$$

leaving  $m = 0$  and  $m = 1$  as the only integer solutions.

Unfortunately, this result does not imply that  $\mathbb{G}_{m,2}$  has infinite global dimension, since  $\mathbb{G}_{m,2}$  is not Noetherian.

## IV.5 : Explicit computations

In this final section we will scetch the contents of a letter from C. Procesi [P2], in which he links the study of the Poincaré series of the trace ring of  $m$  generic 2 by 2 matrices to the rather extensive theory of Grassmannians.

Let us recall the definition of the homogeneous coordinate ring of the Grassmann variety of 2-planes in  $m$ -space (or, equivalently, of lines in  $\mathbf{P}_F^{m-1}$ ),  $Grass_{2,m}$ . For more details we refer the reader to [H-P],[Kl] or [G-H]. Let  $Z = (z_{ij})_{i,j}$  be a 2 by  $m$  matrix of indeterminates. Then  $Grass_{2,m}$  is the  $F$ -subalgebra of  $F[Z] = F[z_{1j}, z_{2j} : 1 \leq j \leq m]$  generated by all the 2 by 2 minors of the matrix  $Z$ .

The Grassmannian can be embedded in projective space using the Plücker coordinates  $\lambda_{ij}$ , for all  $1 \leq i < j \leq m$ , where

$$\lambda_{ij} = \det \begin{pmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{pmatrix}$$

It is well known, cfr. e.g. [D-E-P;Th.2.1] for a modern treatment, that there is a one-to-one correspondence between an  $F$ -vector space basis of  $Grass_{2,m}$  and standard Young tableaux of shape  $\sigma = 2^b$  for all  $b \in \mathbb{N}$ . Of course, the element corresponding to such a Young tableau is the product of  $b$  rows  $[\overline{i} \overline{j}]$  each of which is interpreted as the Plücker coordinate  $\lambda_{ij}$ . Therefore, the Poincaré series is

$$\mathcal{P}(Grass_{2,m}; t) = \sum_{b=0}^{\infty} \mathcal{L}_{2^b} \cdot t^{2b}$$

The crucial observation Procesi makes is that

$$\sum_{(a,b,c)} \mathcal{L}_{3a2b1c} \cdot t^{3a+2b+c} = \frac{1}{(1-t)^m} \cdot \sum_{i=0}^{\infty} \mathcal{L}_{2^i} \cdot t^{2^i}$$

This follows from the Pieri formula, cfr. e.g. [St2], which is a general formula for the tensor product of a representation of  $GL_m(F)$  of a Young tableau  $\sigma$  by a column.

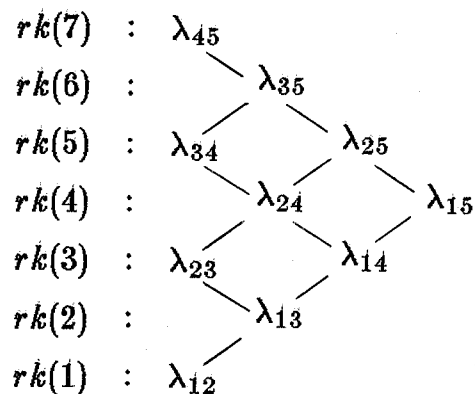
Therefore, the study of the rational expression of  $\mathcal{P}(\mathbb{T}_m^o; t)$  reduces to that of  $\mathcal{P}(\text{Grass}_{2,m}; t)$ . Now, the Grassmannian  $\text{Grass}_{2,m}$  can be realized as a fixed ring in the following way. The group  $GL_2(F)$  acts on  $F[Z]$  by sending for an element  $\alpha \in GL_2(F)$  the variable  $z_{ij}$  to the entry  $(i, j)$  of the matrix  $\alpha.Z$ .

