Proj of Generic Matrices and Trace Rings.

by

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The trace ring of generic matrices appears as the ring of global sections of the central graded structure sheaf on the projective space of the generic matrices. In this paper we aim to derive some consequences of this interpretation as well as some related results. If $\mathcal{G}_{m,n}$ is the ring of m generic $n \times n$-matrices then we show that the localization of $\mathcal{G}_{m,n}$ at a localizable prime of Proj($\mathcal{G}_{m,n}$) is a strongly graded ring and use this in the calculation of certain class groups of the part of degree zero of the graded structured sheaf of the graded $Z_{m,n}$-module $\mathcal{G}_{m,n}$, where $Z_{m,n}$ is the centre of $\mathcal{G}_{m,n}$. At this point it becomes clear that the case $n = 2$ allows, or better, demands a separate treatment. Indeed if $n > 2$ then there are $m \in \mathbb{N}$ such that for suitable $p \in Proj(Z_{m,n})$ the stalk $Q^g_p(\mathcal{G}_{m,n})$ at p does not contain a unit of degree one. We then show that the sheaf in degree zero, $\underline{O}_{m,n}$, is never a sheaf of Azumaya algebras but over a field of characteristic zero it is always a sheaf of reflexive Azumaya algebras. In the final part of the paper we present a method to describe the sheaves $\underline{O}_{m,2}$ and the graded sheaves $\underline{O}^g_{m,2}$ over the typical open set assicated to the ideal $(X_iX_j - X_jX_i)^2$, $i, j = 1, \ldots, m$.
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Let $\mathcal{G}_{m,n}$ be the ring of m generic $n \times n$-matrices over a field of characteristic zero k and let $Z_{m,n}$ denote the center of $\mathcal{G}_{m,n}$. If we give every generic matrix degree one then $\mathcal{G}_{m,n}$ and $Z_{m,n}$ are both positively graded k-algebras. For any positively graded graded ring R we let $Proj(R)$ be the set of all graded prime ideals of R not containing the positive part $R_+ = \bigoplus_{n>0} R_n$ equipped with the topology induced by the Zariski topology of $Spec(R)$. In this way we define $Proj(\mathcal{G}_{m,n})$ and $Proj(Z_{m,n})$.

Let $Proj^g(Z_{m,n})$ be the (commutative) stucture sheaf of graded k-algebras defined in the usual way, i.e. if c is a homogeneous element of $Z_{m,n}$ then the ring of sections of the graded sheaf $\underline{O}^g_{Z_{m,n}}$ over the open set $X_+(c)$ is precisely the graded ring $Q^g_c(Z_{m,n})$. There is a canonical continuous map

$$\phi : \mathcal{U}_{m,n} \to Proj(Z_{m,n})$$

where $\mathcal{U}_{m,n}$ is the open set of prime ideals P of $Proj(\mathcal{G}_{m,n})$ such that $(Z_{m,n})_+ \not\subset P$. Let $\underline{O}^g_{\mathcal{G}_{m,n}}$ be the usual structure sheaf of the graded $Z_{m,n}$-module $\mathcal{G}_{m,n}$. In the sequel the following observation is fundamental : the Formanek center of $\mathcal{G}_{m,n}$ is equal to $(Z_{m,n})_+$. Consequently for every graded prime ideal p of $Proj(Z_{m,n})$ it follows that $(\mathcal{G}_{m,n})_p$ is an Azumaya algebra over $(Z_{m,n})_p$. Let us sum up some consequences of this in the following lemma.

Lemma 1. :

(1) : For every $P \in \mathcal{U}_{m,n}$, $\mathcal{G}_{m,n}$ satisfies the left and right Öre conditions with respect to the multiplicative set $\mathcal{C}(P)$ associated to P.

(2) For every $P \in \mathcal{U}_{m,n}$, $Q_P(\mathcal{G}_{m,n})$ may be obtained as a central localization of $\mathcal{G}_{m,n}$ at $p = P \cap Z_{m,n}$. Moreover, the kernel functor κ_P associated to P is central and it coincides with κ_p where $p = P \cap Z_{m,n}$ on $\mathcal{G}_{m,n}$-mod.

(3) The map $\phi : \mathcal{U}_{m,n} \to Proj(Z_{m,n})$ is a homeomorphism.

Proof. (1) Localize at $p = P \cap Z_{m,n}$. Since $(\mathcal{G}_{m,n})_p$ is an Azumaya algebra over $(Z_{m,n})_p$ it follows that $P_p = (\mathcal{G}_{m,n})_p.p$. Therefore, P_p is the Jacobson radical of $(\mathcal{G}_{m,n})_p$. From this one may derive that $\mathcal{G}_{m,n}$ satisfies the left and right Öre conditions with respect to $\mathcal{C}(P)$ in a straightforward way.

(2) The first part is obvious from the foregoing. Since $(\mathcal{G}_{m,n})_P = ((\mathcal{G}_{m,n})_p)_{(P)_p}$ it follows from the fact that $(\mathcal{G}_{m,n})_p$ is an Azumaya algebra that localization at P is obtained by localizing at $(P)_p$ and both localizations correspond to central kernel functors. Hence κ_P is central too.

(3) Easy from the foregoing.

Theorem 1.

(1) $\underline{\mathcal{O}}^g_{\mathcal{G}_{m,n}}$ is a sheaf of regular k-algebras.

(2) The ring of global sections $\Gamma(Proj(Z_{m,n}), \underline{\mathcal{O}}^g_{\mathcal{G}_{m,n}})$ is the trace ring of m generic $n \times n$-matrices, denoted by $\mathcal{T}_{m,n}$, except if $(m,n) = (2,2)$ then the ring of global sections is a localization of $\mathcal{T}_{2,2}$.

Proof. (1). $Proj^g(Z_{m,n})$ is a scheme and it can be covered by affine open sets of the form $X_+(f)$, f homogeneous in $Z_{m,n}$. Now clearly :

$$\Gamma(X_+(f), \underline{\mathcal{O}}^g_{\mathcal{G}_{m,n}}) = Q^g_f(\mathcal{G}_{m,n})$$

is a graded Azumaya algebra with center

$$\Gamma(X_+(f), \underline{\mathcal{O}}^g_{Z_{m,n}}) = Q^g_f(Z_{m,n})$$

On $X_+(f)$, $Q^g_f(\mathcal{G}_{m,n})$ is an Azumaya algebra whence it follows that it is equal to its trace ring which is also equal to $Q^g_f(\mathcal{T}_{m,n})$. Therefore $Q^g_f(Z_{m,n}) = Q^g_f(\mathcal{R}_{m,n})$ is the center of $\mathcal{T}_{m,n}$ which is smooth on the open set of Azumaya stalks of $\mathcal{T}_{m,n}$ by a result of M. Artin [1] and Procesi [8]. This establishes regularity of the sheaf $\underline{\mathcal{O}}^g_{\mathcal{G}_{m,n}}$.

