THE ARTIN - SCHOFIELD THEOREM AND SOME APPLICATIONS

by

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0. Introduction.

The purpose of this note is threefold :

First we aim to survey two of the main results on trace rings of generic matrices. Both of them have their roots in the influencial M. Artin - paper [ARTIN] of 1969. The first is concerned with classifying finite dimensional semi-simple representations of the free algebras up to equivalence. It turns out that the affine varieties associated to the centers of the trace rings are the wanted classifying spaces. The second result deals with the invariant theory of $n \times n$-matrices. The center of the trace ring of m generic $n \times n$-matrices turns out to be the ring of invariant polynomial mappings from m copies of $M_n(k)$ to k under componentswise conjugation of $GL_n(k)$ whereas the trace ring itself is the ring of matrixconcomitants.

Secondly we aim to describe the proof of a recent result due to M. Artin and A. Schofield (unpublished) stating that trace rings of generic matrices are maximal orders. Their proof is a beautiful application of both the representation- and invariant theoretic description of trace rings.

Finally we aim to give some noteworthy applications of their result : the centers of the trace rings of generic matrices are Gorenstein unique factorization domains, height one prime ideals of the trace rings are cyclic and the Montgomery result on centrally fixed automorphisms of generic matrices follows immediatly from it.

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1. Some definitions.

Throughout this paper, k will be a (commutative) field of characteristic zero. $\mathcal{F}_m = k < x_1, ..., x_m >$ will be the free k-algebra in m variables, i.e. $\mathcal{F}_m$ is the tensor algebra of a vector space of dimension m over k.

If $I_{m,n}$ is the ideal of $\mathcal{F}_m$ consisting of all identities satisfied by $n \times n$ - matrices in m variables, then

$$\mathcal{G}_{m,n} = k < x_1, ..., x_m > / I_{m,n}$$

is the ring of m generic $n \times n$-matrices.

A more convenient description of this ring is obtained as follows. Let $\mathcal{R}$ be the commutative polynomial ring :

$$\mathcal{R} = k[t_{ij}(l); 1 \leq i, j \leq n; 1 \leq l \leq m]$$

and consider the matrices :

$$X_l = (t_{ij}(l))_{i,j} \text{ in } M_n(\mathcal{R})$$

$\mathcal{G}_{m,n}$ is then the subring of $M_n(\mathcal{R})$ generated as a k-algebra by the elements $\{X_l : 1 \leq l \leq m\}$, the so called generic $n \times n$ - matrices.

With $\mathcal{R}_{m,n}$ we will denote the subring of $\mathcal{R}$ generated as a k-algebra by the elements :

$$\{Tr(\alpha_1 ... \alpha_j) : j \in \mathbb{N}; \alpha_i \in \mathcal{G}_{m,n}\}$$

The subring $\mathcal{G}_{m,n}.\mathcal{R}_{m,n}$ of $M_n(\mathcal{R})$ will be denoted by $\mathcal{T}_{m,n}$ and is called the trace ring of m generic $n \times n$-matrices.

It is fairly easy to verify that $\mathcal{R}_{m,n}$ is the center of $\mathcal{T}_{m,n}$ whenever $m \geq 2$.

From [PROCESI,III.1.3] we retain that $\mathcal{G}_{m,n}$ is a domain and its classical ring of quotients ,$\Delta_{m,n}$, which exists by Posner's theorem is a division ring of dimension n^2 over its center, $\mathcal{K}_{m,n}$. Clearly, $\mathcal{K}_{m,n}$ is the field of fractions of $\mathcal{R}_{m,n}$.

In the next two sections we will briefly recall the two main motivations for studying trace rings of generic matrices : finite dimensional representations of the free algebra $\mathcal{F}_m$ and the invariant theory of $n \times n$ - matrices.

2. Representation theory, [ARTIN],[PROCESI 2]

For simplicities' sake we will assume in this section that k is algebraically closed. For arbitrary fields similar results hold, see for example [ARTIN SCHELTER].