$\text{Grass}_{2,m}$  is the fixed ring of  $F[Z]$  under this action. From this fact one deduces by using the Hochster-Roberts theorem (III.2), that  $\text{Grass}_{2,m}$  is a Cohen-Macaulay domain. Further, one can explicitly determine the generators  $\theta_i$  of the polynomial subring of  $\text{Grass}_{2,m}$  over which it is a free module of finite rank.

To do so, let  $\mathcal{H}_m$  be the set of the  $\frac{m(m-1)}{2}$  Plücker coordinates  $\lambda_{ij}$  for  $1 \leq i < j \leq m$ . One can make  $\mathcal{H}_m$  into a partially ordered set by defining

$$\lambda_{ij} \leq \lambda_{kl} \text{ iff } i \leq k \text{ and } j \leq l$$

For example if  $m = 5$  we get the picture (the Hasse diagram)



For general  $m$ ,  $\mathcal{H}_m$  is a ranked partially ordered set of rank  $2m - 3$ . Combining Th.8.1 and Th.11.1 of [D-E-P] one can prove that  $\text{Grass}_{2,m}$  is a free module of finite rank over the polynomial ring

$$\mathcal{F}_m = F[\theta_1, \dots, \theta_{2m-3}]$$

where

$$\theta_k = \sum_{rk(\lambda_{ij})=k} \lambda_{ij}$$



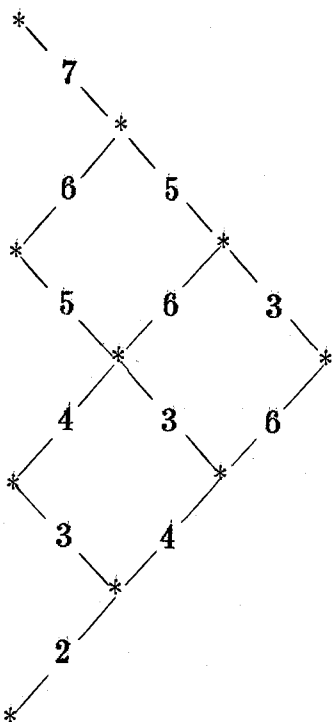
Moreover, one can give an homogeneous basis for  $Grass_{2,m}$  over  $\mathcal{F}_m$ . To begin, note that  $\mathcal{H}_m$  is graded, i.e. it is finite, bounded (it has a minimal and maximal element) and it is pure (all maximal chains have the same length). Further,  $\mathcal{H}_m$  is lexicographic shellable, i.e. one can assign to every edge in the Hasse diagram of  $\mathcal{H}_m$ ,  $\lambda_{ij} \prec \lambda_{kl}$ , a natural number  $\mu(\lambda_{ij}, \lambda_{kl})$  such that in every interval  $[\lambda_{ij}, \lambda_{kl}]$  of  $\mathcal{H}_m$  there is a unique unrefinable chain

$$\lambda_{ij} = \lambda_{i_0 j_0} \prec \lambda_{i_1 j_1} \prec \dots \prec \lambda_{i_n j_n} = \lambda_{kl}$$

which is rising, i.e. such that

$$\mu(\lambda_{i_0 j_0}, \lambda_{i_1 j_1}) \leq \mu(\lambda_{i_1 j_1}, \lambda_{i_2 j_2}) \leq \dots \leq \mu(\lambda_{i_{n-1} j_{n-1}}, \lambda_{i_n j_n})$$

One assigns to an edge in the  $n$ -th main diagonal corridor the number  $2n$  and to an edge in the other  $n$ -th diagonal corridor the number  $2n+1$ . For example, for  $m = 5$  we get the following picture



There is a one-to-one correspondence between a basis of  $Grass_{2,m}$  over  $\mathcal{F}_m$  and the set of maximal chains in  $\mathcal{H}_m$ , cfr. e.g. [Bj] or [Ga] for more details.