(2) Because all stalks are Azumaya algebras the global sections are equal to

$$\cap_{\mathfrak{p} \in Proj(Z_{m,n})} (\mathcal{G}_{m,n})_{\mathfrak{p}} = \cap_{\mathfrak{p} \in Proj(Z_{m,n})} (\mathcal{T}_{m,n})_{\mathfrak{p}}$$

Evidently, $\mathcal{T}_{m,n}$ is contained in the right hand side. Furthermore, $Proj(Z_{m,n})$ is homeomorphic to the open set of $Proj(\mathcal{R}_{m,n})$ determined by the Formanek center of $\mathcal{T}_{m,n}$, say $X_+(I)$. So

$$\cap_{p\in Proj(Z_{m,n})} Q_p^g(\mathcal{T}_{m,n}) \subset \cap_{q\in X_+(I)} Q_p^g(\mathcal{T}_{m,n})$$

If $(m,n) \neq (2,2)$, then $\mathcal{T}_{m,n}$ is a graded reflexive Azumaya algebra (this is a theorem due to Artin and Schofield [5]). In this case

$$\cap_{q\in X_+(I)} Q_q^g(\mathcal{T}_{m,n}) \subset \cap_{q\in X_g^{(1)}(\mathcal{R}_{m,n})} Q_q^g(\mathcal{T}_{m,n}) = \mathcal{T}_{m,n}$$

whence $\mathcal{T}_{m,n}$ is equal to all of the intersections mentioned in the process, finishing the proof.

The foregoing result implies that $\mathcal{G}_{m,n}$ is a projective Azumaya algebra in the sense of [10]. In order to establish that $\mathcal{G}_{m,n}$ represents an element in the Brauer group of $Proj(Z_{m,n})$ we have to verify that $\mathcal{G}_{m,n}$ is also a κ_+-Azumaya algebra where κ_+ is the localization assicated to the Gabriel filter $\{(Z_{m,n})^\mu, \mu \in \mathbb{N}\}$. By Theorem 1, (2), the localization of $\mathcal{G}_{m,n}$ at κ_+ is exactly $T_{m,n}$ except if $(m,n)=(2,2)$ and therefore the verification mentioned above comes down to verifying whether $T_{m,n}$ is a κ_U-Azumaya algebra where κ_U is the localization associated to the open set $U_{m,n}(T)$ in $Proj(Z(T_{m,n}))$ determined by the Formanek center of $T_{m,n}$ if $(m,n) \neq (2,2)$ but κ_U is associated to $U_{2,2}(T) \cap X_+(X_1X_2 - X_2X_1)$ if $(m,n)=(2,2)$. Since $T_{m,n}$ is finitely presented as a $Z(T_{m,n})$-module the relative Azumaya properties follow at once from the fact that the localizations at the primes of the open sets involved are known to be Azumaya algebras. Since some of the relative finiteness properties of $\mathcal{G}_{m,n}$ may have an independent interest we include a more detailed proof here.

Proposition 1. Let κ_+ be the central kernel functor defined on $\mathcal{G}_{m,n}$-mod as above; then :

(1). κ_+ has finite type i.e. for every $\mu \in \mathbb{N}$ there is a $J(\mu)$ such that $(Z_{m,n})_+^\mu \supset J(\mu)$, where $J(\mu)$ is a finitely generated ideal such that $J(\mu)$ is κ_+-dense in $(Z_{m,n})_+$.

(2). $\mathcal{G}_{m,n}$ is κ_+-finitely generated as a $Z_{m,n}$-module, i.e. there is a finitely generated $Z_{m,n}$- submodule S of $\mathcal{G}_{m,n}$ such that $\mathcal{G}_{m,n}/S$ is a κ_+-torsion module.

Proof. If $(m,n) \neq (2,2)$ or not, in any case we have that $Q_{\kappa_+}(\mathcal{G}_{m,n})$ is a finite module over its Noetherian centre $Z(T_{m,n})$. The ideal $Z(T_{m,n})(Z_{n,m})_+$ may be written as $Z(T_{m,n})x_1 + \ldots + Z(T_{m,n})x_r$ with $x_1,\ldots,x_r \in (Z_{m,n})_+$. Consider the $Z_{m,n}$-module $M = Z_{m,n}x_1 + \ldots + Z_{m,n}x_r \subset (Z_{m,n})_+$. From $T_{m,n} \subset Q_{\kappa_+}(\mathcal{G}_{m,n})$ it follows that M is κ_+-dense in $Z(T_{m,n})(Z_{m,n})_+$ and also in $(Z_{m,n})_+$. Obviously, in (1) one may take $J(\mu)$ equal to M^μ for every $\mu \in \mathbb{N}$. Furthermore let $y_1,\ldots,y_m$ generate $T_{m,n}$ as a $Z(T_{m,n})$-module and fix $N \in \mathbb{N}$ such that $M^N y_i \subset \mathcal{G}_{m,n}$ for $i = 1,\ldots,m$. Let S be the $Z_{m,n}$-module generated by the set $\{m(\underline{x})y_j, j = 1,\ldots,m, m(\underline{x})$ *a monomial in* $x_1,\ldots,x_r$ *of degree* $N\}$. It is clear that $T_{m,n}/S$ is κ_+-torsion as a $Z_{m,n}$-module, hence $\mathcal{G}_{m,n}/S$ is κ_+-torsion or S is κ_+-dense in $\mathcal{G}_{m,n}$. Consequently $\mathcal{G}_{m,n}$ is of κ_+-finite type as a $Z_{m,n}$- module. □

The relation between graded structure sheaves and the more common structure sheaves in degree zero will be particularly nice in case the stalks contain homogeneous units of degree one. We are about to prove that this situation arises when $(m,n) = (m,2)$ but we also give a counterexample showing how this fails for $n > 2$. First we prove a general results which yields as a particular case the property that the stalks of $\underline{O}^g_{\mathcal{G}_{m,n}}$ are strongly graded rings even if they need not contain units of degree one. We need :

Lemma 2. Let A be a positively graded ring and let P be a localizable prime ideal not containing A_+ such that A/P is a gr-Goldie ring (in the sense of [7]). Then A satisfies the Öre conditions with respect to the set $h(G(P))$ of homogeneous elements in A which are regular modulo P.