An n-dimensional representation of $\mathcal{F}_m$ is an algebra morphism :

$$\phi : \mathcal{F}_m \rightarrow M_n(k)$$

Note that this is equivalent to giving m $n \times n$ - matrices $\phi(X_1), ..., \phi(X_n)$. Therefore $REP_n(\mathcal{F}_m)$, the set of all n-dimensional representations of $\mathcal{F}_m$, can be identified to the affine variety associated to $\mathcal{R}$, i.e. to $A_k^{mn^2}$.

Two n-dimensional representations ϕ_1 and ϕ_2 are said to be equivalent if they differ only up to a k-automorphism of $M_n(k)$:

$$\begin{array}{ccc} \mathcal{F}_m & \rightarrow & M_n(k) \\ & \phi_1 & \\ \| & & \downarrow \\ & \phi_2 & \\ \mathcal{F}_m & \rightarrow & M_n(k) \end{array}$$

Therefore, $Aut_k(M_n(k)) = PGL_n(k) = GL_n(k)/k^*$ acts on $REP_n(\mathcal{F}_m) = A_k^{mn^2}$ and the orbits under this action are the equivalence classes of representations. To classify representations up to equivalence is therefore essentially the study of the orbit space of $A_k^{mn^2}$ under the action of $PGL_n(k)$.

If $\phi : \mathcal{F}_m \rightarrow M_n(k)$ is a representation, then $k^n = k \oplus ... \oplus k$ (n times) becomes an $\mathcal{F}_m$-module via ϕ. If k^n is completely reducible as a $\mathcal{F}_m$- module , then ϕ is said to be semi-simple. In general we can find a decomposition series :

$$0 = V_t \subset ... \subset V_1 \subset V_0 = k^n$$

for k^n as $\mathcal{F}_m$-module.

Then $W = \bigoplus (V_i/V_{i+1})$ is a completely reducible $\mathcal{F}_m$-module and $dim_k(W) = n$.

Choose a k-basis for W as follows : the first $dim_k(V_{t-1})$- vectors form a basis for V_{t-1}, the next $dim_k(V_{t-2}/V_{t-1})$- vectors form a basis for V_{t-2}/V_{t-1} etc. . With respect to this basis, ϕ will in matrixnotation have the form :

$$\phi = \begin{pmatrix} \phi_1 & & N \\ & \ddots & \\ 0 & & \phi_t \end{pmatrix}$$

where $\phi_j : \mathcal{F}_m \rightarrow M_{n_j}(k)$ $(n_j = dim_k(V_{j-1}/V_j))$ are the irreducible quotients of ϕ , so that ϕ_j is epimorphic, $n = n_1 + ... + n_t$ and where N is the nilradical of ϕ. With ϕ we can therefore associate a semi-simple representation :

$$\phi^{ss} = \phi_1 \oplus ... \oplus \phi_t$$

Artin [ARTIN] has proved that ϕ^{ss} lies in the closure of the orbit of ϕ under $PGL_n(k)$, i.e. we will not be able, in any type of quotient variety of $A_k^{mn^2}$ by $PGL_n(k)$, to distinguish between ϕ and ϕ^{ss}.

This motivates us to construct an affine variety whose k-points correspond to the equivalence classes of semi-simple representations of $\mathcal{F}_m$ in $M_n(k)$.

Let V be the variety associated to the affine k-algebra $\mathcal{R}_{m,n}$ (see [PROCESI 3]) , then the natural inclusion $\mathcal{R}_{m,n} \subset \mathcal{R}$ induces a map between the varieties :

$$p : A_k^{mn^2} \rightarrow V$$

which satisfies :

(a) : p is onto

(b) : $p(\phi) = p(\psi)$ iff $\phi^{ss} \simeq \psi^{ss}$

These facts together yield :

Theorem 1 [ARTIN],[PROCESI 2]

The points of $V = AFF(\mathcal{R}_{m,n})$ are in one-to-one correspondence with the equivalence classes of semi-simple n-dimensional representations of $\mathcal{F}_m$.

