THE TRACE RING OF 2 GENERIC 3x3-MATRICES IS NOT REGULAR

by

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0.Introduction.

Throughout this note, k will denote a field of characteristic zero. The ring of m generic $n \times n$-matrices, $\mathcal{G}_{m,n}$, is the quotient of the free k-algebra on m variables modulo the ideal generated by the elements which are polynomial identities for $n \times n$-matrices in m variables.

$\mathcal{G}_{m,n}$ is an Öre domain with classical division ring of quotients $\Delta_{m,n}$ which is of dimension n^2 over its center. The sub k-algebra $\mathcal{T}_{m,n}$ of $\Delta_{m,n}$ generated by $\mathcal{G}_{m,n}$ and the traces of its elements is called the trace ring of m generic $n \times n$-matrices. Whereas $\mathcal{G}_{m,n}$ has rather odd ringtheoretical properties, $\mathcal{T}_{m,n}$ is Noetherian,affine and finite as a module over its center $\mathcal{R}_{m,n}$. Further, Artin and Schofield (unpublished) showed that $\mathcal{T}_{m,n}$ is a maximal order and as a consequence of their proof one gets that $\mathcal{T}_{m,n}$ is a unique factorization ring (i.e. all height one primes are cyclic).

So, trace rings of generic matrices have many properties in common with commutative polynomial rings.This observation motivates the following question

Question : Determine all $m, n \in \mathbb{N}$ such that the global dimension of $\mathcal{T}_{m,n}$ is finite.

The first (noncommutative) result is due to L.Small and J.T.Stafford [SMALL STAFFORD].They proved that the trace ring of 2 generic 2×2-matrices is an iterated Öre-extension having global dimension 5.

In [LE BRUYN],the author showed that the trace ring of m generic 2×2-matrices has finite global dimension if and only if $m \leq 3$. So, our present knowledge can be visualised by

4	+	?	?	?	?...
3	+	?	?	?	?...
2	+	+	+	−	−...
1	+	+	+	+	+...
	1	2	3	4	5...

These results led to some fairly optimistic conjectures ranging from $gldim(\mathcal{T}_{2,3})$ is finite to $gldim(\mathcal{T}_{m,n}) < \infty$ if and only if $m \leq n^2 - 1$ for all $m, n \geq 2$.

However, thinking of trace rings as being fixed ring under an action of $GL_n(k)$, it seems unlikely that for $m \geq 3$ any $\mathcal{T}_{m,n}$ will be regular. Intuitively speaking the singularity in the origin of the central variety becomes so bad that it cannot be resolved by a central p.i.-algebra of the appropiate p.i.-degree.

In this note we like to present an algorith to test (at least in principle) whether the trace ring of m generic $n \times n$- matrices is regular. This test is based on the fact that the Poincaré series of a Noetherian positively graded regular ring whose part of degree zero is k is a pure inverse. Combining Formanek's computation of the Poincaré series of $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$,[FORMANEK], with Procesi's proof that $\mathcal{R}_{m,n}$ is an affine k-algebra one can compute this rational expression. As an illustration of this general method we prove that $\mathcal{T}_{2,3}$ has infinite global dimension.

1. A general strategy .

In this section we will outline an algorithm to find (at least in principle) the rational expression of the Poincaré series for the trace ring of m generic $n \times n$-matrices, $\mathcal{T}_{m,n}$, and for its center $\mathcal{R}_{m,n}$. This method also enables us to test trace rings of generic matrices for regularity (i.e. having finite global dimension).

1.1. Formanek's description of the Poincaré series.

It is known that $\mathcal{R}_{m,n}$ (resp. $\mathcal{T}_{m,n}$) are fixed rings of an action of $GL_n(k)$ on $\mathcal{R}$ (resp. on $M_n(\mathcal{R})$), where $\mathcal{R} = k[t_{ij}(l); 1 \leq i,j \leq n; 1 \leq l \leq m]$. Using this fact, Formanek applied the general theory developed by H. Weyl and I. Schur to obtain formulas for the Poincaré series

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) \text{ and } \mathcal{P}(\mathcal{T}_{m,n}; y_1, ..., y_m)$$

of the center (resp. trace ring) in a multigradation, i.e. by giving each indeterminate $t_{ij}(l) \in \mathcal{R}$ the degree $(0, ..., 0, 1, 0, ..., 0)$ with 1 on spot l. To describe the results of [FORMANEK] we must recall first some basic definitions and results on the ring of symmetric functions.

A degree sequence of lenght n is a sequence

$$\alpha = (\alpha_1, ..., \alpha_n)$$

of non-negative integers. The total degree of α is

$$| \alpha | = \alpha_1 + ... + \alpha_n$$

A partition of lenght $\leq n$ is a degree sequence $\lambda = (\lambda_1, ..., \lambda_n)$ satisfying

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$$

For any partition $\lambda = (\lambda_1, ..., \lambda_n)$ of lenght $\leq n$ we define the element in $\mathbb{Z}[x_1, ..., x_n]$

$$a_\lambda = a_\lambda(x_1, ..., x_n) = \sum_{\pi \in S_n} (sign(\pi)) x_{\pi(1)}^{\lambda_1} ... x_{\pi(n)}^{\lambda_n}$$

where S_n is the group of all permutations on n letters.

In the special case that

$$\delta = (n-1, n-2, ..., 1, 0)$$

we get

$$a_\delta = \prod_{i<j} (x_i - x_j)$$

In $\mathbb{Z}[x_1, ..., x_n]$, $a_{\delta+\lambda}$ is divisible by a_δ and the quotient $s_\lambda = a_{\delta+\lambda}/a_\delta$ is invariant under the natural action of S_n on $\mathbb{Z}[x_1, ..., x_n]$, i.e. by permuting the indeterminates. $s_\lambda = s_\lambda(x_1, ..., x_n)$ is said to be the Schur function associated to the partition of lenght $\leq n$, λ.