Proof. If A is Noetherian then the result is known, cf. Theorem I.1.14 in [7]. In this more general situation the proof has to be modified somewhat. By Corollary I.1.7. of [7], a prime positively graded gr-Goldie ring admits a gr-simple gr-Artinian graded ring of fractions (in particular it is also a Goldie ring in the ungraded sense), so we are in a situation where the graded version of Goldie's theorem does hold. Consequently, the graded essential left (or right) ideals of A/P contain **homogeneous** regular elements of A/P. If $a \in A$ and $as = 0$ for some $s \in h(G(P))$ then $a_i s = 0$ for every homogeneous part a_i of a. By the localizability of P $s'_i a_i = 0$ for some $s'_i \in G(P)$ and also $(s'_i)_\lambda a_i = 0$ for every homogeneous component $(s'_i)_\lambda$ of s'_i. Since $\sum_\lambda A(s'_i)_\lambda$ is left essential modulo P and graded too there must be an $s''_i \in h((P))$ such that $s''_i a_i = 0$. Now let i_1 be the highest degree appearing in the decomposition of a then s''_{i_1} still satisfies $s''_{i_1} a = 0$ but the decomposition of $s''_{i_1} a$ is shorter than the decomposition of a. Repetition of the argument leads to $s''_{i_1}, \ldots, s''_{i_t} \in hG(P)$ such that $s''_{i_t} \ldots s''_{i_1} a = 0$, hence $s' a = 0$ for some $s' \in h((P))$. Let us now check the least obvious part of the left Öre conditions. Pick $s \in h((P)), r \in A$ and look at $(As : r)$. Write $r = r_1 + \ldots + r_n$ with $deg(r_1) < \ldots < deg(r_n)$, and consider $L = \cap_{i=1}^{n}(As : r_i)$ in $(As : r)$. The assumptions on of P entails that L is in the Gabriel filter of the localization at P and therefore $L \not\subset ann_A(\overline{x})$ for every $\overline{x} \in A/P$ if $\overline{x} \neq 0$. Take $\overline{y} \in A/P$, $\overline{y} \neq 0$. From the foregoing remark it follows that we can find some $\lambda \in (L : y)$ in A such that $\lambda \notin ann_A(\overline{y})$.

Hence $\lambda y \in L$ and $\lambda \overline{y} \neq 0$, but this means that the image $\overline{L}$ of L in A/P is an essential left ideal because $\overline{L}$ intersects every $(A/P)\overline{y}$ nontrivially if $\overline{y} \neq 0$. Since $\overline{L}$ is also graded it follows that $\overline{L}$ contains a regular homogeneous element of A/P, i.e. $L \cap h(G(P)) \neq \emptyset$. If $s' \in L \cap h(G(P))$, then $s'r = as$ for some $a \in A$. The right Öre conditions may be verified in exactly the same way.

Corollary 1. If we form the graded ring of fractions of A with respect to $h(G(P))$, say $Q^g_P(A)$ then $Q^g_P(A)P$ is an ideal of $Q^g_P(A)$ and actually, $Q^g_p(A)$ is a gr-local ring with gr- maximal ideal $Q^g_P(P)$. This follows in a straightforward way from the Öre conditions with respect to $h(G(P))$ just as in the ungraded case.

Proposition 2. Let A be a positively graded ring generated by its part of degree 1. Let P be a localizable graded prime ideal not containing A_+ and such that A/P

is a gr-Goldie ring then $Q^g_P(A)$ is strongly graded (recall that R is strongly graded by $\mathbb{Z}$ iff $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in \mathbb{Z}$).

Proof. Write B for $Q^g_P(A)$. By the lemma B is a gr-local ring with BP as gr-maximal ideal. Since $B/BP = Q^g(A/P)$ is gr- simple gr-Artinian and since it contains elements of degree 1 ($A_1 \not\subset P$ because $A_+ \subset P$ and A is generated over A_0 by A_1) the structure theorem for these rings (cf. Theorem I.5.8., Corollary I.4.3. of [7]), entails that B/BP is strongly graded.

Put $I = B_{-1}B_1$. From $(B/BP)_{-1}(B/BP)_1 = (B/BP)_0$ it follows that $I + (BP)_0 = B_0$. It is clear that B_0 is "local" with maximal ideal $(BP)_0$ since B is gr-local with gr-maximal ideal P. Note that $B_0(BP)_0$ is semisimple Artinian because B/BP is gr-simple gr-Artinian. Therefore, if L' is a maximal left ideal containing I then $L' \supset I + (BP)_0$ or $L' = B_0$. Consequently $I = B_0$ and B is strongly graded.

Corollary 2. For every $P \in Proj(\mathcal{G}_{m,n})$ with $P \supset (Z_{m,n})_+$ the stalk at P of $\overline{O}^g_{\mathcal{G}_{m,n}}$ is a strongly graded ring.

Proof. By Lemma 1 (2) P is a localizable prime ideal and it is clear that $\mathcal{G}_{m,n}/P$ is a gr-Goldie ring because for a P.I. ring this is obvious. □

Let us write $\underline{O}^g_{m,n}$ for $\underline{O}^g_{\mathcal{G}_{m,n}}$ and $\underline{O}_{m,n}$ for its part of degree zero and let $\mathcal{U}_{m,n}(T)$ be the open set of $Proj(Z(T_{m,n}))$ determined by the Formanek center of $T_{m,n}$. The results in the first part of this paper may then be reinterpreted as :

$$\underline{O}_{m,n} \simeq \underline{O}_{T_{m,n}} \mid U_{m,n}(T)$$

$$\underline{O}^g_{m,n} \simeq \underline{O}^g_{T_{m,n}} \mid U_{m,n}(T)$$

(isomorphisms of ringed spaces). Furthermore, each graded prime in $Proj(Z_{m,n})$ lies under a unique localisable prime graded ideal of $\mathcal{G}_{m,n}$. Hence $\underline{O}^g_{m,n}$ is a strongly graded sheaf over $\underline{O}_{m,n}$ and we can write $\underline{O}^g_{m,n} \simeq \sum_{n\in\mathbb{Z}} \underline{Q}^n X^n$ for some invertible $\underline{O}_{m,n}$-bimodule. One can show that $\underline{Q}$ is the structure sheaf of

the invertible graded bimodule $\underline{G}_{m,n}$ (1). One can ask whether $\underline{Q}$ is necessarily locally trivial or equivalently, is every stalk of $\underline{O}^g_{m,n}$ of the form $B[X, X^{-1}, \phi]$ with deg $X = 1$ and ϕ an automorphism of B. This will be the case if $(m,n) = (m, 2)$ but not in general. On the other hand we may derive that $\underline{O}_{m,n}$ is a sheaf of reflexive Azumaya algebras (but not one of Azumaya algebras) for every m, n. In the study of the properties of $\underline{Q}$ the class group of $\underline{O}_{m,n}$ appears in a very natural way, actually the class of $\underline{Q}$ generates the class group of $\underline{O}_{m,n}$. Let us recall here the definition of the central class group and the normal class group of an order; the sheaf theoretic equivalents of these are easily written down. If Λ is a maximal R-order in a central dsimple algebra Σ, such that R is a Krull domain, then a divisorial Λ-ideal is a divisorial R-lattice (in the classical sense) which is also a Λ-bimodule. On the set $D(\Lambda)$ of divisorial Λ-ideals there exists a multiplication given by $(A.B) = (AB)^{**}$ where $()^*$ denotes the R-dual, actually $(AB)^{**} = ((AB : \Lambda) : \Lambda)$ where $(X : \Lambda)$ stands for $\{x \in \Sigma, xX \subset \Lambda\}$. The set $D(\Lambda)$ equipped with this multiplication is isomorphic to the free abelian group generated by the set of height one prime ideals of Λ.
Define the central class group $CCl(\Lambda)$ of Λ to be the abelian group $D(\Lambda)/P_c(\Lambda)$ where $P_c(\Lambda) = \{\Lambda u, u \in K^*\}$ where $K = Z(\Sigma)$; the normalizing class group is $D(\Lambda)/P_n(\Lambda)$ where $P_n(\Lambda) = \{\Lambda x, x \in \Sigma, \Lambda x = x\Lambda\}$. We do not need to develop much theory about these class groups here because the facts we use are either straightforward generalizations of the commutative case or else imediate consequences of (the sheaf version of) the definition.