To the affine k-algebra $\mathcal{T}_{m,n}$ (see [PROCESI 3]) we can associate its affine variety $Y = AFF(\mathcal{T}_{m,n})$ consisting of all maximal twosided ideals equipped with the induced Zariski topology.

Because $\mathcal{T}_{m,n}$ is integral over $\mathcal{R}_{m,n}$ the natural map :

$$\delta : Y = AFF(\mathcal{T}_{m,n}) \to V = AFF(\mathcal{R}_{m,n})$$

is onto. It is easy to describe the fibers of this map. Let ϕ be a k-point in V , i.e. ϕ corresponds to an equivalence class of a semi-simple representation :

$$\phi = \phi_1 \oplus ... \oplus \phi_t$$

then $\delta^{-1}(\phi)$ consists of precisely t elements each corresponding to an irreducible component of ϕ.

Theorem 2 [ARTIN SCHELTER]

The k-points of $Y = AFF(\mathcal{T}_{m,n})$ correspond to couples (ϕ, ϕ_j) where ϕ is a representant of an equivalence class of semi-simple n-dimensional representations of $\mathcal{F}_m$ and ϕ_j is an irreducible component of ϕ. The natural map $\delta : Y \to V$ sends (ϕ, ϕ_j) to ϕ.

We will now show that the equivalence classes of irreducible n-dimensional representations form an open subvariety of $V = AFF(\mathcal{R}_{m,n})$.

Let us consider the set of all elements of $\mathcal{T}_{m,n}$ obtained by evaluation of n-central polynomials subject to the restriction of being linear in at least one variable. Call $F_n(\mathcal{T}_{m,n})$ this set, the "Formanek center" because it is an ideal of the center. If S is a multiplicative set of central elements of $\mathcal{T}_{m,n}$, then $(\mathcal{T}_{m,n})_S$ is an Azumaya algebra over $(\mathcal{R}_{m,n})_S$ iff $S \cap F_n(\mathcal{T}_{m,n}) \neq \emptyset$. With V_{irr} we denote the open subscheme $X(F_n(\mathcal{T}_{m,n}))$, on which $F_n(\mathcal{T}_{m,n})$ does not vanish identically, of $V = AFF(R_{m,n})$.

Theorem 3 [PROCESI]

(1) : The k-points of V_{irr} are in one-to-one correspondence with the equivalence classes of irreducible n-dimensional representations of $\mathcal{F}_m$.

(2) : The variety V_{irr} is smooth of dimension $(m-1).n^2+1$.

3. Invariant theory , [PROCESI 3].

In the foregoing section we noticed that there exists a group action of $GL_n(k)$ actually of $PGL_n(k)$, on $\mathcal{R}$. This action is defined as follows : if $P \in GL_n(k)$ and if $X_l = (t_{ij}(l))_{i,j}$ is the l-th generic matrix, let

$$P.X_l.P^{-1} = (\psi_{ij}(l))_{i,j}$$

where the $\psi_{ij}(l)$ are k-linear combinations of the $t_{ij}(l)$. Then, sending $t_{ij}(l)$ to $\psi_{ij}(l)$ induces a k-automorphism on $\mathcal{R}$ and on its field of fractions $\mathcal{K} = k(t_{ij}(l); 1 \leq i,j \leq n; 1 \leq l \leq m)$ which we denote by α_P.

A polynomial $f(t_{ij}(l)) \in \mathcal{R}$ is said to be an invariant of m copies of $n \times n$-matrices iff $\alpha_P(f) = f$ for all $P \in GL_n(k)$. A rational function $g(t_{ij}(l)) \in \mathcal{K}$ is called a rational invariant of m copies of $n \times n$-matrices iff $\alpha_P(g) = g$ for all $P \in GL_n(k)$.

The set of all invariants (resp. rational invariants) is called the ring (resp. field) of invariants of m copies of $n \times n$-matrices. They are denoted, respectively, $\mathcal{R}^{GL_n(k)}$ and $\mathcal{K}^{GL_n(k)}$.

