An Explicit Description of $\mathbb{T}_{3,2}$

by

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Let F be a field of characteristic different from 2. Consider the polynomial ring

$$P_{m,n} = F[X_{ij}(l) : 1 \leq i,j \leq n, i \leq l \leq m]$$

and the so called generic n by n matrices :

$$X_l = (X_{ij}(l))_{ij} \in M_n(P_{m,n})$$

The ring of m generic n by n matrices, $G_{m,n}$, is the F-subalgebra of $M_n(P_{m,n})$ generated by $\{X_1, \ldots, X_m\}$. The trace ring of m generic n by n matrices, $\mathbb{T}_{m,n}$ is the F-subalgebra of $M_n(P_{m,n})$ generated by $G_{m,n}$ and the elements $Tr(Y)$ where $Y \in G_{m,n}$.

Herstein [1] and Formanek, Halpin and Lie [2] have given an explicit description of the trace ring of 2 generic 2 by 2-matrices. It turned out that $R_{2,2}$, the center of $\mathbb{T}_{2,2}$, is the polynomial ring

$$R_{2,2} = F[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1X_2)]$$

and $\mathbb{T}_{2,2}$ is the free $R_{2,2}$-module of rank four with generators $\{1, X_1, X_2, X_1X_2\}$. In [5] Small and Stafford proved that $\mathbb{T}_{2,2}$ has finite global dimension. We give here a shorter proof of this result :

PROPOSITION : $\text{gldim}(\mathbb{T}_{2,2}) = 5$

PROOF : It is sufficient to prove that for any maximal ideal m in $R_{2,2}$, $\text{gldim}((\mathbb{T}_{2,2})_m) \leq 5$. . Consider first the case that m contains $(X_1X_2 - X_2X_1)^2$. It is easy to verify that $X_1X_2 - X_2X_1$ is a normalizing element of $\mathbb{T}_{2,2}$ and the quotient

$$\mathbb{T}_{2,2}/\mathbb{T}_{2,2}(X_1X_2 - X_2X_1) = F[\overline{X}_1, \overline{X}_2, \overline{Tr(X_1)}, \overline{Tr(X_2)}]$$

because $\overline{D(X_i)} = \overline{Tr(X_i)}.\overline{X}_i - \overline{X}_i^2$ and $\overline{Tr(X_1X_2)} = 2\overline{X}_1.\overline{X}_2 + \overline{Tr(X_1)} \quad \overline{Tr(X_2)} - \overline{Tr(X_1)}.\overline{X}_2 - \overline{Tr(X_2)}.\overline{X}_1$. Therefore

$$\text{gldim}((\mathbb{T}_{2,2})_m/(\mathbb{T}_{2,2})_m(X_1X_2 - X_2X_1)) = 4$$

and by a standard argument $\text{gldim}((\mathbb{T}_{2,2})_m) = 5$. Now, let m be a maximal ideal in $R_{2,2}$, not containing $(X_1X_2 - X_2X_1)^2$. Because $R_{2,2}(X_1X_2 - X_2X_1)^2$ is the Formanek center of $\mathbb{T}_{2,2}$,$(\mathbb{T}_{2,2})_m$ is an Azumaya algebra over the regular domain $(R_{2,2})_m$ whence

$$\text{gldim}((\mathbb{T}_{2,2})_m) = \text{gldim}((R_{2,2})_m) = 5$$

finishing the proof.

In the remaining part of this note we will give an explicit description of the trace ring of 3 generic 2 by 2 matrices. The center of this ring, $R_{3,2}$, was described by Formanek [3] in a rather laborious way. Working with 2 by 2-matrices, one uses basicaly only two indentities

$(1) : A^2 - Tr(A).A + D(A) = 0$

$(2) : A.B + B.A = Tr(A.B) - Tr(A)Tr(B) + Tr(A)B + Tr(B).A$

Consider the F-subalgebra R of $\Delta_{3,2}$, the generic division algebra of 3 generic 2 by 2-matrices, generated by the elements

$\{Tr(X_1), Tr(X_2, Tr(X_3), D(X_1), D(X_2), D(X_3), Tr(X_1X_2), Tr(X_1X_3), Tr(X_2X_3)\}$

Using the identities (1) and (2), one verifies that the F-subalgebra of $\Delta_{3,2}$, $R\{X_1, X_2, X_3\}$ is a finite module over R generated by the elements

$$(*) = \{1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, X_1X_2X_3\}$$

Because $\mathbb{T}_{3,2} \subset R\{X_1, X_2, X_3\} \subset \Delta_{3,2}$, we get that $K\dim(R) = \text{trdeg}_F(Z(\Delta_{3,2})) = 9$ by [4]. Therefore, the generating elements of R are algebraically independent i.e. R is the polynomial ring

$F[Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3), Tr(X_1X_2), Tr(X_1.X_3), Tr(X_2X_3)]$