Proposition 3. a. If $(m,n) \neq (2,2)$ then $NCl(\underline{O}_{m,n})$ and $CCl(\underline{O}_{m,n})$ are both equal to $\mathbb{Z}$.
b. If $(m,n) = (2,2)$ then $CCl(\underline{O}_{m,n}) = \mathbb{Z}/4\mathbb{Z}$.
The generator of $CCl(\underline{O}_{m,n})$ is the class of $\underline{Q}$.

Proof. If $(m,n) \neq (2,2)$ then $T_{m,n}$ is a reflexive Azumaya algebra by a result usually attributed to M. Artin and A. Schofield cf. [5], and $CCl(T_{m,n}) = 1$. A straightforward modification (cf. Hartshorne [4], p. 147 exercise 6.3.) of a commutative argument yields the exactness of :

$$0 \to \mathbb{Z} \to Cl(\underline{O}_{m,n}) \to CCl(T_{m,n}) \to 0$$

where Cl may be taken to the NCl or CCl. Hence, $NCl(\underline{O}_{m,n}) = CCl(\underline{O}_{m,n})$ the exactness of the sequence leads to the statement in **a.**

If $(m,n) = (2,2)$ then, exactly as in the commutative case, we obtain an exact diagram :

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathbb{Z} & & & & \\
 & & \downarrow & & & & \\
 & \phi & & & & & \\
\mathbb{Z} & \to & CCl(\underline{O}_{T_{2,2}}) & \to & CCl(\underline{O}_{2,2} & \to & 0 \\
 & & \downarrow & & & & \\
 & & CCl(T_{2,2}) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where ϕ maps 1 to the structure sheaf of the ideal $(X_1X_2 - X_2X_1)$ over $\underline{O}_{T_{2,2}}$ cf. Proposition 6.5. p. 133 in [4]). Furthermore, $CCl(T_{2,2}) = \mathbb{Z}/2\mathbb{Z}$ where $(X,X_2 - X_2X_1)$ represents a transversal of the unit element $\bar{1}$. Because $(X_1X_2 - X_2X_1)^2$ is central in $T_{2,2}$, we obtain $CCl(\underline{O}_{T_{2,2}}) \simeq (\mathbb{Z} + \mathbb{Z}a)/(2a-4)$, where $\phi(1) = a$.

It follows that $CCl(\overline{O}^g_{T_{2,2}})/\phi(\mathbb{Z}) = (\mathbb{Z} + \mathbb{Z}a)/(2a-4, a)$ hence it is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. From the cyclicity of the class group involved and from $\underline{O}^g_{m,n} \simeq \sum_{n \in \mathbb{Z}} \underline{O}^n X^n$, it follows that the class of $\underline{Q}$ generates the class group $CCl(\underline{O}_{m,n})$.

□

As pointed out earlier, we may complete these results in the case $(m,2)$ by establishing that $\underline{Q}$ is locally trivial then.

Theorem 2. Let P be a graded prime ideal of $\mathcal{G}_{m,2}$ not containing $(Z_{m,2})_+$, then the homogeneous ring of fractions $Q^g_P(\mathcal{G}_{m,2})$ contains a homogeneous unit of degree one, i.e. $Q^g_P(\mathcal{G}_{m,2}) = B[x_1, x_1^{-1}]$ for some x_1 with deg $x_1 = 1$ which need not commute with $B = (Q^g_P(\mathcal{G}_{m,2}))_0$.

Proof. By Theorem 1 we know that $Q^g_P(\mathcal{G}_{m,2}) = Q^g_{P'}(T_{m,2})$ for some graded prime ideal P' of $T_{m,2}$ not containing $(Z(T_{m,2}))_+$. We have to establish the existence of a homogeneous regular element of degree one in $T_{m,2}/P'$. Clearly, we may assume that the traces $T(X_1), \ldots, T(X_m)$ of the generating 2×2 matrices are contained in P' because otherwise there is nothing to prove. Moreover, we may also assume all $D(X_i^0)$, $i = 1, \ldots, m$, to be in P because otherwise X_i^0 mod P' is a regular element in $T_{m,2}/P'$, again proving the claim. Here $X_i^0 = X_i - \frac{1}{2}T_2(X_i)$.
Look at a linear combination $Y = \Sigma_{i=1}^m \alpha_i X_i^o$. Then Y is an element of $T_{m,2}$ with trace zero, whence $D(Y) = Y^2 = \sum_{i<j} \alpha_i\alpha_j(X_i^o X_j^o + X_j^o X_i^o) + \sum_{i=1}^m \alpha_i^2 (X_i^o)^2 = \sum_{i<j} \alpha_i\alpha_j T(X_i^o X_j^o) + \sum_{i=1}^m \alpha_i^2 D(X_i^o)$.
Therefore, in $T_{m,2}/P'$ we get : $\overline{D(Y)} = \sum_{i<j} \alpha_i\alpha_j \overline{T(X_i^o X_j^o)}$. Whenever some $T(X_i^o X_j^o) \notin P'$, $X_i^o + X_j^o$ is the required element. On the other hand, the assumption that $\{T(X_i), D(X_i^o), T(X_j^o X_j^o), i = 1, \ldots, m, j = 1, \ldots, m\}$ is contained in P' leads to $(Z(T_{m,2}))_+ \subset P'$, a contradiction. $\square$

The above situation is exceptional for $n = 2$ in general it fails as the following lemma learns.

Lemma 3. If $n > 2$ and m is large enough then there are $p \in Proj(Z_{m,n})$ such that $Q^g_P(\mathcal{G}_{m,n})$ is not of the form $R[x, x^{-1}]$ with deg $x = 1$ and R the part of degree zero.

Proof. Take $m \geq 2n-2$. Let A_n be the following graded $n \times n$-matrix ring :

$$\begin{pmatrix} k[X^2,X^{-2}] & Xk[X^2,X^{-2}] & \cdots & Xk[X^2,X^{-2}] \\ X^{-1}k[X^2,X^{-2}] & & & \\ & & k[X^2,X^{-2}] & \\ X^{-1}k[X^2,X^{-2}] & & & \end{pmatrix}$$

Define $\psi : \mathcal{G}_{m,n} \to A_n$, by $X_i \to Xe_{1,i+1}$ for $1 \leq i \leq n-1$, and $X_j \to Xe_{j+1,1}$ for $n \leq j \leq 2n-2$, with $\Psi(X_\nu)$ arbitrary for $\nu > 2n-2$. It is readily verified that $\Psi(\mathcal{G}_{m,n})Z(A_n) = A_n$, hence Ψ is a central extension of rings. Since Ψ is obviously graded too, we find that $P = Ker\Psi$ is a graded prime ideal of $\mathcal{G}_{m,n}; P \in Proj(\mathcal{G}_{m,n})$ and P lies over a $p \in Proj(Z_{m,n})$. From $Q^g_p(\mathcal{G}_{m,n})/pQ^g_p(\mathcal{G}_{m,n}) \simeq \overline{H}_n$ and the fact that A_n cannot contain a regular element of degree one, it follows that $Q^g_p(\mathcal{G}_{m,n}) = Q^g_P(\mathcal{G}_{m,n})$ is not of the form $R[x,x^{-1}]$ with deg $x = 1$. □