It is not hard to show that the field of invariants is the field of fractions of the ring of invariants of m copies of $n \times n$- matrices. Artin [ARTIN] conjectured that any invariant is a polynomial in the elements :

$$\{Tr(X_{i_1}...X_{i_r}); r \in \mathbb{N}\}$$

For $n = 2$, this fact was proved as far back as 1903 by J.H. Grace and A. Young [GRACE YOUNG].

For arbitrary n, Artins conjecture was proved independently by Gurevich [GUREVICH,Th.16.2], Siberskii [SIBERSKII,Th.1] and Procesi [PROCESI 3,Th.1.3]. The proof of this result relies heavily on the so called "first fundamental theorem" on vector invariants [GUREVICH,Th.16.2] which gives a generating set for the invariants of m vectors and m covectors , i.e. invariants of $GL_n(k)$ acting on the symmetric algebra of

$$(V^{\otimes m}) \otimes (V^*)^{\otimes m}$$

where V is an n-dimensional k-vector space and V^* is its dual. This theorem is quite old but the fist complete proof seems to be that of Gurevich. The solution of Artins conjecture is a translation of the first fundamental theorem, originally due to Siberskii and later rediscovered by Procesi, using the dictionary :

$$V \otimes V^* \simeq M_n(k)$$

$$\text{vector invariant} = \text{trace}$$

for details see [PROCESI 3,pp 310-313].

Theorem 4 [GUREVICH],[SIBERSKII],[PROCESI 3]

The ring of invariants of m copies of $n \times n$-matrices , $\mathcal{R}^{GL_n(k)}$, is equal to $\mathcal{R}_{m,n}$.

The field of invariants of m copies of $n \times n$-matrices , $\mathcal{K}^{GL_n(k)}$, is equal to $\mathcal{K}_{m,n}$.

We will now define an action of $GL_n(k)$ on $M_n(\mathcal{R})$ and $M_n(\mathcal{K})$. Let $P \in GL_n(k)$ and $(a_{ij})_{i,j} \in M_n(\mathcal{R})$, then there is an action by conjugation :

$$(a_{ij})_{i,j} \to P.(a_{ij})_{i,j}.P^{-1}$$

and an action extending $\alpha_P : \mathcal{R} \to \mathcal{R}$:

$$(a_{ij})_{i,j} \to (\alpha_P(a_{ij}))_{i,j}$$

If we regard $M_n(\mathcal{R})$ as $M_n(k) \otimes \mathcal{R}$ then the first action is on the first factor, fixing the second, whereas the second action is vice-versa, thus the two actions commute. Note that the two actions agree on generic matrices.Define :

$$\beta_P : M_n(\mathcal{R}) \to M_n(\mathcal{R})$$

sending a matrix $(a_{ij})_{i,j}$ to $P^{-1}.(\alpha_P(a_{ij}))_{i,j}.P$. This defines an action of $GL_n(k)$ on $M_n(\mathcal{R})$ since the two actions used to define β_P commute. Clearly, this action extends to $M_n(\mathcal{K})$.

The ring of matrix concomitants (resp. of rational matrix concomitants) is the fixed ring of $M_n(\mathcal{R})$ (resp. of $M_n(\mathcal{K})$) under this action. It will be denoted by $M_n(\mathcal{R})^{GL_n(k)}$ (resp. by $M_n(\mathcal{K})^{GL_n(k)}$).

Theorem 5 [PROCESI 3],[FORMANEK]
The ring of matrix concomitants, $M_n(\mathcal{R})^{GL_n(k)}$, is equal to $\mathcal{T}_{m,n}$.
The ring of rational matrix concomitants, $M_n(\mathcal{K})^{GL_n(k)}$, is equal to $\Delta_{m,n}$.