Take a maximal chain

$$\lambda_{12} = \lambda_{i_0 j_0} \prec \lambda_{i_1 j_1} \prec \dots \prec \lambda_{i_{2m-4} j_{2m-4}} = \lambda_{m-1 m}$$

then the element of  $Grass_{2,m}$  corresponding to it is

$$\prod_{k \in S} \lambda_{i_k j_k}$$

where  $S$  is the subset of  $\{0, \dots, 2m - 4\}$  consisting of the indices  $k$  such that

$$\mu(\lambda_{i_k - 1 j_{k-1}}, \lambda_{i_k j_k}) > \mu(\lambda_{i_k j_k}, \lambda_{i_k + 1 j_{k+1}})$$

In the case that  $m = 5$  we get

<i>chains</i>	<i>elements</i>
234567	1
234657	$\lambda_{25}$
243567	$\lambda_{14}$
243657	$\lambda_{14} \cdot \lambda_{25}$
246357	$\lambda_{15}$

Combining all this information we get

**Theorem 5.1** : The Poincaré series of the homogeneous coordinate ring of the Grassmann variety of 2-planes in  $m$ -space is

$$\mathcal{P}(Grass_{2,m}; t) = \sum_{i=0}^{\infty} L_{2^i} \cdot t^{2^i} = \frac{g_m(t^2)}{(1-t^2)^{2m-3}}$$

where  $g_m(t^2)$  is a polynomial in  $\mathbb{N}[t]$ . The coefficient of  $t^{2j}$  in  $g_m(t^2)$  is the number of maximal chains in  $\mathcal{X}_m$  having precisely  $j$  descents.

In our example, we have

$$\mathcal{P}(Grass_{2,5}; t) = \frac{1 + 3t^2 + t^4}{(1-t^2)^7}$$

consistent with the computation of  $\mathcal{P}(\Pi_{5,2}; t)$  or  $\mathcal{P}(\Pi_5^0; t)$  carried out in section three. In general we have

**Corollary 5.2** :  $f_m(t) = (1 - t^2)^{\frac{(m-2)(m-3)}{2}} \cdot g_m(t)$

Another immediate consequence of the theory expounded above is the following elegant proof of the functional equation due to C.Procesi [P2] :

Since  $Grass_{2,m}$  is a Gorenstein, Cohen-Macaulay domain, cfr. [St], its Poincaré series satisfies the functional equation

$$\mathcal{P}(Grass_{2,m}; \frac{1}{t}) = -t^\alpha \cdot \mathcal{P}(Grass_{2,m}; t)$$

for some  $\alpha \in \mathbb{Z}$  since  $Kdim(Grass_{2,m}) = 2m - 3$  is odd, cfr. [St]. Finally, since

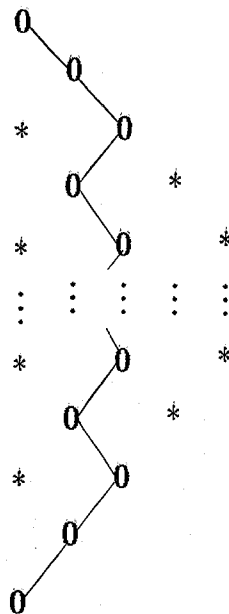
$$\mathcal{P}(\Pi_{m,2}; t) = \frac{1}{(1-t)^{2m}} \cdot \mathcal{P}(Grass_{2,m}; t)$$

the functional equation follows.

Further, it is easy to verify that

$$\alpha = 4m - 6 - deg(g_m) = 2m$$

since  $deg(g_m)$  is equal to twice the maximal number of descents possible for a chain in  $\mathcal{H}_m$ . The unique chain having a maximal number of descents can be visualized as



so it has precisely  $m - 3$  descents. This finishes the proof of the fact that  $\mathcal{P}(\Pi_{m,2}; t)$  satisfies the functional equation

$$\mathcal{P}(\Pi_{m,2}; \frac{1}{t}) = -t^{4m} \cdot \mathcal{P}(\Pi_{m,2}; t)$$

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