In general we know that $\underline{Q}^g_{m,n}$ is a sheaf of graded Azumaya algebras and one may wonder whether $\underline{O}_{m,n}$ has similar nice properties. Even in the case $n = 2$ we see that the parts of degree zero of the Azumaya algebras $Q^g_P(\mathcal{G}_{m,n})$ with $P \in U_{m,n}$ need not be Azumaya because the invertible element of degree one constructed before need not be central, in fact it **never** is! Since $Q^g_P(\mathcal{G}_{m,n})$ is strongly graded it determines an automorphism of the centre of its parts of degree zero (cf. [6]), the fact whether the Azumaya property descends to degree zero is related to properties of this automorphism e.g. to it being trivial. The latter kind of property would contradict the generic nature of $\mathcal{G}_{m,n}$, hence the following result cannot come as a big surprise :

Lemma 4. $O_{m,n}$ is never a sheaf of Azumaya algebras (of course : $n > 1$).

Proof. In the philosophy of the remarks preceeding this lemma we look for a "bad" specialization of $\mathcal{G}_{m,n}$. The role of this villain is played by the following graded matrix ring : (take $m \geq 2n-2$).

$$S_n = \begin{pmatrix} k[X^n, X^{-n}] & Xk[X^n, X^{-n} & \cdots & X^{n-1}k[X^n, X^{-n} \\ & k[X^n, X^{-n}] & & \\ & & & \\ X^{-n+1}k[X^n, X^{-n}] & X^{-n+2}k[X^n, X^{-n}] & & k[X^n, X^{-n}] \end{pmatrix}$$

Extending on the proof of a similar fact in [8] we may actually show that S_n is generated by two elements of degree one as an algebra over its centre. Suppose that $g, h \in (S_n)_1$ are the algebra generators for S_n. Define $\Psi : \mathcal{G}_{m,n} \to S_n$ by $\Psi(X_1) = g$, $\Psi(X_2) = h$ and $\Psi(X_i)$ is arbitrary for $i > 2$. It is fairly obvious that Ψ is a central extension which is a graded map of degree zero. Therefore $P = Ker\Psi$ is in $U_{m,n}$ because the centre of S_n is not trivially graded. Put $B = Q^g_P(\mathcal{G}_{m,n}) = Q^g_p(\mathcal{G}_{m,n}$, where $p = P \cap Z_{m,n}$.. We have : $B/BP \simeq S_n$ and $B_o/(BP)_o \simeq (S_n)_o = kI_n$. Knowing that an epimorphic image of an Azumaya algebra is again an Azumaya algebra of the same p.i. degree, we have to conclude that B_o cannot be an Azumaya algebra. □

Remark. We believe that a gr-central simple algebra A which is generated by A_1 over A_o may in fact be generated by two elements of degree one as an algebra over its centre. If this claim holds then we can take $m = 2$ in Lemma 3, but as yet we have no proof available.

The specific properties of $\underline{O}^g_{m,n}$ and $\mathcal{G}_{m,n}$ invite an explicit description of the local structure of these. Before we deal with this description, we derive an important general property. We have seen that $\underline{O}_{m,n}$ is certainly not a sheaf of Azumaya algebras. On the other hand $\underline{O}_{m,n}$ is not too bad because it is the part of degree zero of a strongly graded Azumaya Algebra and it follows from [6] that $\underline{O}_{m,n}$ is a tame order, in particular it is divisorial. From this we want to arrive at the fact that $\underline{O}_{m,n}$ is a reflexive Azumaya Algebra, we need the following general lemma :

Lemma 5. Consider a scheme S and a quasicoherent sheaf of commutative $\mathbb{Z}$-graded algebras $\underline{O}_A$ such that $(\underline{O}_A)_{\ddot{o}} = \underline{O}_S$.

a. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a map of graded quasicoherent $\underline{O}_A$- modules and suppose that $\mathcal{N}$ is coherent over $\underline{O}_A$. The set $U = \{x \in S, \mathcal{M}_x/m_x\mathcal{M}_x \to$

$\mathcal{N}_x/m_x\mathcal{N}_x$ *is surjective* $\}$ is open.

b. Let $p : T \to S$ be a morphism of schemes. Then $p^*\mathcal{M} \to p^*\mathcal{N}$ is surjective if and only if $p(T) \subset U$.

Proof. a. If $x \in U$ then $\phi_x(\mathcal{M}_x) + m_x\mathcal{N}_x = \mathcal{N}_x$. Now $m_x\underline{O}_A$ is contained in the graded Jacobson radical of $(\underline{O}_A)_x$ and since $\mathcal{N}_x$ is finitely generated over $(\underline{O}_A)_x$ we may apply Nakayama's lemma and conclude that $\phi_x(\mathcal{M}_x) = \mathcal{N}_x$ (conversely, the surjectivity of ϕ_x would entail $x \in U$). The coherence condition on $\mathcal{N}$ entails that $\mathcal{N}/V$ is generated by a finite number of (graded) global sections $t_1, \ldots, t_n$. Choose $s_1, \ldots, s_n \in h(\mathcal{M}_x)$ such that $\phi_x(s_i) = t_i, i = 1, \ldots, n$. These sections live on an open subset $V' \subset V$ and $\phi_{V'}(s_i/V')$ coincides with t_i on some open set $U_i \subset V'$. Put $U_x = \bigcap_{i=1}^n U_i$. Clearly $\phi_{U_x}(s_i/U_x) = t_i/U_x$ and consequently $\phi_{U_x} : \mathcal{M}/U_x \to \mathcal{N}/U_x$ is surjective. For every x we have found an open neighbourhood U_x of x such that for every $y \in U_x$, $\phi_\phi : \mathcal{M}_y \to \mathcal{N}_y$ is surjective, hence U is open.

b. Consider $x' \in T$ and $p(x') = x$. We then have :

$$(p^*\mathcal{M})_{x'} = \mathcal{M}_x \otimes_{(\underline{O}_S)_{x'}}; (p^*\mathcal{N})_{x'} = \mathcal{N}_x \otimes_{(\underline{O}_S)_x} (\underline{O}_T)_{x'}$$

If k, resp. k', is the residue field of x, resp. x', then $(p^*\mathcal{M})_{x'} \otimes_k k' \simeq \mathcal{M}_x \otimes_{(\underline{O}_S)_S} k \otimes_k k'$, and $(p^*\mathcal{N})_{x'} \otimes k' \simeq \mathcal{N}_x \otimes_{(\underline{O}_S)_x} k \otimes_k k'$.
If we assume that $(p^*\mathcal{M})_x \to (p^*\mathcal{N})_x$ is surjective then $(p^*\mathcal{M})_x \otimes k' \to (p^*\mathcal{N})_x \otimes k'$ is surjective. Since k'/k is faithfully flat the map $\mathcal{M}_x \otimes_{(\underline{O}_S)_x} k \to \mathcal{N}_x \otimes_{(\underline{O}_S)_x} k$ is surjective i.e. $x \in U$ and also we obtain that $\mathcal{M}_x \to \mathcal{N}_x$ is surjective. Conversely, it is clear that the surjectivity of $\mathcal{M}_x \to \mathcal{N}_x$ yields surjectivity of $(p^*\mathcal{M})_x \to (p^*\mathcal{N})_x$. $\square$

Theorem 3. Suppose that $char(k) = 0$. Then $O_{\ddot{m},\ddot{n}}$ is a refexive Azumaya algebra.