4. Factoriality of $\mathcal{R}_{m,n}$.

In this section we aim to prove that the ring of invariants of m copies of $n \times n$-matrices is always a unique factorization domain. Further, we will show that the natural inclusion :

$$\mathcal{R}_{m,n} \subset \mathcal{R}$$

satisfies no blowing up.

Proposition 6

Let B be a unique factorization domain and G a group of automorphisms of B such that $H^1(G, U(B)) = 1$, where $U(B)$ is the group of units of B. Let A be the fixed ring of B under G, then :

(1) : $A \subset B$ satisfies no blowing up

(2) : A is a unique factorization domain.

Proof : (1) : Let $P = B.p$ be an height one prime ideal of B such that $P \cap A \neq 0$. Then P has a finite orbit under G, say $\{B.p, B.p_1, ..., B.p_k\}$. For, take an element $a \in P \cap A$ and write it as a product of irreducible elements in B , say

$$a = p^{l_0}.p_1^{l_1}...p_m^{l_m}$$

then for every $\sigma \in G$ we have that $\sigma(B.p)$ belongs to $\{B.p, B.p_1, ..., B.p_m\}$. This shows that for every $\sigma \in G$ there exists a unit $f_\sigma \in U(B)$ such that $\sigma(p.p_1.....p_k) = f_\sigma.p.p_1.....p_k$. Now, $\{f_\sigma : \sigma \in G\}$ is clearly a 1-cocycle so by our assumption there exists a unit $\alpha \in U(B)$ such that :

$$f_\sigma = \sigma(\alpha).\alpha^{-1}$$

for every $\sigma \in G$. Replace p by $p' = \alpha^{-1}.p$, then $p'.p_1.....p_k \in A$. Therefore, any nonzero element $a \in P \cap A$ can be written as :

$$a = (p'.p_1.....p_k)^l.q_1^{m_1}.....q_l^{m_l}$$

i.e. $a \in (p'.p_1.....p_k)A$, because clearly $\sigma(q_1^{m_1}.....q_l^{m_l}) = q_1^{m_1}.....q_l^{m_l}$ for all $\sigma \in G$. So, $P \cap A = (p'.....p_k)A$, whence $A \cap B$ satisfies no blowing up.

(2) : By part (1) we know that the natural map between the classgroups :

$$Cl(A) \to Cl(B)$$

is a groupmorphism. Suppose Q is an height one prime ideal of A, then $(B.Q)^{**} = B.p_1^{l_1}.....p_m^{l_m}$ for irreducible elements $p_i \in B$. Clearly, $Q = B.p_i \cap A$ which is a principal ideal by the proof of part (1).

Theorem 7

The natural inclusion $\mathcal{R}_{m,n} \subset \mathcal{R}$ satisfies no blowing up and $\mathcal{R}_{m,n}$ is a unique factorization domain.

Proof :

In the foregoing section we have seen that $PGL_n(k)$ acts as a group of automorphisms on $\mathcal{R}$ such that $\mathcal{R}_{m,n}$ is its fixed ring. Further, $PGL_n(k)$ acts trivially on k whence :

$$H^1(PGL_n(k), U(\mathcal{R})) = H^1(PGL_n(k), U(k)) = Hom(PGL_n(k), U(k))$$

which is trivial because $PGL_n(k)$ is a simple algebraic group.

As a first consequence of this result we obtain that $\mathcal{R}_{m,n}$ is normal. For, the foregoing result entails that $\mathcal{R}_{m,n}$ is a Krull domain and $\mathcal{R}_{m,n}$ being an affine k-algebra it is clearly Noetherian.

Because $\mathcal{R}_{m,n}$ is an affine k-algebra there exists a polynomial ring $\mathcal{P}_{m,n}$ and a gradation-preserving epimorphic map :

$$\omega : \mathcal{P}_{m,n} \to \mathcal{R}_{m,n}$$

where the gradation on $\mathcal{R}_{m,n}$ is given by $deg(t_{ij}(l)) = 1$.