Further, $\mathbb{T}_{3,2} \subset R\{X_1, X_2, X_3\} \subset \mathbb{T}_{3,2}$ and $Tr(R\{X_1, X_2, X_3\}) \subset R\{X_1, X_2, X_3\}$. This entails that $\mathbb{T}_{3,2} = R\{X_1, X_2, X_3\}$. Now, let K be the field of fractions of R, then

$$\dim_K(\Delta_{3,2}) = \dim_K(K\{X_1, X_2, X_3\}) \leq 8$$

because $K\{X_1, X_2, X_3\}$ has generating set $(*)$. Further $\dim_K(Z(\Delta_{3,2})) \geq 2$ because $Tr(X_1X_2X_3) \notin K$. For otherwise, because $Tr(X_1X_2X_3)$ is linear in each of the generic matrices, this would entail that

$$\begin{aligned} &Tr(X_1X_2X_3) \\ &\quad = \alpha Tr(X_1)Tr(X_2)Tr(X_3) \\ &\quad + \beta(Tr(X_1)Tr(X_2X_3) + Tr(X_2) + Tr(X_1X_3) + Tr(X_3)Tr(X_1X_2)) \end{aligned}$$

and by specializing $X_1 \vdash\!\!- \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $X_2 \vdash\!\!- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $X_3 \vdash\!\!- \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ one obtains a contradiction. Combining this, we get that

$$8 \leq \dim_K(Z(\Delta_{3,2}))\dim_{Z(\Delta_{3,2})}(\Delta_{3,2}) = dim_K(\Delta_{3,2}) \leq 8$$

i.e. the set $(*)$ is linearly independent over K or R.

Taking traces in the identity

$$(X_1X_2X_3)^2 - Tr(X_1X_2X_3)X_1X_2X_3 + D(X_1)D(X_2)D(X_3) = 0$$

and simplifying the first term we get that $Tr(X_1X_2X_3)$ satisfies the quadratic equation :

$(**) : X^2 - AX + B = 0$

where

$$A = Tr(X_1)Tr(X_2X_3) + Tr(X_2)Tr(X_1X_3) + Tr(X_3)Tr(X_1X_2) - Tr(X_1)Tr(X_2)Tr(X_3)$$

$$\begin{aligned} B = {} & D(X_1)Tr(X_2X_3)^2 + D(X_2)Tr(X_1X_3)^2 + D(X_3)Tr(X_1X_2)^2 \\ & - Tr(X_1)Tr(X_2)Tr(X_1X_2)D(X_3) - Tr(X_1)Tr(X_3)Tr(X_1X_3)D(X_2) \\ & - Tr(X_2)Tr(X_3)Tr(X_2X_3)D(X_1) \\ & + Tr(X_1)^2D(X_2)D(X_3) + Tr(X_2)^2D(X_1)D(X_3) + Tr(X_3)^2D(Xr)D(X_2) \\ & - 4D(X_1)D(X_2)D(X_3) + Tr(X_1X_2)Tr(X_1X_2)Tr(X_2X_3) \end{aligned}$$

So we proved the

THEOREM : If R is the polynomial ring :

$$F[Tr(X_1), Tr(X_2), Tr(X_3), D(X_1), D(X_2), D(X_3), T(X_1X_2), T(X_1X_3), T(X_2X_3)]$$

(1) : $R_{3,2}$ is the free R-module of rank 2 generated by 1 and $T(X_1X_2X_3)$. $T(X_1X_2X_3)$ satisfies the quadratic equation $(**)$ over R.

(2): $\mathbb{T}_{3,2}$ is the free R-module of round 8 generated by

$$\{1, X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, X_1X_2X_3\}$$

Being free over a polynomial subring of the center, $\mathbb{T}_{3,2}$ is a reflexive module over $R_{3,2}$. Further, we claim that the localization of $\mathbb{T}_{3,2}$ at a central height one prime is an Azumaya algebra. For, such a prime p cannot contain simultaneously the elements $(X_1X_2 - X_2X_1)^2, (X_1X_3 - X_3X_1)^2, (X_2X_3 - X_3X_2)^2$ belonging to the Formanek center, whence $(\mathbb{T}_{3,2})_p$ is a localization of an Azumaya algebra. This proves that $\mathbb{T}_{3,2}$ is a reflexive Azumaya algebra.

This implies that $\mathbb{T}_{3,2}$ is not a free module over $R_{3,2}$, since this would entail that $\mathbb{T}_{3,2}$ is an Azumaya algebra and dividing out the commutator ideal one finds an epimorphic image of smaller p.i. degree.

From m generic 2 by 2 - matrices one would similarly like to consider the F-subalgebra R of $\Delta_{m,2}$ generated by the elements $Tr(X_1), D(X_i), Tr(X_iX_j)$. But from $m \geq 4$ R can never be a polynomial ring because the number of generators in $2m + \binom{m}{2}$ whereas the $K\dim(R) = 4m - 3$. Therefore, a similar approach fails for $m \geq 4$.

The description of $R_{3,2}$ and $\mathbb{T}_{3,2}$ can also be applied to determine the Poincaré series. Clearly,

$$P(R,t) = \frac{1}{(1-t)^3(1-t^2)^6}$$

since $\deg(Tr(X_i)) = 1$ and $\deg(Tr(X_iX_j)) = \deg(D(X_i)) = 2$.

Therefore,

$$P(R_{3,2},t) = \frac{1+t^3}{(1-t)^3(1-t)^6}$$

whence $R_{3.2}$ cannot have finite global dimension, for otherwise the Poincaré series should have the form $\frac{1}{f(t)}$ for some $f(t) \in \mathbb{Z}[t]$. Finally,

$$P(\mathbb{T}_{3,2},t) = \frac{1+3t+3t^2+t^3}{(1-t)^3(1-t^2)^6} = \frac{1}{(1-t)^6(1-t^3)^3}$$

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