The set

$$\{s_\lambda \mid \text{a partition of lenght } \leq n\}$$

forms a $\mathbb{Z}$-basis for Λ_n , the ring of symmetric functions in n variables i.e. the ring of invariants of $\mathbb{Z}[x_1, ..., x_n]$ under action of S_n.

One can define an inner product $<,>$ on Λ_n such that the s_λ form an ortonormal basis and it can be extended to

$$\Gamma_n = \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]^{S_n}$$

Formanek [FORMANEK] defines this inproduct intrinsically in the following way. Let

$$(-)^* : \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \rightarrow \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$$

be the involution defined by $(x_i)^* = x_i^{-1}$. Now, define the linear functional

$$\int : \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}] \rightarrow \mathbb{Z}$$

by $\int 1 = 1$ and $\int x_1^{\alpha_1}...x_n^{\alpha_n} = 0$ if $\alpha_1, ..., \alpha_n$ are not all zero. For any $a, b \in \Gamma_n$ the inproduct is defined to be

$$< a, b >= \frac{1}{n!} \int a.(b)^*.a_\delta.(a_\delta)^*$$

Now consider the finite dimensional $GL_n(k) \times GL_m(k)$- module $M_n(k) \otimes U_m$ where U_m is the standard $GL_m(k)$-module of dimension m and the action of $GL_n(k)$ on $M_n(k)$ is given by conjugation. Then it is clear that $M_n(k) \otimes U_m$ is rational as a $GL_n(k)$-module and polynomial as a $GL_m(k)$- module.

This action of $GL_n(k) \times GL_m(k)$ extends in the natural way to the symmetric algebra of $M_n(k) \otimes U_m$ which is just

$$\mathcal{R} = k[t_{ij}(l); 1 \leq i, j \leq n; 1 \leq l \leq m]$$

By giving each of the generators $t_{ij}(l)$ degree one, $\mathcal{R}$ is a positively graded k-algebra

$$\mathcal{R} = k \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus ...$$

where each homogeneous part $\mathcal{R}_i$ is a finite dimensional $GL_n(k) \times GL_m(k)$-module , rational in the first factor and polynomial in the second.Therefore the Poincaré series of $\mathcal{R}$ as a $GL_n(k) \times GL_m(k)$-module is

$$\mathcal{P}(\mathcal{R}; x_i, x_i^{-1}, y_j) = 1 + \chi(\mathcal{R}_1) + \chi(\mathcal{R}_2) + ...$$

where χ is the isomorphism between the Grothendieck ring $Mod(n, m)$ of all finite dimensional $GL_n(k) \times GL_m(k)$-modules which are rational in the first factor and polynomial in the second, and

$$\mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}, y_1, ..., y_m]^{S_n \times S_m}$$

see for example [FORMANEK,lemma 11].

Therefore, $\mathcal{P}(\mathcal{R})$ is a formal power series over $\Gamma_n \otimes \Lambda_m$. It is fairly easy to see that $\chi(\mathcal{R}_i)$ is the i-th complete symmetric function of

$$\{x_i.x_j^{-1}.y_k \mid 1 \leq i, j \leq n; 1 \leq k \leq m\}$$

i.e. the coefficient of t^i in the power series expansion of

$$\prod_{i,j,k}(1 - x_i.x_j^{-1}.y_k.t)^{-1}$$

Further, we have

$$\prod(1 - x_i.x_j^{-1}.y_k)^{-1} = \sum_\lambda s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

where the sum is taken over all partitions λ of lenght $\leq min(n,m)$. Therefore

$$\mathcal{P}(\mathcal{R}) = \sum_\lambda s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

$$\chi(\mathcal{R}_i) = \sum_{|\lambda|=1} s_\lambda(x_i.x_j^{-1}).s_\lambda(y_k)$$

Here, $s_\lambda(x_i.x_j^{-1})$ is the image of $s_\lambda(z_{ij}; 1 \leq i,j \leq n)$ under the homomorphism

$$\mathbb{Z}[z_{ij}; 1 \leq i,j \leq n] \to \mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$$

sending z_{ij} to $x_i.x_j^{-1}$.

Now, for any $GL_n(k) \times GL_m(k)$-module of the form $N \otimes M$ where N is a rational $GL_n(k)$-module and M is a polynomial $GL_m(k)$-module we have :

$$\chi((N \otimes M)^{GL_n(k)}) = \chi(N^{GL_n(k)} \otimes M) = < \chi_n(N), 1 > .\chi_m(M)$$

where χ_n is the isomorphism between the Grothendieck ring of all finite dimensional rational $GL_n(k)$-modules and Γ_n and χ_m the natural isomorphisms between the Grothendieck ring of all finite dimensional polynomial $GL_m(k)$-modules and Λ_m and the inproduct $<,>$ is taken in $\mathbb{Z}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]^{S_n}$.

Theorem 1 : [FORMANEK,Theorem 12]

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) = \sum_\lambda < s_\lambda(x_i.x_j^{-1}), 1 > .s_\lambda(y_l)$$

where λ varies over all partitions of lenght $\leq min(n,m)$.

Similarly, Formanek computes the Poincaré series of the trace ring $\mathcal{T}_{m,n}$ as fixed ring of $M_n(\mathcal{R})$ under action of $GL_n(k)$.