$\mathcal{R}_{m,n}$ is said to be Cohen-Macauley if :

$$Ext^i_{\mathcal{P}_{m,n}}(\mathcal{R}_{m,n}, \mathcal{P}_{m,n}) = 0$$

for all $i \neq 0, Kdim(\mathcal{P}_{m,n}) - Kdim(\mathcal{R}_{m,n})$.

$\mathcal{R}_{m,n}$ is said to be Gorenstein if it is Cohen-Macauley and if

$$Ext^j_{\mathcal{P}_{m,n}}(\mathcal{R}_{m,n}, \mathcal{P}_{m,n}) \simeq \mathcal{R}_{m,n}$$

where $j = Kdim(\mathcal{P}_{m,n}) - Kdim(\mathcal{R}_{m,n})$.

Theorem 8

$\mathcal{R}_{m,n}$ is a Gorenstein domain.

Proof :

Because $\mathcal{R}_{m,n}$ is the fixed ring under the reductive group $GL_n(k)$ of the regular domain $\mathcal{R}$, it follows from the Hochster-Roberts theorem [HOCHSTER ROBERTS] that $\mathcal{R}_{m,n}$ is Cohen-Macauley.

Further, $\mathcal{R}_{m,n}$ being factorial and an epimorphic image of the regular domain $\mathcal{P}_{m,n}$ id follows from Murthy's theorem, see for example [FOSSUM], that $\mathcal{R}_{m,n}$ is Gorenstein.

5. The Artin - Schofield theorem .

M.Auslander and O.Goldman defined in [AUSLANDER GOLDMAN] an order over a normal domain R to be a subring Λ of a central simple algebra Σ over the field of fractions K of R such that Λ is a finitely generated R-module which spans Σ over K. An order Λ in the central simple algebra Σ is said to be maximal if Λ is not properly contained in any order of Σ. From the results stated above it is clear that $\mathcal{T}_{m,n}$ is an order over $\mathcal{R}_{m,n}$ in $\Delta_{m,n}$. M.Artin and A.Schofield (unpublished) proved that $\mathcal{T}_{m,n}$ is actually a maximal order.

One of the many equivalent characterizations of maximal orders is the following :

An R order Λ in Σ is maximal iff :

(1) : Λ_p is a maximal R_p-order for every height one prime ideal p of R.

(2) : Λ is a reflexive R-module , i.e. $\Lambda = \Lambda^{**}$ where $(-)^{**}$ denotes the bidual module $Hom_R(Hom_R(-, R))$.

We will first proof (2). Recall that a finitely generated torsion free R-module is reflexive if and only if $\Gamma(U, \mathcal{O}_M) = M$ for any Zariski open subset of $Spec(R)$ containing all height one prime ideals. As usual, $\mathcal{O}_M$ denotes the structure sheaf of the R-module M.

Theorem 9 (Artin Schofield)

$\mathcal{T}_{m,n}$ is a reflexive $\mathcal{R}_{m,n}$-module.

Proof :

Let U be an open subset of $Spec(\mathcal{R}_{m,n})$ containing all height one prime ideals, then it follows from :

$$\begin{array}{ccc} \mathcal{T}_{m,n} & \rightarrow & M_n(\mathcal{R}) \\ \uparrow & & \uparrow \\ \mathcal{R}_{m,n} & \rightarrow & \mathcal{R} \end{array}$$

that there is an inclusion :

$$\Gamma(U, \mathcal{O}_{\mathcal{T}_{m,n}}) \rightarrow \Gamma(\gamma^{-1}(U), \mathcal{O}_{M_n(\mathcal{R})})$$

where γ is the induced morphism on the spectra. It follows from Theorem 7 that $\gamma^{-1}(U)$ is an open subset of $Spec(\mathcal{R})$ containing all height one prime ideals of $\mathcal{R}$. Now, $M_n(\mathcal{R})$ being R-free it is clearly R-reflexive and therefore :

$$\Gamma(\gamma^{-1}(U), \mathcal{O}_{M_n(\mathcal{R})}) = M_n(\mathcal{R})$$

On the other hand, $\Gamma(U, \mathcal{O}_{\mathcal{T}_{m,n}}) \subset \Delta_{m,n}$ consists of $GL_n(k)$-invariant elements of $M_n(\mathcal{K})$.So,

$$\Gamma(U, \mathcal{O}_{\mathcal{T}_{m,n}}) \subset M_n(\mathcal{R})^{GL_n(k)} = \mathcal{T}_{m,n}$$

The other inclusion being trivial finishes the proof.