Proof. A simple descent argument allows us to assume that k is algebraically closed. Let $W \subset Proj(Z_{n,m})$ be the set of closed points where $O_{m,n}$ does not have an Azumaya stalk. A closed point P in $Proj(Z_{m,n}$ corresponds to a graded map $\phi : \mathcal{G}_{m,n} \to A$, where A is a graded matrix algebra over $k[X^{\nu}, X^{-\nu}]$ and $Q^g(\phi(\mathcal{G}_{m,n})) = A, \nu \in \mathbb{N}$. Clearly, W is the disjoint union of sets W_i; corresponding to the different graded isomorphism classes $A^{(i)}$ of graded matrix algebras. Let Sch/k be the category of k-schemes and consider the functor $F : Sch/k \to$ sets defined by sending a k-scheme $p : S \to k$ to the set $\{$ *graded ring maps* $\phi : p^*(\mathcal{G}_{m,n}) \to p^*(A)$ *such that* $\phi(p^*(\mathcal{G}_{m,n})_1) \neq 0.\}$

Step 1. F is representable by the k-scheme $W' = (A_1)^m - \{(0,\ldots,0)\}$. We have to establish that for each $S \to k$ we have : $(*) Hom_k(S, W') \simeq F(S)$. It is clear that the presheaf defined by associating $F(U)$ to $U \subset S$ is in fact a sheaf, therefore we only have to verify $(*)$ for affine S, (see E.G.A. [], p. 106), say $S = Spec(B)$. The B-points of W' correspond to $(a_1, \ldots, a_m) \in A_1 \otimes B$ not all equal to zero and such an m-tuple corresponds to a B-map $\phi : B \otimes_k \mathcal{G}_{m,n} \to B \otimes_k A$, such that $\phi(B \otimes_k \mathcal{G}_{m,n}) \neq 0$. This proves the claim.
We now define the subfunctor $G \subset F, Sch/k \to$ Sets which associate to a k-scheme $p : S \to k$ the set $\{$ *graded ring maps* $\phi : P^*(\mathcal{G}_{m,n}) \to p^*(A)$, such that $\phi(p^*(\mathcal{G}_{m,n}))Z(p^*(A)) = p^*(A)\}$.

Step 2. The functor G is representable by an open subscheme V' of W'.
Let $q : W' \to k$ be the structure map and let $\phi_c, \phi_c : q^*(\mathcal{G}_{m,n}) \to q^*(A)$ be the canonical morphism corresponding to the identity map $W' \to W'$. If $t : S \to W'$ is a map of k-schemes then the element of $G(S)$ corresponding to t is : $t^*(\phi_c) : (qt)^*(\mathcal{G}_{m,n}) \to (qt)^*(A)$. Let us now consider $\mathcal{M} = \phi_c(q^*(\mathcal{G}_{m,n}))Z(q^*(A))$ and $\mathcal{N} = q^*(A)$. Then it is clear that the $\mathcal{M}$ and $\mathcal{N}$ are graded modules over $q^*(Z(A))$ whilst $\mathcal{N}$ is of finite type. Let V' be the open set such that $\mathcal{M}_x = \mathcal{N}_x$ (see Lemma 5.a.). We have to verify that $t(S) \subset V'$ if and only if $t^*(\phi_c)((qt)^*(\mathcal{G}_{m,n})Z((qt)^*(A)) = (qt)^*(A)$. The functoriality of t^* reduces the problem to proving that $t^*(\mathcal{M}) = t^*(\mathcal{N})$ if and only if $t(S) \subset V'$. The latter is just Lemma 5.b., so the claim follows.

Next we define a natural transformation $\eta : G \to Hom(-, Proj(Z_{m,n}))$. As in Step 1. we may define η_S for affine S first and extend it to arbitrary k-schemes in the obvious way. So let $p : Spec(B) \to Spec(k)$ be any affine k-scheme and let $\phi \in G(B)$. Associated to this is a graded central extension $\tilde{\phi} : B \otimes_k \mathcal{G}_{m,n} \to B \otimes_k A$, hence also a graded map $B \otimes_k Z_{m,n} \to B \otimes_k Z(A)$. By the definition of A we have that $B \otimes_k Z(A) \simeq B[X^{\nu}, X^{-\nu}]$.

Now a prime ideal of B corresponds to a graded prime ideal of $B[X^{\nu}, X^{-\nu}]$, in the obvious way, and the latter extends to a graded prime ideal of $B \otimes_k A$ because this is an Azumaya algebra over $B \otimes_k Z(A)$. The obtained graded prime ideal of $B \otimes_k A$ has an inverse image under $\tilde{\phi}$ which is again a graded prime ideal of $B \otimes_k \mathcal{G}m, n$ because $\tilde{\phi}$ is a graded central extension. Finally we look at the restriction of this graded prime ideal to $B \otimes_k Z_{m,n}$ and we see that localization at this prime yields an Azumaya algebra, hence it has to be of maximal p.i. degree in $B \otimes_k Z_{m,n}$. This explains how we arrive at a map $Spec(B) \to Proj(B \otimes_k Z_{m,n})$ which represents a certain $Spec(B)$-point of $Proj(Z_{m,n})$, say $\eta_B(q)$. All of this means that there is a map of k-schemes $\theta : V' \to Proj(Z_{m,n})$ which is of finite type since $V' \to k$ is of finite type and V' is Noetherian, cf. [3] p. 305. The image of the closed points of V in $Proj(Z_{m,n})$ is some W_j.