Theorem 2 : [FORMANEK,Theorem 12]

$$\mathcal{P}(\mathcal{T}_{m,n}; y_1,...,y_m) = \sum_{\lambda} < s_\lambda(x_i.x_j^{-1}), s_{(1)}(x_i.x_j^{-1}) > .s_\lambda(y_k)$$

where λ varies over all partitions of lenght $\leq min(n,m)$ and $(1) = (1,...,1)$, i.e. $s_{(1)}(x_i.x_j^{-1}) = \sum_{i,j} x_i.x_j^{-1}$.

1.2. Towards a rational expression.

It is known [PROCESI] that $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ are both affine algebras over k and that $\mathcal{T}_{m,n}$ is a finite module over $\mathcal{R}_{m,n}$.

Therefore, it follows from the Hilbert-Serre theorem that the Poincaré series of $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ are rational, i.e. there exist polynomials $f, g, h, j \in \mathbb{Z}[y_1,...,y_m]$ such that

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1,...,y_m) = \frac{f(y_1,...,y_m)}{g(y_1,...,y_m)}$$

$$\mathcal{P}(\mathcal{T}_{m,n}; y_1,...,y_m) = \frac{h(y_1,...,y_m)}{j(y_1,...,y_m)}$$

The main problem in determining the rational expressions is to find out which polynomials can occur in the numerator.

In the special case that $n = 2$, this is easy because there exists an epimorphism

$$\pi_m : \Gamma_m \to \mathcal{T}_{m,2}$$

where Γ_m is the iterated Öre-extension

$$\Gamma_m = k[a_{ij}; 1 \leq i,j \leq m][a_1][a_2,\sigma_2,\delta_2]...[a_m,\sigma_m,\delta_m][b_1,...,b_m]$$

where $\sigma_j(a_i) = -a_i$ and $\delta_j(a_i) = a_{ij}$ for all $i < j$ and on all other variables both σ_j and δ_j act trivially.

The epimorphism π_m is defined by sending a_i to $X_i - Tr(X_i)$, b_i to $Tr(X_i)$ and a_{ij} to $Tr((X_i - Tr(X_i))(X_j - Tr(X_j))$, cfr. [LE BRUYN] for more details.

The Poincaré series of Γ_m is easy to determine

$$\mathcal{P}(\Gamma_m; y_1, ..., y_m) = \frac{1}{\prod_{i<j}(1 - y_i.y_j).\prod_i(1 - y_i)^2}$$

and because $\mathcal{T}_{m,2}$ has a finite resolution as Γ_m-module one can write

$$\mathcal{P}(\mathcal{T}_{m,2}; y_1, ..., y_m) = \frac{f(y_1, ..., y_m)}{\prod_{i<j}(1 - y_i.y_j).\prod_i(1 - y_i)^2}$$

Comparing the power series expansion of $\mathcal{P}(\Gamma; y_1, ..., y_m)$ with that of $\mathcal{P}(\mathcal{T}_{m,2}; y_1, ..., y_m)$ as can be calculated using (1.1) it is fairly easy to calculate the functions $f(y_1, ..., y_m)$.

In the general case, however, one has to find another approach. Our starting point will be the following theorem of Procesi

Theorem 3 [PROCESI,Theorem 3.4]

$\mathcal{R}_{m,n}$ is generated as a k-algebra by the elements $Tr(X_{i_1}...X_{i_j})$ with $j \leq 2^n - 1$.

Therefore, $\mathcal{R}_{m,n}$ has a finite resolution over the ring

$$\mathcal{S}_{m,n} = k[a_{i_1...i_j}; j \leq 2^n - 1, i_k \in \{1, ..., m\}]$$

whose Poincaré series is

$$\frac{1}{\prod_i(1 - y_i).\prod_{i_1,i_2}(1 - y_{i_1}.y_{i_2})...\prod_{i_1,..,i_{2^n-1}}(1 - y_{i_1}...y_{i_{2^n-1}})}$$

Therefore, $\mathcal{R}_{m,n}$ and $\mathcal{T}_{m,n}$ being finite modules over $\mathcal{S}_{m,n}$ we get

$$\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m) = f(y_1, ..., y_m).\mathcal{P}(\mathcal{S}_{m,n}; y_1, ..., y_m)$$

$$\mathcal{P}(\mathcal{T}_{m,n}; y_1, ..., y_m) = h(y_1, ..., y_m).\mathcal{P}(\mathcal{S}_{m,n}; y_1, ..., y_m)$$

and , again, comparing the power series expansion of $\mathcal{P}(\mathcal{R}_{m,n}; y_1, ..., y_m)$ (resp. $\mathcal{T}_{m,n}; y_1, ..., y_m)$) and that of $\mathcal{P}(\mathcal{S}_{m,n}; y_1, ..., y_m)$ gives us an algorithm to compute the functions $f(y_1, ..., y_m)$ and $h(y_1, ..., y_m)$.

Of course, this is a very laborous method and usually we will contend ourselves with computing the rational expression of the Poincaré series in one variable. These are obtained from the multi-graded ones by setting

$$y_1 = y_2 = ... = y_m = t$$

and we have :

Theorem 4 :
There exist polynomials $f(t), h(t) \in \mathbb{Z}[t]$ such that

$$\mathcal{P}(\mathcal{R}_{m,n}; t) = \frac{f(t)}{(1-t)^m.(1-t^2)^{m^2}...(1-t^{2^n-1})^{m^{2^n-1}}}$$

$$\mathcal{P}(\mathcal{T}_{m,n}; t) = \frac{g(t)}{(1-t)^m.(1-t^2)^{m^2}...(1-t^{2^n-1})^{m^{2^n-1}}}$$

A direct consequence of this result is the determination of all possible rational expressions of the Poincaré series of $\mathcal{R}_{m,n}$ (resp. of $\mathcal{T}_{m,n}$) providing it has finite global dimension.