Theorem 10 (Artin - Schofield)
$\mathcal{T}_{m,n}$ is a maximal order over $\mathcal{R}_{m,n}$.

Proof : In view of the foregoing result we are left to prove that $(\mathcal{T}_{m,n})_p$ is a maximal order over $(\mathcal{R}_{m,n})_p$ for every height one prime ideal p of $\mathcal{R}_{m,n}$.

Because $\mathcal{R}_{m,n}$ is a normal domain, this is true for every p such that $(\mathcal{T}_{m,n})_p$ is an Azumaya algebra, or equivalently such that $p.i.deg(\mathcal{T}_{m,n}/p.\mathcal{T}_{m,n}) = n$.

Let us assume that p is an height one prime of $\mathcal{R}_{m,n}$ such that $p.i.deg(\mathcal{T}_{m,n}/p.\mathcal{T}_{m,n}) = r < n$. View p as a closed codimension one subvariety of $V = AFF(\mathcal{R}_{m,n})$. The points on p have to correspond to reducible n-dimensional semi-simple representations of $\mathcal{F}_m$.

An open subset of p corresponds to n-dimensional semi-simple representations of $\mathcal{F}_m$ of the form :

$$\phi = \phi_1 \oplus \phi_2$$

where $dim(\phi_1) = r$ and $dim(\phi_2) = n - r$ and both are irreducible representations.

Therefore, the dimension of p is smaller or equal to the sum of the dimensions of the varieties classifying equivalence classes of irreducible r (resp. $n-r$)-dimensional representations of $\mathcal{F}_m$. These dimensions were computed in section 2

$$dim(p) \leq (m-1).r^2 + 1 + (m-1).(n-r)^2 + 1$$

On the other hand, p has to be of codimension one in V which is of dimension $(m-1).n^2 + 1$, i.e. we have to investigate :

$$(m-1).n^2 \leq (m-1).(r^2 + (n-r)^2) + 2$$

or

$$0 \leq (r-n).r + (m-1)^{-1}$$

Because $r < n$, this inequality can only be satisfied if $m = 2$. Then we have :

$$(n-r).r \leq 1$$

leaving only $r = 1$ and $n = 2$ as possible solution. Thus, the only trace ring $\mathcal{T}_{m,n}$ having non-Azumaya central height one primes is $\mathcal{T}_{2,2}$. In the next proposition we will show that $\mathcal{T}_{2,2}$ is also a maximal order, finishing the proof.

An explicit description of the trace ring of two generic 2×2- matrices is given in [FORMANEK HALPIN LI]. They showed :

(1) : $\mathcal{R}_{2,2} = k[Tr(X_1), Tr(X_2), D(X_1), D(X_2), Tr(X_1X_2)]$

(2) : $\mathcal{T}_{2,2}$ is a free $\mathcal{R}_{2,2}$-module with basis $1, X_1, X_2, X_1X_2$.

Further, the following relations are satisfied :

$$D(X_1) = -X_1^2 + Tr(X_1)X_1$$

$$D(X_2) = -X_2^2 + Tr(X_2)X_2$$

$$(X_1X_2 - X_2X_1)X_1 = (-X_1 + Tr(X_1))(X_1X_2 - X_2X_1)$$

$$(X_1X_2 - X_2X_1)X_2 = (-X_2 + Tr(X_2))(X_1X_2 - X_2X_1)$$

$$Tr(X_1X_2) = X_1X_2 + X_2X_1 + Tr(X_1)Tr(X_2) - Tr(X_2)X_1 - Tr(X_1)X_2$$

Theorem 11 [SMALL STAFFORD]
$\mathcal{T}_{2,2}$ is the iterated Ore extension :

$$k[X_1X_2 - X_2X_1, Tr(X_1), Tr(X_2)][X_1, \sigma_1, \delta_1][X_2, \sigma_2, \delta_2]$$

where σ_i and δ_i are obtained from the above relations.