Let W'' be the image of V in $Proj(Z_{m,n})$. Then W'' is a constructible set, cf. [4]. We aim to show that the closure $\overline{W''}$ has codimension larger than two ! This would entail that the height one primes of $Proj(Z_{m,n})$ yield Azumaya stalks for $\underline{Q}_{m,n}$; note that it will then also follow that height one primes of $Z(\underline{Q}_{m,n}$ yield Azumaya stalks for $\underline{Q}_{m,n}$ (by localization of course) and hence $\underline{Q}_{m,n}$ will be a reflexive Azumaya algebra because we already know that it is divisorial. By definition, W'' is a disjoint union of locally closed sets W_i, $i = 1, \ldots, n$. Since $\overline{W''} \subset U_i \overline{W_i}$ it suffices to establish that each of the $\overline{W_i}$ has codimension bigger than two. Let V_i be the inverse image of W_i in V'. We then obtain a surjective map $\theta_i : V_i \to W_i$. That this is indeed an algebraic map is a consequence of the results in step 1 and step 2. Two maps $\mathcal{G}_{m,n} \overset{\Psi_{1,2}}{\to} A^{(j)}$, where $A^{(j)}$ is in some graded isomorphism class of the possible gradations on $M_n(k[X^{\nu}, X^{-\nu}])$ (note that there is a finite number of such isomorphism classes (also the possible ν are bounded by n, cf. [7]), represent the same point of $Proj(\bar{Z}_{m,n})$ if there is an element γ of $Aut_k^g(A)$ such that $\Psi_2 = \gamma\Psi_1$. This states that the closed fibres of θ_i are all isomorphic to $Aut_k^g(A)$. The dimension of the latter algebraic group may be calculated.

Step 3. Calculation of the dimension of $Aut_k^g(A)$ and codim $\overline{W}_i$. First one verifies that every $\gamma' \in Aut_{A_o-A_o}(A_1)$ extends to a graded automorphism of A over k. Indeed, A_1 represents an invertible A_o- bimodule since A is strongly graded, hence $Aut_{A_o-A_o}(A_1) \simeq Z(A_o)^*$. Let γ' be left multiplication by some $\lambda \in Z(A_o)^*$, then we extend γ' to a k-algebra automorphism of A by putting $\gamma(a_{-1}) = a_{-1}\lambda^{-1}$ for all $a_{-1} \in A_{-1}$ and γ is defined on $A_1^{\otimes n}$ in the obvious way, i.e. $\gamma(a_1 \ldots .a_n) = \lambda a_1 \lambda a_2 \ldots \lambda a_n$ for $a_1, \ldots, a_n \in A_1$, while on $A_{-1}^{\otimes n}$ it is given by $\gamma(b_1, \ldots, b_n) = b_1\lambda^{-1} \ldots .b_n\lambda^{-1}$ for$b_1, \ldots, b_n \in A_{-1}$. Then one easily verifies the exactness of the following sequence :

$$1 \to Aut_{A_o-A_o}(A_1) \to Aut_k^g(A) \to Aut_k(A_o) \to F \to 1$$

where F is a finite group. Indeed, if $A_o = M_{t_1}(k) \oplus \ldots \oplus M_{t_s}(k)$ then the subgroup of $Aut_k(A_o)$ consisting of those k-algebra automorphisms of A_o which fix the blocks globally has finite index. Since such a τ is inner in A_o it is certainly in the image of $Aut_k^g(A)$, hence the cokernel F is finite as claimed. The dimension of $Aut_k(A_o)$ is equal to the dimension of the described subgroup of finite index and the latter obviously has dimension $\sum_{i=1}^{s}(t_i^2 - 1)$. On the other hand dim $Z(A_o)^* = s$ because $Z(A)^* = k^* \oplus \ldots \oplus k^*$, s-times.Consequently, the dimension formula yields :

$$dim(Aut_k^g(A)) = \sum_{i=1}^{s}(t_i^2 - 1) + s = \sum_{i=1}^{s} t_i^2.$$

A map $\mathcal{G}_{m,n} \xrightarrow{b} A$ is completely determined by the images $\delta(X_1), \ldots, \delta(X_m) \in A_1$. of the generic matrices $X_1, \ldots, X_m$. Since we have

$V_i \subset V$ and by Step 1, V is an open k-subscheme of $A_1^{(m)}$, we obtain $dim V_i \leq m \quad dim \quad A_1 = m\sum_{i=1}^{s} t_i t_{\sigma(i)}$, where σ is some cyclic permutation of $\{1,\ldots,s\}$, cf [7]. Since $\theta_i : V_i \to W_i$ has fibres isomorphic to $Aut_k^g(A)$, we derive from the foregoing relations that :

$$dim \quad W_j \leq m \sum_{i=1}^{s} t_i t_{\sigma(i)} - \sum_{i=1}^{s} t_i^2$$

On the other hand $dim \quad Proj(Z_{m,n}) = mn^2 - n^2$, so we are looking to arrive at the relation :

$$mn^2 - n^2 - m\sum_{i=1}^{s} t_i t_{\sigma(i)} + \sum_{i=1}^{s} t_i^2 \geq 2. \qquad (**)$$

Since

$$\sum_{i=1}^{s} t_i t_{\sigma(i)} \geq (\sum_{i=1}^{s} t_i^2)^{1/2} (\sum_{i=1}^{s} t_{\sigma(i)}^2)^{1/2} = \sum_{i=1}^{s} t_i^2$$

we see that the left hand side of $(**)$ is at least $(m-1)(n^2 - \sum_{i=1}^{s} t_i^2)$. However we also have :

$$n^2 - \sum_{i=1}^{s} t_i^2 = (\sum_{i=1}^{s} t_i)^2 - \sum_{i=1}^{s} = \sum_{i\neq j} t_i t_j,$$

and the latter is larger than 2 if s is at least two. Since $m \geq 2$, this proves the relation $(**)$ if s is at least two. Now s must be at least two if we are considering non-Azumaya stalks in degree zero, so the theorem is proved.

Now we present here a method to describe the sheaves $\underline{O}_{m,n}^{g}$ and $\underline{O}_{m,2}$ over the typical open set associated to $\Delta_{i,j} = (X_i X_j - X_j X_i)^2$. The sections over this open set are the localization of $\mathcal{G}_{m,2}$ at the central homogeneous polynomial $\Delta_{i,j}$ resp. its part of degree zero in case we consider $\underline{O}_{m,2}$; this coincides with the localization of the trace ring $T_{m,2}$ at $\Delta_{i,j}$. From Procesi [9] we recall that $T_{m,2} = T_m^0[Tr(X_1),\ldots,Tr(X_m)]$ where T_m^0 is the k-subalgebra of $T_{m,n}$ generated by the elements $X_i^0 = X_i - \frac{1}{2}Tr(X_i)$, $i = 1,\ldots,m$. Clearly, we have $\Delta_{i,j} = (X_i^0 X_j^0 - X_j^0 X_i^0)^2$ so we write $X = X_i^0$, $Y = X_j^0$ for notational simplicity, also we put $\Delta = \Delta_{i,j}$. First we write the generic division algebra $D_{m,2} = Q(\mathcal{G}_{m,2})$ as a quaternion algebra over its centre $K_{m,2}$.

Lemma 6.

$$D_{m,2} = \begin{pmatrix} \Delta & & D(X) \\ & K_{m,2} & \end{pmatrix}$$

Proof. We calculate :

$X(XY - YX) = -(XY - YX)X$

$X^2 = D(X)$

$(XY - YX)^2 = T(XY)^2 - 4D(X)D(Y) = \Delta.$

We claim that Y belongs to

$$K_{m,2}.1 \oplus K_{m,2}X \oplus K_{m,2}(XY - YX) \oplus K_{m,2}X(XY - YX)$$

This is clear from $X(XY - YX) = 2D(X)Y - T(XY)X$ and $Y = \frac{1}{2D(X)}(T(XY)X + X(XY - YX))$. □

We put $R_{m,2} = Z(T_{m,2})$, then :

Proposition 4. The ring of sections of $\underline{O}^g_{m,2}$ over $X_+(\Delta)$ is $Q^g_\Delta(R_{m,2})1 \oplus Q^g_\Delta(R_{m,2})X \oplus Q^g_\Delta(R_{m,2})Y \oplus Q^g_\Delta(R_{m.2})XY$.