For, in that case, the Poincaré series has to have the form $\frac{1}{g(t)}$ and comparing this with the foregoing theorem we get

Corollary 5 :

If $\mathcal{R}_{m,n}$ or $\mathcal{T}_{m,n}$ has finite global dimension, then its Poincaré series has the form

$$\frac{1}{F_1^{\alpha_1}...F_k^{\alpha_k}}$$

where the F_j are irreducible factors (in $\mathbb{Z}[t]$) of $1-t^l$ for some $1 \leq l \leq 2^n-1$ and $\sum_1^k \alpha_i$ is clearly bounded by

$$m + 2.m^2 + ... + (2^n-1).m^{2^n-1}$$

As an application of this general method we will prove in the next two sections that $gldim(\mathcal{R}_{2,3}) = \infty$ and $gldim(\mathcal{T}_{2,3}) = \infty$. In fact, we conjecture for $m, n \geq 2$

CONJECTURE :

$gldim(\mathcal{R}_{m,n}) < \infty \Leftrightarrow (m,n) = (2,2)$

$gldim(\mathcal{T}_{m,n}) < \infty \Leftrightarrow (m,n) = (2,2)$ or $(3,2)$.

2. Computation of the Poincaré series of $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$.

In this section we will explicitate the foregoing general results in the special case of 2 generic 3×3-matrices. The computation of the first terms in the power series expansion of the Poincaré series will enable us in the next section to prove that $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$ both have infinite global dimension.

Our first job is to calculate the Schur functions in 9 variables , $z_1, ..., z_9$, associated to partitions λ of lenght ≤ 2, i.e. λ has the form

$$\lambda = (a, b, 0, 0, 0, 0, 0, 0, 0)$$

where $a, b \in \mathbb{N}$ such that $a \geq b$.

By definition the Schur function s_λ is

$$s_\lambda(z_1, ..., z_9) = \frac{det(A)}{\prod_{i<j}(z_i - z_j)}$$

where A is the following 9×9-matrix

$$\begin{pmatrix} z_1^{8+a} & z_1^{7+b} & z_1^6 & \cdots\cdots & z_1 & 1 \\ z_2^{8+a} & z_2^{7+b} & z_2^6 & \cdots\cdots & z_2 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ z_9^{8+a} & z_9^{7+b} & z_9^6 & \cdots\cdots & z_9 & 1 \end{pmatrix}$$

by elementary row alterations on A it is easy to schow that the Schur functions becomes

$$s_\lambda(z_1, ..., z_9) = \det \begin{pmatrix} F & G \\ H & I \end{pmatrix}$$

where

$$F = \sum_{|i|=a+1} z_1^{i_1} ... z_8^{i_8}$$

$$G = \sum_{|j|=b} z_1^{j_1} ... z_8^{j_8}$$

$$H = \sum_{|k|=a} z_1^{k_1}...z_9^{k_9}$$

$$I = \sum_{|l|=b-1} z_1^{l_1}...z_9^{l_9}$$

where i, j are 8-tuples and k, l are 9-tuples of nonnegative integers. If $b \leq 1$ then the under right corner becomes 0 by definition.

Next, we calculate the image of $s_\lambda(z_1, ..., z_9)$ under the map

$$\mathbb{Z}[z_1, ..., z_9]^{S_9} \to \mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]^{S_3}$$

defined by

$$\begin{array}{ccc} z_1 \to 1 & z_4 \to x_2 x_1^{-1} & z_7 \to x_3 x_1^{-1} \\ z_2 \to x_1 x_2^{-1} & z_5 \to 1 & z_8 \to x_3 x_2^{-1} \\ z_3 \to x_1 x_3^{-1} & z_6 \to x_2 x_3^{-1} & z_9 \to 1 \end{array}$$

Therefore, $s_\lambda(x_i.x_j^{-1})$ is the determinant of the following 2×2-matrix

$$\begin{pmatrix} F_1 & G_1 \\ H_1 & I_1 \end{pmatrix}$$

where :

$$F_1 = \sum_{|i|=a+1} 1^{i_1+i_5}.x_1^{i_2+i_3-i_4-i_7}.x_2^{i_4+i_6-i_2-i_8}.x_3^{i_7+i_8-i_3-i_6}$$

$$G_1 = \sum_{|j|=b} 1^{j_1+j_5}.x_1^{j_2+j_3-j_4-j_7}.x_2^{j_4+j_6-j_2-j_8}.x_3^{j_7+j_8-j_3-j_6}$$

$$H_1 = \sum_{|k|=a} 1^{k_1+k_5+k_9}.x_1^{k_2+k_3-k_4-k_7}.x_2^{k_4+k_6-k_2-k_8}.x_3^{k_7+k_8-k_3-k_6}$$

$$I_1 = \sum_{|l|=b-1} 1^{l_1+l_5+l_9}.x_1^{l_2+l_3-l_4-l_7}.x_2^{l_4+l_6-l_2-l_8}.x_3^{l_7+l_8-l_3-l_6}$$

Our next job is to compute the inproducts in $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]^{S_3}$

$$< s_\lambda(y_i.y_j^{-1}), 1 >= \frac{1}{6} \int s_\lambda(y_i.y_j^{-1}).a_\delta.a_\delta^*$$

$$< s_\lambda(y_i.y_j^{-1}), s_{(1)}(y_i.y_j^{-1}) >= \frac{1}{6}\int s_\lambda(y_i.y_j^{-1}).\sum_{i,j} y_i.y_j^{-1}.a_\delta.a_\delta^*$$

where we have :

$$a_\delta = (y_1 - y_2).(y_2 - y_3).(y_1 - y_3)$$

$$a_\delta^* = (y_1^{-1} - y_2^{-1}).(y_2^{-1} - y_3^{-1}).(y_1^{-1} - y_3^{-1})$$