By a result of Chamarie's [CHAMARIE] we get that $\mathcal{T}_{2,2}$ is a maximal order. The only non-Azumaya height one prime ideal of the center is generated by $(X_1X_2 - X_2X_1)^2$ which is a central element. The height one prime ideal of $\mathcal{T}_{2,2}$ lying above this prime is generated by the normalizing element $X_1X_2 - X_2X_1$.

6. Factoriality of $\mathcal{T}_{m,n}$.

A reflexive order Λ over a normal domain R is said to be a reflexive Azumaya algebra , cfr. [YUAN], if the natural map :

$$\delta : (\Lambda \otimes_R \Lambda^{opp})^{**} \to End_R(\Lambda)$$

is an isomorphism. Obviously, a reflexive Azumaya algebra is a maximal order if it is finitely generated. Further, it is fairly easy to see that for every divisorial Λ-ideal I (i.e. a twosided fractional Λ-ideal which is reflexive as an R-module) we have $I = (\Lambda(I \cap R))^{**}$.

Chatters and Jordan [CHATTERS JORDAN] call a left and right Noetherian prime ring Λ a unique factorization ring if every nonzero prime ideal of Λ contains a nonzero principal prime ideal.

Theorem 12 The trace ring of m generic $n \times n$-matrices is a unique factorization ring.

Proof :

Because $\mathcal{T}_{m,n}$ is affine and a finite module over its affine center $\mathcal{R}_{m,n}$ we only need to show that every height one prime ideal of $\mathcal{T}_{m,n}$ is principal.

The proof of the Artin-Schofield theorem shows that the localization of $\mathcal{T}_{m,n}$ at every central height one prime p is an Azumaya algebra (except if $m = n = 2$). This shows that δ_p is an isomorphism for all p. $\mathcal{T}_{m,n}$ being a reflexive $\mathcal{R}_{m,n}$-module, this yields that $\mathcal{T}_{m,n}$ is a reflexive Azumaya algebra. So, for any height one prime ideal P we have :

$$P = (\mathcal{T}_{m,n}(P \cap \mathcal{R}_{m,n})^{**} = \mathcal{T}_{m,n}.r$$

for some irreducible element $r \in \mathcal{R}_{m,n}$ because $\mathcal{R}_{m,n}$ is a unique factorization domain.

If $m = n = 2$, then the only height one prime which is not centrally generated is $\mathcal{T}_{2,2}(X_1X_2 - X_2X_1)$ which is principal,done.

As an immediate consequence of this result we obtain a new proof for a result due to Montgomery.

Theorem 13 [MONTGOMERY]
Every automorphism of $\mathcal{G}_{m,n}$ which leaves the center fixed is the identity.

Proof :
By Skolem-Noether such an automorphism is given by conjugation with a normalizing element, say h, of $\mathcal{G}_{m,n}$. $\mathcal{G}_{m,n} \subset \mathcal{T}_{m,n}$ being a central extension , h is also a normalizing element of $\mathcal{T}_{m,n}$, i.e. $\mathcal{T}_{m,n}h$ is a divisorial ideal.

If m or n is not equal to 2, this ideal must be centrally generated, i.e. $h = \gamma.c$ for some $\gamma \in U(\mathcal{T}_{m,n}) = U(k)$ and $c \in \mathcal{K}_{m,n}$,done.

If $m = n = 2$, the only noncentral normalizing element of $\mathcal{T}_{m,n}$ is $X_1X_2 - X_2X_1$. This element does not normalize $\mathcal{G}_{m,n}$,done.

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