Proof. Obviously $\{1, X, Y, XY\}$ is $Q^g_\Delta(R_{m,2})$-independent. The module generated by $\{I, X, Y, XY\}$ over $Q^g_\Delta(R_{m,2}$ is an algebra A because $YX = T(XY) - XY, X^2 = D(X), Y^2 = D(Y)$. Since $T(X_j) \in R_{m,2}$ for every j it suffices to show that every matrix of trace zero X^0_l is in A. Now by a result of [9], $\frac{1}{3}S_3(X, Y, X^0_l) \in R_{m,2}$ and $\frac{1}{3}S_3(X, Y, X^0_l) = (XY - YX)X^0_l + T(XX^0_l)Y - T(YX^0_l)X$, i.e. $X^0_l = \frac{1}{\Delta}(\frac{1}{3}(XY - YX)S_3(X, Y, X^0_l) + T(XX^0_l)Y - T(YX^0_l)X)$, and this finishes the proof. Furthermore, we denote by Δ^g_m the graded localization of $\mathcal{G}_{m,2}$ just as well at the nonzero central homogeneous elements. □

It is clear that D^g_m is an Azumaya algebra over K^g_m which is the graded localization of $Z_{m,2}$ (or $R_{m,2}$ at the set of homogeneous elements different from zero).

Lemma. $\Delta^g_m \otimes_{K^g_m} K^g_m[\sqrt{\Delta}] \simeq M_2(K^g_m[\sqrt{\Delta}])(0,1)$.

Proof. The isomorphism is determined by :

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X \to \begin{pmatrix} 0 & 1 \\ D(X) & 0 \end{pmatrix}$$

$$XY - YX \to \begin{pmatrix} \sqrt{\Delta} & 0 \\ 0 & -\sqrt{\Delta} \end{pmatrix}$$

$$X(XY - YX) \to \begin{pmatrix} 0 & -\sqrt{\Delta} \\ D(X)\sqrt{\Delta} & 0 \end{pmatrix}$$

The morphism preserves degrees in the $(0,1)$-gradation on $M_2(K^g_m[\sqrt{\Delta}])$. □

This presents a method to visualize $\Gamma(X_+(\Delta, \underline{O}^g_{m,2})$ as a graded subring of $M_2(K^g_m[\sqrt{\Delta}](0,1)$. The image of Y is easy to compute :

$$2D(X)Y = T(XY)X + X(XY - YX) = \begin{pmatrix} 0 & T(XY) - \sqrt{\Delta} \\ D(X)(T(XY) + \sqrt{\Delta} & 0 \end{pmatrix}$$

whence it follows that Y maps to :

$$\frac{1}{2}\begin{pmatrix} 0 & (T(XY) - \sqrt{\Delta})/D(X) \\ T(XY) + \sqrt{\Delta} & 0 \end{pmatrix}$$

and therefore :

$$XY \to \frac{1}{2}\begin{pmatrix} T(XY) + \sqrt{\Delta} & 0 \\ 0 & T(XY) - \sqrt{\Delta} \end{pmatrix}$$

Now let $Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]$ be the (graded) integral closure of the ring $Q^g_{m,2}(R_{m,2})$ in $K^g_m(\sqrt{\Delta})$, then :

$$(***) Q^g_\Delta(T_{m,2} \otimes_{Q^g_\Delta(R_{m,2})} Q^g_\Delta(R_{m,2}[\sqrt{\Delta}])$$

is a graded Azumaya Algebra in $M_2(K^g_m[\sqrt{\Delta}])(0,1)$. Since $Q^g_\Delta(R_{m,2})$ and hence $Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]$, is a regular domain we know that $(***)$ is the graded endomorphism ring of a graded projective module. Actually, more is true :

Theorem 4. The ring $(***)$ is the graded endomorphism ring of a graded free $Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]$-module.

Proof. A general element of $(***)$ is if the form :

$$(X_1 + X_2\sqrt{\Delta})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (X_3 + X_4\sqrt{\Delta})\begin{pmatrix}0 & 1\\ D(X) & 0\end{pmatrix} +$$

$$+(X_5 + X_6\sqrt{\Delta})\begin{pmatrix}0 & \frac{1}{D(X)}(T(XY) - \sqrt{\Delta})\\ T(XY) + \sqrt{\Delta} & 0\end{pmatrix}$$

$$+(X_7 + X_8\sqrt{\Delta})\begin{pmatrix}T(XY) - \sqrt{\Delta} & 0\\ 0 & T(XY) + \sqrt{\Delta}\end{pmatrix}$$

$$= \begin{pmatrix}F & G\\ H & I\end{pmatrix},$$

all $X_i \in Q^g_\Delta(R_{m,2})$, where

$F = (X_1 + X_7T(XY) - X_8\Delta) + (X_2 - X_7 + X_8T(XY))\sqrt{\Delta}$

$I = (X_1 + X_7T(XY) + X_8\Delta) + (X_2 + X_7 + X_8T(XY))\sqrt{\Delta}$

$G = (X_3 + \frac{1}{D(X)}(X_5T(XY) - X_6\Delta)) + (X_4 + \frac{1}{D(X)}(X_6T(XY) - X_5))\sqrt{\Delta}$

$H = D(X)\{(X_3 + \frac{1}{D(X)}(X_5T(XY) + X_6\Delta)) + (X_4 + \frac{1}{D(X)}(X_6T(XY) - X_5))\sqrt{\Delta}\}$

In order to prove the theorem it will be sufficient to find two idempotents in the tensorproduct $(***)$. We claim that $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ are in $(***)$.
For the first, we have to solve the set of equations :

$$X_1 + X_7T(XY) - X_8\Delta = 1$$
$$X_1 + X_7T(XY) + X_8\Delta = 0$$

$$X_2 - X_7 + X_8T(XY) = 0$$
$$X_2 + X_7 + X_8T(XY) = 0$$

This yields $X_8 = -\frac{2}{\Delta}$, $X_7 = 0$, $X_1 = -1$ and $X_2 = \frac{2T(XY)}{\Delta}$, and these belong to $Q^g_\Delta(R_{m,2}$.

A similar argument (by symmetry) yields that the second idempotent is also in $(***)$. Therefore

$$(***) \simeq END_{Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]}(P \oplus Q)$$

with P and Q graded projective fractional ideals. Since $Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]$ is factorial both P and Q are isomorphic to $Q^g_\Delta(R_{m,2})[\sqrt{\Delta}]$, hence $(***)$ is isomorphic as a graded ring to $M_2(Q^g_\Delta(R_{m,2})[\sqrt{\Delta}])(0,1)$.

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