If we denote $y_1^{\alpha_1}.y_2^{\alpha_2}.y_3^{\alpha_3}$ by $(\alpha_1, \alpha_2, \alpha_3)$, we find that $a_\delta.a_\delta^*$ is equal to

$$6.(0,0,0)$$

$$-2.[(1,-1,0)+(-1,1,0)+(1,0,-1)+(-1,0,1)+(0,1,-1)+(0,-1,1)]$$

$$+2.[(2,-1,-1)+(-2,1,1)+(-1,2,-1)+(1,-2,1)+(-1,-1,2)+(1,1,-2)]$$

$$-1.[(2,-2,0)+(-2,2,0)+(2,0,-2)+(-2,0,2)+(0,2,-2)+(0,-2,2)]$$

Therefore, we have all the necessary ingredients to compute the inproduct $< s_\lambda(y_i.y_j^{-1}), 1 >$ for partitions of the form $\lambda = (a, b, 0, 0, 0, 0, 0, 0, 0)$. In appendix 1, a listing is given of a Pascal program which computes this inproduct as well as the obtained values for $a + b \leq 10$.

The Schur functions in 2 variables are easy to compute

$$s_{(k,0)}(y_1, y_2) = \sum_{i=0}^{k} y_1^i.y_2^{k-i}$$

$$s_{(k,l)}(y_1, y_2) = (y_1.y_2)^{k-l}.s_{(k-l,0)}(y_1, y_2)$$

Therefore, we have all the necessary information to calculate the first terms of the Poincaré series of $\mathcal{R}_{2,3}$.

$$1+$$

$$y_1+y_2+$$

$$2y_1^2+2y_1.y_2+2y_2^2+$$

$$3y_1^3+4y_1^2.y_2+4y_1.y_2^2+3y_2^3+$$

$$4y_1^4+6y_1^3.y_2+9y_1^2.y_2^2+6y_1.y_2^3+4y_2^4+$$

$$5y_1^5+9y_1^4.y_2+14y_1^3.y_2^2+14y_1^2.y_2^3+9y_1.y_2^4+5y_2^5$$

$$7y_1^6+12y_1^5.y_2+22y_1^4.y_2^2+25y_1^3.y_2^3+22y_1^2.y_2^4+12y_1.y_2^5+7y_2^6+$$

$$.....$$

From this we deduce the Poincaré series in one variable

$$\mathcal{P}(\mathcal{R}_{2,3};t)=1+2t+6t^2+14t^3+29t^4+56t^5+107t^6+...$$

For the trace ring, we have to compute

$$\sum_{i,j} y_i.y_j^{-1}.a_\delta.a_\delta^*$$

which is equal to

$$6.(0,0,0)$$

$$-1.[(1,-1,0)+(-1,1,0)+(1,0,-1)+(-1,0,1)+(0,1,-1)+(0,-1,1)]$$

$$-1.[(2,-2,0)+(-2,2,0)+(2,0,-2)+(-2,0,2)+(0,2,-2)+(0,-2,2)]$$

$$-1.[(3,-3,0)+(-3,3,0)+(3,0,-3)+(-3,0,3)+(0,3,-3)+(0,-3,3)]$$

$$+1.[(3,-2,-1)+(-3,2,1)+(3,-1,-2)+(-3,1,2)+(-1,3,-2)+(1,-3,2)+$$

$$(-2,3,-1)+(2,-3,1)+(-1,-2,3)+(1,2,-3)+(-2,-1,3)+(2,1,-3)]$$

In appendix 2 we give a listing of a Pascal program which computes the inproduct

$$< s_\lambda(y_i.y_j^{-1}), s_{(1)}(y_i.y_j^{-1}) >$$

for partitions $\lambda = (a, b, 0, 0, 0, 0, 0, 0, 0)$. Also contained in this appendix are the values for $a + b \leq 10$.

They enable us to compute the first terms in $\mathcal{P}(\mathcal{T}_{2,3}; y_1, y_2)$

$$1+$$

$$2y_1 + 2y_2 +$$

$$4y_1^2 + 6y_1.y_2 + 4y_2^2 +$$

$$6y_1^3 + 13y_1^2.y_2 + 13y_1.y_2^2 + 6y_2^3 +$$

$$9y_1^4 + 22y_1^3.y_2 + 31y_1^2.y_2^2 + 22y_1.y_2^3 + 9y_2^4 +$$

$$12y_1^5 + 34y_1^4.y_2 + 56y_1^3.y_2^2 + 56y_1^2.y_2^3 + 34y_1.y_2^4 + 12y_2^5 +$$

$$16y_1^6 + 48y_1^5.y_2 + 91y_1^4.y_2^2 + 109y_1^3.y_2^3 + 91y_1^2.y_2^4 + 48y_1.y_2^5 + 16y_2^6 +$$

$$.....$$

Hence, the Poincaré series in one variable is

$$\mathcal{P}(\mathcal{T}_{2,3}; t) = 1 + 4t + 14t^2 + 38t^3 + 93t^4 + 204t^5 + 419t^6 + 806t^7 + 1480t^8 ...$$

3. The global dimension of $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$.

In this section we will prove that $\mathcal{R}_{2,3}$ and $\mathcal{T}_{2,3}$ both have infinite global dimension.For, if one of them has finite global dimension, then by corollary 5 its Poincaré series should have the form

$$\frac{1}{F_1^{\alpha_1}...F_k^{\alpha_k}}$$

where the F_j are irreducible factors in $\mathbb{Z}[t]$ of $1-t^l$ for $l \leq 7$, i.e. there are nonnegative integers a,b,c,d,e,f,g and h such that

$$\mathcal{P} = \frac{1}{(1-t)^a A_1^b A_2^c A_3^d A_4^e A_5^f (1+t^2)^g (1+t^3)^h}$$

Where

$A_1 = 1+t$

$A_2 = 1+t+t^2$

$A_3 = 1-t+t^2$

$A_4 = 1+t+t^2+t^3+t^4$

$A_5 = 1+t+t^2+t^3+t^4+t^5+t^6$

We will first consider the trace ring

Theorem 6 :

The trace ring of two generic 3×3-matrices has infinite global dimension.

Proof :

Because $A_1.A_3 = 1+t^3$ we will examine two cases :

<u>$[C1] : b < d$</u> :

Then the Poincaré series is of the form :

$$\frac{1}{(1-t)^a(1-t+t^2)^v A_2^c A_4^e A_5^f (1+t^2)^g (1+t^3)^h}$$

Again, we will examine two cases :

[C1A] : $a \geq c+e+f$, then $\mathcal{P}$ has the form

$$\frac{1}{(1-t)^u(1-t+t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^7(1+t^2)^g(1+t^3)^h}$$

and the first two terms in the power series expansion of this rational expression are

$$1+(u+v)t+(\frac{u(u+1)}{2}+\frac{v(v+1)}{2}-v+uv-g)t^2+...$$

Comparing this with $\mathcal{P}(\mathcal{T}_{2,3},t)$ as computed in the foregoing section we get

$$u+v=4$$

$$u-g=8$$

which has no solution in nonnegative integers.

[C1B] : $a < c+e+f$, then $\mathcal{P}$ is of the form

$$\frac{(1-t)^u}{(1-t+t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t^2)^g(1+t^3)^h}$$

The first two terms in the power series expansion are

$$1+(v-u)t+(\frac{u(u-1)}{2}+\frac{v(v+1)}{2}-v-uv-g)t^2+...$$

and we get $v-u=4$ and $u+g=-8$, a contradiction.

[$C2$] : $b \geq d$:

In this case the rational expression of the Poincaré series is of the form

$$\frac{1}{(1-t)^a A_1^b A_2^c A_4^e A_5^f(1+t^2)^g(1+t^3)^h}$$

First we claim that $a \geq b+c+e+f$.For otherwise

$$\mathcal{P}=\frac{(1-t)^u}{(1-t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t^2)^g(1+t^3)^h}$$

and the power series expansion of this expression has a negative coefficient for t which is impossible for Poincaré series.Therefore

$$\frac{1}{(1-t)^u(1-t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t^2)^g(1+t^3)^h}$$

Comparing the coefficient of t in the power series expansion with that of $\mathcal{P}(\mathcal{T}_{2,3},t)$ gives $u=4$.Computing the first terms we get

$$1+4t+(10+v-g)t^2+(20+4(v-g)+w-h)t^3+$$

$$(35+10(v-g)+4(w-h)+\frac{v(v+1)}{2}+\frac{g(g+1)}{2}-vg)t^4+\ldots$$

Comparing this with $\mathcal{P}(\mathcal{T}_{2,3},t)$ gives

$v-g=4$

$w-h=2$

which gives us $v^2+v+g^2+g-2vg=20$ giving $v=4$, so $g=0$.The coefficient of t^5 then becomes

$$176+14(w-h)+x$$

and comparing with $\mathcal{P}(\mathcal{T}_{2,3},t)$ gives $x=0$.Now, computing the coefficient of t^6 gives

$$344+20(w-h)+\frac{w(w+1)}{2}+\frac{h(h+1)}{2}-wh$$

and comparing this with $\mathcal{P}(\mathcal{T}_{2,3},t)$ gives us $h=32$ and $w=34$. Computing the seventh term gives us $744+y$ whence $y=32$. Using these values the eight term of the Poincaré series should be equal to 1816 , a contradiction, finishing the proof.

Theorem 7 :

The center of the trace ring of 2 generic 3×3-matrices has infinite global dimension.

Proof :

The different cases we will consider here are the same ones as in the proof of Theorem 6.

[$C1A$] : Here we get $u+v=2$ and $u-g=5$ which has no nonnegative solution.

[$C1B$] : Then we get $v-u=2$ and $u+g=-5$,a contradiction.

[$C2$] : As in the proof of Theorem 6 we may assume that the rational expression of $\mathcal{P}$ has the form

$$\frac{1}{(1-t)^u(1-t^2)^v(1-t^3)^w(1-t^5)^x(1-t^7)^y(1+t^2)^g(1+t^3)^h}$$

comparing the coefficients of t gives $u=2$. The first terms in the power series expansion become

$$(5+3(v-g)+2(w-h)+\frac{v(v+1)}{2}+\frac{g(g+1)}{2}-vg)t^4+...$$

This gives us $v-g=3$ and $w-h=4$.Comparing the coefficients of t^4

$$1+2t+(3+v-g)t^2+(4+2(v-g)+w-h)t^3+$$

we get $g=1$ hence $v=4$.Using these facts the coefficient of t^5 becomes $64+x$.Comparing this with $\mathcal{P}(\mathcal{R}_{2,3},t)$ we get $x=-8$,a contradiction.

APPENDIX 1

```
program centertr(input,output);

{this program computes the coefficient in the Poincare-series}
{of the center of T of 2 generic 3x3-matrices for the Schur-function}
{associated to the partition (x,y,0,0,0,0,0,0,0)}

var x,y,z,t,v,result :integer;

function hulp(a,b : integer):integer;

var i1,i2,i3,i4,i5,i6,i7,i8 : integer;
    j1,j2,j3,j4,j5,j6,j7,j8,j9 : integer;
    c,d,e,k  : integer;

begin{hulp}
k:=0;
for i1:=0 to a do begin
  for i2:=0 to (a-i1) do begin
    for i3:=0 to (a-i1-i2) do begin
      for i4:=0 to (a-i1-i2-i3) do begin
        for i5:=0 to (a-i1-i2-i3-i4) do begin
          for i6:=0 to (a-i1-i2-i3-i4-i5) do begin
            for i7:=0 to (a-i1-i2-i3-i4-i5-i6) do
   begin
      i8:=(a-i1-i2-i3-i4-i5-i6-i7);
      for j1:=0 to b do begin
        for j2:=0 to (b-j1) do begin
          for j3:=0 to (b-j1-j2) do begin
            for j4:=0 to (b-j1-j2-j3) do begin
              for j5:=0 to (b-j1-j2-j3-j4) do begin
                for j6:=0 to (b-j1-j2-j3-j4-j5) do begin
                  for j7:=0 to (b-j1-j2-j3-j4-j5-j6) do begin
                    for j8:=0 to (b-j1-j2-j3-j4-j5-j6-j7) do
      begin
         j9:=(b-j1-j2-j3-j4-j5-j6-j7-j8);
         c:=(i2+j2+i3+j3-i4-j4-i7-j7);
         d:=(i4+j4+i6+j6-i2-j2-i8-j8);
         e:=(i7+j7+i8+j8-i3-j3-i6-j6);
         if (c*c+d*d+e*e)=0 then k:=k+6
                              else begin
                                if (c*d*e)=0 then begin
                                   if (c*c+d*d+e*e)=2 then k:=k-2
                                             else begin
                                   if (c*c+d*d+e*e)=8 then k:=k-1;
                                                  end;
                                                  end
                                              else begin
                                   if (c*c+d*d+e*e)=6 then k:=k+2;
                                                   end;
                                   end;
        end;
                                                                  end;
                                                              end;
                                                         end;
                                                    end;
                                               end;
                                          end;
                                      end;
                                  end;
                                                         end;
                                                    end;
                                               end;
                                          end;
                                      end;
                                  end;
 hulp:=trunc(k/6);
 end{hulp};

 begin{main program}
 read(x);read(y);
 z:=hulp(y,x);
 if (y-1)<0 then result:=z
           else begin
                   t:=hulp(x+1,y-1);
                   result:=(z-t);
                end;
 writeln('coefficient of s(',x:3,',',y:3,',0,0,0,0,0,0,0) = ',result:3);
 end.
```

```
coefficient of s(  1,  0,0,0,0,0,0,0,0) =   1
coefficient of s(  2,  0,0,0,0,0,0,0,0) =   2
coefficient of s(  1,  1,0,0,0,0,0,0,0) =   0
coefficient of s(  3,  0,0,0,0,0,0,0,0) =   3
coefficient of s(  2,  1,0,0,0,0,0,0,0) =   1
coefficient of s(  4,  0,0,0,0,0,0,0,0) =   4
coefficient of s(  3,  1,0,0,0,0,0,0,0) =   2
coefficient of s(  2,  2,0,0,0,0,0,0,0) =   3
coefficient of s(  5,  0,0,0,0,0,0,0,0) =   5
coefficient of s(  4,  1,0,0,0,0,0,0,0) =   4
coefficient of s(  3,  2,0,0,0,0,0,0,0) =   5
coefficient of s(  6,  0,0,0,0,0,0,0,0) =   7
coefficient of s(  5,  1,0,0,0,0,0,0,0) =   5
coefficient of s(  4,  2,0,0,0,0,0,0,0) =  10
coefficient of s(  3,  3,0,0,0,0,0,0,0) =   3
coefficient of s(  7,  0,0,0,0,0,0,0,0) =   8
coefficient of s(  6,  1,0,0,0,0,0,0,0) =   8
coefficient of s(  5,  2,0,0,0,0,0,0,0) =  14
coefficient of s(  4,  3,0,0,0,0,0,0,0) =   9
coefficient of s(  8,  0,0,0,0,0,0,0,0) =  10
coefficient of s(  7,  1,0,0,0,0,0,0,0) =  10
coefficient of s(  6,  2,0,0,0,0,0,0,0) =  21
coefficient of s(  5,  3,0,0,0,0,0,0,0) =  15
coefficient of s(  4,  4,0,0,0,0,0,0,0) =  10
coefficient of s(  9,  0,0,0,0,0,0,0,0) =  12
coefficient of s(  8,  1,0,0,0,0,0,0,0) =  13
coefficient of s(  7,  2,0,0,0,0,0,0,0) =  27
coefficient of s(  6,  3,0,0,0,0,0,0,0) =  27
coefficient of s(  5,  4,0,0,0,0,0,0,0) =  18
coefficient of s( 10,  0,0,0,0,0,0,0,0) =  14
coefficient of s(  9,  1,0,0,0,0,0,0,0) =  16
coefficient of s(  8,  2,0,0,0,0,0,0,0) =  36
coefficient of s(  7,  3,0,0,0,0,0,0,0) =  36
coefficient of s(  6,  4,0,0,0,0,0,0,0) =  37
coefficient of s(  5,  5,0,0,0,0,0,0,0) =  10
```

APPENDIX 2

```
program tracering(input,output);

{this program computes the coefficient in the Poincare-series}
{of the trace ring of 2 generic 3x3-matrices for the Schur-function}
{associated to the partition (x,y,0,0,0,0,0,0,0)}

var x,y,z,t,v,result :integer;

function hulp(a,b : integer):integer;

var i1,i2,i3,i4,i5,i6,i7,i8 : integer;
    j1,j2,j3,j4,j5,j6,j7,j8,j9 : integer;
    c,d,e,k  : integer;

begin{hulp}
k:=0;
for i1:=0 to a do begin
  for i2:=0 to (a-i1) do begin
    for i3:=0 to (a-i1-i2) do begin
      for i4:=0 to (a-i1-i2-i3) do begin
        for i5:=0 to (a-i1-i2-i3-i4) do begin
          for i6:=0 to (a-i1-i2-i3-i4-i5) do begin
            for i7:=0 to (a-i1-i2-i3-i4-i5-i6) do
   begin
      i8:=(a-i1-i2-i3-i4-i5-i6-i7);
      for j1:=0 to b do begin
        for j2:=0 to (b-j1) do begin
          for j3:=0 to (b-j1-j2) do begin
            for j4:=0 to (b-j1-j2-j3) do begin
              for j5:=0 to (b-j1-j2-j3-j4) do begin
                for j6:=0 to (b-j1-j2-j3-j4-j5) do begin
                  for j7:=0 to (b-j1-j2-j3-j4-j5-j6) do begin
                    for j8:=0 to (b-j1-j2-j3-j4-j5-j6-j7) do
      begin
         j9:=(b-j1-j2-j3-j4-j5-j6-j7-j8);
         c:=(i2+j2+i3+j3-i4-j4-i7-j7);
         d:=(i4+j4+i6+j6-i2-j2-i8-j8);
         e:=(i7+j7+i8+j8-i3-j3-i6-j6);
         if (c*c+d*d+e*e)=0 then k:=k+6
                           else begin
                             if (c*d*e)=0 then begin
                               if (c*c+d*d+e*e)=2 then k:=k-1
                                        else begin
                               if (c*c+d*d+e*e)=8 then k:=k-1
                                            else begin
                               if (c*c+d*d+e*e)=18 then k:=k-1;
                                               end;
                                           end;
                                           end
                                       else begin
                               if (c*c+d*d+e*e)=14 then k:=k+1;
                                           end;
                               end;
      end;
                                                  end;
                                               end;
                                            end;
                                         end;
                                      end;
                                   end;
                                end;
                             end;
                                         end;
                                      end;
                                   end;
                                end;
                             end;
                          end;
hulp:=trunc(k/6);
end{hulp};

begin{main program}
read(x);read(y);
z:=hulp(y,x);
if (y-1)<0 then result:=z
         else begin
                 t:=hulp(x+1,y-1);
                 result:=(z-t);
              end;
writeln('coefficient of s(',x:3,',',y:3,',0,0,0,0,0,0,0) = ',result:3);
end.
```

```
coefficient of s(  1,  0,0,0,0,0,0,0,0) =   2
coefficient of s(  2,  0,0,0,0,0,0,0,0) =   4
coefficient of s(  1,  1,0,0,0,0,0,0,0) =   2
coefficient of s(  3,  0,0,0,0,0,0,0,0) =   6
coefficient of s(  2,  1,0,0,0,0,0,0,0) =   7
coefficient of s(  4,  0,0,0,0,0,0,0,0) =   9
coefficient of s(  3,  1,0,0,0,0,0,0,0) =  13
coefficient of s(  2,  2,0,0,0,0,0,0,0) =   9
coefficient of s(  5,  0,0,0,0,0,0,0,0) =  12
coefficient of s(  4,  1,0,0,0,0,0,0,0) =  22
coefficient of s(  3,  2,0,0,0,0,0,0,0) =  22
coefficient of s(  6,  0,0,0,0,0,0,0,0) =  16
coefficient of s(  5,  1,0,0,0,0,0,0,0) =  32
coefficient of s(  4,  2,0,0,0,0,0,0,0) =  43
coefficient of s(  3,  3,0,0,0,0,0,0,0) =  18
coefficient of s(  7,  0,0,0,0,0,0,0,0) =  20
coefficient of s(  6,  1,0,0,0,0,0,0,0) =  45
coefficient of s(  5,  2,0,0,0,0,0,0,0) =  68
coefficient of s(  4,  3,0,0,0,0,0,0,0) =  52
coefficient of s(  8,  0,0,0,0,0,0,0,0) =  25
coefficient of s(  7,  1,0,0,0,0,0,0,0) =  59
coefficient of s(  6,  2,0,0,0,0,0,0,0) = 101
coefficient of s(  5,  3,0,0,0,0,0,0,0) =  97
coefficient of s(  4,  4,0,0,0,0,0,0,0) =  46
coefficient of s(  9,  0,0,0,0,0,0,0,0) =  30
coefficient of s(  8,  1,0,0,0,0,0,0,0) =  76
coefficient of s(  7,  2,0,0,0,0,0,0,0) = 138
coefficient of s(  6,  3,0,0,0,0,0,0,0) = 159
coefficient of s(  5,  4,0,0,0,0,0,0,0) = 114
coefficient of s( 10,  0,0,0,0,0,0,0,0) =  36
coefficient of s(  9,  1,0,0,0,0,0,0,0) =  94
coefficient of s(  8,  2,0,0,0,0,0,0,0) = 183
coefficient of s(  7,  3,0,0,0,0,0,0,0) = 232
coefficient of s(  6,  4,0,0,0,0,0,0,0) = 215
coefficient of s(  5,  5,0,0,0,0,0,0,0) =  83
```

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